

Time-Periodic Solutions of the Einstein's Field Equations

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In this paper, we develop a new method to find the exact solutions of the Einstein's field equations by using which we construct time-periodic solutions. The singularities of the time-periodic solutions are investigated and some new physical phenomena, such as the time-periodic event horizon, are found. The applications of these solutions in modern cosmology and general relativity are expected.

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1. Introduction. The Einstein's field equations are the fundamental equations in general relativity and cosmology. The general version of the gravitational field equations or the Einstein's field equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (1)$$

where $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) is the unknown Lorentzian metric, $R_{\mu\nu}$ is the Ricci curvature tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, Λ is the cosmological constant, G stands for the Newton's gravitational constant, c is the velocity of the light and $T_{\mu\nu}$ is the energy-momentum tensor. In a vacuum, i.e., in regions of space-time in which $T_{\mu\nu} = 0$, the Einstein's field equations (1) reduce to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0, \quad (2)$$

or equivalently,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \quad (3)$$

In particular, if the cosmological constant Λ vanishes, i.e., $\Lambda = 0$, then the equation (2) becomes

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \quad (4)$$

or equivalently,

$$R_{\mu\nu} = 0. \quad (5)$$

Each of the equations (2)-(5) can be called the vacuum Einstein's field equations.

The mathematical study on the Einstein's field equations includes, roughly speaking, the following two aspects: (i) establishing the well-posedness theory of solutions; (ii) finding exact solutions with physical background. Up to now, very few results on the well-posedness for the Einstein's field equations have been established. In their classical monograph [4], Christodoulou and Klainerman proved the global nonlinear stability of the Minkowski space for the vacuum Einstein's field equations, i.e. they showed the nonlinear stability of the trivial solution of the vacuum Einstein's field equations. Recently, by using wave coordinates, Lindblad and Rodnianski gave a simpler proof (see

[12]). In her Ph.D. thesis [18], Zipser generalizes the result of Christodoulou and Klainerman [4] to the Einstein-Maxwell equations. As for finding exact solutions, many works have been done and many interesting results have been obtained (see, e.g., [3], [16], [2]). In what follows, we will briefly recall some basic facts about the exact solutions of the Einstein's field equations.

The exact solutions are very helpful to understand the theory of general relativity and the universe. The typical examples are the Schwarzschild solution and the Kerr solution (see [3]). These solutions provide two important physical space-times: the Schwarzschild solution describes a stationary, spherically symmetric and asymptotically flat space-time, while the Kerr solution provides a stationary, axisymmetric and asymptotically flat space-time.

The study on exact solutions of the Einstein's field equations has a long history. In December 1915, Karl Schwarzschild discovered the first non-trivial solution to the vacuum Einstein's field equations which is a static solution with zero angular momentum (see [15]). The unique two-parameter family of solutions which describes the space-time around black holes is the Kerr family discovered by Roy Partrick Kerr in July 1963 (see [10]). These solutions are very important in studying black holes in the Nature which is just the study of these solutions (see [3]). Various generalizations of the Kerr solution have been done (see, e.g., [16] and [2]). Gowdy [6]-[7] constructed a new kind of solutions of the vacuum Einstein's equations, these solutions provide a new type of cosmological model. This model describes a closed inhomogeneous universe, space sections of these universes have either the three-sphere topology S^3 or the wormhole (hypertorus) topology $S^1 \otimes S^2$. Recently, Ori [13] presented a class of curved-spacetime vacuum solutions which develop closed timelike curves at some particular moment, and used these vacuum solutions to construct a time-machine model. The Ori model is regular, asymptotically flat, and topologically trivial.

From the above discussions, we see that the exact solutions play a crucial role in general relativity and cosmology, so it is always interesting to find new exact solutions for the Einstein's equations. Although many interesting and important solutions have been obtained, there are still many fundamental but open problems. One inter-

esting open problem is *if there exists a “time-periodic” solution to the Einstein’s field equations*. One of the main results in this paper is a solution of this problem.

In this paper, we focus on finding the exact solutions of the vacuum Einstein’s field equations (3) and (5). We will present a new method to find exact solutions. Using this method we can construct interesting and important exact solutions, for example, the time-periodic solution of the vacuum Einstein’s field equations. We analyze the singularities of time-periodic solutions and investigate some new physical phenomena enjoyed by these new space-times. We find that the new time-periodic solutions have time-periodic event horizon, which is a new phenomenon in the space-time geometry. The applications of these solutions and their new properties in modern cosmology and general relativity may be expected.

More precisely, our time-periodic solution to the vacuum Einstein’s field equations in the spherical coordinates (t, r, θ, φ) can be written in the following form

$$ds^2 = (dt, dr, d\theta, d\varphi)(\eta_{\mu\nu})(dt, dr, d\theta, d\varphi)^T, \quad (6)$$

where

$$(\eta_{\mu\nu}) = \begin{pmatrix} G & -G + \frac{Mr}{r-m} & QK & 0 \\ -G + \frac{Mr}{r-m} & G - \frac{2Mr}{r-m} & -QK & 0 \\ QK & -QK & -K^2 & 0 \\ 0 & 0 & 0 & -K^2 \sin^2 \theta \end{pmatrix} \quad (7)$$

in which

$$\begin{cases} G = 1 + 2\varepsilon\Omega^+ \sin \theta \cos(t-r), \\ K = r + m \ln|r-m| + \varepsilon \sin(t-r), \\ M = \Omega^+ \sin \theta, \\ Q = -\frac{1}{2}(1 + 2 \sin \theta)\Omega^-. \end{cases} \quad (8)$$

In the above, $\varepsilon \in (-\frac{1}{8}, \frac{1}{8})$ and $m \in \mathbb{R}$ are two parameters, and Ω^\pm are defined by

$$\Omega^\pm = |\tan \theta/2|^{\frac{1}{2}} \pm |\tan \theta/2|^{-\frac{1}{2}}. \quad (9)$$

In Section 5, we analyze the singularity behaviors and find some new physical phenomena.

According to the authors’ knowledge, (6) gives the first time-periodic solution to the Einstein’s field equations. Here we would like to point out that, by using our method, we can re-derive almost all known exact solutions to the vacuum Einstein’s field equations, for examples, Gödel’s solution [5], Khan-Penrose’s solution [11], Gowdy’s solution [6]-[7], etc. Our method can also be used to find exact solutions of the Einstein’s field equations in higher dimensions which will be of interests in string theory.

2. Lorentzian metrics. The Einstein’s field equations are a second order global hyperbolic system of highly nonlinear partial differential equations with respect to the Lorentzian metric $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$). To solve

the Einstein’s field equations, one key point is to choose a suitable coordinate system. A good coordinate system can simplify the equations and make them easier to solve. In the study on the Einstein’s field equations, there are three famous coordinate systems: harmonic coordinates, wave coordinates and the Gaussian coordinates. In this paper, we consider the metric of the following form

$$(g_{\mu\nu}) = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & 0 & 0 \\ g_{20} & 0 & g_{22} & 0 \\ g_{30} & 0 & 0 & g_{33} \end{pmatrix}, \quad (10)$$

where $g_{\mu\nu}$ are smooth functions of the coordinates (x^0, x^1, x^2, x^3) and satisfy $g_{0i} = g_{i0}$. In the coordinates (x^0, x^1, x^2, x^3) , the line element reads $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. For simplicity of notations, we denote the coordinates (x^0, x^1, x^2, x^3) by (t, x, y, z) and rewrite (10) as

$$(g_{\mu\nu}) = \begin{pmatrix} u & v & p & q \\ v & w & 0 & 0 \\ p & 0 & \rho & 0 \\ q & 0 & 0 & \sigma \end{pmatrix}, \quad (11)$$

where u, v, p, q, w, ρ and σ are smooth functions of the coordinates (t, x, y, z) . It is easy to verify that the determinant of $(g_{\mu\nu})$ is given by

$$g \triangleq \det(g_{\mu\nu}) = uw\rho\sigma - v^2\rho\sigma - p^2w\sigma - q^2w\rho \quad (12)$$

Throughout this paper, we assume that

$$g < 0 \quad (H)$$

On the other hand, it is easy to see that at least two of the functions w, ρ, σ have the same sign. Without loss of generality, we may suppose that ρ and σ keep the same sign, for example,

$$\rho < 0 \quad \text{and} \quad \sigma < 0. \quad (13)$$

We can easily prove the following theorem.

Theorem 1 *Under the assumptions (H) and (13), the metric $(g_{\mu\nu})$ is Lorentzian.*

We now consider the solutions of the Einstein’s field equations. By Bianchi identities, we believe that the general form of the solutions of the Einstein’s field equations takes one of the following forms

$$(\eta_{\mu\nu}) \triangleq \begin{pmatrix} u & v & p & q \\ v & 0 & 0 & 0 \\ p & 0 & \rho & 0 \\ q & 0 & 0 & \sigma \end{pmatrix}, \quad (\text{Type I})$$

$$(\eta_{\mu\nu}) \triangleq \begin{pmatrix} 0 & v & p & q \\ v & w & 0 & 0 \\ p & 0 & \rho & 0 \\ q & 0 & 0 & \sigma \end{pmatrix} \quad (\text{Type II})$$

or

$$(\eta_{\mu\nu}) \triangleq \begin{pmatrix} u & v & p & 0 \\ v & w & 0 & 0 \\ p & 0 & \rho & 0 \\ 0 & 0 & 0 & \sigma \end{pmatrix}. \quad (\text{Type III})$$

For type I, the assumption (H) is equivalent to $v \neq 0$. Therefore, by Theorem 1 we have

Conclusion 1 If

$$\rho < 0, \quad \sigma < 0 \quad \text{and} \quad v \neq 0, \quad (14)$$

then the metric $(\eta_{\mu\nu})$ is Lorentzian.

For type II, we have

Conclusion 2 If w, ρ, σ are all negative functions and $v^2 + p^2 + q^2 \neq 0$, then the hypotheses (H) and (13) are satisfied, and the metric $(\eta_{\mu\nu})$ is Lorentzian.

Similarly, for type III we have

Conclusion 3 If u is positive and w, ρ, σ are negative, then the hypotheses (H) and (13) are satisfied, and the metric $(\eta_{\mu\nu})$ is Lorentzian.

We are interested in finding exact solutions of the Einstein's field equations of the above types I-III.

3. *New method to find exact solutions.* In order to illustrate our method, as an example, we use the Lorentzian metric type I to construct some interesting exact solutions, in particular time-periodic solution for the vacuum Einstein's field equations (2).

For type I metrics, our method can be described by the following algorithm:

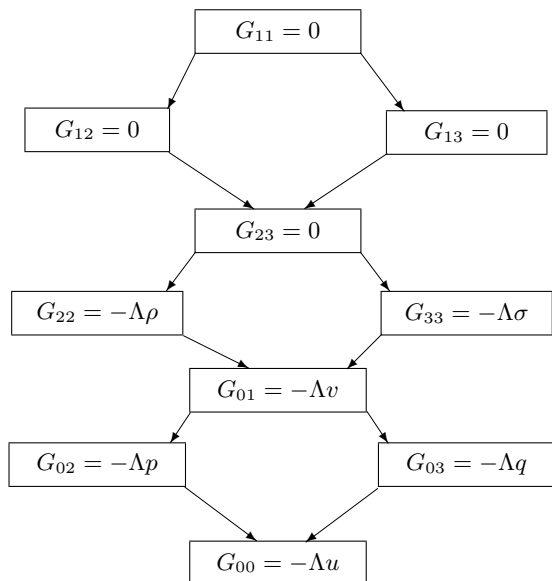


FIG. 1: The algorithm to construct the exact solutions

In fact, the equations (2) can be rewritten as

$$G_{\mu\nu} \triangleq R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = -\Lambda\eta_{\mu\nu}, \quad (15)$$

where $G_{\mu\nu}$ is the Einstein tensor. Noting the special form of type I, we have

$$G_{11} = -\frac{1}{2} \left\{ \frac{v_x}{v} \left(\frac{\rho_x}{\rho} + \frac{\sigma_x}{\sigma} \right) + \frac{1}{2} \left[\left(\frac{\rho_x}{\rho} \right)^2 + \left(\frac{\sigma_x}{\sigma} \right)^2 \right] - \left(\frac{\rho_{xx}}{\rho} + \frac{\sigma_{xx}}{\sigma} \right) \right\}. \quad (16)$$

One of the equations (15) reads

$$G_{11} = -\Lambda\eta_{11}. \quad (17)$$

Solving the ODE (17) gives

$$v = v_0 \exp \left\{ \int \left[\frac{\rho_{xx}}{\rho} + \frac{\sigma_{xx}}{\sigma} - \frac{1}{2} \left(\frac{\rho_x}{\rho} \right)^2 - \frac{1}{2} \left(\frac{\sigma_x}{\sigma} \right)^2 \right] \frac{\rho\sigma}{(\rho\sigma)_x} dx \right\}, \quad (18)$$

where $v_0 = v_0(t, y, z)$ is an integral function depending on t, y and z , provided that $(\rho\sigma)_x \neq 0$.

In particular, by taking the ansatz

$$\rho = \tilde{\rho}(t, y, z) \exp\{2f(t, x)\}, \quad \sigma = \tilde{\sigma}(t, y, z) \exp\{2f(t, x)\}, \quad (19)$$

(18) becomes

$$v = v_0 f_x e^f. \quad (20)$$

According to the algorithm shown in Fig 1, in a similar way we can solve other equations in (2), and then we can construct many exact solutions of (2).

Remark 1 Our method can be used to solve the Einstein's field equations with physically relevant energy-momentum tensors, e.g., the tensor for perfect fluid: $T_{\gamma\delta} = (\mu + p)u_\gamma u_\delta + pg_{\gamma\delta}$, where $\mu > 0$ is the density, p is the pressure, u stands for the space-time velocity of the fluid with $u_\gamma u^\gamma = -1$.

Remark 2 The method presented in this paper can also be used to solve the Einstein's field equations in higher space-time dimensions.

4. *Time-periodic solutions.* By the method presented in the last section, we can construct many new exact solutions to the vacuum Einstein's field equations

$$G_{\mu\nu} = 0. \quad (21)$$

For example, in the coordinates $(\tau, \tilde{r}, \tilde{\theta}, \tilde{\varphi})$, taking the ansatz in (19) as follows

$$\tilde{\rho} = -1, \quad \tilde{\sigma} = -\sin^2 \tilde{\theta}, \quad f = \ln[\tilde{r} + \tau + \varepsilon \sin \tau],$$

one can obtain an interesting solution of the form

$$\tilde{\eta}_{\mu\nu} = \begin{pmatrix} \tilde{\eta}_{00} & \tilde{\eta}_{01} & \tilde{\eta}_{02} & 0 \\ \tilde{\eta}_{01} & 0 & 0 & 0 \\ \tilde{\eta}_{02} & 0 & \tilde{\eta}_{22} & 0 \\ 0 & 0 & 0 & \tilde{\eta}_{33} \end{pmatrix}, \quad (22)$$

where

$$\begin{cases} \tilde{\eta}_{00} = 1 + 2\varepsilon\tilde{\Omega}^+ \sin\tilde{\theta} \cos\tau + \frac{2\tilde{\Omega}^+ \sin\tilde{\theta}}{m}\tilde{\Omega}, \\ \tilde{\eta}_{01} = \frac{\tilde{\Omega}^+ \sin\tilde{\theta}}{m}\tilde{\Omega}, \\ \tilde{\eta}_{02} = \frac{1}{2}(1 + 2\sin\tilde{\theta})\tilde{\Omega}^- [\tilde{r} + \tau + \varepsilon \sin\tau], \\ \tilde{\eta}_{22} = -[\tilde{r} + \tau + \varepsilon \sin\tau]^2, \\ \tilde{\eta}_{33} = -[\tilde{r} + \tau + \varepsilon \sin\tau]^2 \sin^2\tilde{\theta}, \end{cases} \quad (23)$$

in which ε is a parameter, $m \neq 0$ is a constant, and

$$\tilde{\Omega} = \exp\left(\frac{\tilde{r} + \tau}{m}\right) + m, \quad \tilde{\Omega}^\pm = |\tan\tilde{\theta}/2|^\pm \pm |\tan\tilde{\theta}/2|^{-\frac{1}{2}}. \quad (24)$$

The solution (22) belongs to the class of type I, also belongs to the class of type III.

Making the transformation

$$\begin{cases} t = \tau + \tilde{r}, \\ r = m + \exp\left(\frac{\tau + \tilde{r}}{m}\right), \\ \theta = \tilde{\theta}, \\ \varphi = \tilde{\varphi}. \end{cases} \quad (25)$$

Then the solution to (21) becomes, in the coordinates (t, r, θ, φ) ,

$$ds^2 = (dt, dr, d\theta, d\varphi)(\eta_{\mu\nu})(dt, dr, d\theta, d\varphi)^T, \quad (26)$$

where

$$(\eta_{\mu\nu}) = \begin{pmatrix} G & -G + \frac{Mr}{r-m} & QK & 0 \\ -G + \frac{Mr}{r-m} & G - \frac{2Mr}{r-m} & -QK & 0 \\ QK & -QK & -K^2 & 0 \\ 0 & 0 & 0 & -K^2 \sin^2\theta \end{pmatrix}, \quad (27)$$

in which

$$\begin{cases} G = 1 + 2\varepsilon\Omega^+ \sin\theta \cos(t-r), \\ K = r + m \ln|r-m| + \varepsilon \sin(t-r), \\ M = \Omega^+ \sin\theta, \\ Q = -\frac{1}{2}(1 + 2\sin\theta)\Omega^-. \end{cases} \quad (28)$$

In (28), Ω^\pm are defined by

$$\Omega^\pm = |\tan\theta/2|^\pm \pm |\tan\theta/2|^{-\frac{1}{2}}. \quad (29)$$

An important property of the space-time described by (26) is given by the following theorem.

Theorem 2 *When ε takes its value in the interval $(-\frac{1}{8}, \frac{1}{8})$, i.e., $\varepsilon \in (-\frac{1}{8}, \frac{1}{8})$, the solution (26) to the vacuum Einstein's field equations is time-periodic.*

Proof. Noting $\varepsilon \in (-\frac{1}{8}, \frac{1}{8})$, we have

$$\begin{aligned} \eta_{00} &= G = 1 + 2\varepsilon\Omega^+ \sin\theta \cos(t-r) \\ &= 1 + 4\varepsilon \left(\sqrt{\left| \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \right|} + \sqrt{\left| \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \right|} \right) \times \\ &\quad \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos(t-r) \\ &\geq 1 - 4|\varepsilon| \left(\sqrt{\left| \sin^3\frac{\theta}{2} \right| \left| \cos\frac{\theta}{2} \right|} + \sqrt{\left| \cos^3\frac{\theta}{2} \right| \left| \sin\frac{\theta}{2} \right|} \right) \times \\ &\quad |\cos(t-r)| \\ &\geq 1 - 8|\varepsilon| > 1 - 8 \times \frac{1}{8} = 0. \end{aligned} \quad (30)$$

On the other hand, by calculations we have

$$\begin{vmatrix} G & -G + \frac{Mr}{r-m} \\ -G + \frac{Mr}{r-m} & G - \frac{2Mr}{r-m} \end{vmatrix} = -\frac{(Mr)^2}{(r-m)^2} < 0 \quad (31)$$

for $r \neq 0, m$ and $\theta \neq 0, \pi$,

$$\begin{vmatrix} G & -G + \frac{Mr}{r-m} & QK \\ -G + \frac{Mr}{r-m} & G - \frac{2Mr}{r-m} & -QK \\ QK & -QK & -K^2 \end{vmatrix} = K^2 \frac{(Mr)^2}{(r-m)^2} > 0 \quad (32)$$

$$\begin{vmatrix} G & -G + \frac{Mr}{r-m} & QK & 0 \\ -G + \frac{Mr}{r-m} & G - \frac{2Mr}{r-m} & -QK & 0 \\ QK & -QK & -K^2 & 0 \\ 0 & 0 & 0 & -K^2 \sin^2\theta \end{vmatrix} = -K^4 \sin^2\theta \frac{(Mr)^2}{(r-m)^2} < 0 \quad (33)$$

for $r \neq 0, m$, $\theta \neq 0, \pi$ and $K \neq 0$. In next section, we will show that $\theta = 0, \pi$, $r = 0, m$ and $K = 0$ are not essential singularities, in other words, they are not physical singularities.

The above discussion implies that the variable t is a time coordinate. Therefore, it follows from (27) and (28) that the Lorentzian metric (26) is indeed a time-periodic solution of the vacuum Einstein's field equations. This proves Theorem 2. ■

Remark 3 *The time-periodic solution (26) was first obtained by us in early 2007. The authors have presented this solution in several conferences for the past year. As mentioned above, according to the authors' knowledge,*

this is the first time-periodic solution to the Einstein's field equations.

Remark 4 In July 2006, the first author discussed with Dr Y.-Q. Gu about some ideas presented in this paper. We noted that for the case of type I metric, the author of [8] constructed some exact solutions, none of which is time-periodic, because the variable t in these solutions can not be taken as the time coordinate.

Direct computations give us the following property of the time-periodic solutions.

Property 1 In the geometry of the space-time (26), it holds that

$$\frac{\partial K}{\partial t} = \frac{G-1}{2M}, \quad \frac{\partial^2 K}{\partial t^2} = \frac{1}{2M} \frac{\partial G}{\partial t}, \quad (34)$$

$$\frac{\partial \Omega^+}{\partial \theta} = \frac{\Omega^-}{2 \sin \theta}, \quad \frac{\partial \Omega^-}{\partial \theta} = \frac{\Omega^+}{2 \sin \theta}, \quad \frac{\partial M}{\partial \theta} = Q, \quad (35)$$

$$\frac{\partial G}{\partial r} = -\frac{\partial G}{\partial t}, \quad \frac{\partial^2 G}{\partial r \partial t} = -\frac{\partial^2 G}{\partial t^2}, \quad \frac{\partial^2 G}{\partial \theta \partial r} = -\frac{\partial^2 G}{\partial \theta \partial t}, \quad (36)$$

$$\frac{\partial K}{\partial t} + \frac{\partial K}{\partial r} = \frac{r}{r-m}, \quad \frac{\partial^2 K}{\partial t \partial r} = -\frac{\partial^2 K}{\partial t^2}, \quad \frac{\partial^2 K}{\partial \theta \partial r} = -\frac{\partial^2 K}{\partial t \partial \theta}, \quad (37)$$

$$\frac{\partial Q}{\partial \theta} = Q \cot \theta - \frac{3\Omega^+}{4 \sin \theta}, \quad Q \frac{\partial K}{\partial t} = \frac{1}{2} \frac{\partial G}{\partial \theta}, \quad \frac{\partial^2 G}{\partial \theta^2} = 2 \frac{\partial K}{\partial t} \frac{\partial Q}{\partial \theta}. \quad (38)$$

$$\frac{2Q \cos \theta}{\Omega^+} - \frac{3}{4} - \frac{Q^2}{(\Omega^+)^2} - \frac{1}{(\Omega^+)^2} = -\sin^2 \theta. \quad (39)$$

$$2Q \cot \theta - \frac{3}{4} \frac{\Omega^+}{\sin \theta} - \frac{1+Q^2}{M} = -M. \quad (40)$$

The relations given above will play an important role in the future study on the geometry of the time-periodic space-time (26).

There are several important questions which deserve further study: (a) what is the topological structure of the time-periodic space-time (26)? (b) does the space-time (26) have a compact Cauchy surface? (c) an important point is to consider the structure of the maximal globally hyperbolic part of the space-time (26), the question is whether this also exhibits time periodicity. Problems (b) and (c) were suggested by Andersson [1].

5. Singularity behaviors and physical properties. This section is devoted to the analysis of singularities and the physical properties of the time-periodic solution (26).

For the metric (27), by a direct calculation, we have

$$g \triangleq \det(\eta_{\mu\nu}) = -M^2 K^4 \frac{r^2}{(r-m)^2} \sin^2 \theta. \quad (41)$$

Combing (28) and (41) gives

$$g = -(\Omega^+)^2 \sin^4 \theta [r + m \ln |r - m| + \varepsilon \sin(t - r)]^4 \frac{r^2}{(r - m)^2}. \quad (42)$$

Noting (24), we obtain from (42) that

$$\begin{aligned} \mathcal{S}_{r=0,m} &\triangleq \{(t, r, \theta, \varphi) | r = 0, m\}, \\ \mathcal{S}_{K=0} &\triangleq \{(t, r, \theta, \varphi) | r + m \ln |r - m| + \varepsilon \sin(t - r) = 0\}, \\ \mathcal{S}_{\theta=0,\pi} &\triangleq \{(t, r, \theta, \varphi) | \theta = 0, \pi\} \end{aligned} \quad (43)$$

are singularities for the solution metric (26).

By a direct calculation, we have

$$R_{\alpha\beta\gamma\delta} = 0 \quad \text{and} \quad R^{\alpha\beta\gamma\delta} = 0 \quad (\alpha, \beta, \gamma, \delta = 0, 1, 2, 3). \quad (44)$$

This gives

$$R_{\mu\nu} = 0 \quad (\mu, \nu = 0, 1, 2, 3) \quad (45)$$

and

$$\|\mathbf{R}\| \triangleq R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = 0. \quad (46)$$

(45) implies that the Lorentzian metric (26) is indeed a solution of the vacuum Einstein's field equation (21), and (46) implies that this solution does not have any essential singularity.

According to the definition of event horizon (see e.g., Wald [17]), it is easy to show that $\mathcal{S}_{\theta=0,\pi}$ and $\mathcal{S}_{K=0}$ are the event horizons of the space-time (26). If (t, r, θ, φ) are the polar coordinates, then the $\mathcal{S}_{r=0}$ can be regarded as a degenerate event horizon. However, $\mathcal{S}_{r=m}$ is a new kind of singularity which is neither event horizon nor black hole. This differs from that in Schwarzschild space-time, in which $r = 0$ corresponds to the black hole and $r = m$ corresponds to the event horizon. In other words, the solution (26) describes an essentially regular space-time, it does not contain any essential singularity like black hole. It is an interesting topic to see how to cancel these kinds of singularities by making some coordinate transformation. Therefore, we have

Property 2 The Lorentzian metric (26) describes a regular space-time, this space-time is Riemannian flat in the sense of (44), it does not contain any essential singularity. However it contains some non-essential singularities which correspond to event horizons and some other new physical phenomena.

Property 3 The non-essential singularities of the space-time (26) consist of three parts $\mathcal{S}_{r=0,m}$, $\mathcal{S}_{K=0}$ and $\mathcal{S}_{\theta=0,\pi}$. $\mathcal{S}_{\theta=0,\pi}$ are two steady event horizons; and $\mathcal{S}_{K=0}$ is the "time-periodic" event horizon; $\mathcal{S}_{r=0,m}$ are two new kinds of singularities which are neither event horizons nor black holes. In particular, if (t, r, θ, φ) are the polar coordinates, then $\mathcal{S}_{r=0}$ can be regarded as a degenerate event horizon.

According to the authors' knowledge, the degenerate event horizon and the time-periodic event horizon are two new phenomena in the space-time geometry.

We next consider the time-periodic event horizon in detail. Without loss of generality, we may assume that ε and m are positive constants. We have two cases: $0 \leq r \leq m$ and $r > m$.

Case I: $0 \leq r \leq m$.

Let

$$f(r; t) = r + m \ln |m - r| + \varepsilon \sin(t - r). \quad (47)$$

For any fixed $t \in \mathbb{R}$, it holds that

$$f(r; t) \longrightarrow -\infty \quad \text{as } r \rightarrow m \quad (48)$$

and

$$f(r; t) \longrightarrow \infty \quad \text{as } r \rightarrow \infty. \quad (49)$$

At $r = 0$, we consider

$$m \ln m + \varepsilon \sin t = 0, \quad (50)$$

i.e.,

$$\sin t = -\frac{m \ln m}{\varepsilon}. \quad (51)$$

It is obvious that (51) has a solution if and only if

$$\left| \frac{m \ln m}{\varepsilon} \right| \leq 1. \quad (52)$$

In what follows, we always assume condition (52). Therefore, it follows from (51) that

$$f(0; t_k) = 0, \quad (53)$$

where

$$t_k = 2k\pi + \arcsin \left\{ -\frac{m \ln m}{\varepsilon} \right\}, \quad (54)$$

in which $k \in \mathbb{Z}$.

We now divide the discussion into two cases.

Case I-1: $0 < m \leq 1$.

In this case, we have

$$-\frac{m \ln m}{\varepsilon} \geq 0, \quad (55)$$

and then

$$\arcsin \left\{ -\frac{m \ln m}{\varepsilon} \right\} \geq 0. \quad (56)$$

Therefore,

$$t_k = 2k\pi + \arcsin \left\{ -\frac{m \ln m}{\varepsilon} \right\} \quad (k = 0, 1, 2, \dots). \quad (57)$$

Case I-2: $1 < m$.

In this case,

$$-\frac{m \ln m}{\varepsilon} < 0, \quad (58)$$

and

$$\arcsin \left\{ -\frac{m \ln m}{\varepsilon} \right\} < 0. \quad (59)$$

Thus, we shall take

$$t_k = 2(k+1)\pi + \arcsin \left\{ -\frac{m \ln m}{\varepsilon} \right\} \quad (k = 0, 1, 2, \dots). \quad (60)$$

In both case I-1 and case I-2, by fixing $k \in \{0, 1, 2, \dots\}$ and noting (48), we see that there exists a maximum $r_- \in [0, m)$ such that, for any given $r \in [0, r_-]$, the equation for t

$$f(r; t) = 0 \quad (61)$$

has solutions. When $r = r_-$, we denote the solution by t_k^- . It holds that

$$f(r_-; t_k^-) = 0. \quad (62)$$

Summarizing the above discussion, we observe that, for case I, the time-periodic event horizons are given in Fig. 2.

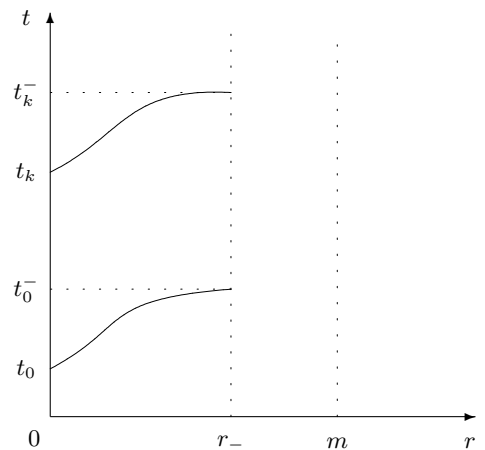


FIG. 2: Time-periodic event horizons for case I

Case II: $r > m$.

Similar to the discussion of case I, in this case the time-periodic event horizons are given in Fig. 3.

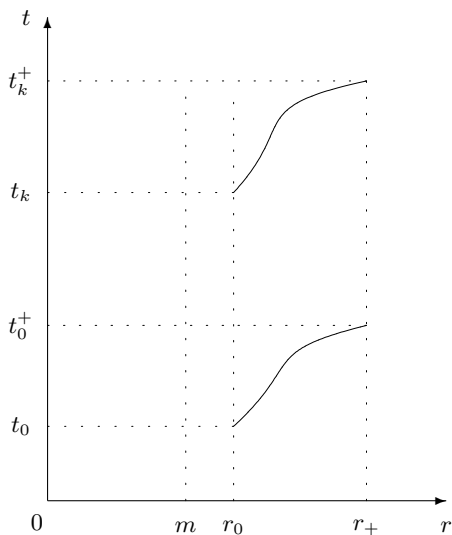


FIG. 3: Time-periodic event horizons for case II

In Fig. 3, r_0 and r_+ are defined in the following way: noting (48) and (49), we see that there exists a minimum $r_0 \in (m, \infty)$ and a maximum $r_+ \in (m, \infty)$ such that, for any given $r \in [r_0, r_+]$, the equation for t

$$f(r; t) = 0 \quad (63)$$

has solutions. In particular, when $r = r_0$ (resp. $r = r_+$), we denote the solution by t_k (resp. by t_k^+). That is to say, it holds that

$$f(r_0; t_k) = 0 \quad \text{and} \quad f(r_+; t_k^+) = 0. \quad (64)$$

Therefore, we have proved the following property.

Property 4 *The non-essential singularities of the space-time (26) consists of three parts $r = 0$, $r = m$ and $r + m \ln |r - m| + \varepsilon \sin(t - r) = 0$. $r = 0$ is a degenerate event horizon, $r = m$ is a steady event horizon, and $r + m \ln |r - m| + \varepsilon \sin(t - r) = 0$ are the “time-periodic” event horizons. Time-periodic event horizons form and disappear in finite times, they propagate time-periodically.*

On the other hand, by some elementary matrix transformations, the metric $(\eta_{\mu\nu})$ can be reduced to

$$(\hat{\eta}_{\mu\nu}) = \text{diag} \left\{ G, -\frac{M^2 r^2}{G(r-m)^2}, -K^2, -K^2 \sin^2 \theta \right\}. \quad (65)$$

Noting that (28) gives

$$(\hat{\eta}_{\mu\nu}) \sim \text{diag} \left\{ 1 + 2\varepsilon\Omega^+ \sin \theta \cos(t - r), -\frac{(\Omega^+)^2 \sin^2 \theta}{1 + 2\varepsilon\Omega^+ \sin \theta \cos(t - r)}, -r^2, -r^2 \sin^2 \theta \right\}. \quad (66)$$

In (66), we have made use of the fact that, when r is large enough, it holds that $K \sim r$ because of the second

equation in (28). (66) implies that the space-time (26) is not homogenous and not asymptotically flat, more precisely not asymptotically Minkowski, because the first two components in (66) depend strongly on the angle θ . Therefore, we have

Property 5 *The space-time (26) is not homogenous and not asymptotically flat.*

Remark 5 *Property 5 perhaps has some new applications in cosmology due to the recent WMAP data, since the recent WMAP data show that our Universe exists anisotropy (see [9]). This inhomogenous property of the new space-time (26) may provide a way to give an explanation of this phenomena.*

Summarizing the above discussion gives the following theorem.

Theorem 3 *The vacuum Einstein’s field equations have a time-periodic solution (26), this solution describes a regular space-time, which has vanishing Riemann curvature tensor but is not homogenous and not asymptotically flat. This space-time does not contain any essential singularity, but contains some non-essential singularities which correspond to steady event horizons, time-periodic event horizon and some other new physical phenomena.*

6. *Summary and discussion.* In this paper we describe a new method to find exact solutions to the Einstein’s field equations (1). Using our method, we can construct some important exact solutions including the time-periodic solutions of the vacuum Einstein’s field equations. We also analyze the singularities of the time-periodic solutions and investigate some new physical phenomena enjoyed by these new space-times.

We remark that, by using our method, we can obtain almost all known solutions to the Einstein’s field equations. Our method can also be used to find exact solutions of the higher dimensional Einstein’s field equations, which play an important role in string theory. The structures of these new space-times, the behaviors of their singularities and some new nonlinear phenomena appeared in the time-periodic solutions are very interesting and important. We expect some applications of these new phenomena and the time-periodic solutions in modern cosmology and general relativity.

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