

From local Torelli to global Torelli

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Abstract

We introduce the notions of strong local Torelli and T-class for polarized manifolds, and prove that strong local Torelli implies global Torelli theorem on the Torelli spaces for polarized manifolds in the T-class. We discuss many new examples of projective manifolds for which such global Torelli theorem holds. As applications we prove that, in these cases, a canonical completion of the Torelli space is a bounded pseudoconvex domain in complex Euclidean space, and show that the global Torelli theorem holds on moduli space with certain level structure.

0 Introduction

Let (X, L) be a polarized manifold with X a projective manifold and L an ample line bundle on X . Let \mathcal{M} denote the moduli space of polarized manifolds, or certain smooth cover of the moduli space of polarized manifolds, which contains (X, L) . Consider the period map

$$\Phi_{\mathcal{M}} : \mathcal{M} \rightarrow D/\Gamma,$$

from the moduli space \mathcal{M} to the period domain D , modulo the action of the monodromy group Γ .

Among the central problems in Hodge theory are the local Torelli problem and the global Torelli problem. As discussed in [21], the local Torelli problem is the question of deciding when the Hodge structure on $H^n(X_p, \mathbb{C})$ separates points in any local neighborhood in \mathcal{M} . In case that \mathcal{M} is smooth, the local Torelli problem is equivalent to that the tangent map of $\Phi_{\mathcal{M}}$ is injective at any point in \mathcal{M} , which is also called infinitesimal Torelli problem.

The global Torelli problem is the question about when the Hodge structure on $H^n(X_p, \mathbb{C})$ uniquely characterizes the polarized algebraic structure on X_p , which is equivalent to the question that when the period map $\Phi_{\mathcal{M}}$ is globally injective.

The infinitesimal Torelli problem is solved for a large class of compact complex manifolds, for examples, algebraic curves X , if $g(X) = 2$ or if $g(X) > 2$ and X is not hyper-elliptic, K3-Surfaces, Calabi-Yau manifolds, hyperkähler manifolds, hypersurfaces and complete intersections in $\mathbb{C}P^n$ with a few exceptions. We refer the reader to Chapter VIII in [22] for a detailed survey. Moreover, many criteria have been found to decide when the infinitesimal Torelli theorem holds.

The global Torelli problem is solved only for several special cases, and there seems to be no systematic methods to study the global Torelli problem. In many cases, only generic Torelli type theorem can be proved, which means that the period map is injective on some open and dense subset of \mathcal{M} . From our work in this paper, one will see that this difficulty is mostly due to the complicatedness of the moduli space and the monodromy group Γ .

To avoid such difficulties in proving the global Torelli theorem, we lift the period map to the universal cover of the moduli space, which we call the Teichmüller space, and consider the lifted period map. Furthermore, we consider the moduli space of polarized manifolds with level m structure, as well as the Torelli space which is an irreducible component of the moduli space of marked and polarized manifolds. We notice that Torelli space is also called the moduli space of framed polarized manifolds in the literature, for example [3].

More precisely, let (X, L) be a polarized manifold. For simplicity we also denote by L its first Chern class. We fix a lattice Λ with a pairing Q_0 , where Λ is isomorphic to $H^n(X_0, \mathbb{Z})/\text{Tor}$ for some polarized manifold X_0 in \mathcal{M} , and Q_0 is defined by the cup-product. For a polarized manifold $(X, L) \in \mathcal{M}$, we define a marking γ as an isometry of the lattices

$$\gamma : (\Lambda, Q_0) \rightarrow (H^n(X, \mathbb{Z})/\text{Tor}, Q). \quad (1)$$

Let m be any integer ≥ 3 . We follow the definition of Szendrői [64] to define an m -equivalent relation of two markings on (X, L) by

$$\gamma \sim_m \gamma' \text{ if and only if } \gamma' \circ \gamma^{-1} - \text{Id} \in m \cdot \text{End}(H^n(X, \mathbb{Z})/\text{Tor}),$$

and denote by $[\gamma]_m$ the set of all the m -equivalent classes of γ . Then we call $[\gamma]_m$ a level m structure on the polarized manifold (X, L) .

Let \mathcal{L}_m be the moduli space of polarized algebraic manifolds with level m structure that contains (X, L) . We then introduce the notion of T -class, where the letter T stands for Torelli. The purpose of introducing T -class is for us to work on smooth coverings of the moduli spaces.

Definition 0.1. *We say that the polarized manifold (X, L) belongs to T -class if there exists an integer $m_0 \geq 3$ such that the irreducible component \mathcal{Z}_m of \mathcal{L}_m containing (X, L) is a complex manifold, on which there is an analytic family $f_m : \mathcal{U}_m \rightarrow \mathcal{Z}_m$ for any $m \geq m_0$.*

Clearly we can assume that $m_0 = 3$ without loss of generality.

Let \mathcal{T} be the universal cover of \mathcal{Z}_m with the pull-back family $g_m : \mathcal{U} \rightarrow \mathcal{T}$. It will be proved that, in our setting, \mathcal{T} is independent of the choice of m . See Lemma 1.3. We will call \mathcal{T} the Teichmüller space of the polarized manifold (X, L) .

Let \mathcal{T}' be an irreducible component of the moduli space of marked and polarized manifolds containing (X, L) in the T -class, which we will call the Torelli space in this paper. We will see that \mathcal{T}' is a connected complex manifold which is a covering space of \mathcal{Z}_m for each $m \geq 3$. Therefore as the universal cover of \mathcal{Z}_m , the Teichmüller space \mathcal{T} is also a universal cover of \mathcal{T}' . See Section 1 for detailed discussions about moduli and Torelli spaces.

There are many examples of polarized manifolds that belong to the T -class, such as Calabi-Yau manifolds, hyperkähler manifolds, many hypersurfaces and complete intersections in projective spaces.

With the above notations, we can define the period map

$$\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \rightarrow D/\Gamma$$

and the lifted period map $\Phi : \mathcal{T} \rightarrow D$ such that the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\Phi} & D \\ \downarrow \pi_m & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{\Phi_{\mathcal{Z}_m}} & D/\Gamma \end{array}$$

is commutative, where $\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$ is the universal covering map, and

$$\pi_D : D \rightarrow D/\Gamma$$

denotes the natural quotient map. We can also define the period map

$$\Phi' : \mathcal{T}' \rightarrow D$$

from the Torelli space \mathcal{T}' , such that the following diagram commutes

$$\begin{array}{ccc}
 \mathcal{T} & \xrightarrow{\Phi} & D \\
 \downarrow \pi_m & \searrow \pi & \nearrow \Phi' \\
 & \mathcal{T}' & \\
 \downarrow \pi'_m & \swarrow & \downarrow \pi_D \\
 \mathcal{Z}_m & \xrightarrow{\Phi_{\mathcal{Z}_m}} & D/\Gamma,
 \end{array}$$

where the maps $\pi'_m : \mathcal{T}' \rightarrow \mathcal{Z}_m$ and $\pi : \mathcal{T} \rightarrow \mathcal{T}'$ are the natural covering maps.

Since \mathcal{Z}_m is smooth, the local Torelli is equivalent to that the tangent maps of $\Phi_{\mathcal{Z}_m}$ and Φ are everywhere nondegenerate. To proceed further, we introduce the notion of *strong local Torelli*. The purpose of introducing this new notion is to construct an affine structure on certain natural completion space of the Torelli space \mathcal{T}' . The affine structure is one of the most crucial ingredients for our arguments.

More precisely we say that strong local Torelli holds for the polarized manifold (X, L) , if there exists a holomorphic subbundle \mathcal{H} of the Hodge bundle

$$\bigoplus_{k=1}^n \text{Hom}(\mathcal{F}^k / \mathcal{F}^{k+1}, \mathcal{F}^{k-1} / \mathcal{F}^k)$$

on the period domain D , such that the holomorphic tangent map of the period map induces an isomorphism from the tangent bundle $T^{1,0}\mathcal{T}$ to the Hodge subbundle \mathcal{H} on \mathcal{T}

$$d\Phi : T^{1,0}\mathcal{T} \xrightarrow{\sim} \mathcal{H}. \quad (2)$$

Here we still denote by \mathcal{H} the pull-back of Hodge subbundle on \mathcal{T} for convenience.

Strong local Torelli is satisfied for various examples, including Calabi–Yau manifolds, hyperkähler manifolds, some hypersurfaces in projective space or weighted projective space, and certain complete intersections in complex projective space. We will show that they all satisfy strong local Torelli, and have explicit Hodge subbundle \mathcal{H} as required in (2) above. Among them, we should mention the smooth hypersurfaces of degree d in \mathbb{P}^{n+1} satisfying

$$d|(n+2) \text{ and } d \geq 3.$$

Till now, generic Torelli theorem for such hypersurfaces is still open, except the case of quintic threefold which is proved by Voisin [68].

By using eigenperiod map, we will get more non-trivial examples admitting Hodge subbundles in (2), which contain the arrangements of hyperplanes in \mathbb{P}^n , cubic surface and cubic threefold.

In this paper, we will prove that strong local Torelli implies global Torelli on the Torelli space \mathcal{T}' of the polarized manifolds in the T-class, which is the main theorem of this paper.

Theorem 0.2 (Main theorem). *Suppose that the polarized manifold (X, L) belongs to the T-class, and that strong local Torelli holds for (X, L) , then the global Torelli theorem holds on the Torelli space \mathcal{T}' , i.e. the period map $\Phi' : \mathcal{T}' \rightarrow D$ is injective.*

Our proof mainly uses the construction of certain natural completion space of the Torelli space, which we will prove to be the universal cover of the Griffiths completion of the moduli space \mathcal{Z}_m , the affine structure on the Teichmüller space and the boundedness of the period maps that we proved in [37].

The above theorem has several interesting corollaries.

Corollary 0.3. *Let the polarized manifold (X, L) be one of the following cases,*

- (i) *K3 surface;*
- (ii) *Calabi-Yau manifold;*
- (iii) *hyperkähler manifold.*

Then global Torelli theorem holds on the Torelli space \mathcal{T}' for (X, L) .

Global Torelli theorem for K3 surfaces is proved in algebraic and Kählerian cases in many famous papers, see [4], [14], [42], [52] and so on. In [66], Verbitsky proved the global Torelli theorem for hyperkähler manifolds on the irreducible components of the birational Teichmüller space and the birational moduli space as defined [66]. Corollary 0.3 in case (i) and (iii) can be viewed as a different version of their theorems on the Torelli space as defined in our paper. Corollary 0.3 in case (ii) was first proved in [38].

Corollary 0.4. *Let the polarized manifold (X, L) be one of the following cases,*

- (i) *the smooth hypersurface of degree d in \mathbb{P}^{n+1} satisfying $d|(n+2)$ and $d \geq 3$;*
- (ii) *the polarized manifold in \mathbb{P}^{m-1} associated to an arrangement of m hyperplanes in \mathbb{P}^n with $m \geq n$, defined as in (20) of Section 2.2;*
- (iii) *smooth cubic surface or cubic threefold.*

Then global Torelli theorem holds on the corresponding Torelli space \mathcal{T}' for (X, L) .

We should mention that the Hodge subbundles satisfying (2) in case (ii) and (iii) of Corollary 0.4 are constructed by the eigenperiod maps, see Example 2.4 and 2.6. Also note that the moduli space in case (ii) of Corollary 0.4 is precisely the moduli space of hyperplane arrangements as discussed in [12]. See Section 2.2 for more details.

We remark that case (i) of Corollary 0.4 is interesting and new. Voisin in [68] proved the generic Torelli theorem on the moduli space of quintic threefold. Case (ii) of Corollary 0.4 with $n = 1$ can be considered as a version on the Torelli space of the main result of [11], which is the famous Deligne-Mostow theory. The general case for (ii) in Corollary 0.4 with $n > 1$ is new.

In [1] and [2], Allcock, Carlson and Toledo proved the global Torelli theorem for cubic surfaces and cubic threefolds respectively, as one of their main

results. They proved that the refined period map on the moduli space of smooth cubic surfaces or cubic threefolds is an isomorphism of complex analytic orbifolds onto its image, and the refined period map on the moduli space of framed smooth cubic surfaces or cubic threefolds is an isomorphism of complex manifolds onto its image. Case (iii) of Corollary 0.4 can be considered as a different version of their results in [1] and [2] for the ordinary period map on the Torelli space.

Let \mathcal{Z}_m^H denote the Hodge metric completion of the moduli space \mathcal{Z}_m of polarized manifolds with level m structure with respect to the induced Hodge metric on \mathcal{Z}_m , and \mathcal{T}^H be the universal cover of \mathcal{Z}_m^H .

We will prove that \mathcal{T}^H is the Hodge metric completion of the Torelli space \mathcal{T}' . Moreover we will prove the following theorem.

Theorem 0.5. *Suppose that the polarized manifold (X, L) belongs to the T -class and the strong local Torelli holds for (X, L) . Then the Hodge metric completion \mathcal{T}^H of the Torelli space \mathcal{T}' is a bounded pseudoconvex domain in \mathbb{C}^N . In particular, there exists a unique complete Kähler-Einstein metric on \mathcal{T}^H with Ricci curvature -1 .*

Let $\Gamma = \rho(\pi_1(\mathcal{Z}_m^H))$ denote the global monodromy group, where

$$\rho : \pi_1(\mathcal{Z}_m^H) \rightarrow \text{Aut}(H_{\mathbb{Z}}, Q)$$

is the monodromy representation. As an application, we will prove that the global Torelli theorem holds on the moduli space with certain level structure. More precisely, we have the following theorem.

Theorem 0.6. *Suppose that the polarized manifold (X, L) belongs to the T -class and the strong local Torelli holds for (X, L) . Then the extended period map $\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow D/\Gamma$ is injective. As a consequence, the global Torelli theorem holds on the moduli space \mathcal{Z}_m with level m structure.*

More applications of affine structures on the Teichmüller and Torelli spaces can be found in [39].

The paper is divided into the following sections, which we will describe briefly. In Section 1 we review the basics of Hodge theory and introduce the definitions of various moduli spaces and period maps. In Section 2, we first

define the notion of strong local Torelli. Then we discuss various examples and study the explicit forms of the Hodge subbundle \mathcal{H} as required in (2) for them. In Section 3 we discuss the three completions of the moduli spaces and prove their equivalence. Extended period maps are also introduced in this section.

In Section 4, we review certain results in [37], which include the boundedness of the images of the period maps and the existence of affine structures on the Teichmüller spaces for polarized manifolds satisfying strong local Torelli. In Section 5, we prove the main result of this paper, Theorem 0.2, and apply it to some examples in Corollary 0.3 and Corollary 0.4. We also prove Theorem 0.5 at the end of this section. Section 6 contains the proof of Theorem 0.6 as an application of the main result of this paper.

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1 Moduli spaces and Hodge theory

In this section, we introduce the notions of moduli spaces, level m structure, Teichmüller spaces and Torelli spaces. We also review Hodge theory and define the period maps from the moduli spaces, Teichmüller spaces and Torelli spaces. We present the review partially as given in [37] for the reader's convenience. All the results in this section are standard and well-known. For example, one can refer to [54] for the knowledge of moduli space, and [33], [60] for the knowledge of deformation theory.

1.1 Analytic families

Let X be a projective manifold with an ample line bundle L . We call the pair (X, L) a polarized manifold.

An analytic family of polarized manifolds is a proper morphism $f : \mathfrak{X} \rightarrow S$ between complex analytic spaces with the following properties

- (1) the complex analytic spaces \mathfrak{X} and S are smooth and connected, and

the morphism f is nondegenerate, i.e. the tangent map df is of maximal rank at each point of \mathfrak{X} ;

(2) there is a line bundle \mathcal{L} over \mathfrak{X} ;

(3) $(X_s = f^{-1}(s), L_s = \mathcal{L}|_{X_s})$ is a polarized manifold for any $s \in S$.

An analytic family $f : \mathfrak{X} \rightarrow S$ is called universal at a point $s \in S$ if the following two conditions are satisfied:

(1) Given an analytic family $f' : \mathfrak{X}' \rightarrow S'$ with $s' \in S'$ such that $\iota : f'^{-1}(s') \rightarrow f^{-1}(s)$ is biholomorphic, then there exists a neighborhood $U \subseteq S'$ around s' and holomorphic maps $g : U \rightarrow S$ and $h : f'^{-1}(U) \rightarrow \mathfrak{X}$ such that the following diagram commutes

$$\begin{array}{ccc} f'^{-1}(U) & \xrightarrow{h} & \mathfrak{X} \\ f' \downarrow & & \downarrow f \\ U & \xrightarrow{g} & S \end{array}$$

where $g(s') = s$, and for any $t' \in U$ with $t = g(t')$, the restricted map

$$h|_{f'^{-1}(t')} : f'^{-1}(t') \rightarrow f^{-1}(t)$$

is biholomorphic with $h|_{f'^{-1}(s)}$ identified with the biholomorphic map ι .

(2) The map g is uniquely determined.

A universal family is an analytic family $f : \mathfrak{X} \rightarrow S$ which is universal at every point in S .

1.2 Moduli, Teichmüller and Torelli spaces

Let (X, L) be a polarized manifold. The moduli space \mathcal{M} of polarized manifolds is the complex analytic space parameterizing the isomorphism class of polarized manifolds with the isomorphism defined by

$$(X, L) \sim (X', L') \iff \exists \text{ biholomorphic map } f : X \rightarrow X' \text{ s.t. } f^*L' = L.$$

We fix a lattice Λ with a pairing Q_0 , where Λ is isomorphic to $H^n(X_0, \mathbb{Z})/\text{Tor}$ for some X_0 in \mathcal{M} and Q_0 is defined by the cup-product. For a polarized manifold $(X, L) \in \mathcal{M}$, we define a marking γ as an isometry of the lattices

$$\gamma : (\Lambda, Q_0) \rightarrow (H^n(X, \mathbb{Z})/\text{Tor}, Q). \quad (3)$$

For any integer $m \geq 3$, we follow the definition of Szendrői [64] to define an m -equivalent relation of two markings on (X, L) by

$$\gamma \sim_m \gamma' \text{ if and only if } \gamma' \circ \gamma^{-1} - \text{Id} \in m \cdot \text{End}(H^n(X, \mathbb{Z})/\text{Tor}),$$

and denote $[\gamma]_m$ to be the set of all the m -equivalent classes of γ . Then we call $[\gamma]_m$ a level m structure on the polarized manifold (X, L) .

Two polarized manifolds with level m structure $(X, L, [\gamma]_m)$ and $(X', L', [\gamma']_m)$ are said to be isomorphic, or equivalent, if there exists a biholomorphic map $f : X \rightarrow X'$ such that

$$f^*L' = L \text{ and } f^*\gamma' \sim_m \gamma,$$

where $f^*\gamma'$ is given by

$$\gamma' : (\Lambda, Q_0) \rightarrow (H^n(X', \mathbb{Z})/\text{Tor}, Q)$$

composed with the induced map

$$f^* : (H^n(X', \mathbb{Z})/\text{Tor}, Q) \rightarrow (H^n(X, \mathbb{Z})/\text{Tor}, Q).$$

We denote by $[M, L, [\gamma]_m]$ the isomorphism class of the polarized manifolds with level m structure $(M, L, [\gamma]_m)$.

The moduli space of polarized manifolds with level m structure is the analytic space which parameterizes the isomorphism class of polarized manifolds with level m structure, where $m \geq 3$. Let \mathcal{L}_m be the moduli space of polarized manifolds with level m structure, $m \geq 3$, which contains the given polarized manifold (X, L) .

Definition 1.1. *A polarized manifold (X, L) is said to belong to the T -class, if the irreducible component \mathcal{Z}_m of \mathcal{L}_m containing (X, L) is a complex manifold, on which there is an analytic family $f_m : \mathcal{U}_m \rightarrow \mathcal{Z}_m$ for all $m \geq m_0$, where $m_0 \geq 3$ is some integer.*

In this case, we may simply take $m_0 = 3$ without loss of generality.

In this paper we only consider the polarized manifold (X, L) belonging to the T-class, and the smooth moduli space \mathcal{Z}_m for $m \geq 3$. Our purpose of introducing this notion is for us to work on the smooth covers of the moduli spaces.

Let \mathcal{T}^m be the universal covering of \mathcal{Z}_m with covering map $\pi_m : \mathcal{T}^m \rightarrow \mathcal{Z}_m$. Then we have an analytic family $g_m : \mathcal{V}^m \rightarrow \mathcal{T}^m$ such that the following diagram is cartesian

$$\begin{array}{ccc} \mathcal{V}^m & \longrightarrow & \mathcal{U}_m \\ \downarrow g_m & & \downarrow f_m \\ \mathcal{T}^m & \longrightarrow & \mathcal{Z}_m \end{array}$$

i.e. $\mathcal{V}^m = \mathcal{U}_m \times_{\mathcal{Z}_m} \mathcal{T}^m$. The family g_m is called the pull-back family.

The proof of the following lemma is obvious.

Lemma 1.2. *The space \mathcal{T}^m is a connected complex manifold with the pull-back family $g_m : \mathcal{V}^m \rightarrow \mathcal{T}^m$.*

The following lemma proves that the space \mathcal{T}^m is independent of the level m .

Lemma 1.3. *The space \mathcal{T}^m does not depend on the choice of m . From now on, we simply denote \mathcal{T}^m by \mathcal{T} , the analytic family by $g : \mathcal{V} \rightarrow \mathcal{T}$ and the covering map by $\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$.*

Proof. We give two proofs of this lemma. The first proof uses the construction of moduli space with level m structure, see Lecture 10 of [54], or pages 692 – 693 of [64].

Let m_1 and m_2 be two different integers, and

$$f_{m_1} : \mathcal{U}_{m_1} \rightarrow \mathcal{Z}_{m_1}, \quad f_{m_2} : \mathcal{U}_{m_2} \rightarrow \mathcal{Z}_{m_2}$$

be two analytic families with level m_1 structure and level m_2 structure respectively. Let \mathcal{T}^{m_1} and \mathcal{T}^{m_2} be the universal covering space of \mathcal{Z}_{m_1} and \mathcal{Z}_{m_2} with the pull back family

$$g_{m_1} : \mathcal{V}^{m_1} \rightarrow \mathcal{T}^{m_1}, \quad g_{m_2} : \mathcal{V}^{m_2} \rightarrow \mathcal{T}^{m_2}$$

of f_{m_1} and f_{m_2} respectively. Let $m = m_1 m_2$ and consider the analytic family

$$f_m : \mathcal{U}_m \rightarrow \mathcal{Z}_m.$$

From the discussion in Page 130 of [54] or 692 – 693 of [64], we know that \mathcal{Z}_m is a covering space of both \mathcal{Z}_{m_1} and \mathcal{Z}_{m_2} , and the analytic family f_m over \mathcal{Z}_m is the pull-back family of both f_{m_1} and f_{m_2} via the corresponding covering maps.

Let \mathcal{T} be the universal covering space of \mathcal{Z}_m with the pull-back family

$$g : \mathcal{V} \rightarrow \mathcal{T}$$

of f_m . Since \mathcal{Z}_m is a covering space of both \mathcal{Z}_{m_1} and \mathcal{Z}_{m_2} , we conclude that \mathcal{T} is the universal cover of both \mathcal{Z}_{m_1} and \mathcal{Z}_{m_2} , i.e.

$$\mathcal{T}^{m_1} \simeq \mathcal{T}^{m_2} \simeq \mathcal{T},$$

and that the analytic family g is identified with the analytic families g_{m_1} and g_{m_2} via the above isomorphisms.

If the analytic family $f_m : \mathcal{U}_m \rightarrow \mathcal{Z}_m$ is universal, as defined in Section 1.1, then we have a second proof.

Let m_1, m_2 be two different integers ≥ 3 , and let \mathcal{T}^{m_1} and \mathcal{T}^{m_2} be the corresponding two Teichmüller spaces with the universal families

$$g_{m_1} : \mathcal{V}^{m_1} \rightarrow \mathcal{T}^{m_1}, \quad g_{m_2} : \mathcal{V}^{m_2} \rightarrow \mathcal{T}^{m_2}$$

respectively. Then for any point $p \in \mathcal{T}^{m_1}$ and the fiber $X_p = g_{m_1}^{-1}(p)$ over p , there exists $q \in \mathcal{T}^{m_2}$ such that $Y_q = g_{m_2}^{-1}(q)$ is biholomorphic to X_p . By the definition of universal family, we can find a local neighborhood U_p of p and a holomorphic map $h_p : U_p \rightarrow \mathcal{T}^{m_2}$, $p \mapsto q$ such that the map h_p is uniquely determined.

Since \mathcal{T}^{m_1} is simply-connected, all the local holomorphic maps

$$\{h_p : U_p \rightarrow \mathcal{T}^{m_2}, p \in \mathcal{T}^{m_1}\}$$

patches together to give a global holomorphic map $h : \mathcal{T}^{m_1} \rightarrow \mathcal{T}^{m_2}$ which is well-defined. Moreover h is unique since it is unique on each local neighborhood of \mathcal{T}^{m_1} . Similarly we have a holomorphic map $h' : \mathcal{T}^{m_2} \rightarrow \mathcal{T}^{m_1}$ which is also unique. Then h and h' are inverse to each other by the uniqueness of h and h' . Therefore \mathcal{T}^{m_1} and \mathcal{T}^{m_2} are biholomorphic. \square

Due to the above lemma, we can denote $\mathcal{T} = \mathcal{T}^m$ for any $m \geq 3$, and call \mathcal{T} the Teichmüller space of polarized manifolds.

We define the Torelli space \mathcal{T}' to be an irreducible component of the complex analytic space consisting of biholomorphically equivalent triples of (X, L, γ) , where γ is a marking defined in (3). To be more precise, for two triples (X, L, γ) and (X', L', γ') , if there exists a biholomorphic map

$$f : X \rightarrow X'$$

such that $f^*L' = L$ and

$$f^*\gamma' = \gamma, \tag{4}$$

where $f^*\gamma'$ is given by

$$\gamma' : (\Lambda, Q_0) \rightarrow (H^n(X', \mathbb{Z})/\text{Tor}, Q)$$

composed with

$$f^* : (H^n(X', \mathbb{Z})/\text{Tor}, Q) \rightarrow (H^n(X, \mathbb{Z})/\text{Tor}, Q),$$

then $[X, L, \gamma] = [X', L', \gamma'] \in \mathcal{T}'$, where $[X, L, \gamma]$ is the equivalent class of the triple (X, L, γ) .

To summarize, we have the definition of the Torelli space \mathcal{T}' as follows.

Definition 1.4. *The Torelli space \mathcal{T}' is an irreducible component of the moduli space of marked and polarized manifolds containing (X, L) in the T -class.*

By construction, the Torelli space \mathcal{T}' is a covering space of \mathcal{Z}_m , with the natural covering map $\pi'_m : \mathcal{T}' \rightarrow \mathcal{Z}_m$ given by

$$[X, L, \gamma] \mapsto [X, L, [\gamma]_m].$$

Then we have the pull-back family $g' : \mathcal{U}' \rightarrow \mathcal{T}'$ of $f_m : \mathcal{U}_m \rightarrow \mathcal{Z}_m$.

Proposition 1.5. *For the polarized manifold (X, L) in the T -class, the Torelli space \mathcal{T}' containing (X, L) is a connected complex manifold.*

Proof. One notices that there is a natural covering map $\pi'_m : \mathcal{T}' \rightarrow \mathcal{Z}_m$ for any $m \geq 3$ according to the definition of \mathcal{T}' . Therefore \mathcal{T}' is a connected complex manifold, as \mathcal{Z}_m is a connected complex manifold by the definition of T-class. \square

By the universal property of the universal covering space, we know that the Teichmüller space \mathcal{T} is also the universal cover of the Torelli space \mathcal{T}' .

1.3 Variation of Hodge structure

Let (X, L) be polarized manifold defined in Section 1.1 with $\dim_{\mathbb{C}} X = n$. The n -th primitive cohomology groups $H_{pr}^n(X, \mathbb{C})$ of X is defined as

$$H_{pr}^n(X, \mathbb{C}) = \ker\{L : H^n(X, \mathbb{C}) \rightarrow H^{n+2}(X, \mathbb{C})\},$$

where the action of L on the cohomology group is defined by the wedge product with the first Chern class of L .

Let us denote

$$H_{\mathbb{Z}} = H_{pr}^n(X) \cap H^n(X, \mathbb{Z})$$

and denote

$$H_{pr}^{k, n-k}(X) = H^{k, n-k}(X) \cap H_{pr}^n(X, \mathbb{C})$$

with its complex dimension denoted by $h^{k, n-k}$. Since L is defined over \mathbb{Z} , we have that

$$H_{pr}^n(X, \mathbb{C}) = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}.$$

We then have the Hodge decomposition

$$H_{pr}^n(X, \mathbb{C}) = H_{pr}^{n,0}(X) \oplus \cdots \oplus H_{pr}^{0,n}(X), \quad (5)$$

such that $H_{pr}^{n-k, k}(X) = \overline{H_{pr}^{k, n-k}(X)}$. Therefore the data

$$\{H_{\mathbb{Z}}, H_{pr}^{k, n-k}(X)\}$$

is a Hodge structure of weight n .

On this Hodge structure there is a bilinear form

$$Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \rightarrow \mathbb{Z}$$

defined by

$$Q(u, v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any d -closed n -forms u, v on X . It is well-known that Q is nondegenerate and can be extended to $H_{pr}^n(X, \mathbb{C})$ bilinearly. Moreover, it also satisfies the Hodge-Riemann relations

$$Q(H_{pr}^{n-k,k}(X), H_{pr}^{n-l,l}(X)) = 0 \text{ unless } k+l = n; \quad (6)$$

$$(\sqrt{-1})^{2k-n} Q(v, \bar{v}) > 0 \text{ for } v \in H_{pr}^{k,n-k}(X) \setminus \{0\}. \quad (7)$$

Let $f^k = \sum_{i=k}^n h^{i,n-i}$, $m = f^0$, and

$$F^k = F^k(X) = H_{pr}^{n,0}(X) \oplus \dots \oplus H_{pr}^{k,n-k}(X),$$

from which we have the decreasing filtration

$$H_{pr}^n(X, \mathbb{C}) = F^0 \supset \dots \supset F^n.$$

We know that

$$\dim_{\mathbb{C}} F^k = f^k, \quad (8)$$

$$H_{pr}^n(X, \mathbb{C}) = F^k \oplus \overline{F^{n-k+1}}, \text{ and } H_{pr}^{k,n-k}(X) = F^k \cap \overline{F^{n-k}}.$$

In terms of Hodge filtrations, the Hodge-Riemann relations (6) and (7) are

$$Q(F^k, F^{n-k+1}) = 0; \quad (9)$$

$$Q(Cv, \bar{v}) > 0 \text{ if } v \neq 0, \quad (10)$$

where C is the Weil operator given by

$$Cv = (\sqrt{-1})^{2k-n} v \text{ for } v \in H_{pr}^{k,n-k}(X).$$

The period domain D for polarized Hodge structures is defined by the space of all such Hodge filtrations

$$D = \{F^n \subset \dots \subset F^0 = H_{pr}^n(X, \mathbb{C}) \mid (8), (9) \text{ and } (10) \text{ hold}\}.$$

The compact dual \check{D} of D is defined by

$$\check{D} = \{F^n \subset \dots \subset F^0 = H_{pr}^n(X, \mathbb{C}) \mid (8) \text{ and } (9) \text{ hold}\}.$$

The compact dual \check{D} is a projective manifold and the period domain $D \subseteq \check{D}$ is an open submanifold.

From the definition of period domain we naturally get the Hodge bundles on \check{D} , by associating to each point in \check{D} the vector spaces $\{F^k\}_{k=0}^n$ in the Hodge filtration of that point. We will denote the Hodge bundles by $\mathcal{F}^k \rightarrow \check{D}$ with $\mathcal{F}^k|_p = F_p^k$ as the fiber for any $p \in \check{D}$ and each $0 \leq k \leq n$.

For the family $f_m : \mathcal{U}_m \rightarrow \mathcal{Z}_m$, we denote each fiber by

$$[X_s, L_s, [\gamma_s]_m] = f_m^{-1}(s)$$

and $F_s^k = F^k(X_s)$ for $0 \leq k \leq n$. With some fixed point $s_0 \in \mathcal{Z}_m$, the period map is defined as a morphism

$$\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \rightarrow D/\Gamma$$

by

$$s \mapsto \tau^{[\gamma_s]}(F_s^n \subseteq \cdots \subseteq F_s^0) \in D, \quad (11)$$

where $F_s^k = F^k(X_s)$ and $\tau^{[\gamma_s]}$ is an isomorphism between \mathbb{C} -vector spaces

$$\tau^{[\gamma_s]} : H^n(X_s, \mathbb{C}) \rightarrow H^n(X_{s_0}, \mathbb{C}),$$

which depends only on the homotopy class $[\gamma_s]$ of the curve γ_s between s and s_0 .

Recall that the monodromy group Γ is the image of representation of the fundamental group of \mathcal{Z}_m in $\text{Aut}(H_{\mathbb{Z}}, Q)$, the automorphism group of $H_{\mathbb{Z}}$ preserving Q . Then the period map from \mathcal{Z}_m is well-defined with respect to the monodromy representation

$$\rho : \pi_1(\mathcal{Z}_m) \rightarrow \Gamma \subseteq \text{Aut}(H_{\mathbb{Z}}, Q).$$

It is well-known that the period map has the following properties:

1. locally liftable;
2. holomorphic, i.e. $\partial F_z^i / \partial \bar{z} \subset F_z^i$, $0 \leq i \leq n$;
3. Griffiths transversality: $\partial F_z^i / \partial z \subset F_z^{i-1}$, $1 \leq i \leq n$.

Let X be any polarized manifold with local coordinate chart $(U; w)$, and $\theta \in H^1(X, \Theta_X)$ with the local representation

$$\theta = \sum_{i\bar{j}} \theta_{i\bar{j}} \frac{\partial}{\partial w^i} \otimes d\bar{w}^j$$

under the identification

$$H^1(X, \Theta_X) \simeq \frac{Z^{0,1}(X, \mathbb{T}^{1,0}X)}{\bar{\partial}(A^0(X, \mathbb{T}^{1,0}X))},$$

where $Z^{0,1}(X, \mathbb{T}^{1,0}X)$ denotes the space of all the closed $(0, 1)$ forms on X with values in $\mathbb{T}^{1,0}X$, and $A^0(X, \mathbb{T}^{1,0}X)$ denotes the space of the smooth sections of $\mathbb{T}^{1,0}X$ on X .

For any (p, q) form $\alpha = \sum_{|I|=p, |J|=q} a_{I, \bar{J}} dw^I \wedge d\bar{w}^J$, the contraction map $\theta \lrcorner$ is given by

$$\theta \lrcorner \alpha = \sum_{k=1}^p \sum_{\substack{i_1, \dots, i_p \\ |J|=q, \bar{j}}} (-1)^{k-1} \theta_{i_k \bar{j}} a_{i_1, \dots, i_p, \bar{J}} d\bar{w}^j \wedge dw^{i_1} \cdots \widehat{dw^{i_k}} \cdots dw^{i_p} \wedge d\bar{w}^J. \quad (12)$$

Hence the contraction map $\theta \lrcorner$ maps $H^{p,q}(X)$ to $H^{p-1, q+1}(X)$.

From [18], we know that the derivative of the period map is precisely given by the contraction $KS(v) \lrcorner$, which is a linear map. Here

$$KS : \mathbb{T}_q^{1,0} \mathcal{Z}_m \rightarrow H^1(X_q, \Theta_{X_q})$$

is the Kodaira-Spencer map and $v \in \mathbb{T}_q \mathcal{Z}_m$ is a tangent vector at any point $q \in \mathcal{Z}_m$.

By (1) of the properties of the period map as above, we can lift the period map onto the universal cover \mathcal{T} of \mathcal{Z}_m , to get the lifted period map $\Phi : \mathcal{T} \rightarrow D$ such that the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\Phi} & D \\ \downarrow \pi_m & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{\Phi_{\mathcal{Z}_m}} & D/\Gamma \end{array}$$

is commutative.

From the definition of marking in (3), we have a well-defined period map $\Phi' : \mathcal{T}' \rightarrow D$ from the Torelli space \mathcal{T}' by

$$p \mapsto \gamma_p^{-1}(F_p^n \subseteq \cdots \subseteq F_p^0) \in D, \quad (13)$$

where the triple $[X_p, L_p, \gamma_p]$ is the fiber over $p \in \mathcal{T}'$ of the analytic family $\mathcal{U}' \rightarrow \mathcal{T}'$, and the marking γ_p is an isometry from a fixed lattice Λ to $H^n(X_p, \mathbb{Z})/\text{Tor}$, which extends \mathbb{C} -linearly to an isometry from $H = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ to $H^n(X_p, \mathbb{C})$. Here

$$\gamma_p^{-1}(F_p^n \subseteq \cdots \subseteq F_p^0) = \gamma_p^{-1}(F_p^n) \subseteq \cdots \subseteq \gamma_p^{-1}(F_p^0) = H$$

denotes a Hodge filtration of H .

Now we summarize the period maps defined as above in the following commutative diagram

$$\begin{array}{ccc}
 \mathcal{T} & \xrightarrow{\Phi} & D \\
 \pi \searrow & & \nearrow \Phi' \\
 \mathcal{T}' & & \\
 \pi_m \swarrow & & \searrow \pi_D \\
 \mathcal{Z}_m & \xrightarrow{\Phi_{\mathcal{Z}_m}} & D/\Gamma
 \end{array}$$

where the maps π_m , π'_m and π are all natural covering maps between the corresponding spaces as discussed in Section 1.

Before closing this section, we prove a lemma concerning the monodromy group Γ on \mathcal{Z}_m for $m \geq 3$.

Lemma 1.6. *Let γ be the image of some element of $\pi_1(\mathcal{Z}_m)$ in Γ under the monodromy representation. Suppose that γ is finite, then γ is trivial. Therefore for $m \geq 3$, we can assume that Γ is torsion-free and D/Γ is smooth.*

Proof. Let us look at the period map locally as $\Phi_{\mathcal{Z}_m} : \Delta^* \rightarrow D/\Gamma$. Assume that γ is the monodromy action corresponding to the generator of the fundamental group of Δ^* .

We lift the period map to $\Phi : \mathbb{H} \rightarrow D$, where \mathbb{H} is the upper half plane and the covering map from \mathbb{H} to Δ^* is

$$z \mapsto \exp(2\pi\sqrt{-1}z).$$

Then $\Phi(z + 1) = \gamma\Phi(z)$ for any $z \in \mathbb{H}$. Since $\Phi(z + 1)$ and $\Phi(z)$ correspond to the same point in \mathcal{Z}_m , by the definition of \mathcal{Z}_m we have

$$\gamma \equiv I \pmod{(m)}.$$

But γ is also in $\text{Aut}(H_{\mathbb{Z}})$, applying Serre's lemma [58] or Lemma 2.4 in [64], we have $\gamma = I$. \square

2 Strong local Torelli and Examples

In this section, we first introduce the notion of strong local Torelli, which is given in Definition 2.1 below. In particular we remark that the identification (14) below of the tangent bundle of the Teichmüller space to certain Hodge subbundle is crucial for our proof of global Torelli theorem from strong local Torelli.

Then we give some examples including hyperkähler manifolds, Calabi–Yau manifolds, some hypersurfaces in projective spaces or weighted projective spaces, and some complete intersections in complex projective space. They all satisfy strong local Torelli, and have explicit forms of the Hodge subbundle \mathcal{H} in (14). We also introduce the notion of eigenperiod map to get more nontrivial examples admitting Hodge subbundles in (14), which contain the arrangements of hyperplanes in \mathbb{P}^n , cubic surface and cubic threefold.

2.1 Strong local Torelli and examples

We first introduce a stronger notion of local Torelli, which we will prove to be a sufficient condition for global Torelli on the Torelli spaces for projective manifolds in the T-class. Let \mathcal{Z}_m be the smooth component of moduli space of polarized manifolds with level m structure containing (X, L) , and let \mathcal{T} be the corresponding Teichmüller space.

Definition 2.1. *The period map $\Phi : \mathcal{T} \rightarrow D$ is said to satisfy the strong local Torelli, provided that the local Torelli holds, i.e. the tangent map of the period map Φ is injective everywhere on \mathcal{T} , and furthermore, there exists a*

holomorphic subbundle \mathcal{H} of the Hodge bundle

$$\bigoplus_{k=1}^n \text{Hom}(\mathcal{F}^k / \mathcal{F}^{k+1}, \mathcal{F}^{k-1} / \mathcal{F}^k)$$

on the period domain D , such that the tangent map of the period map induces an isomorphism on \mathcal{T} of the holomorphic tangent bundle $\mathbb{T}^{1,0}\mathcal{T}$ to the Hodge subbundle \mathcal{H} ,

$$d\Phi : \mathbb{T}^{1,0}\mathcal{T} \xrightarrow{\sim} \mathcal{H}. \quad (14)$$

Here we still denote by \mathcal{H} the pull-back of Hodge subbundle on \mathcal{T} for convenience.

For simplicity we will also say that strong local Torelli holds for the polarized manifold (X, L) , if the corresponding period map Φ satisfies the above definition.

The purpose of introducing the notion of strong local Torelli is for us to extend affine structure to the Hodge metric completion space \mathcal{T}^H to be studied in detail later.

The following special case of condition (14) is interesting, which includes many well-known examples.

There exist holomorphic subbundles \mathcal{F}_0^l of \mathcal{F}^l , $0 \leq l \leq n$, and there exists some integer k from 0 to n , such that $\mathcal{F}_0^l = 0$ for any $l > k$ and the holomorphic subbundle \mathcal{H} of Definition 2.1 is of the form

$$\mathcal{H} = \text{Hom}(\mathcal{F}_0^k, \mathcal{F}_0^{k-1} / \mathcal{F}_0^k). \quad (15)$$

Now we discuss some examples satisfying the strong local Torelli to illustrate the Hodge subbundle \mathcal{H} of Definition 2.1.

First note that, from the arguments before Corollary 5.15 and Corollary 5.16 in Section 5, one sees that the moduli spaces \mathcal{Z}_m for m large enough, as well as the Teichmüller space \mathcal{T} , are smooth in the following examples. Hence we only need to describe the Hodge subbundle \mathcal{H} locally, or equivalently, from the point of view of deformation theory.

Example 2.2 (K3 surfaces and Calabi–Yau manifolds). Let M be a Calabi–Yau manifold of complex dimension $n \geq 3$, which means that M is a compact projective manifold with a trivial canonical bundle

$$K_M = \Omega_M^n \simeq \mathcal{O}_M$$

and satisfies $H^i(M, \mathcal{O}_M) = 0$ for $0 < i < n$. Then the contraction map induces an isomorphism of the sheaves

$$\Theta_M = \mathcal{O}_M(\mathbb{T}_M) \simeq \Omega_M^n(\mathbb{T}_M) \simeq \Omega_M^{n-1}.$$

Hence

$$H^1(M, \Theta_M) \simeq H^1(M, \Omega_M^{n-1}) \simeq H^{n-1,1}(M),$$

which together with the unobstructedness of the deformations of compact Kähler manifolds with trivial canonical bundle implies that local Torelli theorem holds for Calabi–Yau manifolds such that

$$H^1(M, \Theta_M) \xrightarrow{\sim} \text{Hom}(F^n, F^{n-1}/F^n),$$

is an isomorphism. Hence the subbundle \mathcal{H} of Definition 2.1 exists and is

$$\mathcal{H} = \text{Hom}(\mathcal{F}^n, \mathcal{F}^{n-1}/\mathcal{F}^n).$$

Therefore Calabi–Yau manifolds satisfy Definition 2.1. In fact, global Torelli theorem on the Torelli space for Calabi–Yau manifolds was already proved in [38].

Since K3 surfaces are 2-dimensional analogues of Calabi–Yau manifolds, we can see that K3 surfaces also satisfy Definition 2.1 by a similar argument. Global Torelli theorem for K3 surfaces are proved in algebraic and Kählerian cases in many famous papers, see [4], [14], [42], [52] and so on.

Example 2.3 (Hyperkähler manifolds). Let X be a hyperkähler manifold, which means that X is an irreducible simply-connected compact Kähler manifold such that $H^0(X, \Omega_X^2)$ is generated by an everywhere nondegenerate holomorphic two form σ .

A direct consequence of the definition of X is that the cohomology $H^2(X, \mathbb{C})$ has the following Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

such that $H^{2,0}(X) = \mathbb{C}\sigma$ and $H^{0,2}(X) = \mathbb{C}\bar{\sigma}$. Moreover, X has complex dimension $2n$ and the canonical bundle $K_X = \Omega_X^{2n}$ is trivialized by σ^n . The everywhere nondegenerate holomorphic two form σ also induces an isomorphism

$$\sigma : \Theta_X = \mathcal{O}(T_X) \rightarrow \Omega_X.$$

The unobstructedness of the deformations of compact Kähler manifolds with trivial canonical bundles and the isomorphism above imply that the infinitesimal Torelli theorem holds for hyperkähler manifolds such that

$$H^1(X, \Theta_X) \xrightarrow{\sim} \text{Hom}(F^2, F^1/F^2),$$

is an isomorphism. Therefore the subbundle \mathcal{H} of Definition 2.1 exists and is

$$\mathcal{H} = \text{Hom}(\mathcal{F}^2, \mathcal{F}^1/\mathcal{F}^2).$$

Example 2.4 (Hypersurfaces in projective spaces). Let X be a smooth hypersurface of degree d in \mathbb{P}^{n+1} , which means that X is an algebraic subvariety of \mathbb{P}^{n+1} determined by a polynomial $F(x_0, \dots, x_{n+1})$ with nondegenerate Jacobian. Define the graded Jacobian quotient ring $R(F)$ by

$$R(F) = \mathbb{C}[x_0, \dots, x_{n+1}] / \left\langle \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_{n+1}} \right\rangle.$$

As proved in [19], the primitive cohomology of X is

$$H_{pr}^{k, n-k}(X) \simeq R(F)^{d(n+1-k)-n-2},$$

where $R(F)^l$ is the graded piece of degree l of $R(F)$, $l \geq 0$.

From Lemma 5.4.4 and Example 8.1.1 of [6], we have that the infinitesimal Torelli theorem holds for the smooth hypersurface

$$X = (F = 0) \subset \mathbb{P}^{n+1}$$

of degree d if the product

$$R(F)^d \times R(F)^{d(n+1-k)-n-2} \rightarrow R(F)^{d(n+2-k)-n-2}$$

is nondegenerate in the first factor for some k between 1 and n . Macaulay's theorem tells that this product map is nondegenerate in each factor as long as

$$0 \leq d(n+2-k) - n - 2 \leq (n+2)(d-2).$$

Therefore the only exceptions for the infinitesimal Torelli are quadric and cubic curves, and cubic surfaces.

Let us consider the following case. Suppose there exists some integer $1 \leq k \leq n$ such that

$$R(F)^d \simeq R(F)^{d(n+2-k)-n-2}. \quad (16)$$

One sufficient condition of (16) is that

$$d = d(n+2-k) - n - 2, \text{ i.e. } d(n+1-k) = n+2,$$

or equivalently

$$d|(n+2).$$

Taking infinitesimal Torelli into consideration, we require that $d \leq (n+2)(d-2)$, and hence

$$d \geq 3.$$

In this case we have

$$H_{pr}^{k-1, n-k+1}(X) \simeq R(F)^d, \quad H_{pr}^{k, n-k}(X) \simeq R(F)^0 \simeq \mathbb{C},$$

and $H_{pr}^{l, n-l}(X) = 0$ for $l > k$. Then the Hodge subbundle \mathcal{H} of Definition 2.1 exists and is

$$\mathcal{H} = \text{Hom}(\mathcal{F}^k, \mathcal{F}^{k-1}/\mathcal{F}^k).$$

We remark that generic Torelli theorem for smooth hypersurface of degree d in \mathbb{P}^{n+1} satisfying $d|(n+2)$ is still open, except for quintic threefold, i.e. $d = 5$ and $n = 3$, see [68]. Still our method gives the global Torelli theorem on the Torelli space for such class of projective manifolds.

We now look at some well-known examples satisfying $d|(n+2)$ and $d \geq 3$.

If $n = 2$, then the nontrivial case is $d = 4$, i.e. the quartic surface, which is a special case of K3 surface.

If $n = 3$, then $d = 5$, i.e. the quintic threefold, which is a special case of Calabi–Yau threefold.

If $n = 4$, then the nontrivial cases are $d = 3$ and $d = 6$. The former is cubic fourfold and the later is a Calabi–Yau manifold.

Note that the condition $d|(n+2)$, $d \geq 3$ is only one of the sufficient conditions from Definition 2.1 in the case of hypersurfaces. We believe that many more examples of hypersurfaces satisfying Definition 2.1 can be found, just like cubic surfaces and cubic threefolds in Example 2.5 below.

We also remark that we can generalize the consideration in Example 2.4 to other cases, including k -sheeted branched covering of \mathbb{P}^n , Veronese double cone and some complete intersections in complex projective space, since there are similar theories studying infinitesimal Torelli theorems in terms of the Jacobian quotient rings in these cases, see, for examples, [35] and [55].

2.2 Eigenperiod map and examples

From the examples above, we can see that in general the canonically polarized hypersurfaces in \mathbb{P}^n seem not to satisfy strong local Torelli, since the rank of the Hodge bundles $\text{Hom}(\mathcal{F}^k, \mathcal{F}^{k-1}/\mathcal{F}^k)$, $1 \leq k \leq n$ are too big when compared with the dimension of the Teichmüller space \mathcal{T} . But the notion of eigenperiod map gives us many new examples in such cases. Below we will give a brief review of the eigenperiod map. One can refer to [12] for more details.

Let \mathcal{T} be the Teichmüller space of (X, L) , and $g : \mathcal{U} \rightarrow \mathcal{T}$ be the analytic family. Let G be a finite abelian group acting holomorphically on \mathcal{U} , preserving the line bundle \mathcal{L} on \mathcal{U} . Recall that the line bundle \mathcal{L} defines a polarization $L_q = \mathcal{L}|_{X_q}$ on $X_q = g^{-1}(q)$ for any $q \in \mathcal{T}$. We assume that $g(X_q) = X_q$ for any $q \in \mathcal{T}$ and $g \in G$. Fix $p \in \mathcal{T}$ and $o = \Phi(p) \in D$ as the base points. Let $H_{\mathbb{Z}} = H^n(X_p, \mathbb{Z})/\text{Tor}$ with the Poincaré pairing Q . Then the simply-connectedness of \mathcal{T} implies that we can identify

$$(H^n(X_q, \mathbb{Z})/\text{Tor}, Q) \xrightarrow{\sim} (H_{\mathbb{Z}}, Q)$$

for any $q \in \mathcal{T}$. Since G preserves the line bundle \mathcal{L} on \mathcal{U} , we have a induced action of G on $H_{\mathbb{Z}}$ preserving Q , i.e. we have a representation $\rho : G \rightarrow \text{Aut}(H_{\mathbb{Z}}, Q)$.

Let $H = H_{\mathbb{Z}} \otimes \mathbb{C}$. Define D^ρ by

$$D^\rho = \{(F^n \subset \cdots \subset F^0) \in D : \rho(g)(F^k) = F^k, 0 \leq k \leq n, \text{ for any } g \in G\}.$$

Let $\chi \in \text{Hom}(G, \mathbb{C}^*)$ be a character of G . Let

$$H_\chi = \{v \in H : \rho(g)(v) = \chi(g)v, \forall g \in G\}.$$

For any $(F^n \subset \cdots \subset F^0) \in D^\rho$, we define

$$F_\chi^k = F^k \cap H_\chi \text{ and } H_\chi^{k, n-k} = F^k \cap \overline{F}^{n-k} \cap H_\chi, \quad 0 \leq k \leq n.$$

Then we have the decomposition

$$H_\chi = H_\chi^{n,0} \oplus \cdots \oplus H_\chi^{0,n}. \quad (17)$$

Note that decomposition (17) is a Hodge decomposition if and only if χ is real.

Although decomposition (17) is not Hodge decomposition for general χ , it still has the restricted polarization Q on H_χ such that

$$Q(Cv, \bar{v}) > 0 \text{ for any } v \in H_\chi \setminus \{0\}.$$

Hence the sub-domain D_χ^ρ defined by

$$D_\chi^\rho = \{(F_\chi^n \subset \cdots \subset F_\chi^0) : (F^n \subset \cdots \subset F^0) \in D^\rho\},$$

has a well-defined metric, which is the restriction of the Hodge metric on D .

Since the action of G on any fiber X_q is holomorphic and preserves the polarization L_q on X_q , the period map $\Phi : \mathcal{T} \rightarrow D$ takes values in D^ρ . Then we define the eigenperiod map by

$$\Phi_\chi : \mathcal{T} \rightarrow D_\chi^\rho,$$

which is the composition of Φ with the projection map $D^\rho \rightarrow D_\chi^\rho$. We can also define the Hodge subbundles \mathcal{F}_χ^k , $0 \leq k \leq n$, on D_χ^ρ , and pull them back via the eigenperiod map to get the Hodge subbundles on \mathcal{T} , which are still denoted by \mathcal{F}_χ^k , $0 \leq k \leq n$.

Historically, the eigenperiod map is an effective way to reduce the dimension of the period domain, so that the eigenperiod map could still be locally isomorphic. In this paper we will study the ordinary period maps Φ and Φ' on \mathcal{T} and \mathcal{T}' respectively, while the eigenperiod map only serves as a tool to find the Hodge subbundle in (15). More precisely, we are interested in the following case, which includes many known examples.

There exist some character $\chi \in \text{Hom}(G, \mathbb{C}^*)$ and some integer k from 0 to n , such that $\mathcal{F}_\chi^l = 0$ for any $l > k$, and that the Hodge subbundles $\mathcal{F}_\chi^k, \mathcal{F}_\chi^{k-1}$ satisfy (15), which is equivalent to that the Hodge subbundle \mathcal{H} of Definition 2.1 is of the form

$$\mathcal{H} = \text{Hom}(\mathcal{F}_\chi^k, \mathcal{F}_\chi^{k-1}/\mathcal{F}_\chi^k). \quad (18)$$

We remark that the global Torelli theorem will be proved in the above case for the original period map $\Phi' : \mathcal{T}' \rightarrow D$, and the global Torelli theorem for the eigenperiod map follows in a similar way.

Example 2.5 (Cubic surfaces and cubic threefolds). We follow the notations of [10]. Let $X \subset \mathbb{P}^3$ be a smooth cubic surface with defining equation $F(x_0, x_1, x_2, x_3)$. Since $H^2(X, \mathbb{C}) = H^{1,1}(X)$ and the period map is trivial, we have to consider other ways to define the period map.

In their papers [1], Allcock, Carlson, and Toledo consider the cyclic triple covering \tilde{X} of \mathbb{P}^3 branched along X , with \tilde{X} defined by

$$\tilde{X} = (F(x_0, x_1, x_2, x_3) + x_4^3 = 0) \subset \mathbb{P}^4.$$

The Hodge structure of \tilde{X} is

$$H^3(\tilde{X}, \mathbb{C}) = H^{2,1}(\tilde{X}) \oplus H^{1,2}(\tilde{X}),$$

where $h^{2,1}(\tilde{X}) = h^{1,2}(\tilde{X}) = 5$. Moreover in [1], they show that the Hodge structure carries an action of the group μ_3 of cubic roots of unity, which is induced by the action of the group μ_3 on \tilde{X} such that

$$[x_0, x_1, x_2, x_3, x_4] \mapsto [x_0, x_1, x_2, x_3, \omega x_4], \text{ where } \omega = e^{2\pi i/3}.$$

The eigenspace $H_{\bar{\omega}}^3(\tilde{X}, \mathbb{C})$ for the eigenvalue $\bar{\omega} = e^{\pi i/3}$ has the Hodge structure

$$H_{\bar{\omega}}^3(\tilde{X}, \mathbb{C}) = H_{\bar{\omega}}^{2,1}(\tilde{X}) \oplus H_{\bar{\omega}}^{1,2}(\tilde{X}),$$

where $\dim_{\mathbb{C}}(H_{\bar{\omega}}^{2,1}(\tilde{X})) = 1$ and $\dim_{\mathbb{C}}(H_{\bar{\omega}}^{1,2}(\tilde{X})) = 4$. In this case, the infinitesimal Torelli theorem holds in the form

$$H^1(X, \Theta_X) \xrightarrow{\sim} \text{Hom}(F_{\bar{\omega}}^2, F_{\bar{\omega}}^1/F_{\bar{\omega}}^2),$$

where

$$F_{\bar{\omega}}^2 = H_{\bar{\omega}}^{2,1}(\tilde{X}) \text{ and } F_{\bar{\omega}}^1 = H_{\bar{\omega}}^{2,1}(\tilde{X}) \oplus H_{\bar{\omega}}^{1,2}(\tilde{X}).$$

Hence the subbundle \mathcal{H} of Definition 2.1 exists and is

$$\mathcal{H} = \text{Hom}(\mathcal{F}_\omega^2, \mathcal{F}_\omega^1/\mathcal{F}_\omega^2).$$

Similarly if Y is a smooth cubic threefold, in [2], they consider the cyclic triple covering \tilde{Y} of \mathbb{P}^4 branched along Y , such that $H^4(\tilde{Y}, \mathbb{C})$ has an eigenspace $H_\omega^4(\tilde{Y}, \mathbb{C})$ for the eigenvalue $\omega = e^{2\pi i/3}$ with the Hodge structure

$$H_\omega^4(\tilde{Y}, \mathbb{C}) = H_\omega^{3,1}(\tilde{Y}) \oplus H_\omega^{2,2}(\tilde{Y}),$$

where $H_\omega^{3,1}(\tilde{Y}) = H^{3,1}(\tilde{Y})$, and $\dim_{\mathbb{C}}(H_\omega^{3,1}(\tilde{Y})) = 1$ and $\dim_{\mathbb{C}}(H_\omega^{2,2}(\tilde{Y})) = 10$. In this case, the infinitesimal Torelli theorem holds in the form

$$H^1(Y, \Theta_Y) \xrightarrow{\sim} \text{Hom}(F_\omega^3, F_\omega^2/F_\omega^3),$$

where $F_\omega^3 = H_\omega^{3,1}(\tilde{Y})$ and $F_\omega^2 = H_\omega^4(\tilde{Y}, \mathbb{C})$, and hence the subbundle \mathcal{H} of Definition 2.1 exists and is

$$\mathcal{H} = \text{Hom}(\mathcal{F}_\omega^3, \mathcal{F}_\omega^2/\mathcal{F}_\omega^3).$$

Example 2.6 (Arrangements of hyperplanes). Let $m \geq n$ be positive integers. Consider the complementary set

$$U = \mathbb{P}^n \setminus \bigcup_{i=1}^m H_i,$$

where H_i is a hyperplane of \mathbb{P}^n for each $1 \leq i \leq m$ such that H_1, \dots, H_m are in general positions. That is to say that, if H_i is defined by the linear forms

$$f_i(z_0, \dots, z_n) = \sum_{j=0}^n a_{ij} z_j, \quad 1 \leq i \leq m, \quad (19)$$

then the matrix $(a_{ij})_{1 \leq i \leq m, 0 \leq j \leq n}$ is of full rank.

The fundamental group $\pi_1(U)$ of U is generated by the basis g_1, \dots, g_m with the relation that

$$g_1 \cdots g_m = 1.$$

One can see Section 8 of [12] for the geometric meaning of each generator g_i .

Choose a set of rational numbers $\mu = (\mu_1, \dots, \mu_m)$ satisfying

$$0 < \mu_i < 1, 1 \leq i \leq m;$$

$$|\mu| := \sum_{i=1}^m \mu_i \in \mathbb{Z}.$$

Define \mathcal{L}_μ to be the local system on U by the homomorphism

$$\chi : \pi_1(U) \rightarrow \mathbb{C}^*, g_i \mapsto e^{-2\pi\sqrt{-1}\mu_i}.$$

Since $|\mu| \in \mathbb{Z}$, the above homomorphism is well-defined. We are interested in the cohomology $H^*(U, \mathcal{L}_\mu)$ with Hodge decomposition defined as follows.

Let d be the least common denominator of μ_1, \dots, μ_m . We define X to be the smooth projective variety of \mathbb{P}^{m-1} by the equations

$$a_{1j}z_1^d + \dots + a_{mj}z_m^d = 0, 0 \leq j \leq n, \quad (20)$$

where the coefficients a_{ij} , $1 \leq i \leq m, 0 \leq j \leq n$ determine the hyperplanes as (19), and $[z_1, \dots, z_m]$ is the homogeneous coordinate of \mathbb{P}^{m-1} .

Define the finite group G to be

$$G = \pi_1(U)/\pi_1(U)^d,$$

which is isomorphic to the additive group

$$(\mathbb{Z}/d)^m / \langle \sum_{i=1}^m e_i \rangle,$$

where e_i is the generator of the i -th component of $(\mathbb{Z}/d)^m$. The group G acts on X as an automorphism induced by the well-defined action on \mathbb{P}^{m-1} :

$$[z_1, \dots, z_m] \mapsto [\tilde{g}_1 z_1, \dots, \tilde{g}_m z_m],$$

where \tilde{g}_i is the image of g_i in G , $1 \leq i \leq m$. The action of G on X also induces an action on $H^*(X, \mathbb{Z})$. By the definition of d and $\chi : \pi_1(U) \rightarrow \mathbb{C}^*$, χ can also be considered as a character in $\text{Hom}(G, \mathbb{C}^*)$. Let $H_\chi^n(X, \mathbb{C})$ be defined in the construction of eigenperiod map, and

$$H_\chi^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H_\chi^{p,q}(X)$$

be the decomposition as in (17).

Lemma 8.1 of [12] implies that

$$H^i(U, \mathcal{L}_\mu) = 0 \text{ if } i \neq n, \text{ and } H^n(U, \mathcal{L}_\mu) \simeq H_\chi^n(X, \mathbb{C}).$$

Then the isomorphism $H^n(U, \mathcal{L}_\mu) \simeq H_\chi^n(X, \mathbb{C})$ gives a Hodge structure on $H^n(U, \mathcal{L}_\mu)$ by defining

$$H^n(U, \mathcal{L}_\mu)^{p,q} = H_\chi^{p,q}(X), \text{ for any } p + q = n.$$

Denote $h_\chi^{p,q}(X) = \dim_{\mathbb{C}} H_\chi^{p,q}(X)$. Then Lemma 8.2 of [12] implies that

$$h_\chi^{p,q}(X) = \binom{|\mu| - 1}{p} \binom{m - 1 - |\mu|}{q},$$

and

$$\dim_{\mathbb{C}} H_\chi^n(X, \mathbb{C}) = \binom{m - 2}{n}.$$

Now we consider the case that $|\mu| = \mu_1 + \cdots + \mu_m = n + 1$. Then by direct computations

$$\begin{aligned} h_\chi^{n,0}(X) &= \binom{n}{n} \binom{m - n - 2}{0} = 1, \\ h_\chi^{n-1,1}(X) &= \binom{n}{n-1} \binom{m - n - 2}{1} = n(m - n - 2). \end{aligned}$$

We know that the moduli space $\mathcal{P}_{m,n}$ of the ordered sets of m hyperplanes in general linear position in \mathbb{P}^n is a quasi-projective algebraic variety of dimension $n(m - n - 2)$.

From [12], we can define the eigenperiod map

$$\Phi_\chi : \tilde{\mathcal{P}}_{m,n} \rightarrow D_\chi$$

as before on the universal covering space $\tilde{\mathcal{P}}_{m,n}$ of $\mathcal{P}_{m,n}$. Moreover from Theorem 8.3 of [12], [65] and [29], we know that the eigenperiod map $\Phi_\chi : \tilde{\mathcal{P}}_{m,n} \rightarrow D_\chi$ is a local isomorphism, if $|\mu| = n + 1$. Hence the Hodge subbundle \mathcal{H} of Definition 2.1 exists and is

$$\mathcal{H} = \text{Hom}(\mathcal{F}_\chi^n, \mathcal{F}_\chi^{n-1} / \mathcal{F}_\chi^n).$$

When $n = 1$, hyperplanes in \mathbb{P}^1 are points in \mathbb{P}^1 , and the arrangement of m hyperplanes in \mathbb{P}^1 with local system \mathcal{L}_μ , $|\mu| = 2$ is equivalent to the hypergeometric form

$$\omega_\mu = (z - z_1)^{-\mu_1} \cdots (z - z_{m-1})^{-\mu_{m-1}} dz.$$

This is Deligne-Mostow theory studied in their famous paper [11].

3 Extensions of moduli spaces

In this section, we study three completions, or extensions, of the moduli spaces and prove the equivalence of them. Then we prove that the extended period maps over the extended moduli spaces still satisfy the Griffiths transversality. Most of the results in this section are either review or extensions of the results from [37]. For reader's convenience we give proofs of some of the key results.

3.1 Extensions of moduli spaces

In this section we assume that \mathcal{Z}_m is quasi-projective. By Hironaka's resolution of singularity theorem, \mathcal{Z}_m admits a compactification $\overline{\mathcal{Z}}_m$ such that $\overline{\mathcal{Z}}_m$ is a smooth projective variety, and that $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ is a divisor with simple normal crossings.

Let $\mathcal{Z}'_m \supseteq \mathcal{Z}_m$ be the maximal subset of $\overline{\mathcal{Z}}_m$ to which the period map $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \rightarrow D/\Gamma$ extends continuously and let

$$\Phi_{\mathcal{Z}'_m} : \mathcal{Z}'_m \rightarrow D/\Gamma$$

be the extended map. Then one has the commutative diagram

$$\begin{array}{ccc} & \Phi_{\mathcal{Z}_m} & \\ & \curvearrowright & \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}'_m \xrightarrow{\Phi_{\mathcal{Z}'_m}} D/\Gamma. \end{array}$$

with $i : \mathcal{Z}_m \rightarrow \mathcal{Z}'_m$ the inclusion map.

The following result is a simple corollary of Griffiths [20].

Proposition 3.1. *The space \mathcal{Z}'_m is a Zariski open subset of $\overline{\mathcal{Z}}_m$ with*

$$\text{codim}_{\mathbb{C}}(\overline{\mathcal{Z}}_m \setminus \mathcal{Z}'_m) \geq 1,$$

and $\mathcal{Z}'_m \setminus \mathcal{Z}_m$ consists of the points of $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ around which the Picard-Lefschetz transformations are trivial. Moreover the extended period map

$$\Phi_{\mathcal{Z}'_m} : \mathcal{Z}'_m \rightarrow D/\Gamma$$

is a proper holomorphic map.

Proof. Since $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ is a divisor with simple normal crossings, for any point p in $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ we can find a neighborhood U of p in $\overline{\mathcal{Z}}_m$, which is isomorphic to a polycylinder Δ^n , such that

$$U \cap \mathcal{Z}_m \simeq (\Delta^*)^k \times \Delta^{N-k},$$

where N is the complex dimension of $\overline{\mathcal{Z}}_m$ and $0 \leq k \leq N$.

Let T_i , $1 \leq i \leq k$ be the image of the i -th fundamental group of $(\Delta^*)^k$ under the representation of monodromy. Then the T_i 's are called the Picard-Lefschetz transformations.

Let us define the subspace $\mathcal{Z}''_m \subset \overline{\mathcal{Z}}_m$ which contains \mathcal{Z}_m and the points in $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ around which the Picard-Lefschetz transformations are of finite order, hence trivial by Lemma 1.6. Now we claim that $\mathcal{Z}'_m = \mathcal{Z}''_m$.

From Theorem 9.6 in [20] and its proof, or Corollary 13.4.6 in [6], we know that \mathcal{Z}''_m is open and dense in $\overline{\mathcal{Z}}_m$ and the period map $\Phi_{\mathcal{Z}_m}$ extends to a holomorphic map

$$\Phi_{\mathcal{Z}''_m} : \mathcal{Z}''_m \rightarrow D/\Gamma$$

which is proper.

In fact, as proved in Theorem 3.1 of [64], which follows directly from Propositions 9.10 and 9.11 of [20], $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}''_m$ consists of the components of divisors in $\overline{\mathcal{Z}}_m$ whose Picard-Lefschetz transformations are of infinite order, and therefore \mathcal{Z}''_m is a Zariski open subset in $\overline{\mathcal{Z}}_m$. Hence by the definition of \mathcal{Z}'_m , we know that $\mathcal{Z}''_m \subseteq \mathcal{Z}'_m$.

Conversely, let q be any point in \mathcal{Z}'_m with image $u = \Phi_{\mathcal{Z}'_m}(q) \in D/\Gamma$. By the definition of \mathcal{Z}'_m , we can choose the points

$$q_k \in \mathcal{Z}_m, \quad k = 1, 2, \dots$$

such that $q_k \rightarrow q$ with images $u_k = \Phi_{\mathcal{Z}_m}(q_k) \rightarrow u$ as $k \rightarrow \infty$. Since

$$\Phi_{\mathcal{Z}_m''} : \mathcal{Z}_m'' \rightarrow D/\Gamma$$

is proper, the sequence

$$\{q_k\}_{k=1}^\infty \subset (\Phi_{\mathcal{Z}_m''})^{-1}(\{u_k\}_{k=1}^\infty)$$

has the limit point q in \mathcal{Z}_m'' , that is to say $q \in \mathcal{Z}_m''$ and $\mathcal{Z}_m' \subseteq \mathcal{Z}_m''$.

Therefore we have proved that $\mathcal{Z}_m' = \mathcal{Z}_m''$ and $\mathcal{Z}_m' \setminus \mathcal{Z}_m$ consists of the points around which the Picard-Lefschetz transformations are trivial.

Griffiths [20] proved that the extended period map $\Phi_{\mathcal{Z}_m'} : \mathcal{Z}_m' \rightarrow D/\Gamma$ is a proper holomorphic map. See also [61] in which $\Phi_{\mathcal{Z}_m'}$ is called the Griffiths extension. \square

3.2 Hodge metric completion

In Section 3.1 we have discussed the Griffiths extension

$$\Phi_{\mathcal{Z}_m'} : \mathcal{Z}_m' \rightarrow D/\Gamma.$$

In this section, we use the local Torelli theorem to give a geometric interpretation of \mathcal{Z}_m' , which is the Hodge metric completion of \mathcal{Z}_m .

In [23], Griffiths and Schmid studied the Hodge metric on the period domain D , which we denote by h . The Hodge metric is a complete homogeneous metric. In the case that local Torelli holds, the tangent maps of the period map $\Phi_{\mathcal{Z}_m}$ and the lifted period map Φ both are injective. It follows that the pull-backs of h by $\Phi_{\mathcal{Z}_m}$ and Φ on \mathcal{Z}_m and \mathcal{T} respectively are well-defined Kähler metrics. For convenience we will still call these pull-back metrics the Hodge metrics.

Let us denote by \mathcal{Z}_m^H the completion of \mathcal{Z}_m in $\overline{\mathcal{Z}_m}$ with respect to the Hodge metric. Then \mathcal{Z}_m^H is the smallest complete space with respect to the Hodge metric that contains \mathcal{Z}_m .

Now we recall some basic properties about metric completion space we are using in this paper. First we know that the metric completion space of a connected space is still connected. Therefore, \mathcal{Z}_m^H is connected.

Suppose (X, d) is a metric space with the metric d . Then the metric completion space of (X, d) is unique in the following sense: if \overline{X}_1 and \overline{X}_2 are complete metric spaces that both contain X as a dense set, then there exists an isometry

$$f : \overline{X}_1 \rightarrow \overline{X}_2$$

such that $f|_X$ is the identity map on X . Moreover, the metric completion space \overline{X} of X is the smallest complete metric space containing X in the sense that any other complete space that contains X as a subspace must also contain \overline{X} as a subspace. Hence the Hodge metric completion space \mathcal{Z}_m^H is unique up to isometry, although the compact space $\overline{\mathcal{Z}}_m$ may not be unique. This means that our definition of \mathcal{Z}_m^H is intrinsic.

Moreover, suppose \overline{X} is the metric completion space of the metric space (X, d) . If there is a continuous map $f : X \rightarrow Y$ which is a local isometry with Y a complete space, then there exists a continuous extension $\overline{f} : \overline{X} \rightarrow Y$ such that $\overline{f}|_X = f$. Since D/Γ with the Hodge metric h is complete, we can extend the period map to a continuous map

$$\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow D/\Gamma.$$

With the above preparations, we can prove the following proposition.

Proposition 3.2. *We have $\mathcal{Z}'_m = \mathcal{Z}_m^H$, and the extended period map*

$$\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow D/\Gamma$$

is proper and holomorphic.

Proof. Since we have the continuously extended map

$$\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow D/\Gamma,$$

we see that $\mathcal{Z}_m^H \subseteq \mathcal{Z}'_m$ by the definition of \mathcal{Z}'_m .

Conversely, let q be any point in $\mathcal{Z}'_m \setminus \mathcal{Z}_m$. Fix a point p in \mathcal{Z}_m . Since q is mapped into D/Γ via the extended period map $\Phi_{\mathcal{Z}'_m}$, we get that q has finite Hodge distance from p . Here the Hodge distance is defined by the pull-back Hodge metric on \mathcal{Z}_m . Therefore q lies in the Hodge metric completion \mathcal{Z}_m^H of \mathcal{Z}_m . So we get that $\mathcal{Z}'_m \subseteq \mathcal{Z}_m^H$.

We now have proved that $\mathcal{Z}'_m = \mathcal{Z}_m^H$, which, together with Proposition 3.1, completes the proof of the proposition. \square

From now on we will use \mathcal{Z}_m^H for our discussions, since it has explicit geometric structure which is convenient for our discussion.

3.3 Properties of the extended period map

Let \mathcal{T}_m^H be the universal cover of \mathcal{Z}_m^H with the universal covering map

$$\pi_m^H : \mathcal{T}_m^H \rightarrow \mathcal{Z}_m^H.$$

We then have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_m} & \mathcal{T}_m^H & \xrightarrow{\Phi_m^H} & D \\ \downarrow \pi_m & & \downarrow \pi_m^H & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m^H}} & D/\Gamma, \end{array} \quad (21)$$

where i_m is the lifting of $i \circ \pi_m$ with respect to the covering map π_m^H and Φ_m^H is the lifting of $\Phi_{\mathcal{Z}_m^H} \circ \pi_m^H$ with respect to the covering map

$$\pi_D : D \rightarrow D/\Gamma.$$

As the lifts of the holomorphic maps i and $\Phi_{\mathcal{Z}_m^H}$ to universal covers, both i_m and Φ_m^H are easily seen to be holomorphic maps. There are different choices of i_m and Φ_m^H , but the elementary topological argument as given in Lemma A.1 in the Appendix of [37] shows that we can choose i_m and Φ_m^H such that

$$\Phi = \Phi_m^H \circ i_m.$$

Let $\mathcal{T}_m \subseteq \mathcal{T}_m^H$ be defined by

$$\mathcal{T}_m = i_m(\mathcal{T}).$$

Then we have the following lemma.

Lemma 3.3. $\mathcal{T}_m = (\pi_m^H)^{-1}(\mathcal{Z}_m)$, and $i_m : \mathcal{T} \rightarrow \mathcal{T}_m$ is a covering map.

Proof. The proof is an elementary argument in basic topology. First, from diagram (21), we see that

$$\pi_m^H(\mathcal{T}_m) = \pi_m^H(i_m(\mathcal{T})) = i(\pi_m(\mathcal{T})) = \mathcal{Z}_m,$$

hence $\mathcal{T}_m \subseteq (\pi_m^H)^{-1}(\mathcal{Z}_m)$.

Conversely, for any $q \in (\pi_m^H)^{-1}(\mathcal{Z}_m)$, we need to prove that $q \in \mathcal{T}_m$. Let $p = \pi_m^H(q)$. If there exists a $r \in \pi_m^{-1}(q)$ such that $i_m(r) = q$, then we are done. Otherwise, since \mathcal{T}_m^H is connected and thus path connected, we can draw a curve γ from $i_m(r)$ to q for some $r \in \pi_m^{-1}(q)$. Then we get a circle

$$\Gamma = \pi_m^H(\gamma)$$

in \mathcal{Z}_m^H . But Lemma A.2 in the Appendix of [37] implies that we can choose Γ contained in \mathcal{Z}_m .

Note that $p \in \Gamma$. Since $\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$ is covering map, we can lift Γ to a unique curve $\tilde{\gamma}$ from r to some $r' \in \pi_m^{-1}(p)$. Now both γ and $i_m(\tilde{\gamma})$ map to Γ via the covering map

$$\pi_m^H : \mathcal{T}_m^H \rightarrow \mathcal{Z}_m^H,$$

that is γ and $i_m(\tilde{\gamma})$ are both the lifts of Γ . By the uniqueness of homotopy lifting, $i_m(r') = q$, i.e. $q \in i_m(\mathcal{T}) = \mathcal{T}_m$. Therefore we have proved that

$$\mathcal{T}_m = (\pi_m^H)^{-1}(\mathcal{Z}_m),$$

and \mathcal{T}_m is a smooth complex submanifold of \mathcal{T}_m^H .

To show that i_m is a covering map, note that for any small enough open neighborhood U in \mathcal{T}_m , the restricted map

$$\pi_m^H|_U : U \rightarrow V = \pi_m^H(U) \subset \mathcal{Z}_m$$

is biholomorphic, and there exists a disjoint union $\cup_i V_i$ of open subsets in \mathcal{T} such that

$$\cup_i V_i = (\pi_m)^{-1}(V)$$

and $\pi_m|_{V_i} : V_i \rightarrow V$ is biholomorphic. Then from the commutativity of diagram (21), we have that $\cup_i V_i = (i_m)^{-1}(U)$ and

$$i_m|_{V_i} : V_i \rightarrow U$$

is biholomorphic. Therefore $i_m : \mathcal{T} \rightarrow \mathcal{T}_m$ is also a covering map. \square

Lemma 3.3 also implies that \mathcal{T}_m is an open dense complex submanifold of \mathcal{T}_m^H and $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is an analytic subvariety of \mathcal{T}_m^H with $\text{codim}_{\mathbb{C}}(\mathcal{T}_m^H \setminus \mathcal{T}_m) \geq 1$.

Indeed, from Proposition 3.1 and Proposition 3.2, we have that $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ is an analytic subvariety of \mathcal{Z}_m^H . On the other hand, from Lemma 3.3, we know that $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is the inverse image of $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ under the covering map

$$\pi_m^H : \mathcal{T}_m^H \rightarrow \mathcal{Z}_m^H.$$

This implies that $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is an analytic subvariety of \mathcal{T}_m^H .

The identification $\mathcal{T}_m = (\pi_m^H)^{-1}(\mathcal{Z}_m)$ implies that \mathcal{T}_m is a covering of \mathcal{Z}_m . Since \mathcal{T} is the universal cover, we have proved that \mathcal{T} is also the universal cover of \mathcal{T}_m for each m .

Lemma 3.4. *The extended holomorphic map $\Phi_m^H : \mathcal{T}_m^H \rightarrow D$ satisfies the Griffiths transversality.*

Proof. Let $T_h^{1,0}D$ be the horizontal subbundle. Since $\Phi_m^H : \mathcal{T}_m^H \rightarrow D$ is a holomorphic map, the tangent map

$$d\Phi_m^H : T^{1,0}\mathcal{T}_m^H \rightarrow T^{1,0}D$$

is at least continuous. We only need to show that the image of $d\Phi_m^H$ is contained in the horizontal tangent bundle $T_h^{1,0}D$.

Since $T_h^{1,0}D$ is closed in $T^{1,0}D$, the inverse image $(d\Phi_m^H)^{-1}(T_h^{1,0}D)$ is a closed subset in $T^{1,0}\mathcal{T}_m^H$. But $\Phi_m^H|_{\mathcal{T}_m}$ satisfies the Griffiths transversality, i.e.

$$(d\Phi_m^H)^{-1}(T_h^{1,0}D)$$

contains $T^{1,0}\mathcal{T}_m$, which is open in $T^{1,0}\mathcal{T}_m^H$. Hence $(d\Phi_m^H)^{-1}(T_h^{1,0}D)$ contains the closure of $T^{1,0}\mathcal{T}_m$, which is $T^{1,0}\mathcal{T}_m^H$. \square

4 Boundedness of period maps and affine structures

In this section we review the definitions and basic properties of period domains from Lie theory point of views. We consider the nilpotent Lie subalgebra \mathfrak{n}_+ of \mathfrak{g} and define the corresponding unipotent group to be

$$N_+ = \exp(\mathfrak{n}_+).$$

Then we recall some results from our previous paper [37], including the boundedness of period maps which asserts, as given in Theorem 4.1, that the image of the period map $\Phi : \mathcal{T} \rightarrow D$ lies in $N_+ \cap D$ as a bounded subset, and the result that there exists a global affine structure on \mathcal{T} which is defined by the holomorphic map in (24), see Theorem 4.4.

First we fix a point p in \mathcal{T} and its image $o = \Phi(p)$ as the reference points or base points. Let us introduce the notion of adapted basis for the given Hodge decomposition or Hodge filtration. For the fixed point $p \in \mathcal{T}$ and $f^k = \dim F_p^k$ for any $0 \leq k \leq n$, we call a basis

$$\xi = \{\xi_0, \dots, \xi_{f^{n-1}}, \xi_{f^n}, \dots, \xi_{f^{n-1}-1}, \dots, \xi_{f^{k+1}}, \dots, \xi_{f^{k-1}}, \dots, \xi_{f^1}, \dots, \xi_{f^0-1}\}$$

of $H_{pr}^n(X_p, \mathbb{C})$ an adapted basis for the given Hodge decomposition

$$H_{pr}^n(X_p, \mathbb{C}) = H_p^{n,0} \oplus H_p^{n-1,1} \oplus \dots \oplus H_p^{1,n-1} \oplus H_p^{0,n},$$

if it satisfies $H_p^{k,n-k} = \text{Span}_{\mathbb{C}}\{\xi_{f^{k+1}}, \dots, \xi_{f^{k-1}}\}$ with $h^{k,n-k} = f^k - f^{k+1}$.

We call a basis

$$\zeta = \{\zeta_0, \dots, \zeta_{f^{n-1}}, \zeta_{f^n}, \dots, \zeta_{f^{n-1}-1}, \dots, \zeta_{f^{k+1}}, \dots, \zeta_{f^{k-1}}, \dots, \zeta_{f^1}, \dots, \zeta_{f^0-1}\}$$

of $H_{pr}^n(X_p, \mathbb{C})$ an adapted basis for the given filtration

$$F_p^n \subseteq F_p^{n-1} \subseteq \dots \subseteq F_p^0$$

if it satisfies $F_p^k = \text{Span}_{\mathbb{C}}\{\zeta_0, \dots, \zeta_{f^{k-1}}\}$ with $\dim_{\mathbb{C}} F_p^k = f^k$. For the convenience of notations, we set $f^{n+1} = 0$ and $m = f^0$.

The blocks of an $m \times m$ matrix T are set as follows: for each $0 \leq \alpha, \beta \leq n$, the (α, β) -th block $T^{\alpha, \beta}$ is

$$T^{\alpha, \beta} = (T_{ij})_{f^{-\alpha+n+1} \leq i \leq f^{-\alpha+n-1}, f^{-\beta+n+1} \leq j \leq f^{-\beta+n-1}}, \quad (22)$$

where T_{ij} is the entries of the matrix T . In particular, $T = [T^{\alpha, \beta}]_{0 \leq \alpha, \beta \leq n}$ is called a block lower triangular matrix if $T^{\alpha, \beta} = 0$ whenever $\alpha < \beta$.

Let $H_{\mathbb{F}} = H_{pr}^n(X, \mathbb{F})$, where \mathbb{F} can be chosen as $\mathbb{Z}, \mathbb{R}, \mathbb{C}$. We define the complex Lie group

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

and the real one

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\}.$$

Griffiths in [17] showed that $G_{\mathbb{C}}$ acts on \check{D} transitively and so does $G_{\mathbb{R}}$ on D . The stabilizer of $G_{\mathbb{C}}$ on \check{D} at the fixed point o is

$$B = \{g \in G_{\mathbb{C}} \mid gF_p^k = F_p^k, 0 \leq k \leq n\},$$

and the one of $G_{\mathbb{R}}$ on D is $V = B \cap G_{\mathbb{R}}$. Thus we can realize \check{D} as

$$\check{D} = G_{\mathbb{C}}/B, \text{ and } D = G_{\mathbb{R}}/V$$

so that \check{D} is an algebraic manifold and $D \subseteq \check{D}$ is an open complex submanifold.

The Lie algebra \mathfrak{g} of the complex Lie group $G_{\mathbb{C}}$ is

$$\mathfrak{g} = \{X \in \text{End}(H_{\mathbb{C}}) \mid Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\},$$

and the real subalgebra

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid XH_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$$

is the Lie algebra of $G_{\mathbb{R}}$. Note that \mathfrak{g} is a simple complex Lie algebra and contains \mathfrak{g}_0 as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0$. Let us denote the complex conjugation of \mathfrak{g} with respect to the real form \mathfrak{g}_0 by τ_0 .

On the Lie algebra \mathfrak{g} we can give a Hodge structure of weight zero by

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k, -k} \quad \text{with} \quad \mathfrak{g}^{k, -k} = \{X \in \mathfrak{g} \mid XH_p^{r, n-r} \subseteq H_p^{r+k, n-r-k}, \forall r\}.$$

By definition of B the Lie algebra \mathfrak{b} of B has the form $\mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k, -k}$. Then the Lie algebra \mathfrak{v}_0 of V is

$$\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \bar{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}.$$

With the above isomorphisms, the holomorphic tangent space of \check{D} at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra $\mathfrak{n}_+ := \bigoplus_{k \geq 1} \mathfrak{g}^{-k,k}$. Then one gets the isomorphism $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. We denote the corresponding unipotent Lie group to be

$$N_+ = \exp(\mathfrak{n}_+).$$

As $\text{Ad}(g)(\mathfrak{g}^{k,-k})$ is in $\bigoplus_{i \geq k} \mathfrak{g}^{i,-i}$ for each $g \in B$, the subspace $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \subseteq \mathfrak{g}/\mathfrak{b}$ defines an $\text{Ad}(B)$ -invariant subspace. By left translation via $G_{\mathbb{C}}$, $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$ gives rise to a $G_{\mathbb{C}}$ -invariant holomorphic subbundle of the holomorphic tangent bundle. It will be denoted by $T_h^{1,0}\check{D}$, and will be referred to as the horizontal tangent subbundle. One can check that this construction does not depend on the choice of the base point.

The horizontal tangent subbundle, restricted to D , determines a subbundle $T_h^{1,0}D$ of the holomorphic tangent bundle $T^{1,0}D$ of D . The $G_{\mathbb{C}}$ -invariance of $T_h^{1,0}\check{D}$ implies the $G_{\mathbb{R}}$ -invariance of $T_h^{1,0}D$. Note that the horizontal tangent subbundle $T_h^{1,0}D$ can also be constructed as the associated bundle of the principle bundle $V \rightarrow G_{\mathbb{R}} \rightarrow D$ with the adjoint representation of V on the space $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$.

Let \mathcal{F}^k , $0 \leq k \leq n$ be the Hodge bundles on D with fibers $\mathcal{F}_s^k = F_s^k$ for any $s \in D$. As another interpretation of the horizontal bundle in terms of the Hodge bundles $\mathcal{F}^k \rightarrow \check{D}$, $0 \leq k \leq n$, one has

$$T_h^{1,0}\check{D} \simeq T^{1,0}\check{D} \cap \bigoplus_{k=1}^n \text{Hom}(\mathcal{F}^k/\mathcal{F}^{k+1}, \mathcal{F}^{k-1}/\mathcal{F}^k). \quad (23)$$

We remark that the elements in N_+ can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; elements in B can be realized as nonsingular block upper triangular matrices. If $c, c' \in N_+$ such that $cB = c'B$ in \check{D} , then

$$c'^{-1}c \in N_+ \cap B = \{I\},$$

i.e. $c = c'$. This means that the matrix representation in N_+ of the unipotent orbit $N_+(o)$ is unique. Therefore with the fixed base point $o \in \check{D}$, we can identify N_+ with its unipotent orbit $N_+(o)$ in \check{D} by identifying an element $c \in N_+$ with $[c] = cB$ in \check{D} . Then

$$N_+ \subseteq \check{D}$$

is meaningful. In particular, when the base point o is in D , we have $N_+ \cap D \subseteq D$.

We define a submanifold of the Teichmüller space

$$\check{\mathcal{T}} = \Phi^{-1}(N_+ \cap D).$$

In Proposition 1.3 of [37], we showed that $\mathcal{T} \setminus \check{\mathcal{T}}$ is an analytic subvariety of \mathcal{T} with $\text{codim}_{\mathbb{C}}(\mathcal{T} \setminus \check{\mathcal{T}}) \geq 1$.

At the base point

$$\Phi(p) = o \in N_+ \cap D,$$

we have identifications of the tangent spaces

$$T_o^{1,0} N_+ = T_o^{1,0} D \simeq \mathfrak{n}_+,$$

and the exponential map

$$\exp : \mathfrak{n}_+ \rightarrow N_+$$

is an isomorphism.

The Hodge metric on $T_o^{1,0} D$, which is the restriction of the natural homogeneous metric on D at the base point, induces an Euclidean metric on N_+ so that $\exp : \mathfrak{n}_+ \rightarrow N_+$ is an isometry.

In Theorem 4.1 of [37], we prove the following theorem, which was raised by Griffiths as Conjecture 10.1 in his paper [21].

Theorem 4.1. *The image of $\Phi : \mathcal{T} \rightarrow D$ lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+ .*

We remark that Theorem 4.1 is proved in [37] for general period maps and analytic families. To be precise, let $f : \mathfrak{X} \rightarrow S$ be any analytic family of polarized manifolds over a quasi-projective manifold S , and let \tilde{S} be the universal covering space of S with the universal covering map $\pi : \tilde{S} \rightarrow S$. We can define the period map

$$\Phi : S \rightarrow D/\Gamma$$

with the monodromy representation $\rho : \pi_1(S) \rightarrow \Gamma$, and the lifted period map

$$\tilde{\Phi} : \tilde{S} \rightarrow D$$

such that the following diagram commutes

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\Phi}} & D \\ \downarrow \pi & & \downarrow \pi_D \\ S & \xrightarrow{\Phi} & D/\Gamma. \end{array}$$

Then the conclusion of Theorem 4.1 still holds for the lifted period map $\tilde{\Phi} : \tilde{S} \rightarrow D$.

Recall that we have defined the extended period map

$$\Phi_m^H : \mathcal{T}_m^H \rightarrow D$$

in diagram (21). Then the extended period map Φ_m^H is still bounded by Corollary 6.1 in [37], which follows directly from Theorem 4.1 and the Riemann extension theorem.

Corollary 4.2. *The image of the extended period map $\Phi_m^H : \mathcal{T}_m^H \rightarrow D$ lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+ .*

Now we review the definition of complex affine structure on a complex manifold.

Definition 4.3. *Let M be a complex manifold of complex dimension n . If there is a coordinate cover $\{(U_i, \varphi_i); i \in I\}$ of M such that $\varphi_{ik} = \varphi_i \circ \varphi_k^{-1}$ is a holomorphic affine transformation on \mathbb{C}^n whenever $U_i \cap U_k$ is not empty, then $\{(U_i, \varphi_i); i \in I\}$ is called a complex affine coordinate cover on M and it defines a holomorphic affine structure on M .*

Let us consider

$$\mathfrak{a} = d\Phi_p(\mathbb{T}_p^{1,0}\mathcal{T}) \subseteq \mathbb{T}_o^{1,0}D \simeq \mathfrak{n}_+$$

where p is the base point in \mathcal{T} with $\tilde{\Phi}(p) = o$. By Griffiths transversality, $\mathfrak{a} \subseteq \mathfrak{g}^{-1,1}$ is an abelian subspace, therefore $\mathfrak{a} \subseteq \mathfrak{n}_+$ is an abelian subalgebra of \mathfrak{n}_+ determined by the tangent map of the period map

$$d\Phi : \mathbb{T}^{1,0}\mathcal{T} \rightarrow \mathbb{T}^{1,0}D.$$

Consider the corresponding Lie group

$$A \triangleq \exp(\mathfrak{a}) \subseteq N_+.$$

Then A can be considered as a complex Euclidean subspace of N_+ with the induced Euclidean metric from N_+ .

Define the projection map $P : N_+ \cap D \rightarrow A \cap D$ by

$$P = \exp \circ p \circ \exp^{-1}$$

where $\exp^{-1} : N_+ \rightarrow \mathfrak{n}_+$ is the inverse of the isometry $\exp : \mathfrak{n}_+ \rightarrow N_+$, and

$$p : \mathfrak{n}_+ \rightarrow \mathfrak{a}$$

is the projection map from the complex Euclidean space \mathfrak{n}_+ to its Euclidean subspace \mathfrak{a} .

The period map $\Phi : \mathcal{T} \rightarrow N_+ \cap D$ composed with the projection map P gives a holomorphic map

$$\Psi : \mathcal{T} \rightarrow A \cap D, \tag{24}$$

that is $\Psi = P \circ \Phi$.

Moreover in Theorem 6.4 of [37], we proved under the condition of strong local Torelli that the holomorphic map Ψ is bounded and defines a global affine structure on the Teichmüller space \mathcal{T} .

Theorem 4.4. *The holomorphic map Ψ defines a global affine structure on the Teichmüller space \mathcal{T} of polarized manifolds containing (X, L) , provided that the strong local Torelli holds for the polarized manifold (X, L) in the T -class.*

This theorem follows from the boundedness of the period map Φ , which gives the boundedness of Ψ , and strong local Torelli for Φ which gives the nondegeneracy of Ψ , so that we can pull back the affine structure of $A \simeq \mathbb{C}^N$.

5 Strong local Torelli implies global Torelli

In this section, we prove our main theorem in this paper, which asserts that strong local Torelli implies global Torelli on the Torelli space for polarized manifolds in the T -class.

First, we prove the existence of the affine structure on \mathcal{T}_m^H induced by the extended map

$$\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D$$

which follows from the proof that Ψ_m^H is nondegenerate. Then we apply a lemma of Griffiths and Wolf to conclude that the holomorphic map Ψ_m^H is a covering map. Furthermore, by using the affine structures we show that Ψ_m^H is in fact a biholomorphic map, which implies that \mathcal{T}_m^H does not depend on m .

With this we prove that the image \mathcal{T}_m of $i_m : \mathcal{T} \rightarrow \mathcal{T}_m^H$ is identical to the Torelli space \mathcal{T}' . Here the level structures again play important roles. Moreover, we prove that the extended period map

$$\Phi_m^H : \mathcal{T}_m^H \rightarrow D$$

is an embedding. At last we arrive at our theorem asserting that the period map

$$\Phi' : \mathcal{T}' \rightarrow N_+ \cap D$$

is injective.

Note that in this paper, unless otherwise specified, we only consider the polarized manifold (X, L) , which belongs to the T-class and satisfies strong local Torelli.

Let \mathcal{Z}_m be defined as in Definition 1.1 and $\mathcal{Z}_m^H, \mathcal{T}, \mathcal{T}'$ and \mathcal{T}_m^H be defined as before. First recall diagram (21),

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_m} & \mathcal{T}_m^H & \xrightarrow{\Phi_m^H} & D \\ \downarrow \pi_m & & \downarrow \pi_m^H & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m^H}} & D/\Gamma. \end{array} \quad (25)$$

From Corollary 4.2, we can define the holomorphic map

$$\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D$$

by the extended period map $\Phi_m^H : \mathcal{T}_m^H \rightarrow N_+ \cap D$ composed with the projection map

$$P : N_+ \cap D \rightarrow A \cap D.$$

Recall that we have defined

$$\mathcal{T}_m = i_m(\mathcal{T}) \subset \mathcal{T}_m^H$$

in Section 3. Lemma 3.3 proves that $\mathcal{T}_m \subset \mathcal{T}_m^H$ is a Zariski open submanifold, and furthermore $i_m : \mathcal{T} \rightarrow \mathcal{T}_m$ is a covering map, and

$$\mathcal{T}_m = i_m(\mathcal{T}) = (\pi_m^H)^{-1}(\mathcal{Z}).$$

Therefore we have the holomorphic map

$$\Psi_m : \mathcal{T}_m \rightarrow A \cap D$$

by restricting Ψ_m^H to \mathcal{T}_m ,

$$\Psi_m = \Psi_m^H|_{\mathcal{T}_m},$$

or equivalently $\Psi_m = P \circ \Phi_m$.

We can choose a small neighborhood U of any point in \mathcal{T}_m such that

$$\pi_m^H : U \rightarrow V = \pi_m^H(U) \subset \mathcal{Z}_m^H,$$

is a biholomorphic map. We can shrink U and V simultaneously such that $\pi_m^{-1}(V) = \cup_\alpha W_\alpha$ and $\pi_m : W_\alpha \rightarrow V$ is also biholomorphic. Choose any W_α and denote it by $W = W_\alpha$. Then $i_m : W \rightarrow U$ is a biholomorphic map. Since

$$\Psi = \Psi_m^H \circ i_m = \Psi_m \circ i_m,$$

we have $\Psi|_W = \Psi_m|_U \circ i_m|_W$.

Theorem 4.4 implies that $\Psi|_W$ is biholomorphic onto its image, if we shrink W , V and U again. Therefore

$$\Psi_m|_U : U \rightarrow A \cap D$$

is biholomorphic onto its image. By pulling back the affine coordinate chart in $A \simeq \mathbb{C}^N$, we get an induced affine structure on \mathcal{T}_m such that Ψ_m is an affine map.

In conclusion, we have proved the following lemma.

Lemma 5.1. *The holomorphic map $\Psi_m : \mathcal{T}_m \rightarrow A \cap D$ is a local embedding. In particular, Ψ_m defines a global holomorphic affine structure on \mathcal{T}_m .*

Next we will prove that the affine structure $\Psi_m : \mathcal{T}_m \rightarrow A \cap D$ can be extended to a global affine structure on \mathcal{T}_m^H , which is equivalent to show that

$$\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D$$

is nondegenerate.

Definition 5.2. *Let M be a complex manifold and $N \subset M$ a closed subset. Let $E_0 \rightarrow M \setminus N$ be a holomorphic vector bundle. Then E_0 is called holomorphically trivial along N , if for any point $x \in N$, there exists an open neighborhood U of x in M such that $E_0|_{U \setminus N}$ is holomorphically trivial.*

The following elementary lemma will be needed in the following argument.

Lemma 5.3. *The holomorphic vector bundle $E_0 \rightarrow M \setminus N$ can be extended to a unique holomorphic vector bundle $E \rightarrow M$ such that $E|_{M \setminus N} = E_0$, if and only if E_0 is holomorphically trivial along N .*

Proof. The proof is taken from Proposition 4.4 of [44]. We include it here for the convenience of the readers.

If such an extension $E \rightarrow M$ exists uniquely, then we can take a neighborhood U of x such that $E|_U$ is trivial, and hence $E_0|_{U \setminus N}$ is trivial.

Conversely, suppose that for any point $x \in N$, there exists an open neighborhood U_x of x in M such that $E_0|_{U_x \setminus N}$ is trivial. We can extend E_0 trivially to each U_x . Let us cover N by the open sets $\{U_x : x \in N\}$. For any $y \in M \setminus N$, we can choose an open neighborhood V_y of y such that $V_y \cap N = \emptyset$. Then we can cover M by the open sets in

$$\mathcal{C} = \{U_x : x \in N\} \cup \{V_y : y \in M \setminus N\}.$$

We denote Ψ_V^U be the transition function of E_0 on $U \cap V$ if $U \cap V \neq \emptyset$, where U, V are open subsets of $M \setminus N$ and E_0 is trivial on U and V .

If $V_y \cap V_{y'} \neq \emptyset$, then $\Psi_{V_{y'}}^{V_y}$ is well-defined, since $V_y \cap V_{y'} \subset M \setminus N$. Then we define the transition function $\varphi_{V_{y'}}^{V_y} = \Psi_{V_{y'}}^{V_y}$. For the case of U_x and V_y , we can use the same argument to get the transition functions $\varphi_{V_y}^{U_x} = \Psi_{V_y}^{U_x}$ and $\varphi_{U_x}^{V_y} = \Psi_{U_x}^{V_y}$.

The remaining case is that $U_x \cap U_{x'} \neq \emptyset$, where $x, x' \in N$. Then the transition function $\Psi_{U_{x'} \setminus N}^{U_x \setminus N}$ is well-defined and we define the transition function $\varphi_{U_{x'}}^{U_x} = \Psi_{U_{x'} \setminus N}^{U_x \setminus N}$. Clearly the newly defined transition functions φ_V^U , for any $U, V \in \mathcal{C}$ satisfy that

$$\varphi_V^U \circ \varphi_U^V = \text{id}, \text{ and } \varphi_V^U \circ \varphi_W^V \circ \varphi_U^W = \text{id}.$$

Therefore these transition functions determine a unique holomorphic vector bundle $E \rightarrow M$ such that $E|_{M \setminus N} = E_0$. \square

Let \mathcal{H} be the holomorphic Hodge subbundle in Definition 2.1,

$$\mathcal{H} \subset \bigoplus_{k=1}^n \text{Hom}(\mathcal{F}^k / \mathcal{F}^{k+1}, \mathcal{F}^{k-1} / \mathcal{F}^k),$$

such that

$$d\Phi : T^{1,0}\mathcal{T} \xrightarrow{\sim} \Phi^*\mathcal{H}$$

is an isomorphism of holomorphic vector bundles on \mathcal{T} .

Let \mathcal{H}_A be the restriction of \mathcal{H} on $A \cap D$. Then by the definition of A , the holomorphic tangent bundle $T^{1,0}(A \cap D)$ is isomorphic to \mathcal{H}_A .

Theorem 4.4 and Lemma 5.1 give the natural isomorphisms of the holomorphic vector bundles over \mathcal{T} and \mathcal{T}_m respectively

$$\begin{aligned} d\Psi &: T^{1,0}\mathcal{T} \simeq \Psi^*\mathcal{H}_A, \\ d\Psi_m &: T^{1,0}\mathcal{T}_m \simeq \Psi_m^*\mathcal{H}_A. \end{aligned}$$

We remark that the period map $\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow D/\Gamma$ can be lifted to the universal cover to get

$$\Phi_m^H : \mathcal{T}_m^H \rightarrow D,$$

which is due to the fact that around any point in $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$, the monodromy group action is trivial as proved in Proposition 3.1 and Proposition 3.2.

In fact the period map $\Phi_{\mathcal{Z}_m}$ has extension $\Phi_{\mathcal{Z}_m^H}$ as proved by Griffiths, in Theorem 9.6 of [20], therefore $\Psi_m : \mathcal{T}_m \rightarrow A \cap D$ has extension to

$$\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D$$

which implies that the Hodge bundle $\Psi_m^* \mathcal{H}_A$ has the natural extension $(\Psi_m^H)^* \mathcal{H}_A$ over \mathcal{T}_m^H .

The following lemma is proved by using extension of the period map.

Lemma 5.4. *The isomorphism $\mathbb{T}^{1,0} \mathcal{T}_m \simeq \Psi_m^* \mathcal{H}_A$ of holomorphic vector bundles over \mathcal{T}_m has a unique extension to an isomorphism of holomorphic vector bundles over \mathcal{T}_m^H as*

$$\mathbb{T}^{1,0} \mathcal{T}_m^H \simeq (\Psi_m^H)^* \mathcal{H}_A.$$

Proof. For any $p \in \mathcal{T}_m^H \setminus \mathcal{T}_m$, we can choose an open neighborhood $U_p \subset \mathcal{T}_m^H$ of p such that $\Psi_m^H(U_p)$ is contained in an open neighborhood $W \subset A \cap D$ on which \mathcal{H}_A is trivial.

Let

$$\bar{o} = \Psi_m^H(p) \in W \text{ and } H_A = \mathcal{H}_A|_{\bar{o}}.$$

Then we have the following trivializations of the Hodge subbundles

$$\mathcal{H}_A|_W \simeq W \times H_A$$

and

$$(\Psi_m^H)^* \mathcal{H}_A|_{U_p} \simeq (\Psi_m^H)^* (\mathcal{H}_A|_W)|_{U_p} \simeq (\Psi_m^H)^* (W \times H_A)|_{U_p} \simeq U_p \times H_A.$$

Let $U = U_p \cap \mathcal{T}_m$. Then the trivialization of the Hodge subbundle on U is

$$\Psi_m^* \mathcal{H}_A|_U = ((\Psi_m^H)^* \mathcal{H}_A|_{U_p})|_U \simeq U \times H_A,$$

which implies that $\Psi_m^* \mathcal{H}_A$ is holomorphically trivial along $\mathcal{T}_m^H \setminus \mathcal{T}_m$. Therefore by Lemma 5.3, the Hodge subbundle $\Psi_m^* \mathcal{H}_A$ over \mathcal{T}_m has a unique extension over \mathcal{T}_m^H , which is $(\Psi_m^H)^* \mathcal{H}_A$ by continuity.

Note that Lemma 5.3 also implies that the holomorphic tangent bundle $\mathbb{T}^{1,0} \mathcal{T}_m$, which is isomorphic to $\Psi_m^* \mathcal{H}_A$, is holomorphically trivial along $\mathcal{T}_m^H \setminus \mathcal{T}_m$. Hence $\mathbb{T}^{1,0} \mathcal{T}_m$ has a unique extension, which is obviously $\mathbb{T}^{1,0} \mathcal{T}_m^H$. By the uniqueness of such extension we can conclude that

$$\mathbb{T}^{1,0} \mathcal{T}_m^H \simeq (\Psi_m^H)^* \mathcal{H}_A.$$

□

With the above preparations, we can prove the following theorem, which is key to our main theorem.

Theorem 5.5. *The holomorphic map $\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D$ is nondegenerate. Hence Ψ_m^H defines a global affine structure on \mathcal{T}_m^H .*

Proof. For any point $q \in \mathcal{T}_m^H \setminus \mathcal{T}_m$, we can choose a neighborhood U_q of q in \mathcal{T}_m^H such that

$$\mathbb{T}^{1,0}\mathcal{T}_m^H \simeq (\Psi_m^H)^*\mathcal{H}_A$$

is trivial on U_q . Moreover, we can shrink U_q so that $\Psi_m^H(U_q) \subseteq V$, where $V \subset A \cap D$ on which \mathcal{H}_A is trivial. We choose a basis

$$\{\Lambda_1, \dots, \Lambda_N\}$$

of $\mathcal{H}_A|_V$, which is parallel with respect to the natural affine structure on $A \cap D$. Let

$$\mu_i = ((\Psi_m^H)^*\Lambda_i)|_{U_q}, \quad 1 \leq i \leq N.$$

Then $\{\mu_1, \dots, \mu_n\}$ is a basis of

$$\mathbb{T}^{1,0}\mathcal{T}_m^H|_{U_q} \simeq (\Psi_m^H)^*\mathcal{H}_A|_{U_q}.$$

Let $U = U_q \cap \mathcal{T}_m$ and consider a sequence of points $q_k \in U$ such that $q_k \rightarrow q$ as $k \rightarrow \infty$. Since $\Psi_m : \mathcal{T}_m \rightarrow A \cap D$ defines a global affine structure on \mathcal{T}_m , for any $k \geq 1$ we have

$$d\Psi_m(\mu_i|_{q_k}) = \sum_j A_{ij}(q_k)\Lambda_j|_{o_k},$$

for some non-singular matrix $(A_{ij}(q_k))_{1 \leq i, j \leq N}$, where $o_k = \Psi_m(q_k)$.

Since the affine structure on \mathcal{T}_m is induced from that of $A \cap D$, both the bases

$$\{\mu_1|_U, \dots, \mu_N|_U\} \text{ and } \{\Lambda_1, \dots, \Lambda_N\}$$

are parallel with respect to the affine structures on \mathcal{T}_m and $A \cap D$ respectively. Hence we have that the matrix $(A_{ij}(q_k)) = (A_{ij})$ is constant matrix.

Therefore

$$\begin{aligned}
d\Psi_m^H(\mu_i|_q) &= \lim_{k \rightarrow \infty} d\Psi_m^H(\mu_i|_{q_k}) \\
&= \lim_{k \rightarrow \infty} d\Psi_m(\mu_i|_{q_k}) \\
&= \lim_{k \rightarrow \infty} \sum_j A_{ij} \Lambda_j|_{o_k} \\
&= \sum_j A_{ij} \Lambda_j|_{o_\infty},
\end{aligned}$$

where $o_\infty = \Psi_m^H(q)$. Since the matrix $(A_{ij})_{1 \leq i, j \leq N}$ is nonsingular, the tangent map $d\Psi_m^H$ is nondegenerate. \square

Before moving on, we would like to point out the geometric intuition behind our definition of strong local Torelli. First note that in the proof of Theorem 5.5, we used substantially the identification

$$d\Psi_m : T^{1,0}\mathcal{T}_m \simeq \Psi_m^* \mathcal{H}_A$$

on \mathcal{T}_m , which is explicitly given by the contraction map $KS(v)_\perp$ at a point $(q, v) \in T_q^{1,0}\mathcal{T}_m$ given by (12). Here recall that

$$KS : T_q^{1,0}\mathcal{T}_m \rightarrow H^1(X_q, \Theta_{X_q})$$

is the Kodaira-Spencer map for any $q \in \mathcal{T}_m$. From this explicit expression, one can see that this identification of bundles

$$T^{1,0}\mathcal{T}_m \simeq \Psi_m^* \mathcal{H}_A$$

on \mathcal{T}_m only depends on the tangent vector v . Moreover, since $\Psi_m^* \mathcal{H}_A$ extends trivially to $(\Psi_m^H)^* \mathcal{H}_A$ on \mathcal{T}_m^H due to the trivial monodromy along $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$, it gives the identification of the tangent bundle $T^{1,0}\mathcal{T}_m^H$ with the Hodge subbundle $(\Psi_m^H)^* \mathcal{H}_A$ on \mathcal{T}_m^H by Lemma 5.4.

From this special feature of the period map, one can see that the identification of the tangent bundle of the \mathcal{T}_m with the Hodge subbundle in the definition of strong local Torelli is compatible with both the affine structures on \mathcal{T}_m and A , as well as the affine map Ψ_m .

Now we recall a lemma due to Griffiths and Wolf, which is proved as Corollary 2 in [24].

Lemma 5.6. *Let $f : X \rightarrow Y$ be a local diffeomorphism of connected Riemannian manifolds. Assume that X is complete for the induced metric. Then $f(X) = Y$, f is a covering map and Y is complete.*

Since the Hodge metric on \mathcal{T}_m^H is induced by Ψ_m^H and is complete, we have the following corollary from Lemma 5.6.

Corollary 5.7. *The holomorphic map $\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D$ is a universal covering map, and the image $\Psi_m^H(\mathcal{T}_m^H) = A \cap D$ is complete with respect to the Hodge metric.*

Then by the universal properties of the universal covering

$$\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D,$$

or the uniqueness of the universal cover, we have the following corollary.

Corollary 5.8. *For any $m_1, m_2 \geq 3$, the extended Teichmüller spaces $\mathcal{T}_{m_1}^H$ and $\mathcal{T}_{m_2}^H$ are biholomorphic to each other.*

Since Corollary 5.8 implies that \mathcal{T}_m^H does not depend on m , we can simply denote each \mathcal{T}_m^H by \mathcal{T}^H , rewrite $i_m : \mathcal{T} \rightarrow \mathcal{T}_m^H$ as $i_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}^H$, and denote the corresponding period maps and affine maps by

$$\Phi^H : \mathcal{T}^H \rightarrow D \text{ and } \Psi^H : \mathcal{T}^H \rightarrow A \cap D$$

respectively. Moreover for each \mathcal{Z}_m , one has the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_{\mathcal{T}}} & \mathcal{T}^H & \xrightarrow{\Phi^H} & D \\ \downarrow \pi_m & & \downarrow \pi_m^H & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m^H}} & D/\Gamma. \end{array} \quad (26)$$

Now we prove that the universal covering map $\Psi^H : \mathcal{T}^H \rightarrow A \cap D$ is in fact injective, hence we have

$$\mathcal{T}^H \simeq A \cap D,$$

and $A \cap D$ is simply connected.

We will give two proofs of this result. The first proof is to show directly that $A \cap D$ is simply connected. The second proof uses the affine structures on \mathcal{T}^H and $A \cap D$ in a more substantial way. Both the proofs reflect different geometric structures of the period domain and period map which will be useful for further study, so we include both the proofs in this section.

For the first proof, we introduce some notations following [37]. Let $K \subset G_{\mathbb{R}}$ be the maximal compact subgroup, whose Lie algebra \mathfrak{k}_0 is

$$\mathfrak{k}_0 = (\oplus_{k \text{ even}} \mathfrak{g}^{k, -k}) \cap \mathfrak{g}_0.$$

In fact, we define $\mathfrak{k} = \oplus_{k \text{ even}} \mathfrak{g}^{k, -k}$, then $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$. Similarly, we define

$$\mathfrak{p} = \oplus_{k \text{ odd}} \mathfrak{g}^{k, -k} \text{ and } \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0.$$

Then we have the decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Let

$$\mathfrak{p}_+ = \mathfrak{p} \cap \mathfrak{n}_+ = \oplus_{k \geq 1, k \text{ odd}} \mathfrak{g}^{-k, k}.$$

Then \mathfrak{p}_+ can be viewed as an Euclidean subspace of \mathfrak{n}_+ with the induced metric from \mathfrak{n}_+ . Similarly $\exp(\mathfrak{p}_+)$ can be viewed as an Euclidean subspace of N_+ with the induced metric from N_+ .

From Corollary 3.2 in [37], one gets that the natural projection

$$\pi : D \rightarrow G_{\mathbb{R}}/K,$$

when restricted to the underlying real manifold of $\exp(\mathfrak{p}_+) \cap D$, is given by the diffeomorphism

$$\pi_+ : \exp(\mathfrak{p}_+) \cap D \longrightarrow \exp(\mathfrak{p}_0) \xrightarrow{\cong} G_{\mathbb{R}}/K. \quad (27)$$

Indeed, let \bar{o} be the base point in $\exp(\mathfrak{p}_+) \cap D$ as in Lemma 3.1 of [37]. Given $Y \in \mathfrak{p}_+$, let

$$\exp(Y)\bar{o} \in \exp(\mathfrak{p}_+) \cap D$$

denote the left translation of the base point \bar{o} by $\exp(Y)$ in $\exp(\mathfrak{p}_+) \cap D$. Similarly, let $X \in \mathfrak{p}_0$ with the relation that

$$X = T_0(Y + \tau_0(Y))$$

where τ_0 is the complex conjugate of \mathfrak{g} with respect to the real form \mathfrak{g}_0 , for some real number T_0 determined uniquely in Lemma 3.1 of [37] from the Harish-Chandra argument, and

$$\exp(X)\bar{o} \in \exp(\mathfrak{p}_0) \simeq G_{\mathbb{R}}/K$$

denote the left translation by $\exp(X)$ of the base point \bar{o} in $G_{\mathbb{R}}/K$. Then we have

$$\pi_+(\exp(Y)\bar{o}) = \exp(X)\bar{o}.$$

Theorem 5.9. *Suppose that the polarized manifold (X, L) belongs to T -class and the strong local Torelli holds for (X, L) . Then the holomorphic covering map $\Psi^H : \mathcal{T}^H \rightarrow A \cap D$ is an injection and hence a biholomorphic map.*

Proof. Consider the diffeomorphism $\pi_+ : \exp(\mathfrak{p}_+) \cap D \rightarrow G_{\mathbb{R}}/K$ discussed above. By Griffiths transversality, one has

$$\mathfrak{a} \subset \mathfrak{g}^{-1,1} \subset \mathfrak{p}_+ \text{ and } \mathfrak{a}_0 = \mathfrak{a} + \tau_0(\mathfrak{a}) \subset \mathfrak{p}_0.$$

Then $A \cap D$ is a submanifold of $\exp(\mathfrak{p}_+) \cap D$, and the diffeomorphism π_+ maps $A \cap D \subseteq \exp(\mathfrak{p}_+) \cap D$ diffeomorphically to its image $\exp(\mathfrak{a}_0)$ inside $G_{\mathbb{R}}/K$, from which one has the diffeomorphism

$$A \cap D \simeq \exp(\mathfrak{a}_0)$$

induced by π_+ . Since $\exp(\mathfrak{a}_0)$ is simply connected, one concludes that $A \cap D$ is also simply connected.

Since \mathcal{T}^H is simply connected and $\Psi^H : \mathcal{T}^H \rightarrow A \cap D$ is a covering map, we conclude that Ψ^H must be a biholomorphic map. \square

For the second proof, we will first prove the following elementary lemma, in which we mainly use the completeness of the Hodge metric on \mathcal{T}^H , the holomorphic affine structure on \mathcal{T}^H , the affineness of Ψ^H , and the properties of Hodge metric. We remark that as \mathcal{T}^H is a complex affine manifold, we have the notion of straight lines in it with respect to the affine structure.

Lemma 5.10. *For any point in \mathcal{T}^H , there is a straight line segment in \mathcal{T}^H connecting it to the base point p .*

Proof. The proof uses crucially the facts that the map $\Psi^H : \mathcal{T}^H \rightarrow A \cap D$ is an local isometry with the Hodge metrics on \mathcal{T}^H and $A \cap D$, and that it is also an affine map with the induced affine structure on \mathcal{T}^H .

Let q be any point in \mathcal{T}^H . As \mathcal{T}^H is connected and complete with the Hodge metric, by Hopf-Rinow theorem, there exists a geodesic γ in \mathcal{T}^H connecting the base point p to q . Since Ψ^H is a local isometry with the Hodge metric, $\tilde{\gamma} = \Psi^H(\gamma)$ is also a geodesic in $A \cap D$.

By Griffiths transversality, we have that $\mathfrak{a} \subset \mathfrak{g}^{-1,1} \subset \mathfrak{p}_+$ and $\mathfrak{a}_0 = \mathfrak{a} + \tau_0(\mathfrak{a}) \subset \mathfrak{p}_0$. Restricting the diffeomorphism (27) to $A \cap D$, we know that any geodesic starting from the base point $\tilde{p} = \Psi^H(p)$ in $A \cap D$ is of the form $\exp(tX)\tilde{p}$ with $X \in \mathfrak{a}_0$ and $t \in \mathbb{R}$.

Recall that, from the computation in the proof of Lemma 3.1 in [37] which is due to Harish-Chandra in proving his famous embedding theorem of Hermitian symmetric spaces as bounded domains in complex Euclidean spaces, we have the relation

$$\exp(tX)\tilde{p} = \exp(T(t)Y)\tilde{p}$$

with $Y \in \mathfrak{a}$ satisfying $X = T_0(Y + \tau_0(Y))$ for some $T_0 \in \mathbb{R}$, and $T(t)$ a smooth real valued monotone function of t .

Note that the affine structure on $A \cap D$ is induced from the affine structure on \mathfrak{a} by the exponential map

$$\exp : \mathfrak{a} \rightarrow A,$$

therefore the geodesic

$$\tilde{\gamma} = \exp(T(t)Y)\tilde{p}$$

corresponds to a straight line with respect to the affine structure on $A \cap D$. Hence γ is also a straight line in \mathcal{T}^H with respect to the induced affine structure, since Ψ^H is an affine map. \square

Second Proof of Theorem 5.9. Let $q_1, q_2 \in \mathcal{T}^H$ be any two different points. Suppose that to the contrary, one has $\Psi^H(q_1) = \Psi^H(q_2)$, we will derive a contradiction.

First, suppose that one of q_1, q_2 , say q_1 , is the base point p , then Lemma 5.10 implies that there is a straight line segment $l \subseteq \mathcal{T}^H$ connecting p and

q_2 . Since Ψ^H is an affine map and is locally biholomorphic by local Torelli theorem, the straight line segment l is mapped to a straight line segment $\tilde{l} = \Psi^H(l)$ in

$$A \cap D \subset A \simeq \mathbb{C}^N.$$

But $\Psi^H(p) = \Psi^H(q_2)$, which implies that the straight line segment \tilde{l} is also a cycle in $A \simeq \mathbb{C}^N$, which is a contradiction.

Now suppose that both q_1 and q_2 are different from the base point p , then by Lemma 5.10, there exist two different straight line segments $l_1, l_2 \subseteq \mathcal{T}^H$ connecting p to q_1 and q_2 respectively. Since

$$\Psi^H(q_1) = \Psi^H(q_2),$$

the two straight line segments $\tilde{l}_i = \Psi^H(l_i)$, $i = 1, 2$ in $A \cap D$ intersect at two different points, $\Psi^H(p)$ and $\Psi^H(q_1) = \Psi^H(q_2)$, therefore must coincide. This contradicts to the fact that Ψ^H is locally biholomorphic at the base point p , and the assumption that the two straight line segments l_1 and l_2 are different. \square

Since $\Psi^H = P \circ \Phi^H$ and the corresponding tangent map $d\Psi^H = dP \circ d\Phi^H$, we have the following corollary from Theorem 5.9.

Corollary 5.11. *Let the conditions be as Theorem 5.9. Then the extended period map $\Phi^H : \mathcal{T}^H \rightarrow N_+ \cap D$ is an embedding.*

Now we go on to the Torelli space \mathcal{T}' . Let $\pi'_m : \mathcal{T}' \rightarrow \mathcal{Z}_m$ be the covering map for $m \geq 3$. Then there exists the natural covering map $\pi : \mathcal{T} \rightarrow \mathcal{T}'$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{T} & & \\ \downarrow \pi_m & \searrow \pi & \\ & & \mathcal{T}' \\ & \swarrow \pi'_m & \\ & & \mathcal{Z}_m \end{array}$$

which together with diagram (26) gives the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{T} & \xrightarrow{i_{\mathcal{T}}} & \mathcal{T}^H & \xrightarrow{\Phi^H} & D \\
\downarrow \pi_m & \searrow \pi & \downarrow \pi_m^H & \nearrow \Phi' & \downarrow \pi_D \\
& & \mathcal{T}' & & \\
\downarrow \pi_m & \swarrow \pi'_m & \downarrow \pi_m^H & & \downarrow \pi_D \\
\mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m^H}} & D/\Gamma
\end{array} \tag{28}$$

Let $\mathcal{T}_0 = i_{\mathcal{T}}(\mathcal{T})$ denote the image of $i_{\mathcal{T}}$. Next we will define a map

$$\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$$

such that the following commutative diagram holds,

$$\begin{array}{ccccc}
\mathcal{T} & \xrightarrow{i_{\mathcal{T}}} & \mathcal{T}_0 & \xrightarrow{\Phi^H|_{\mathcal{T}_0}} & D \\
\downarrow \pi_m & \searrow \pi & \swarrow \pi_0 & \nearrow \Phi' & \downarrow \pi_D \\
& & \mathcal{T}' & & \\
\downarrow \pi_m & \swarrow \pi'_m & \downarrow \pi_m^H & & \downarrow \pi_D \\
\mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m^H}} & D/\Gamma
\end{array} \tag{29}$$

The proof of the following proposition is given by explicitly constructing π_0 . Again the level structures plays a crucial role in the argument.

Proposition 5.12. *Let $\mathcal{T}_0 = i_{\mathcal{T}}(\mathcal{T})$ be defined by the image of $i_{\mathcal{T}}$. Then there is a biholomorphic map $\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$ such that diagram (29) is commutative.*

Proof. We start from the definition of π_0 . For any point o in \mathcal{T}_0 , we can choose $p \in \mathcal{T}$ such that $i_{\mathcal{T}}(p) = o$. We define

$$\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$$

such that $\pi_0(o) = \pi(p)$. Then clearly π_0 fits in the above commutative diagram.

We first show that the map $\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$ defined this way is well-defined and satisfies

$$\Phi^H|_{\mathcal{T}_0} = \Phi' \circ \pi_0.$$

Let $p \neq q \in \mathcal{T}$ be two points such that $i_{\mathcal{T}}(p) = i_{\mathcal{T}}(q) \in \mathcal{T}_0$. We choose some $m \geq 3$ such that $\mathcal{T}^H \simeq \mathcal{T}_m^H$. Then

$$i_m(p) = i_m(q) \in \mathcal{T}_m \subset \mathcal{T}_m^H.$$

Let $[X_p, L_p, \gamma_p]$ and $[X_q, L_q, \gamma_q]$ denote the fibers over the points p and q of the analytic family $\mathcal{U}' \rightarrow \mathcal{T}'$ respectively, where γ_p and γ_q are two markings identifying the fixed lattice Λ isometrically with $H^n(X_p, \mathbb{Z})/\text{Tor}$ and $H^n(X_q, \mathbb{Z})/\text{Tor}$ respectively.

Since $i_m(p) = i_m(q)$ and $i \circ \pi_m = \pi_m^H \circ i_m$, we have

$$i \circ \pi_m(p) = i \circ \pi_m(q) \in \mathcal{Z}_m,$$

i.e.

$$\pi_m(p) = \pi_m(q) \in \mathcal{Z}_m.$$

By the definition of \mathcal{Z}_m , there exists a biholomorphic map $f : X_p \rightarrow X_q$ such that $f^*L_q = L_p$ and

$$f^*\gamma_q = \gamma_p \cdot A,$$

where $A \in \text{Aut}(H^n(X_p, \mathbb{Z})/\text{Tor}, \mathbb{Q})$ satisfies

$$A = (A_{ij}) \equiv \text{Id} \pmod{m}, \text{ for } m \geq 3.$$

Let m_0 be an integer such that $m_0 > |A_{ij}|$ for any i, j . Corollary 5.8 implies that

$$\mathcal{T}^H \simeq \mathcal{T}_m^H \simeq \mathcal{T}_{m_0}^H.$$

The same argument as above implies that

$$\pi_{m_0}(p) = \pi_{m_0}(q) \in \mathcal{Z}_{m_0}$$

and hence

$$A = (A_{ij}) \equiv \text{Id} \pmod{m_0}.$$

Since each $m_0 > |A_{ij}|$, we have $A = \text{Id}$.

Therefore, we have found a biholomorphic map

$$f : X_p \rightarrow X_q$$

such that $f^*L_q = L_p$ and $f^*\gamma_q = \gamma_p$. This implies that p and q in \mathcal{T} actually correspond to the same point in the Torelli space \mathcal{T}' , i.e.

$$\pi(p) = \pi(q) \text{ in } \mathcal{T}'.$$

So we have proved that the map $\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$ is well-defined.

From the definitions of the period map and that of the Torelli space, we deduce that the map π_0 satisfies the identity

$$\Phi^H|_{\mathcal{T}_0} = \Phi' \circ \pi_0.$$

Clearly π_0 is surjective, because $\pi : \mathcal{T} \rightarrow \mathcal{T}'$ is surjective. Since $\Phi^H : \mathcal{T}^H \rightarrow D$ is injective, so is the restriction map

$$\Phi^H|_{\mathcal{T}_0} : \mathcal{T}_0 \rightarrow D,$$

which implies the injectivity of the map $\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$. Hence $\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$ is in fact a biholomorphic map. This completes the proof of the proposition. \square

Define the injective map

$$\pi^0 : \mathcal{T}' \rightarrow \mathcal{T}^H, \quad \pi^0 = (\pi_0)^{-1},$$

we then have the relation that

$$\Phi' = \Phi^H \circ \pi^0 : \mathcal{T}' \rightarrow D.$$

From the injectivity of Φ^H and π^0 , we deduce the global Torelli theorem on the Torelli space as follows.

Theorem 5.13 (Global Torelli theorem). *Suppose that the polarized manifold (X, L) belongs to the T -class, and the strong local Torelli holds for (X, L) , then the global Torelli theorem holds on the Torelli space \mathcal{T}' , i.e., the period map $\Phi' : \mathcal{T}' \rightarrow D$ is injective.*

Let

$$\Psi' = P \circ \Phi' : \mathcal{T}' \rightarrow A \cap D$$

be the composite of Φ' with the projection map $P : N_+ \cap D \rightarrow A \cap D$, we then have,

Corollary 5.14. *Under the same conditions in Theorem 5.13, the holomorphic map $\Psi' : \mathcal{T}' \rightarrow A \cap D$ is injective.*

Since the moduli spaces with level m structure of polarized K3 surfaces, Calabi-Yau manifolds and hyperkähler manifolds are smooth for m large, and they have the strong local Torelli property as described in Section 2, we have the following corollary.

Corollary 5.15. *Let the polarized manifold (X, L) be one of the following cases,*

- (i) *K3 surface;*
- (ii) *Calabi-Yau manifold;*
- (iii) *hyperkähler manifold.*

Then the global Torelli theorem holds on the Torelli space \mathcal{T}' for (X, L) .

See introduction and references there for discussions about previous results related to Corollary 5.15.

By Theorem 1.2 in [30] and the result of Benoist as stated in Proposition 2.2 of [29], we know that the moduli spaces of smooth hypersurfaces have smooth covers given by the moduli spaces with level structures, and for the complete intersections used in the hyperplane arrangements, the corresponding moduli spaces also have smooth covers. So when m is large, the moduli space \mathcal{Z}_m containing (X, L) is smooth as required in the definition of T-class. As discussed in Section 2, some of these examples have strong local Torelli property, therefore we have the following corollary.

Corollary 5.16. *Let the polarized manifold (X, L) be one of the following cases,*

- (i) *smooth hypersurface of degree d in \mathbb{P}^{n+1} satisfying $d|(n+2)$ and $d \geq 3$;*
- (ii) *the polarized manifold of \mathbb{P}^{m-1} associated to an arrangement of m hyperplanes in \mathbb{P}^n with $m \geq n$, defined as in (20) of Section 2.2;*
- (iii) *smooth cubic surface or cubic threefold.*

Then global Torelli theorem holds on the corresponding Torelli spaces \mathcal{T}' .

We remark that case (i) of Corollary 5.16 is new, except the work of Voisin in [68] which proves the generic Torelli theorem for the moduli space of quintic threefolds.

Case (ii) of Corollary 5.16 with $n = 1$ can be considered as a version on the Torelli space of the main result of [11], which is the famous Deligne-Mostow theory. Corollary 5.16 in case (ii) with $n > 1$ is new.

As mentioned in the introduction, in [1] and [2], Allcock, Carlson and Toledo proved global Torelli theorems on the moduli space and Torelli space of smooth cubic surfaces or cubic threefolds. Case (iii) of Corollary 5.16 can be considered as a version of their results in [1] and [2] on the Torelli spaces.

Applying Proposition 5.12, we have that \mathcal{T}^H , which is biholomorphic to $A \cap D$, is actually the Hodge metric completion of \mathcal{T}' with respect to the induced Hodge metric. Moreover we have the following theorem.

Theorem 5.17. *Suppose that the polarized manifold (X, L) belongs to T -class and strong local Torelli holds for (X, L) . Then the Hodge metric completion $\mathcal{T}^H \simeq A \cap D$ of \mathcal{T}' is a bounded pseudoconvex domain in $A \simeq \mathbb{C}^N$. In particular, there exists a unique complete Kähler-Einstein metric on $\mathcal{T}^H \simeq A \cap D$ with Ricci curvature -1 .*

First we recall some notions from complex analysis in several variables. One can refer to [27] for details.

An open and connected subset $\Omega \subseteq \mathbb{C}^N$ is called a domain in \mathbb{C}^N . A function u defined on a domain $\Omega \subseteq \mathbb{C}^N$ with values in $[-\infty, +\infty]$ is called plurisubharmonic if

- (1) u is semicontinuous from above;
- (2) For any $z, w \in \mathbb{C}^N$, the function $t \mapsto u(z + tw)$ is subharmonic at the points of \mathbb{C} where it is defined.

A domain $\Omega \subseteq \mathbb{C}^N$ is called pseudoconvex if there exists a continuous plurisubharmonic function u on Ω such that for any $c \in \mathbb{R}$, the set

$$\Omega_c = \{z : z \in \Omega, u(z) < c\}$$

is a relatively compact subset of Ω . Such function u is called an exhaustion function on Ω .

From Theorem 2.6.2 in [27], we know that a C^2 function u on a domain $\Omega \subseteq \mathbb{C}^N$ is plurisubharmornic if and only if its Levi form is positive definite at any point in Ω .

In their paper [23], Griffiths and Schmid proved the following proposition.

Proposition 5.18. *On every manifold D , which is dual to a Kähler C -space, there exists a C^∞ exhaustion function f , whose Levi form, restricted to $T_h^{1,0}(D)$, is positive definite at every point of D .*

Proof of Theorem 5.17. Boundedness of $A \cap D$ comes from the proof of Lemma 3.1 in [37].

Let f be the C^∞ function on the period domain D as in Proposition 5.18. Then the restricted function $g = f|_{A \cap D}$ has positive definite Levi form at any point of $A \cap D$, since the tangent space at any point of $A \cap D$ lies in the fiber of $T_h^{1,0}(D)$.

To complete the proof, we only need to show that g is an exhaustion function on $A \cap D$. In fact, for any $c \in \mathbb{R}$, the set

$$\Omega_c = \{z \in A \cap D : g(z) \leq c\} = \{z \in D : f(z) \leq c\} \cap (A \cap D)$$

is compact in $A \cap D$, since

$$\{z \in D : f(z) \leq c\}$$

is compact in D , see the proof of Theorem 8.1 in [23], and $A \cap D$ with the induced Hodge metric is complete, so is a complete subset of D , and hence closed in D .

The existence and uniqueness of the complete Kähler-Einstein metric on $\mathcal{T}^H \simeq A \cap D$ with Ricci curvature -1 follows directly from the results of [48]. \square

6 Applications

In this section, we still consider polarized manifolds in the T-class for which the strong local Torelli holds. We will use the results in Section 5 to prove

that, in this case, the global Torelli theorem holds on the moduli space with level m structure.

Suppose that the polarized manifold (X, L) belongs to T-class and strong local Torelli holds for (X, L) . Let \mathcal{Z}_m containing (X, L) be defined in Definition 1.1 and $\mathcal{Z}_m^H, \mathcal{T}, \mathcal{T}'$ and \mathcal{T}_m^H be defined as before.

Let $\Gamma = \rho(\pi_1(\mathcal{Z}_m^H))$ denote the global monodromy group, where

$$\rho : \pi_1(\mathcal{Z}_m^H) \rightarrow \text{Aut}(H_{\mathbb{Z}}, Q)$$

denotes the monodromy representation. Then the corresponding period maps can be written in the following commutative diagram,

$$\begin{array}{ccc} \mathcal{T}^H & \xrightarrow{\Phi^H} & D \\ \downarrow \pi_m^H & \searrow \tilde{\Phi}^H & \downarrow \pi_D \\ \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m^H}} & D/\Gamma, \end{array}$$

where

$$\tilde{\Phi}^H = \pi_D \circ \Phi^H.$$

The image of the extended period map $\Phi_{\mathcal{Z}_m^H}$ in general is an analytic subvariety of D/Γ . We refer the reader to page 156 of [20] for details of the analyticity of the image of the period map.

Theorem 6.1. *Suppose that the polarized manifold (X, L) belongs to T-class and strong local Torelli holds for (X, L) . Then the extended period map $\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow D/\Gamma$ is injective. As a consequence the global Torelli theorem holds on the moduli space \mathcal{Z}_m with level m structure.*

Proof of Theorem 6.1. We will give two proofs. For the first proof, we first show that $\Phi_{\mathcal{Z}_m^H}$ is a covering map from \mathcal{Z}_m^H to its image in D/Γ . This follows from the following lemma.

Lemma 6.2. *Let $\tilde{\Phi}^H : \mathcal{T}^H \rightarrow D/\Gamma$ be the composition of Φ^H and the covering map $\pi_D : D \rightarrow D/\Gamma$. Then $\tilde{\Phi}^H$ is a covering map onto its image which is $\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$.*

Proof of Lemma 6.2. First from Theorem (D.2) in page 179 of [20], $\Phi^H(\mathcal{T}^H)$ is invariant under the action of Γ . In our case, Φ^H is a global embedding which implies that $\Phi^H(\mathcal{T}^H)$ is smooth, and π_D is a covering map.

On the other hand D/Γ is smooth, because Γ is torsion-free from Lemma 1.6. As discussed in page 156 of [20], the isotropy group corresponding to the points in the smooth manifold $\Phi^H(\mathcal{T}^H)$ that are mapped to a singular point in $\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$ by the quotient of Γ must be a finite torsion subgroup of Γ . Therefore

$$\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H) = \pi_D \circ \Phi^H(\mathcal{T}^H)$$

is actually a smooth variety, since Γ is torsion-free. This gives that \mathcal{T}^H , which is biholomorphic to $\Phi^H(\mathcal{T}^H)$, is also a covering space of $\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$, and $\tilde{\Phi}^H$ is a covering map. \square

Lemma 6.3. *The extended period map $\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow D/\Gamma$ is a covering map from \mathcal{Z}_m^H onto its image in D/Γ .*

Proof of Lemma 6.3. Note that in the following commutative diagram

$$\begin{array}{ccc} \mathcal{T}^H & \xrightarrow{\Phi^H} & D \\ \downarrow \pi_m^H & \searrow \tilde{\Phi}^H & \downarrow \pi_D \\ \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m^H}} & D/\Gamma, \end{array}$$

all the varieties involved are smooth. Since the map π_m^H is a covering map, π_m^H is locally biholomorphic. Similarly, Φ^H is an embedding and π_D is a covering map.

From Lemma 6.2, we know that the map

$$\tilde{\Phi}^H = \pi_D \circ \Phi^H = \Phi_{\mathcal{Z}_m^H} \circ \pi_m^H : \mathcal{T}^H \rightarrow \Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$$

is a covering map, therefore a locally biholomorphic map, and hence the Jacobians of these maps are all nondegenerate. In particular this implies that the Jacobian of $\Phi_{\mathcal{Z}_m^H}$ is nondegenerate, which implies that

$$\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow \Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$$

is a locally biholomorphic map.

On the smooth complex submanifold

$$\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H) = \pi_D \circ \Phi^H(\mathcal{T}^H)$$

of D/Γ , we have the induced Hodge metric from D/Γ . Since the image of \mathcal{Z}_m^H under $\Phi_{\mathcal{Z}_m^H}$ is precisely $\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$ and the map $\Phi_{\mathcal{Z}_m^H}$ is nondegenerate, the pull-back Hodge metric on \mathcal{Z}_m^H by $\Phi_{\mathcal{Z}_m^H}$ from the submanifold $\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$ in D/Γ is the same as the original Hodge metric induced from D/Γ through the pull-back by $\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow D/\Gamma$.

Since the induced Hodge metric on \mathcal{Z}_m^H is complete as the Hodge metric completion of \mathcal{Z}_m , we can directly apply Lemma 5.6 of Griffiths–Wolf to deduce that

$$\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow \Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$$

is a covering map. □

Proof of Theorem 6.1(continued). In Lemma 6.3 we have already proved that $\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$ is smooth and

$$\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow \Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$$

is a covering map. This implies that $\pi_1(\mathcal{Z}_m^H)$ can be identified as a subgroup of the fundamental group of $\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$.

On the other hand, since \mathcal{T}^H is simply connected and $\Phi^H : \mathcal{T}^H \rightarrow D$ is an embedding, Lemma 6.2 gives that

$$\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H) = \Phi^H(\mathcal{T}^H)/\Gamma$$

which implies that the fundamental group of $\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$ is Γ , where

$$\Gamma = \rho(\pi_1(\mathcal{Z}_m^H)) \simeq \pi_1(\mathcal{Z}_m^H)/\text{Ker}(\rho)$$

with $\rho : \pi_1(\mathcal{Z}_m^H) \rightarrow \text{Aut}(H_{\mathbb{Z}}, Q)$ the monodromy representation. From this we deduce that

$$\pi_1(\mathcal{Z}_m^H) \subset \Gamma \simeq \pi_1(\mathcal{Z}_m^H)/\text{Ker}(\rho),$$

which implies that $\text{Ker}(\rho) = 0$ and $\pi_1(\mathcal{Z}_m^H) \simeq \Gamma$.

Therefore we have proved that $\pi_1(\mathcal{Z}_m^H)$ is isomorphic to the fundamental group of $\Phi_{\mathcal{Z}_m^H}(\mathcal{Z}_m^H)$. From this we deduce that the covering map

$$\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \rightarrow D/\Gamma$$

is a biholomorphic map onto its image. □

The proof of Theorem 6.1 uses Lemma 6.2 and Lemma 6.3, which is proved under the condition of strong local Torelli. In fact, we can also prove Theorem 6.1, once the conclusion of Theorem 5.13 holds.

Second proof of Theorem 6.1. Let p_1 and p_2 be two points in \mathcal{Z}_m such that $\Phi_{\mathcal{Z}_m}(p_1) = \Phi_{\mathcal{Z}_m}(p_2)$ in D/Γ . Let

$$[X_i, L_i, [\gamma_i]_m], i = 1, 2$$

be the fibers over p_1 and p_2 of the analytic family $f_m : \mathcal{U}_m \rightarrow \mathcal{Z}_m$ respectively. From Proposition 5.12 we know that

$$\pi^0 : \mathcal{T}' \rightarrow \mathcal{T}^H$$

identifies \mathcal{T}' to the Zariski open submanifold

$$\mathcal{T}_0 = i_{\mathcal{T}}(\mathcal{T}) \simeq i_m(\mathcal{T}) \subseteq \mathcal{T}_m^H$$

of \mathcal{T}_m^H , and that \mathcal{T}_0 is a cover of \mathcal{Z}_m by Lemma 3.3. In the following discussion we will use freely the identification

$$\mathcal{T}_0 \simeq \mathcal{T}'.$$

There exist two points q_1 and q_2 in \mathcal{T}' over which are the fibers

$$[X_i, L_i, \gamma_i], i = 1, 2$$

of the analytic family $g' : \mathcal{U}' \rightarrow \mathcal{T}'$ respectively, such that

$$\pi'_m(q_i) = p_i, i = 1, 2$$

under the covering map $\pi'_m : \mathcal{T}' \rightarrow \mathcal{Z}_m$.

The condition $\Phi_{\mathcal{Z}_m}(p_1) = \Phi_{\mathcal{Z}_m}(p_2)$ implies that there exists $\gamma \in \Gamma$ such that

$$\Phi'(q_1) = \gamma\Phi'(q_2).$$

Let $q'_1 \in \mathcal{T}'$ correspond to the triple $[X_1, L_1, \gamma_1\gamma]$. Then by the definition of the period map Φ' in (13), we have

$$\begin{aligned} \Phi'(q'_1) &= (\gamma_1\gamma)^{-1}(F^n(X_1) \subseteq \cdots \subseteq F^0(X_1)) \\ &= \gamma^{-1}\gamma_1^{-1}(F^n(X_1) \subseteq \cdots \subseteq F^0(X_1)) \\ &= \gamma^{-1}\Phi'(q_1) \\ &= \Phi'(q_2). \end{aligned}$$

By Theorem 5.13, we see that $q'_1 = q_2$ in \mathcal{T}' .

Now we have the inclusion $\pi^0 : \mathcal{T}' \rightarrow \mathcal{T}^H$ and the inclusion $i : \mathcal{Z}_m \rightarrow \mathcal{Z}_m^H$, therefore we can view the points p_1 and p_2 as points in \mathcal{Z}_m^H and the points q_1 , q'_1 and q_2 as points in \mathcal{T}^H .

Since γ lies in the image of the monodromy representation $\rho : \pi_1(\mathcal{Z}_m^H) \rightarrow \Gamma$, there exists a $\tilde{\gamma} \in \pi_1(\mathcal{Z}_m^H)$ such that

$$\rho(\tilde{\gamma}) = \gamma \text{ and } q'_1 = q_1\tilde{\gamma},$$

where $q_1\tilde{\gamma}$ is defined by the action of $\pi_1(\mathcal{Z}_m^H)$ on the universal cover \mathcal{T}^H , and the action of $\tilde{\gamma}$ on the fiber $[X_1, L_1, \gamma_1]$ over q_1 is defined by

$$[X_1, L_1, \gamma_1]\tilde{\gamma} = [X_1, L_1, \gamma_1\gamma].$$

So we have

$$p_1 = \pi_m^H(q_1) = \pi_m^H(q'_1) = \pi_m^H(q_2) = p_2,$$

which proves the injectivity of $\Phi_{\mathcal{Z}_m}$. □

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