A GEOMETRIC HEAT FLOW FOR VECTOR FIELDS

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Abstract. In this paper we introduce and study a geometric heat flow to find Killing vector fields on closed Riemannian manifolds with positive sectional curvature. We study its various properties, prove the global existence of the solution of this flow, discuss its convergence and possible applications, and its relation to the Navier-Stokes equations on manifolds and Kazdan-Warner-Bourguignon-Ezin identity for conformal Killing vector fields. We also provide two new criterions on the existence of Killing vector fields. The similar flow to finding holomorphic vector fields on Kähler manifolds will be studied in [9].

1. A geometric heat flow for vector fields

Recently, we have witnessed the power of geometric flows in studying lots of problems in geometry and topology. In this paper we introduce a geometric heat flow for vector fields on a Riemannian manifold and study its varies properties.

Throughout this paper, we adopt the Einstein summation and notions as those in [3]. All manifolds and vector fields are smooth; a manifold is said to be *closed* if it is compact and without boundary. We shall often raise and lower indices for tensor fields.

1.1. **Deformation tensor field of a vector field.**

Let (M, g) be a closed and orientable Riemannian manifold. To a vector field X we associate its deformation $(0,2)$-tensor field $\mathbf{Def}(X)$, which is an obstruction of X to be Killing and is locally defined by

$$(\mathbf{Def}(X))_{ij} := \frac{\nabla_i X_j + \nabla_j X_i}{2}, \tag{1.1}$$

where ∇ denotes the Levi-Civita connection of g. Equivalently, it is exactly (up to a constant factor) the Lie derivative of g along the vector field X, i.e., $\mathcal{L}_X g$. We say that X is a *Killing vector field* if $\mathbf{Def}(X) = 0$. Consider the L^2-norm of $\mathbf{Def}(X)$:

$$\mathfrak{L}(X) := \int_M |\mathbf{Def}(X)|^2 dV, \tag{1.2}$$

where dV stands for the volume form of g and $|\cdot|$ means the norm of $\mathbf{Def}(X)$ with respect to g. Clearly that the critical point X of $\mathfrak{L}$ satisfies

$$\Delta X^i + \nabla^i \mathrm{div}(X) + R^i{}_j X^j = 0. \tag{1.3}$$

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Here and henceforth, $\Delta := g^{ij}\nabla_i\nabla_j$ is the Laplace-Beltrami operator of g and R_{ij} denotes the Ricci curvature of g. In fact

$$\begin{aligned}\frac{d}{dt}\mathfrak{L}(X_t) &= \frac{1}{2}\int_M \langle \mathbf{Def}(X_t), \partial_t \mathbf{Def}(X_t)\rangle\, dV \\ &= \frac{1}{2}\int_M \left(\nabla^i (X_t)^j + \nabla^j (X_t)^i\right)\left(\nabla_i \partial_t (X_t)_j + \nabla_j \partial_t (X_t)_i\right) dV \\ &= -\int_M \left[\Delta (X_t)^i \cdot \partial_t (X_t)_i + \nabla^j \nabla^i (X_t)^j \cdot \partial_t (X_t)_i\right] dV \\ &= -\int_M \left[\Delta (X_t)^i + \nabla^i \mathrm{div}(X_t) + R^i{}_j (X_t)^j\right] \partial_t (X_t)^i dV.\end{aligned}$$

1.2. **A geometric heat flow for vector fields.** Motivated by (1.3), we introduce a geometric heat flow for vector fields:

$$\partial_t (X_t)^i = \Delta (X_t)^i + \nabla^i \mathrm{div}(X_t) + R^i{}_j (X_t)^j, \quad X_0 = X, \tag{1.4}$$

where X is a fixed vector field on M and $\partial_t := \frac{\partial}{\partial t}$ is the time derivative. If we define $\mathrm{Ric}^\sharp$, the $(1,1)$-tensor field associated to Ric, by

$$g\left(\mathrm{Ric}^\sharp(X), Y\right) := t\mathrm{Ric}(X, Y),$$

where X, Y are two vector fields, then $\mathrm{Ric}^\sharp$ is an operator on the space of vector fields, denoted by $C^\infty(TM)$, and the flow (1.4) can be rewritten as

$$\partial_t X_t = \Delta X_t + \nabla \mathrm{div}(X_t) + \mathrm{Ric}^\sharp(X_t). \tag{1.5}$$

In 1952, Yano (e.g., [16, 17, 18]) showed that a vector field $X = X^i \frac{\partial}{\partial x^i}$ is a Killing vector field if and only if it satisfies

$$\Delta X^i + R^i{}_j X^j = 0, \quad \mathrm{div}(X) = 0. \tag{1.6}$$

His result depends on an integral formula, now called *Yano's integral formula*,

$$0 = \int_M \left[\mathrm{Ric}(X, X) - |\nabla X|^2 + 2|\mathbf{Def}(X)|^2 - |\mathrm{div}(X)|^2\right] dV, \tag{1.7}$$

which holds for any vector field X. This integral formula lets us define so-called the *Bochner-Yano integral* for every vector field X:

$$\mathcal{E}(X) := \int_M \left[|\nabla X|^2 + |\mathrm{div}(X)|^2 - \mathrm{Ric}(X, X)\right] dV. \tag{1.8}$$

Consequently, Yano's integral formula implies that $\mathcal{E}(X)$ is always nonnegative and $\mathcal{E}(X) = 2\mathfrak{L}(X)$ for every vector field X. On the other hand, Watanabe [14] proved that X is a Killing vector field if and only if $\mathcal{E}(X) = 0$, and hence if and only if $\mathfrak{L}(X) = 0$.

Yano's equations (1.6) induces a system of equations, called the *Bochner-Yano flow*:

$$\partial_t (X_t)^i = \Delta (X_t)^i + R^i{}_j (X_t)^j, \quad \mathrm{div}(X_t) = 0. \tag{1.9}$$

Notice that Yano's equation (1.6) (resp., Bochner-Yano flow (1.9)) is a special case of our equation (1.3) (resp., our flow (1.4)).

Proposition 1.1. *If X_t is the solution of the flow (1.4), then*

$$\mathcal{E}(X_t) \geq 0, \tag{1.10}$$

$$\frac{d}{dt}\mathcal{E}(X_t) = -2\int_M |\partial_t X_t|^2 dV \leq 0, \tag{1.11}$$

$$\mathcal{E}(X_t) = -\frac{d}{dt}\left(\frac{1}{2}\int_M |X_t|^2 dV\right). \tag{1.12}$$

Consequently, $\mathcal{E}(X_t)$ is monotone nonincreasing and $\int_M |X_t|^2 dV$ is also monotone nonincreasing.

Proof. The first one directly follows from (1.7). Since the flow (1.4) is the gradient flow of the functional $\mathcal{E}$, we prove the second one. To prove (1.12), we use the formula $\frac{1}{2}\Delta|X|^2 = \langle X, \Delta X\rangle + |\nabla X|^2$ to deduce that

$$\begin{aligned}
\mathcal{E}(X_t) &= \int_M \left[\frac{1}{2}\Delta|X_t|^2 - (X_t)_i\Delta(X_t)^i + |\mathrm{div}(X_t)|^2 - \mathrm{Ric}(X_t, X_t)\right] dV \\
&= -\int_M \left[(X_t)_i\Delta(X_t)^i - \mathrm{div}(X_t)\cdot\mathrm{div}(X_t) + \mathrm{Ric}(X_t, X_t)\right] dV \\
&= -\int_M \left[(X_t)_i\Delta(X_t)^i + (X_t)_i\nabla^i\mathrm{div}(X_t) + (X_t)_i\cdot R^i{}_j(X_t)^j\right] dV \\
&= -\int_M (X_t)_i\left[\Delta(X_t)^i + \nabla^i\mathrm{div}(X_t) + R^i{}_j(X_t)^j\right] dV \\
&= -\frac{1}{2}\int_M \partial_t|X_t|^2 dV.
\end{aligned}$$

Hence the conclusion is obvious. □

Corollary 1.2. *If X_t is the solution of the flow (1.4) for $t \in [0, T]$, then we have*

$$\int_0^T\int_M |\partial_t X_t|^2 dV dt \leq 2\mathcal{E}(X). \tag{1.13}$$

Proof. For any T, we have

$$-\frac{1}{2}\int_0^T\int_M |\partial_t X_t|^2 dV dt = \mathcal{E}(X_T) - \mathcal{E}(X) \geq -\mathcal{E}(X)$$

since $\mathcal{E}$ is nonnegative. This proves (1.13). □

1.3. **Evolution equations.** To study the long time existence and the convergence of the geometric heat flow (1.4) we prove its several evolution equations.

Lemma 1.3. *If X_t is the solution of (1.4), then*

$$\partial_t|X_t|^2 = \Delta|X_t|^2 - 2|\nabla X_t|^2 + 2\langle X_t, \nabla\mathrm{div}(X_t)\rangle + 2\mathrm{Ric}(X_t, X_t). \tag{1.14}$$

Proof. Calculate

$$\begin{aligned}
\partial_t|X_t|^2 &= 2(X_t)_i\partial_t(X_t)^i \\
&= 2(X_t)_i\left(\Delta(X_t)^i + \nabla^i\mathrm{div}(X_t) + R^i{}_j(X_t)^j\right) \\
&= \Delta|X_t|^2 - 2|\nabla X_t|^2 + 2\langle X_t, \nabla\mathrm{div}(X_t)\rangle + 2\mathrm{Ric}(X_t, X_t)
\end{aligned}$$

which proves (1.4). □

Lemma 1.4. *If X_t is the solution of (1.4), then*

$$\begin{aligned}\partial_t|\nabla X_t|^2 &= \Delta|\nabla X_t|^2 - 2\left|\nabla^2 X_t\right|^2 - 4R_{ijk\ell}\nabla^i(X_t)^k\cdot\nabla^j(X_t)^\ell \\ &\quad - 2R_{ij}\nabla^i(X_t)^k\cdot\nabla^j(X_t)_k + 2R_{ij}\nabla^k(X_t)^i\cdot\nabla_k(X_t)^j \\ &\quad + 2\left\langle \mathbf{Def}(X_t), \nabla\nabla\mathrm{div}(X_t)\right\rangle \\ &\quad + 2\left(\nabla_i R_{jk} - \nabla^\ell R_{\ell ikj}\right)(X_t)^k\nabla^i(X_t)^j. \end{aligned} \tag{1.15}$$

Proof. From the definition of the flow we have

$$\begin{aligned}\partial_t|\nabla X_t|^2 &= 2\nabla^i(X_t)_j\cdot\nabla_i\partial_t(X_t)^j \\ &= 2\nabla^i(X_t)_j\cdot\nabla_i\left(\Delta(X_t)^j + \nabla^j\mathrm{div}(X_t) + R^j{}_k(X_t)^k\right).\end{aligned}$$

We use the Ricci identity to deduce that

$$\begin{aligned}\nabla_i\Delta(X_t)^j &= \nabla_i\left(g^{pq}\nabla_p\nabla_q(X_t)^j\right) \\ &= g^{pq}\nabla_i\nabla_p\nabla_q(X_t)^j \\ &= g^{pq}\left[\nabla_p\nabla_i\nabla_q(X_t)^j - R_{ipq}{}^r\nabla_r(X_t)^j + R_{ipr}{}^j\nabla_q(X_t)^r\right] \\ &= \nabla^q\left[\nabla_q\nabla_i(X_t)^j + R_{iqr}{}^j(X_t)^r\right] - R_{ir}\nabla^r(X_t)^j + R_{ipr}{}^j\nabla^p(X_t)^r \\ &= \Delta\nabla_i(X_t)^j + \nabla^q\left(R_{iqr}{}^j(X_t)^r\right) - R_{ir}\nabla^r(X_t)^j + R_{ipr}{}^j\nabla^p(X_t)^r \\ &= \Delta\nabla_i(X_t)^j + \nabla^q R_{iqr}{}^j\cdot(X_t)^r + 2R_{iqr}{}^j\nabla^q(X_t)^r - R_{ir}\nabla^r(X_t)^j.\end{aligned}$$

Plugging it into the equation for $\partial_t|\nabla X_t|^2$ we arrive at

$$\begin{aligned}\partial_t|\nabla X_t|^2 &= 2\nabla^i(X_t)_j\Big[\Delta\nabla_i(X_t)^j + \nabla^q R_{iqr}{}^j(X_t)^r + 2R_{iqr}{}^j\nabla^q(X_t)^r \\ &\quad - R_{ir}\nabla^r(X_t)^j + \nabla_i\nabla^j\mathrm{div}(X_t) + (X_t)^k\nabla_i R^j{}_k + R^j{}_k\nabla_i(X_t)^k\Big] \\ &= \Delta|\nabla X_t|^2 - 2\left|\nabla^2 X_t\right|^2 + 2\nabla^q R_{iqr}{}^j\nabla^i(X_t)_j\cdot(X_t)^r \\ &\quad + 4R_{iqrj}\nabla^q(X_t)^r\nabla^i(X_t)^j - 2R_{ir}\nabla^r(X_t)^j\nabla^i(X_t)_j \\ &\quad + 2\nabla^i(X_t)_j\cdot\nabla_i\nabla^j\mathrm{div}(X_t) + 2\nabla_i R^j{}_k\cdot(X_t)^k\nabla^i(X_t)_j \\ &\quad 2R^j{}_k\nabla^i(X_t)^k\nabla^i(X_t)_j \\ &= \Delta|\nabla X_t|^2 - 2\left|\nabla^2 X_t\right|^2 - 4R_{qirj}\nabla^q(X_t)^r\nabla^i(X_t)^j \\ &\quad - 2R_{ir}\nabla^r(X_t)^j\nabla^i(X_t)_j + 2R_{jk}\nabla_i(X_t)^k\nabla^i(X_t)^j \\ &\quad + 2\nabla^i(X_t)_j\cdot\nabla_i\nabla^j\mathrm{div}(X_t) + 2\nabla_i R_{jk}\cdot(X_t)^k\nabla^i(X_t)^j \\ &\quad - 2\nabla^q R_{qirj}(X_t)^r\nabla^i(X_t)^j.\end{aligned}$$

Changing the indices yields the desired result. □

By the Bianchi identity, the above lemma can be written as

Corollary 1.5. *If X_t is the solution of the flow (1.4), then*

$$\begin{aligned}\partial_t|\nabla X_t|^2 &= \Delta|\nabla X_t|^2 - 2\left|\nabla^2 X_t\right|^2 - 4R_{ijk\ell}\nabla^i(X_t)^k\nabla^j(X_t)^\ell \\ &\quad - 2R_{ij}\nabla^i(X_t)^k\nabla^j(X_t)_k + 2R_{ij}\nabla^k(X_t)^i\nabla_k(X_t)^j \\ &\quad + 2\left(\nabla_i R_{jk} - \nabla_j R_{ki} + \nabla_k R_{ij}\right)(X_t)^k\nabla^i(X_t)^j \\ &\quad + 2\left\langle\mathbf{Def}(X_t), \nabla\nabla\mathrm{div}(X_t)\right\rangle.\end{aligned}$$

Lemma 1.6. *(1) If X_t is the solution of the flow (1.4), then*

$$\partial_t \mathrm{div}(X_t) = 2\Delta \mathrm{div}(X_t) + \langle X_t, \nabla R\rangle + 2R_{ij}\nabla^i (X_t)^j \tag{1.16}$$

(2) If X_t is the solution of the flow (1.4), then

$$\begin{aligned}\partial_t |\mathrm{div}(X_t)|^2 &= 2\Delta|\mathrm{div}(X_t)|^2 - 4\left|\nabla \mathrm{div}(X_t)\right|^2\\ &\quad + 2\mathrm{div}(X_t)\langle X_t, \nabla R\rangle + 4\mathrm{div}(X_t)\cdot R_{ij}\nabla^i(X_t)^j\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\int_M |\mathrm{div}(X_t)|^2 dV &= -4\int_M |\nabla \mathrm{div}(X_t)|^2 dV\\ &\quad - 4\int_M \mathrm{Ric}\left(X_t, \nabla \mathrm{div}(X_t)\right) dV.\end{aligned} \tag{1.17}$$

In particular, if $\mathrm{Ric} = 0$ *and* $\mathrm{div}(X) \equiv 0$, *then* $\mathrm{div}(X_t) \equiv 0$.

Proof. According to (1.4), one has

$$\begin{aligned}\partial_t \mathrm{div}(X_t) &= \nabla_i\left(\partial_t (X_t)^i\right) = \nabla_i\left(\Delta (X_t)^i + \nabla^i \mathrm{div}(X_t) + R^i{}_j (X_t)^j\right)\\ &= \nabla_i\left(\Delta(X_t)^i\right) + \Delta \mathrm{div}(X_t) + \nabla^i\left(R_{ij}(X_t)^j\right).\end{aligned}$$

Next we compute the first term $\nabla_i\left(\Delta(X_t)^i\right)$ as follows:

$$\begin{aligned}\nabla_i\left(\Delta(X_t)^i\right) &= g^{pq}\nabla_i\nabla_p\nabla_q (X_t)^i\\ &= g^{pq}\left(\nabla_p\nabla_i\nabla_q(X_t)^i - R_{ipq}{}^r\nabla_r(X_t)^i + R_{ipr}{}^i\nabla_q(X_t)^r\right)\\ &= \nabla^q\left(\nabla_q\nabla_i(X_t)^i + R_{iqr}{}^i(X_t)^r\right) - R_{ir}\nabla^r(X_t)^i + R_{pr}\nabla^p(X_t)^r\\ &= \Delta\nabla_i(X_t)^i + \nabla^q\left(R_{qr}(X_t)^r\right).\end{aligned}$$

Combining those two expression gives

$$\begin{aligned}\partial_t \mathrm{div}(X_t) &= 2\Delta \mathrm{div}(X_t) + 2\nabla^i\left(R_{ij}(X_t)^j\right)\\ &= 2\Delta \mathrm{div}(X_t) + 2\nabla^i R_{ij}\cdot (X_t)^j + 2R_{ij}\nabla^i(X_t)^j\\ &= 2\Delta \mathrm{div}(X_t) + \nabla_j R\cdot (X_t)^j + 2R_{ij}\nabla^i(X_t)^j\end{aligned}$$

proving (1.16). For (1.17), the evolution equation for $|\mathrm{div}(X_t)|^2$ is

$$\begin{aligned}\partial_t|\mathrm{div}(X_t)|^2 &= 2\mathrm{div}(X_t)\cdot\partial_t \mathrm{div}(X_t)\\ &= 2\mathrm{div}(X_t)\left(2\Delta \mathrm{div}(X_t) + (X_t)^i\nabla_i R + 2R_{ij}\nabla^i(X_t)^j\right)\\ &= 2\Delta|\mathrm{div}(X_t)|^2 - 4|\nabla \mathrm{div}(X_t)|^2\\ &\quad + 2\mathrm{div}(X_t)\cdot (X_t)^i\nabla_i R + 4\left(\mathrm{div}(X_t)R_{ij}\right)\nabla^i(X_t)^j.\end{aligned}$$

Integrating both sides over M yields

$$\begin{aligned}\frac{d}{dt}\int_M |\mathrm{div}(X_t)|^2 dV &= -4\int_M |\nabla \mathrm{div}(X_t)|^2 dV\\ + 2\int_M \mathrm{div}(X_t)\left((X_t)^i\nabla_i R\right) dV &- 4\int_M \nabla^i\left(\mathrm{div}(X_t)R_{ij}\right)(X_t)^j dV.\end{aligned}$$

Since

$$\begin{aligned}4\nabla^i\left(\mathrm{div}(X_t)R_{ij}\right)(X_t)^j &= 4\left[\nabla^i \mathrm{div}(X_t)\cdot R_{ij} + \mathrm{div}(X_t)\cdot\nabla^i R_{ij}\right](X_t)^j\\ &= 4R_{ij}(X_t)^j\nabla^i \mathrm{div}(X_t) + 2\nabla_j R\cdot (X_t)^j \mathrm{div}(X_t)\end{aligned}$$

it follows that (1.17) is true. When $\mathrm{Ric} = 0$, we obtain

$$\frac{d}{dt}\int_M |\mathrm{div}(X_t)|^2 dV \leq 0$$

which means $\int_M |\mathrm{div}(X_t)|^2 dV \leq \int_M |\mathrm{div}(X)|^2 dV = 0$ and therefore $|\mathrm{div}(X_t)|^2 = 0$. Thus $\mathrm{div}(X_t) \equiv 0$. □

1.4. **Long-time existence.** Now we can state our main results to the flow (1.4).

Theorem 1.7. (Long-time existence) *Suppose that (M, g) is a closed and orientable Riemannian manifold. Given an initial vector field, the flow (1.4) exists for all time.*

The main method on proving above theorem is the standard approach in PDEs and an application of Sobolev embedding theorem. After establishing the long-time existence, we can study the convergence problem of the flow (1.4).

Proof. We now turn to prove the short-time existence of the flow (1.4). Note that (1.4) can be written as

$$\begin{aligned} \partial_t (X_t)^i &= \Delta (X_t)^i + \nabla^i \mathrm{div}(X_t) + R^i{}_j (X_t)^j \\ &= \nabla^k \nabla_k (X_t)^i + \nabla^i \nabla_j (X_t)^j + R^i{}_j (X_t)^j \\ &= \sum_{j=1}^m \left(\delta^i_j \sum_{k=1}^m \nabla^k \nabla_k + \nabla^i \nabla_j \right) (X_t)^j + R^i{}_j (X_t)^j. \end{aligned} \tag{1.18}$$

For any $\xi = (\xi_1, \cdots, \xi_m) \in \mathbb{R}^m$, we have

$$\sum_{i,j=1}^m \left(\delta^i_j \sum_{k=1}^m \xi_k \xi_k + \xi_i \xi_j \right) = \sum_{i,k=1}^m \xi_k \xi_k + \sum_{i,j=1}^m \xi_i \xi_j = m|\xi|^2 + \sum_{i,j=1}^m \xi_i \xi_j, \tag{1.19}$$

where $|\xi| \doteqdot \left(\sum_{k=1}^m \xi_k^2\right)^{1/2}$ denotes the length of ξ in $\mathbb{R}^m$. On the other hand, plugging

$$\sum_{i,j=1}^m (\xi_i + \xi_j)^2 = \sum_{i,j=1}^m \left(\xi_i^2 + \xi_j^2 + 2\xi_i \xi_j \right) = 2m|\xi|^2 + 2\sum_{i,j=1}^m \xi_i \xi_j$$

into (1.19) yields

$$\begin{aligned} \sum_{i,j=1}^m \left(\delta^i_j \sum_{k=1}^m \xi_k \xi_k + \xi_i \xi_j \right) &= \frac{1}{2} \sum_{i,j=1}^m (\xi_i + \xi_j)^2 \\ &= 2\sum_{i=1}^m \xi_i^2 + \frac{1}{2}\sum_{i \neq j} (\xi_i + \xi_j)^2 \\ &\geq 2|\xi|^2. \end{aligned}$$

Then, by the standard theory for partial differential equations of parabolic type, we have that the flow (1.4) exists for a short time.

Since the flow equation is linear, a standard theory in PDEs implies the long-time existence. □

1.5. **Convergence.** In what follows, we always assume that (M, g) is a closed and oriented Riemannian manifold of dimension m. Since M is compact, we can find a constant B such that

$$R_{ij} \leq B g_{ij}. \tag{1.20}$$

Then the energy functional $\mathcal{E}(X_t)$ satisfies

$$\int_M \left[|\nabla X_t|^2 + (\mathrm{div}(X_t))^2 - B|X_t|^2 \right] dV \leq \mathcal{E}(X_t). \tag{1.21}$$

Using Proposition 1.1, we have

$$\int_M |\nabla X_t|^2 dV \leq \mathcal{E}(X_t) + B \int_M |X_t|^2 dV = \mathcal{E}(X_t) + B \cdot u(t) \leq \mathcal{E}(X) + B \cdot u(0),$$

where

$$u(t) := \int_M |X_t|^2 dV.$$

Hence $\nabla X_t \in L^2(M, TM)$. On the other hand $u(t) \leq u(0)$, we conclude that

$$||X_t||_{H^1(M,TM)} \leq C_1(M, g, X). \tag{1.22}$$

By the regularity of parabolic equations and the flow (1.4), we obtain

$$||X_t||_{H^\ell(M,TM)} \leq C_\ell = C_\ell(M, g, X)$$

for each ℓ. Therefore we can find $X_\infty \in H^\ell(M, TM)$ and a subsequence $(X_{t_i})_{i \in \mathbb{N}}$ such that $X_{t_i} \to X_\infty$ a.e. as $i \to \infty$. By Sobolev imbedding theorem, $X_\infty \in C^\infty(M, TM)$ and $X_t \to X_\infty$ as $t \to \infty$.

Corollary 1.2 implies there exists a subsequence, say, without loss of generality, $(X_{t_i})_{i \in \mathbb{N}}$, such that

$$\left|\left| \partial_t X_t \big|_{t=t_i} \right|\right|_{L^2(M,g)} \to 0. \tag{1.23}$$

According to (1.11) and (1.23), $||\partial_t X_t||_{L^2(M,g)}$ decreases and converges to 0 as $t \to \infty$. Therefore the smooth vector field X_∞ satisfies

$$\Delta_{\mathrm{LB}}(X_\infty)^i + \nabla^i \mathrm{div}(X_\infty) + R^i{}_j (X_\infty)^j = 0. \tag{1.24}$$

In summary, we proved

Theorem 1.8. (Convergence) *Suppose that (M, g) is a closed and orientable Riemannian manifold. If X is a vector field, there exists a unique smooth solution X_t to the flow (1.4) for all time t. As t goes to infinity, the vector field X_t converges uniformly to a Killing vector field X_∞.*

Remark 1.9. *As Professor Cliff Taubes remarked that Theorem 1.7 and 1.8 also follow from an eigenfunction expansion for the relevant linear operator that defines the flow (1.4), which gives a short proof of those two theorems.*

Theorem 1.8 does *not* give us a nontrivial Killing vector field. For example, if X is identically zero, then by the uniqueness theorem the limit vector field is also identically zero. When the Ricci curvature is negative, Bochner's theorem implies that there is no nontrivial Killing vector field.

To obtain a nonzero Killing vector field, we have the following criterion.

Proposition 1.10. *Suppose that (M, g) is a closed and orientable Riemannian manifold and X is a vector field on M. If X_t is the solution of the flow (1.4) with the initial value X, then*

$$\int_0^\infty \mathcal{E}(X_t)\,dV < \infty. \tag{1.25}$$

Let

$$\mathbf{Err}(X) := \frac{1}{2}\int_M |X|^2 dV - \int_0^\infty \mathcal{E}(X_t)\,dt. \tag{1.26}$$

Therefore $\mathbf{Err}(X) \geq 0$ and X_∞ is nonzero if and only if $\mathbf{Err}(X) > 0$.

The higher derivatives of $\mathcal{E}(X_t)$ have explicit formulas in terms of the energy functionals of lower derivatives of X_t.

Proposition 1.11. *If X_t is the solution of the flow (1.4), then*

$$\mathcal{E}''(X_t) = 4\mathcal{E}(\partial_t X_t) \geq 0. \tag{1.27}$$

Proof. Using (1.11), we have

$$\begin{aligned}
\mathcal{E}''(X_t) &= -4\int_M \partial_t(X_t)_i \cdot \partial_t\left(\partial_t(X_t)^i\right) dV \\
&= -4\int_M \partial_t(X_t)_i \cdot \partial_t\left(\Delta_{\mathrm{LB}}(X_t)^i + \nabla^i \mathrm{div}(X_t) + R^i{}_j(X_t)^j\right) dV \\
&= -4\int_M \partial_t(X_t)_i\left(\Delta_{\mathrm{LB}}\partial_t(X_t)^i + \nabla^i\mathrm{div}(\partial_t X_t) + R^i{}_j\partial_t(X_t)^j\right) dV \\
&= -4\int_M \left(\frac{1}{2}\Delta|\partial_t X_t|^2 - |\nabla\partial_t X_t|^2\right) dV \\
&\quad - 4\int_M \partial_t(X_t)_i\left(\nabla^i\mathrm{div}(\partial_t X_t) + R^i{}_j\partial_t(X_t)^j\right) dV \\
&= 4\int_M \left[|\nabla\partial_t X_t|^2 + \nabla^i\partial_t(X_t)_i \cdot \mathrm{div}(\partial_t X_t) - \mathrm{Ric}(\partial_t X_t, \partial_t X_t)\right] dV \\
&= 4\int_M | \left[\nabla\partial_t X_t|^2 + \partial_t\mathrm{div}(X_t)\cdot\mathrm{div}(\partial_t X_t) - \mathrm{Ric}(\partial_t X_t, \partial_t X_t)\right] dV \\
&= 4\int_M \left[|\nabla\partial_t X_t|^2 + (\mathrm{div}(\partial_t X_t))^2 - \mathrm{Ric}(\partial_t X_t, \partial_t X_t)\right] dV \\
&= 4\mathcal{E}(\partial_t X_t)
\end{aligned}$$

which is nonnegative according to (1.7). □

1.6. **A connection to the Navier-Stokes equations.** A surprising observation is that our flow (1.4) is very close to the Navier-Stokes equations [2, 13](without the pressure) on manifolds

$$\partial_t X_t + \nabla_{X_t} X_t = \mathrm{div}(S_t), \quad \mathrm{div}(X_t) = 0, \tag{1.28}$$

where $S_t := 2\mathbf{Def}(X_t)$ is the stress tensor of X_t. By an easy computation we can write (1.28) as

$$\partial_t(X_t)^i + (\nabla_{X_t}X_t)^i = \Delta(X_t)^i + \nabla^i\mathrm{div}(X_t) + R^i{}_j X^j, \quad \mathrm{div}(X_t) = 0. \tag{1.29}$$

Compared (1.4) with (1.29), we give a geometric interpolation of the right (or the linear) part of the Navier-Stokes equations on manifolds.

When the Ricci tensor field is identically zero, our flow (1.4) keeps the property that $\mathrm{div}(X_t) = 0$ (see (1.17)).

As a consequence of the non-negativity of $\mathcal{E}$ we can prove that

Theorem 1.12. *Suppose that (M, g) is a closed and orientable Riemannian manifold. If X_t is a solution of the Navier-Stokes equations (1.35), then*

$$\frac{d}{dt}\left(\int_M |X_t|^2 dV\right) = -2\mathcal{E}(X_t) \leq 0. \tag{1.30}$$

In particular

$$\int_M |X_t|^2 dV \leq \int_M |X_0|^2 dV. \tag{1.31}$$

Proof. By multiplying by $(X_t)_i$ the equation (1.29) equals

$$\frac{1}{2}\partial_t|X_t|^2 + \langle\nabla_{X_t}X_t, X_t\rangle = \left\langle \Delta X_t + \nabla\mathrm{div}(X_t) + \mathrm{Rc}^\sharp(X_t), X_t\right\rangle.$$

Integrating on both sides yields

$$\frac{1}{2}\frac{d}{dt}\int_M |X_t|^2 dV + \int_M \langle\nabla_{X_t}X_t, X_t\rangle\, dV = -\mathcal{E}(X_t).$$

From Lemma 1.13 below, we verify (1.30) since $\mathrm{div}(X_t) = 0$. □

Lemma 1.13. *Suppose that (M, g) is closed and oriented Riemannian manifold. Then for any vector field $X \in C^\infty(M, TM)$, we have*

$$\int_M \langle\nabla_X X, X\rangle\, dV = -\frac{1}{2}\int_M \mathrm{div}(X)|X|^2 dV. \tag{1.32}$$

Proof. Indeed, using $(\nabla_X X)^j = X^i\nabla_i X^j$ we have

$$\begin{aligned}
\int_M \langle\nabla_X X, X\rangle\, dV &= \int_M (\nabla_X X)^j X_j\, dV = \int_M X^i\nabla_i X^j \cdot X_j\, dV \\
&= \int_M \nabla_i X^j (X^i X_j)\, dV = -\int_M X^j \nabla_i(X^i X_j)\, dV \\
&= -\int_M X^j\left[\mathrm{div}(X)X_j + X^i\nabla_i X_j\right] dV \\
&= -\int_M \mathrm{div}(X)|X|^2 dV - \int_M X^i X^j \nabla_i X_j\, dV \\
&= -\int_M \mathrm{div}(X)|X|^2 dV - \int_M \langle\nabla_X X, X\rangle\, dV.
\end{aligned}$$

Arranging the terms yields (1.32). □

The similar result was considered by Wilson [15] for the standard metric on $\mathbf{R}^3$.

1.7. **A connection to Kazdan-Warner-Bourguignon-Ezin identity.** If (M, g) is a closed Riemannian manifold with $m \geq 2$ and if X is a Killing vector field, then

$$\int_M \langle\nabla R, X\rangle\, dV = 0. \tag{1.33}$$

This identity (actually holds for any conformal Killing vector fields) was proved by Bourguignon and Ezin [1] and the surface case is the classical Kazdan-Warner identity [6]. For convenience, we call such an identity as KWBE *identity*. For its application to Ricci flow we refer readers to [3]. In this subsection we study the asymptotic behavior of the KWBE identity under the flow (1.4).

For any vector field X, we define the KWBE *functional* as

$$\mathcal{I}(X) := \int_M \langle \nabla R, X \rangle \, dV.$$

Then, where $X_t = X^i \frac{\partial}{\partial x^i}$,

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(X_t) &= \int_M \nabla_i R \left(\Delta X^i + \nabla^i \mathrm{div}(X_t) + R^i_j X^j\right) dV \\ &= \int_M \nabla_i R \cdot \Delta X^i \, dV - \int_M \Delta R \cdot \mathrm{div}(X_t)\, dV + \int_M R_{ij} X^i \nabla^j R \, dV.\end{aligned}$$

Using the commutative formula $\nabla \Delta R = \Delta \nabla R - \mathrm{Rc}(\nabla R, \cdot)$ yields

$$\begin{aligned}\int_M \nabla_i R \cdot \Delta X^i dV &= \int_M \langle X_t, \Delta \nabla R \rangle \, dV \\ &= \int_M \langle X_t, \nabla \Delta R + \mathrm{Rc}(\nabla R, \cdot) \rangle \, dV \\ &= -\int_M \Delta R \cdot \mathrm{div}(X_t)\, dV + \int_M R_{ij} X^i \nabla^j R \, dV\end{aligned}$$

and therefore

$$\frac{d}{dt}\mathcal{I}(X_t) = -2\int_M \Delta R \cdot \mathrm{div}(X_t)\, dV + 2\int_M \mathrm{Rc}(X_t, \nabla R)\, dV. \tag{1.34}$$

The last term on the right-hand side of (1.34) can be simplified by

$$\begin{aligned}\int_M \nabla^i R \left(X^j R_{ij}\right) dV &= -\int_M R \left(\nabla^i X^j \cdot R_{ij} + X^j \cdot \frac{1}{2}\nabla_j R\right) dV \\ &= -\int_M R R_{ij} \nabla^i X^j \, dV - \frac{1}{2}\int_M R X^j \nabla_j R \, dV.\end{aligned}$$

We also have

$$\begin{aligned}\int_M R X^j \nabla_j R \, dV &= -\int_M \nabla_j (R X^j) R \, dV \\ &= -\int_M R X^j \nabla_j R \, dV - \int_M R^2 \mathrm{div}(X_t)\, dV\end{aligned}$$

so that

$$\int_M R X^j \nabla_j R \, dV = -\frac{1}{2}\int_M R^2 \mathrm{div}(X_t) dV. \tag{1.35}$$

From (1.34), (1.35), (1.1) and Theorem 1.8, we arrive at

Proposition 1.14. *If (M, g) is a closed Riemannian manifold and X_t is a solution to (1.4), then*

$$\frac{d}{dt}\mathcal{I}(X_t) = 2\int_M \left(-\Delta + \frac{R}{4}\right) R \cdot \mathrm{div}(X_t)\, dV - 2\int_M R \left\langle \mathrm{Rc}, \mathbf{Def}(X_t) \right\rangle dV. \tag{1.36}$$

In particular,

$$\lim_{t\to\infty} \frac{d}{dt}\mathcal{I}(X_t) = 0. \tag{1.37}$$

This proposition gives the limiting behavior of $\frac{d}{dt}\mathcal{I}(X_t)$. However, the pointwise behavior of $\frac{d}{dt}\mathcal{I}(X_t)$ is very complicated. For example, we can find a compact Riemannian manifold such that $\frac{d}{dt}\mathcal{I}(X_t) > 0$ or < 0 depending on the choice of the initial vector fields.

Corollary 1.15. *Suppose that (M, g) is a closed m-dimensional Einstein manifold with $m \geq 3$.*

(a) *When $m = 4$ or the scalar curvature of g vanishes identically, $\frac{d}{dt}\mathcal{I}(X_t) = 0$ for all t, where X_t is the solution of (1.4) with any given initial vector field X.*
(b) *When $m \neq 4$ and the scalar curvature of g does not vanish identically, there exists a vector field X such that $\frac{d}{dt}\mathcal{I}(X_t) > 0$ for all t, where X_t is the solution of (1.4) with the initial vector field X.*
(c) *When $m \neq 4$ and the scalar curvature of g does not vanish identically, there exists a vector field X such that $\frac{d}{dt}\mathcal{I}(X_t) < 0$ for all t, where X_t is the solution of (1.4) with the initial vector field X.*

Proof. By assumption we have $\mathrm{Ric} = \frac{R}{m}g$ and R is constant. Using (1.36) we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(X_t) &= \int_M \frac{R^2}{2}\cdot \mathrm{div}(X_t)\,dV - 2\int_M \frac{R^2}{m}\mathrm{div}(X_t)\,dV \\ &= \int_M \frac{m-4}{2m}R^2\cdot\mathrm{div}(X_t)\,dV. \end{aligned} \tag{1.38}$$

The part (a) follows immediately.

For parts (b) and (c), we may assume that $m > 4$, otherwise we can consider $-X_t$. According to the evolution equation (1.16) yields

$$\partial_t \mathrm{div}(X_t) = 2\Delta\mathrm{div}(X_t) + \frac{2R}{m}\mathrm{div}(X_t)$$

which can be written as

$$\partial_t\left(e^{-\frac{2R}{m}t}\mathrm{div}(X_t)\right) = 2\Delta\left(e^{-\frac{2R}{m}t}\mathrm{div}(X_t)\right).$$

By the maximum principle,

$$e^{\frac{2R}{m}t}\min_M \mathrm{div}(X) \leq \mathrm{div}(X_t) \leq e^{\frac{2R}{m}t}\max_M \mathrm{div}(X), \quad X := X_0. \tag{1.39}$$

Given any fixed vector field X', let f be a smooth function on M satisfying $\Delta f = \mathrm{div}(X') - 1$. Then

$$\mathrm{div}(X' - \nabla f) = \mathrm{div}(X') - \Delta f = 1 > 0$$

on M. Choose $X := X' - \nabla f$. Then

$$\mathrm{div}(X_t) \geq e^{\frac{2R}{m}t} > 0 \quad \text{on } M$$

for all t. Substituting this into (1.38) we arrive at

$$\frac{d}{dt}\mathcal{I}(X_t) \geq \frac{m-4}{2m}e^{\frac{2R}{m}t}\int_M R^2\,dV$$

where we used $m > 4$. Since the scalar curvature R does not vanishes identically, the L^2-norm of R must be positive and consequently, $\frac{d}{dt}\mathcal{I}(X_t) > 0$ for all t.

Similarly, we can prove part (c). □

2. A conjecture to the flow and its application

Before stating a conjecture to the flow (1.4), we shall look at a simple case that (M, g) is an Einstein manifold with positive sectional curvature and the solution of (1.4) is the sum of the initial vector field and a gradient vector field. That is, we assume

$$R_{ij} = \frac{R}{m} g_{ij}, \quad m \geq 3, \quad X_t = X + \nabla f_t,$$

where f_t are some functions on M. By a theorem of Schur, the scalar curvature R must be a constant. In this case the flow (1.4) is equivalent to

$$\nabla\left(\partial_t f_t - 2\Delta f_t - \frac{2R}{m} f_t\right) = X^{\dagger}, \tag{2.1}$$

where

$$X^{\dagger} := \Delta X + \nabla(\mathrm{div}(X)) + \mathrm{Ric}^{\sharp}(X) \tag{2.2}$$

is the vector field associated to X. Clearly that the operator $\dagger$ is not self-adjoint on the space of vector fields, with respect to the L^2-inner product with respect to (M, g).

2.1. Einstein manifolds with positive scalar curvature.

If (M, g) is an m-dimensional Einstein manifold with positive scalar curvature, then we can prove that the limit vector field converges to a nonzero Killing vector field, provided the initial vector field satisfying some conditions. We first give a L^2-estimate for f_t.

Proposition 2.1. *Suppose that (M, g) is an m-dimensional closed and orientable Einstein manifold with positive scalar curvature R, where $m \geq 3$. Let X be a nonzero vector field satisfying $X^{\dagger} = \nabla \varphi_X$ for some smooth function φ_X on M. Then for any given constant c, the equation*

$$\partial_t f_t = 2\Delta f_t + \frac{2R}{m} f_t + \varphi_X, \quad f_0 = c, \tag{2.3}$$

exists for all time. Moreover,

(i) *we have*

$$\int_M f_t \, dV = \left[c \cdot \mathrm{Vol}(M, g) + \frac{m}{2R}\int_M \varphi_X \, dV\right] e^{\frac{2R}{m}t} - \frac{m}{2R}\int_M \varphi_X \, dV. \tag{2.4}$$

Setting

$$c_X := -\frac{m}{2R \cdot \mathrm{Vol}(M, g)} \int_M \varphi_X \, dV,$$

yields

$$\int_M f_t \, dV = -\frac{m}{2R}\int_M \varphi_X \, dV, \quad \text{if } c = c_X.$$

(ii) *if we choose the nonzero function φ_X so that its integral over M is zero and $f_0 = 0$, then $\int_M f_t \, dV = 0$ and the L^2-norm of f_t is bounded by*

$$||f_t||_2 \leq \frac{||\varphi_X||_2}{2\left(\lambda_1 - \frac{R}{m}\right)} - \frac{||\varphi_X||_2}{2\left(\lambda_1 - \frac{R}{m}\right)} e^{-2\left(\lambda_1 - \frac{R}{m}\right)t}, \tag{2.5}$$

where $||\cdot||_2$ means $||\cdot||_{L^2(M,g)}$ the L^2-norm with respect to (M, g), and λ_1 stands for the first nonzero eigenvalue of (M, g).

For a moment, we put

$$a(t) := \int_M f_t \, dV, \quad b(t) := \int_M |f_t|^2 dV.$$

Then, the equation (2.3) implies that

$$a'(t) = \frac{2R}{m} a(t) + \int_M \varphi_X \, dV,$$

and

$$\begin{aligned} b'(t) &= -4 \int_M |\nabla f_t|^2 dV + \frac{4R}{m} b(t) + 2 \int_M f_t \varphi_X \, dV \\ &\leq -4 \left(\lambda_1 - \frac{R}{m} \right) b(t) + 2b^{1/2}(t) ||\varphi_X||_2. \end{aligned}$$

By a theorem of Lichnerowicz, we have that $\lambda_1 \geq \frac{R}{m-1} > \frac{R}{m}$. Hence (2.3) and (2.4) follow immediately.

Consequently, we have the following

Theorem 2.2. *Suppose that (M, g) is an m-dimensional closed and orientable Einstein manifold with positive scalar curvature R, where $m \geq 3$. Let X be a nonzero vector field satisfying the following two conditions:*

(i) *$X^\dagger$ is a gradient vector field, and*
(ii) *X is not a gradient vector field.*

Then the flow (1.4) with initial value X converges uniformly to a nonzero Killing vector field.

2.2. **A conjecture and its applications.** By Bochner's theorem, any Killing vector field on a closed and orientable Riemmanian manifold with negative Ricci curvature is trivial. Hence, based on a result in the Einstein case, we propose the following conjecture.

Conjecture 2.3. *Suppose that M is a closed Riemannian manifold with positive sectional curvature. For some initial vector field and a certain Riemannian metric g of positive sectional curvature, the flow (1.4) converges uniformly to a nonzero Killing vector field with respect to g.*

Our study shows that we may need to change to a new metric, which still has positive sectional curvature, to get the nonzero limit which is a Killing vector field with respect to this new metric. For this purpose we have computed variations of the functional $\mathfrak{L}$ or $\mathcal{E}$ relative to the new metric, as well as the Perelman-type functional for our flow.

Obviously a solution of this conjecture immediately answers the following long-standing question of Yau [12].

Question 2.4. *Does there exist an effective $\mathbb{S}^1$-action on a closed manifold with positive sectional curvature?*

Assuming Conjecture 2.3, we can deduce several important corollaries. We first recall the well-known Hopf's conjectures.

Conjecture 2.5. *If M is a closed and even dimensional Riemannian manifold with positive sectional curvature, then the Euler characteristic number of M is positive, i.e., $\chi(M) > 0$.*

Conjecture 2.6. *On $\mathbb{S}^2 \times \mathbb{S}^2$ there is no Riemannian metric with positive sectional curvature.*

For the recent development of Hopf's conjectures, we refer to [11, 12]. A simple argument shows that Conjecture 2.5 and 2.6 follow from Conjecture 2.3.

Corollary 2.7. *Conjecture 2.3 implies Conjecture 2.5.*

Proof. From [7] we know that the Killing vector field X must have zero, and the zero sets consist of finite number of totally geodesic submanifolds $\{M_i\}$ of M with the induced Riemannian metrics. Moreover each M_i is even dimensional and has positive sectional curvature. Hence we have $\chi(M) = \sum_i \chi(M_i)$. By induction, we obtain $\chi(M) > 0$. $\square$

Hsiang and Kleiner [5] showed that if M is a 4-dimensional closed Riemannian manifold with positive sectional curvature, admitting a nonzero Killing vector field, then M is homeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$. Consequently, $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit a Riemannian metric, whose sectional curvature is positive, with a nontrivial Killing vector field. Therefore

Corollary 2.8. *Conjecture 2.3 implies Conjecture 2.6.*

3. Variants geometric flows

In the last section, we discuss several new geoemtric flows whose fixed points give Killing vector fields. Recall the notions in [10]. Let (M, g) be a closed and orientable Riemannian manifold of dimension m and ϕ a positive smooth function on M. Define

$$\widetilde{\mathrm{Ric}}_\infty := \mathrm{Ric} - \mathrm{Hess}(\ln\phi) \tag{3.1}$$

the Bakry-Émery Ricci tensor field. For any smooth tensor field T on M consider the weighted L^2-inner product given by

$$\langle T, T\rangle_\phi := \int_M (T, T)\phi\, dV \tag{3.2}$$

and let us denote $\tilde{\delta}$ the formal adjoint of d with respect to this inner product. Then

$$\tilde{\delta} = \delta - i_{(d\ln\phi)^\#} \tag{3.3}$$

where δ is the usual formal adjoint of d and $(d\ln\phi)^\#$ stands for the corresponding vector field of the 1-form $d\ln\phi$.

In [10], Lott obtained the following Bochner formula (where ω is a 1-form):

$$\langle d\omega, d\omega\rangle_\phi + \langle\tilde{\delta}\omega, \tilde{\delta}\omega\rangle_\phi - \langle\nabla\omega, \nabla\omega\rangle_\phi = \langle\widetilde{\mathrm{Ric}}_\infty\omega, \omega\rangle_\phi \tag{3.4}$$

or

$$\langle\nabla\omega, \nabla\omega\rangle_\phi + \langle\tilde{\delta}\omega, \tilde{\delta}\omega\rangle_\phi - \langle\omega, \widetilde{\mathrm{Ric}}_\infty\omega\rangle_\phi = \langle\mathcal{L}_{\omega^\#}g, \mathcal{L}_{\omega^\#}g\rangle_\phi \tag{3.5}$$

where $\mathcal{L}$ means the Lie derivative. Let $X := \omega^\#$ or $X_\flat = \omega$ in (3.5) we obtain

$$\int_M |\mathcal{L}_X g|^2\phi\, dV = \int_M \left[|\nabla X|^2 + |\tilde{\delta}X_\flat|^2 - \widetilde{\mathrm{Ric}}_\infty(X, X)\right]\phi\, dV. \tag{3.6}$$

3.1. **New criterion: I.** Given a smooth function f on M, set

$$\phi := e^f, \quad \ln\phi = f \tag{3.7}$$

and define

$$\begin{aligned} \mathrm{Ric}_f &:= \widetilde{\mathrm{Ric}}_\infty = \mathrm{Ric} - \mathrm{Hess}(f), \\ \mathrm{div}_f &:= -\tilde{\delta} = -\delta + i_{\nabla f} = \mathrm{div} + i_{\nabla f}. \end{aligned}$$

For any smooth vector field X we have

$$e^{-f}\mathrm{div}\left(e^f X\right) = e^{-f}\left(e^f\mathrm{div}(X) + e^f\langle\nabla f, X\rangle\right) = \mathrm{div}(X) + \langle\nabla f, X\rangle$$

which implies that

$$\mathrm{div}_f = \frac{1}{e^f}\mathrm{div}\left(e^f\ \right), \tag{3.8}$$

a weighted divergence in the sense of [4]. Therefore the identity (3.6) can be rewritten as

$$\int_M |\mathcal{L}_X g|^2 e^f dV = \int_M \left[|\nabla X|^2 + |\mathrm{div}_f(X)|^2 - \mathrm{Ric}_f(X,X)\right] e^f dV. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} \int_M |\nabla X|^2 e^f dV &= \int_M \nabla_i X_j \left(e^f \nabla^i X^j\right) dV \\ &= -\int_M X_j\left(\nabla_i f\nabla^i X^j + \Delta X^j\right) e^f dV \\ &= -\int_M \langle X, \Delta_f X\rangle e^f dV \end{aligned}$$

where

$$\Delta_f X^j := \Delta X^j + \nabla_i f \nabla^i X^j.$$

Similarly,

$$\begin{aligned} \int_M |\mathrm{div}_f(X)|^2 e^f dV &= \int_M \mathrm{div}_f(X)\left(e^f\mathrm{div}_f(X)\right) dV \\ &= \int_M e^{-f}\mathrm{div}\left(e^f X\right)\left(e^f\mathrm{div}_f(X)\right) dV \\ &= -\int_M \langle X, \nabla\mathrm{div}_f(X)\rangle e^f dV. \end{aligned}$$

Hence the identity (3.9) implies

$$\int_M |L_X g|^2 e^f dV = -\int_M \langle X, \Delta_f X + \nabla\mathrm{div}_f(X) + \mathrm{Ric}_f(X)\rangle\, e^f dV. \tag{3.10}$$

The above identity shows that the Euler-Lagrange equation for the functional

$$X \mapsto \int_M |L_X g|^2 e^f dV$$

is

$$\Delta_f X + \nabla\mathrm{div}_f(X) + \mathrm{Ric}_f(X) = 0. \tag{3.11}$$

We now simplify the equation (3.11). Compute

$$\begin{aligned}\Delta_f X^i &= \Delta X^i + \nabla_j f \nabla^j X^i,\\ \nabla^i \mathrm{div}_f(X) &= \nabla^i\left(e^{-f}\mathrm{div}\left(e^f X\right)\right)\\ &= \nabla^i\left(\mathrm{div}(X) + \langle \nabla f, X\rangle\right)\\ &= \nabla^i \mathrm{div}(X) + \nabla^i(X^j\nabla_j f)\\ &= \nabla^i \mathrm{div}(X) + \nabla^i X^j \nabla_j f + X^j \nabla^i \nabla_j f.\end{aligned}$$

Consequently,

$$\Delta_f X^i + \nabla^i \mathrm{div}_f(X) = \Delta X^i + \nabla^i \mathrm{div}(X) + \nabla_j f (L_X g)^{ij} + X_j \nabla^i \nabla^j f. \tag{3.12}$$

Plugging (3.12) into (3.11) and noting the definition of Ric_f yields

$$0 = \Delta X^i + \nabla^i \mathrm{div}(X) + R^i{}_j X^j + \nabla_j f (L_X g)^{ij}. \tag{3.13}$$

As in [16], we can prove the following

Theorem 3.1. *Given any smooth function f on a closed orientable Riemanian manifold $(\mathcal{M}, g)$. A smooth vector field X is Killing if and only if it satisfies (3.13). When $f \equiv 0$, it reduces to the classical criterion of Yano.*

Proof. Suppose X is Killing. Then $L_X g = 0$ and $\Delta X + \nabla \mathrm{div}(X) + \mathrm{Ric}(X) = 0$ by Yano's theorem. These two equations immediately imply (3.13). Conversely, if X is a smooth vector field satisfying (3.13), then it also satisfies (3.11) and then

$$\int_M |\mathcal{L}_X g|^2 e^f dV = 0$$

according to (3.10). Since $\mathcal{L}_X g$ is symmetric, by choosing a suitable coordinates we can diagonalize $\mathcal{L}_X g$ into $\mathrm{diag}(\lambda_1, \cdots, \lambda_m)$ so that each λ_i must be identically zero. Hence $\mathcal{L}_X g \equiv 0$ and X is Killing. □

The above theorem suggests us to consider the following flow

$$\partial_t X^i = \Delta X^i + \nabla^i \mathrm{div}(X) + R^i{}_j X^j + \nabla_j f (\mathcal{L}_X g)^{ij} \tag{3.14}$$

for a given smooth function $f \in C^\infty(M)$, or consider a nonlinear flow

$$\partial_t X^i = \Delta X^i + \nabla^i \mathrm{div}(X) + R^i{}_j X^j + \nabla_j \mathrm{div}(X) (\mathcal{L}_X g)^{ij}. \tag{3.15}$$

As in the proof of Theorem 1.8, we have

Theorem 3.2. *Let (M, g) be a closed orientable Riemannian manifold, f a smooth function on M, and X a smooth vector field on M. Then the flow (3.14) starting with the initial data X smoothly converges to a Killing vector field X_∞.*

Proof. By replacing $\mathrm{div}, \Delta, dV, \mathrm{Ric}$ by $\mathrm{div}_f, \Delta_f, e^f dV, \mathrm{Ric}_f$ in the argument of Theorem 1.8, we can show that $\int_M |X_t|^2 e^f dV$ is decreasing, $\int_M |\partial_t X_t|^2 e^f dV \to 0$, and then by the same method X_t smoothly converges to a smooth vector field X_∞ satisfying (3.13). By Theorem 3.1, X_∞ must be Killing. □

3.2. **New criterion: II.** The second new criterion is based on the following identity

$$\int_M \left[(\mathcal{L}_X g)(X,X) + \frac{1}{2}\mathrm{div}(X)|X|^2\right] dV = 0 \tag{3.16}$$

for any smooth vector field X on M. Since $2(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$, to prove (3.16), we suffice to show that

$$\int_M \left[X^i X^j \nabla_i X_j + \frac{1}{2}\mathrm{div}(X)|X|^2\right] dV = 0.$$

Actually,

$$\begin{aligned}\int_M X^i X^j \nabla_i X_j \, dV &= -\int_M X_j \nabla_i (X^i X^j)\, dV \\ &= -\int_M X_j \left[\mathrm{div}(X) X^j + X^i \nabla_i X^j\right] dV \\ &= -\int_M \mathrm{div}(X)|X|^2 dV - \int_M X^i X_j \nabla_i X^j \, dV\end{aligned}$$

which yields

$$\int_M X^i X^j \nabla_i X_j \, dV = -\frac{1}{2}\int_M \mathrm{div}(X)|X|^2 \, dV.$$

The second new criterion can be stated as follows.

Theorem 3.3. *A smooth vector field X on a closed orientable Riemannian manifold (M,g) is a Killing vector field if and only if it satisfies*

$$0 = \Delta X + \nabla \mathrm{div}(X) + \mathrm{Ric}(X,\cdot) + (\mathcal{L}_X g)(X,\cdot) + \frac{1}{2}\mathrm{div}(X)X. \tag{3.17}$$

Proof. If X is Killing, then $\mathrm{div}(X) = \mathcal{L}_X g = 0$ and hence (3.17) reduces to Yano's classical result. Conversely, suppose a smooth vector field X satisfies (3.17). Multiplying (3.17) by X and integrating over M, we obtain

$$\begin{aligned} 0 &= -\int_M [|\nabla X|^2 + |\mathrm{div}(X)|^2 - \mathrm{Ric}(X,X)] dV \\ &\quad + \int_M \left[(\mathcal{L}_X g)_{ij} X^i X^j + \frac{1}{2}\mathrm{div}(X)|X|^2\right] dV. \end{aligned} \tag{3.18}$$

The second integral on the right-hand side equals zero by the identity (3.16), and consequently, (3.18) is equivalent to $\mathcal{E}(X) = 0$, where $\mathcal{E}(X)$ was defined in (1.8). By a result of Watanabe [14], X must be Killing. □

Theorem 3.3 also suggests a nonlinear equation

$$\partial_t X = \Delta X + \nabla \mathrm{div}(X) + \mathrm{Ric}(X,\cdot) + (\mathcal{L}_X g)(X,\cdot) + \frac{1}{2}\mathrm{div}(X)X. \tag{3.19}$$

We note that the flows (1.4) and (3.14) linear, while the flows (3.15) and (3.19) are nonlinear. We will later study those flows and applications to geometry.

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