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The Candelas-de la Ossa-Green-Parkes Formula

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In this note we discuss a recent proof of the formula for the worldsheet instanton prepotential predicted by Mirror Symmetry for the quintics in $\mathbf{P}^4$. One of the key ingredients in the proof is the equivariant cohomology groups on the so-called linear sigma model moduli spaces. We introduce the notion of admissible data on the equivariant cohomology groups of the linear sigma model. An admissible data may be thought of as a sequence of equivariant classes satisfying certain algebraic conditions. It arises naturally from Kontsevich's stable map compactification of moduli spaces of maps from curves into projective manifolds. The structures of admissible data help reduce many counting problems to checking certain combinatorial structure of a compactification. The mirror transformation of Candelas et al turns out to be a transformation between two admissible data associated respectively to the linear and the non-linear sigma models. As an application, we prove the formula for the worldsheet instanton prepotential in terms of hypergeometric series. At the end we also interprete an infinite dimensional transformation group, called the mirror group, acting on admissible data, as a certain *duality group* of the linear sigma model.

1. INTRODUCTION

For the remarkable history of the Mirror Conjecture, we refer the reader to articles in [14].

In 1990, Candelas, de la Ossa, Green, and Parkes conjectured a formula for counting the number n_d of rational curves in every degree d on a general quintic in $\mathbf{P}^4$. Their computation is inspired by an earlier construction of a mirror manifold by Greene-Plesser. It has been conjectured earlier, by Clemens, that the number of rational curves in every degree is finite. The conjectured formula agrees with a classical result in degree 1, an earlier computation by S. Katz in degree 2, and has been verified in degree 3 by Ellingsrud-Stromme. In 1994 following some ideas of Gromov and Witten, Ruan-Tian introduced the notion of a symplectic Gromov-Witten (GW) invariants. Independently Kontsevich proposed an algebraic geometric notion of Gromov-Witten (GW) invariants. Significant generalizations of his definition have been given by Kontsevich-Manin, Li-Tian and Behrend-Fentachi. For an excellent introduction to stable maps, see the paper of Fulton-Pandharipande [16]. A recent paper of Li-Tian shows that the symplectic version and the algebraic geometric version of the GW theory are essentially the same in the projective catetory. Beautiful applications of ideas from quantum cohomology have recently been done by Caporaso-Harris [10], Crauder-Miranda [11], DiFrancesco-Itzykson [12] and others, solving many important enumerative problems. Significant connection between quantum cohomology and the geometry of Frobenius manifolds appears in the work of Dubrovin [13] and Manin [32].

Closer to mirror symmetry, a theorem of Manin says that the degree k GW invariants for $\mathbf{P}^1$ (the so-called multiple-cover contribution) is given by k^{-3}. This was conjectured by Candelas et al, and was justified by Aspinwall-Morrison using a different compactification. See also the recent paper of Voisin [35]. According to the definition of Kontsevich, the number

$$K_d = \sum_{k|d} n_{d/k} k^{-3}$$

is the degree of the Euler class $c_{5d+1}(U_d)$ where

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U_d is the obstruction bundle over the degree d (genus zero with no marked point) stable map moduli space $\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^4)$, induced by the line bundle $\mathcal{O}(5) \rightarrow \mathbf{P}^4$. We shall call K_d the degree d Kontsevich-Manin number.

Using the torus action on $\mathbf{P}^4$ and the Atiyah-Bott localization formula, Kontsevich computes the numbers K_d for $d = 1, 2, 3, 4$, and verifies that they agree with the conjectured formula. In a series of recent papers, Givental and Kim introduce a number of important ideas, among which is an equivariant version of Kontsevich's approach. Givental then proposes to use fixed point formulas on stable moduli to compute equivariant Euler classes, along the lines proposed by Kontsevich. More recently, Graber-Pandharipande have also applied fixed point method to study Gromov-Witten invariants of $\mathbf{P}^n$, generalizing some results of Kontsevich.

In two recent papers [18][19], Givental proposes a proof of the conjectured formula. Since his papers, there has been many seminars devoted to attempts to understand the papers. The papers contain many beautiful ideas which have led to important new insights into the conjectured formula. However the ideas have not been carried through in details. (See for example [32]). Inspired by Givental's achievement, we have recently given a complete proof of the conjectured formula [29]. Our goal in this note is to give sketch of our proof. For details we refer the readers to our forth-coming paper [29].

We now formulate one of our main theorems in this paper. Let M be a projective manifold and $\beta \in H^2(M, \mathbf{Z})$. Let $\bar{\mathcal{M}}_{g,k}(\beta, M)$ be the stable map moduli space of degree β, arithmetic genus g, with k marked points [26]. Throughout this paper, we shall only deal with the case with $g = 0$. Let

$$K_d = \int_{\bar{\mathcal{M}}_{0,0}(d,\mathbf{P}^4)} c_{5d+1}(U_d),$$

$$F(T) = \frac{5T^3}{6} + \sum_{d\geq 1} K_d e^{dT}$$

where $U_d \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^4)$ is the bundle whose fiber at (f, C) is given by the section space $H^0(C, f^*(5H))$. Consider the fourth order hypergeometric differential operator:

$$L := (\frac{d}{dt})^4 - 5e^t(5\frac{d}{dt} + 1)\cdots(5\frac{d}{dt} + 4).$$

By the Frobenius method, it is easy to show that

$$f_i := \frac{1}{i!}(\frac{d}{dH})^i|_{H=0} \sum_{d\geq 0} e^{d(t+H)} \frac{\prod_{m=1}^{5d}(5H + m)}{\prod_{m=1}^{d}(H + m)^5},$$

$i = 0, 1, 2, 3,$

form a basis of solutions to the differential equation $L \cdot f = 0$. Let

$$T = \frac{f_1}{f_0}, \quad \mathcal{F}(T) = \frac{5}{2}(\frac{f_1}{f_0}\frac{f_2}{f_0} - \frac{f_3}{f_0}).$$

Theorem 1.1 *(The Mirror Conjecture)* $F(T) = \mathcal{F}(T)$.

We remark that the definitions of $F, \mathcal{F}$ in [8] differ from those given above by a quadratic polynomial in T. The functions $F(T), \mathcal{F}(T)$ are known respectively as a type IIA and a type IIB *prepotential functions.* The transformation on the functions f_i given by the normalization

$$f_i \mapsto \frac{f_i}{f_0}$$

and the change of variables

$$t \mapsto T = \frac{f_1}{f_0}$$

are the *mirror transformation* of Candelas et al. By their construction, the functions $f_0, .., f_3$ are periods of a family of Calabi-Yau threefolds. By the theorem of Bogomolov-Tian-Todorov, these periods in fact determine the complex structure of the threefold.

A similar Mirror Conjecture formula holds true for a three dimensional Calabi-Yau complete intersection in a toric Fano manifold [30]. This will turn out to agree with the beautiful construction of the mirror manifolds of Batyrev [3], Batyrev-Borisov [4], as well as the many mirror symmetry computations of Morrison [33], Libgober-Teitelboim [28], Batyrev-van Straten [5], Candelas-Font-Katz-Morrison [9], Hosono-Klemm-Theisen-Yau [22] and Hosono-Lian-Yau [23].

The application of the approach outlined in this paper turns out to be rather broad. It applies to manifolds with torus action and many of their submanifolds. We have also obtained the multiple cover formula for $\mathbf{P}^1$, and a formula for the Euler classes of the obstruction bundles induced by the canonical bundle on $\mathbf{P}^2$ in this way. In this note, to make the ideas clear we restrict ourselves to the simplest case, genus 0 curves in some submanifolds of $\mathbf{P}^n$. In a future paper [30], we extend our discussions to toric varieties, homogeneous manifolds, their submanifolds, and for higher genus moduli spaces. We hope to eventually understand from this point of view the far reaching results for higher genus of Bershadsky-Cecotti-Ooguri-Vafa [7], the beautiful computations of Getzler [17] for elliptic GW invariants and of Batyrev-Ciocan Fontanine-Kim-van Straten [6] on Grassmannians.

2. ADMISSIBLE DATA

One of the key ingredients in our approach is the linear sigma model, first introduced by Witten [36], and later used to study mirror symmetry by Morrison-Plesser [21] resulting in new insights into the origin of hypergeometric series. In this paper, we consider the $S^1 \times T$-equivariant cohomology of the linear sigma model.

Throughout this paper, we fix a positive integer n. Let $p, \kappa, \alpha, \lambda_0, .., \lambda_n$ be formal variables. We denote $\lambda = (\lambda_0, .., \lambda_n)$. We will first introduce notations and mention a few facts which will be used throughout this paper.

Let T be an r-dimensional real torus with a complex linear representation on $\mathbf{C}^{N+1}$. Let $\beta_0, .., \beta_N$ be the weights of this action. We consider the induced action of T on $\mathbf{P}^N$, and the T-equivariant cohomology with coefficients in $\mathbf{Q}$, which we shall denote by $H_T(-)$. Now $H_T(pt)$ is given by the symmetric algebra on the dual of the Lie algebra of T. We can now regard the β_i to be elements of $H_T^2(pt)$. By a choice of basis of $H_T^2(pt)$, $H_T(pt)$ becomes a polynomial algebra with r generators of degree 2. Throughout this paper, we shall follow the convention that such generators have degree 1.

It is known that the equivariant cohomology of $\mathbf{P}^N$ is given by [24]

$$H_T(\mathbf{P}^N) = H_T(pt)[\zeta] / \left(\prod_{i=0}^{N} (\zeta - \beta_i) \right).$$

Here ζ, which we shall call the equivariant hyperplane class, is a fixed lifting of the hyperplane class of $\mathbf{P}^N$. Each one-dimensional weight space in $\mathbf{C}^{N+1}$ becomes a fixed point p_i in $\mathbf{P}^N$. *We shall identify the rings* $H_T^*(p_i)$ *and* $H_T^*(pt) = H^*(BT)$. There are $N+1$ canonical restriction maps $\iota_{p_i}^* : H_T(\mathbf{P}^N) \to H_T(pt)$, given by $\zeta \mapsto \beta_i$, $i = 0, .., N$. There is also a push-forward map $H_T(\mathbf{P}^N) \to H_T(pt)$ given by integration along the fiber. By the localization formula, it is given by

$$\omega \mapsto Res_\zeta \frac{\omega}{\prod_{i=0}^{N} (\zeta - \beta_i)}.$$

We now specialize the above to two different situations which will be used frequently in this paper. First consider the standard action of $T = (S^1)^{n+1}$ on $\mathbf{C}^{n+1}$, and let $(\lambda_0, .., \lambda_n)$ denote the weights. In this case, there are obviously $n+1$ isolated fixed points given by the coordinate lines in $\mathbf{C}^{n+1}$. We shall denote the equivariant hyperplane class by p, the canonical restriction maps simply by $\iota_{p_i}^* : \omega(p, \lambda) \mapsto \omega(\lambda_i, \lambda)$, and the push-forward map by

$$pf : H_T(\mathbf{P}^n) \mapsto H_T(pt).$$

We shall often use the evaluation map $\lambda_j \mapsto 0$ on the ring $H_T(\mathbf{P}^n)$, and shall call this the non-equivariant limit. Thus in this limit, p becomes the hyperplane class $H \in H^*(\mathbf{P}^n)$, and the push-forward map becomes the 'degree map' $H(\mathbf{P}^n) = \mathbf{Q}[H]/(H^{n+1}) \to \mathbf{Q}$ given by $H^k \mapsto \delta_{k,n}$.

We now consider the second situation. For each $d = 0, 1, 2, ..$, consider the following complex linear action of the group $S^1 \times T$ on $\mathbf{C}^{(n+1)(d+1)}$. We let the group act on the (ij)-th coordinate line in $\mathbf{C}^{(n+1)(d+1)}$ by the weights $\lambda_i + j\alpha$, $i = 0, .., n$, $j = 0, .., d$. Thus there are $(n+1)(d+1)$ isolated fixed points p_{ij} on the projective space $\mathbf{P}^{(n+1)d+n}$, given by those coordinate lines. In this case, we shall denote the equivariant hyperplane class by κ, the canonical restriction maps by $\iota_{p_{ij}}^* : \omega(\kappa, \alpha, \lambda) \mapsto \omega(\lambda_i + m\alpha, \alpha, \lambda)$, and the push-forward map by pf_d. *Throughout this paper we shall denote*

$\mathcal{R} := H_{S^1\times T}(pt) \cong \mathbf{Q}[\alpha,\lambda]$,

$\mathcal{R}^{-1} :=$ *quotient field of* $\mathcal{R}$.

Then the push-forward map is given by

$pf_d : \mathcal{R}[\kappa]/J_d \to \mathcal{R}$,

$$f(\kappa,\alpha,\lambda) \mapsto Res_\kappa \frac{f(\kappa,\alpha,\lambda)}{\prod_{j=0}^n \prod_{m=0}^d (\kappa - \lambda_j - m\alpha)},$$

where J_d is the ideal

$$J_d = (\prod_{j=0}^{n} \prod_{m=0}^{d} (\kappa - \lambda_j - m\alpha)).$$

Here we have abused the notation κ, using it to represent a class in $\mathcal{R}[\kappa]/J_d$ for every d. But it should present no confusion in the context it arises.

Observe that we have a chain of natural inclusions of ideals in $\mathcal{R}[\kappa]$:

$J_d \subset J_{d-1}, \quad d = 1,2,..$

which gives rise to a chain of linear maps

$$H_{S^1\times T}(\mathbf{P}^{(n+1)d+n}) \cong \mathcal{R}[\kappa]/J_d \to \mathcal{R}[\kappa]/J_{d-1}$$

$$\to \mathcal{R}[\kappa]/J_0 \cong H_T(\mathbf{P}^n)[\alpha].$$

We denote the last isomorphism by $\iota^*_{\mathbf{P}^n}$, which is given by $\iota^*_{\mathbf{P}^n} : \kappa \mapsto p$.

Let N_d be the space of nonzero $(n+1)$-tuple of degree d polynomials in two variables w_0, w_1, modulo scalar. There is a canonical way to identify N_d with $\mathbf{P}^{(n+1)d+n}$. Namely, a point $z \in \mathbf{P}^{(n+1)d+n}$ corresponds to the polynomial tuple $[\sum_r z_{0r} w_0^r w_1^{d-r}, .., \sum_r z_{nr} w_0^r w_1^{d-r}] \in N_d$. *This identification will be used throughout this paper.* It is easy to see that the natural T-action on $(n+1)$-tuples together with a generic S^1-action on $[w_0, w_1] \in \mathbf{P}^1$, induces a $S^1 \times T$-action on N_d which coincides with the $S^1 \times T$-action on $\mathbf{P}^{(n+1)d+n}$ described earlier.

Definition 2.1 *We call the sequence of projective spaces* $\{N_d\}$ *the linear sigma model for* $\mathbf{P}^n$.

Clearly the ground ring $H_{S^1\times T}(pt)$ is canonically a subring of $H_{S^1\times T}(N_d)$. We can localize the latter by tensoring with the quotient field $\mathcal{R}^{-1}$. We shall denote the resulting ring simply by

$$\mathcal{R}^{-1}H_{S^1\times T}(N_d) = \mathcal{R}^{-1}[\kappa]/J_d.$$

2.1. Admissibility

Fix a class $\Omega \in H_T^*(\mathbf{P}^n)$ with the property that its restriction at the i-th fixed point has $\iota^*_{p_i}(\Omega) \neq 0$ for $i = 0,..,n$. If Ω only depends on p, we often denote its restriction simply by $\Omega(\lambda_i)$. Given $\omega \in \mathcal{R}$, let $\bar{\omega}$ be the class obtained from ω by replacing α by $-\alpha$.

Definition 2.2 *(Notations) We denote by* $\mathcal{Q}^\Omega$ *the set of sequences of cohomology classes*

$Q: \quad Q_d(\kappa,\alpha,\lambda) \in \mathcal{R}^{-1}H_{S^1\times T}(N_d)$,

$d = 1,2,...,$ *and* $Q_0 = \Omega$.

Definition 2.3 *We call a sequence of cohomology classes*

$Q: \quad Q_d(\kappa,\alpha,\lambda) \in H^*_{S^1\times T}(N_d)$,

$d = 1,2,..,$ *and* $Q_0 = \Omega$

an Ω*-admissible data if for all* d, *and* $r = 0,..,d$, $i = 0,..,n$,

$$(*) \quad \iota^*_{p_{i,0}}(\Omega)\ \iota^*_{p_{i,r}}(Q_d) = \iota^*_{p_{i,r}}(Q_r)\ \overline{\iota^*_{p_{i,d-r}}(Q_{d-r})}.$$

More explicitly condition (*) can be written as

$$\Omega(\lambda_i,\lambda)Q_d(\lambda_i + r\alpha,\alpha,\lambda) =$$

$$Q_r(\lambda_i + r\alpha,\alpha,\lambda)Q_{d-r}(\lambda_i - (d-r)\alpha,-\alpha,\lambda).$$

We observe that the set of admissible data is a monoid, ie. it is closed under the product $Q_dQ'_d$, and has the unit given by $Q_d = 1$ for all d. Hence the product of an Ω-admissible with an Ω'-admissible data is an $\Omega\Omega'$- admissible data. In a general geometric setting, this multiplicative property corresponds to taking intersection of two suitable projective manifolds. The equivariant class $\Omega \in H_T^*(\mathbf{P}^n)$ will play the role of the equivariant Thom class of the normal bundle of such a projective manifold. *Throughout this paper, we shall deal only with one fixed class* Ω *at* *a time. When no confusion arises, we shall say admissible rather than Ω-admissible.*

Example 1. Let l be a positive integer. Put

$$P: \quad P_d(\kappa, \alpha, \lambda) = \prod_{m=0}^{ld} (l\kappa - m\alpha).$$

It is straightforward to check that P is an Ω-admissible data with $\Omega(\kappa) = l\kappa$. We leave this as an exercise to the reader.

Example 2. Let $M_d^0 := \mathcal{M}_{0,0}((1,d), \mathbf{P}^1 \times \mathbf{P}^n)$ be the moduli space of holomorphic maps $\mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^n$ of bidegree $(1,d)$. Since each such map can represented by $id \times [f_0, .., f_n]$ where f_i are degree d polynomials in two variables w_0, w_1 (which are homogeneous coordinates of $\mathbf{P}^1$), there is an obvious map $\varphi : M_d^0 \to N_d$ which sends a map in M_d^0 to an $(n+1)$-tuple $[f_0, .., f_n] \in N_d$. This map is clearly equivariant with respect to $S^1 \times T$. Since this map is an embedding and since N_d is compact, this shows that N_d is a compactification of M_d^0. We can also compactify M_d^0 by embedding it into the moduli space of stable maps $M_d := \bar{\mathcal{M}}_{0,0}((1,d), \mathbf{P}^1 \times \mathbf{P}^n)$.

Definition 2.4 *(Notations) We call the sequence of the stable moduli spaces M_d the non- linear sigma model for $\mathbf{P}^n$.*

With a bit of work, it can be shown that the map φ has an equivariant regular extension to $\varphi : M_d \to N_d$, by collapsing certain components of the curve C at each point $(f, C) \in M_d$.

The idea of using a collapsing map to relate two moduli problems is not new. The collapsing map φ was known to G. Tian in 1995, and a similar idea also appeared in a paper of J. Li in Donaldson theory [27] in which a collapsing map was used to relate the definitions of Donaldson invariants on two different compactifications of the moduli space of vector bundles. The map φ was also used by Givental in [18].

Now each positive power lH of the hyperplane bundle $H \to \mathbf{P}^n$ induces an equivariant bundle $W_d \to M_d$ similar to the bundle $U_d \to \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$ we have seen earlier. Namely, the fiber of W_d at $(f, C) \in M_d$ will be the section space $H^0(C, (\pi_2 \circ f)^*(lH))$. Here $\pi_2 : \mathbf{P}^1 \times \mathbf{P}^n \to \mathbf{P}^n$ is the projection onto the second factor. Let χ_d be the equivariant Euler class of W_d. We can now push-forward these classes for $d = 1, 2, ..$ via the equivariant φ and obtain a sequence equivariant cohomology classes $\varphi_!(\chi_d) \in H^*_{S^1 \times T}(N_d)$. For $d = 0$, we *define* $\varphi_!(\chi_0)$ to be the class $l\kappa$.

Theorem 2.5 *The sequence $\varphi_!(\chi_d)$ above is an Ω-admissible data with $\Omega = l\kappa$.*

In Example 1, the classes P_d in the first admissible data obviously resembles the coefficients in the hypergeometric series in the Introduction. Whereas in Example 2, the classes $\varphi_!(\chi_d)$ in the second admissible data arise naturally from equivariant Euler classes of the bundle V_d. Recall that the Mirror Conjecture Formula is a formula obtained by transforming certain hypergeometric functions to a generating function of the Euler classes of U_d (ie. the instanton prepotential F). This transformation is the mirror transformation of Candelas et al described in the Introduction. This inspires the following strategy:

1. Relate the admissible data P_d to hypergeometric series.
2. Relate the admissible data $\varphi_!(\chi_d)$ to the instanton prepotential F.
3. Construct an appropriate mirror transformation from the P_d to the $\varphi_!(\chi_d)$.

Definition 2.6 *Two admissible data P, Q are said to be linked if the restrictions $\iota_{p_{i,0}}^*(P_d)$ and $\iota_{p_{i,0}}^*(Q_d)$ agree at $\alpha = (\lambda_i - \lambda_j)/d$ for all distinct $i, j \in \{0, .., n\}$ and all $d = 0, 1, 2, ...$*

Theorem 2.7 *The admissible data $P: \; P_d = \prod_{m=0}^{ld}(l\kappa - m\alpha)$ and $Q: \; Q_d = \varphi_!(\chi_d)$ are linked.*

The above two theorems are proved by carefully studying the localization of χ_d to the fixed points in M_d, and compare with the localization of $\varphi_!(\chi_d)$ in N_d. In fact the computation of $\iota_{p_{i,0}}^*(Q_d)$ at $\alpha = (\lambda_i - \lambda_j)/d$ by using localization gives us the expression:

$$\iota_{p_{i,0}}^*(Q_d) = \prod_{m=0}^{ld} (l\lambda_i - m(\lambda_i - \lambda_j)/d).$$

In fact this is one of our motivations for introducing the (hypergeometric) admissible data

$P_d = \prod_{m=0}^{ld}(l\kappa - m\alpha)$.

This example illustrates the typical way in our approach to solve an enumerative problem. Namely, to study equivariant classes such as χ_d we push them into N_d to obtain an admissible data Q. Then we compute their values by localization method at $\alpha = (\lambda_i - \lambda_j)/d$. We then construct an *explicit* admissible data P which is linked to Q. And finally we try to relate P and Q in some explicit manner. See [29] for many other examples.

Theorem 2.8 *(Uniqueness)*
Suppose P, Q are any two linked admissible data. If $deg_\alpha \iota^*_{p_{i,0}}(P_d - Q_d) \le (n+1)d - 2$ for all i, then $P = Q$.

We now discuss the relationship with hypergeometric series. Given any $Q \in \mathcal{Q}^\Omega$, we define the following *associated hypergeometric series*:

$$HG[P](t) :=$$

$$e^{-pt/\alpha} \sum_{d\ge 0} e^{dt} \frac{\iota^*_{\mathbf{P}^n}(Q_d)}{\prod_{k=0}^n \prod_{m=1}^d (p - \lambda_k - m\alpha)},$$

which is a $H^*_T(\mathbf{P}^n)$ cohomology valued series. Here we use the convention that the denominator in $d = 0$ term in the summation is 1. Note that for $P: \ P_d = \prod_{m=0}^{ld}(l\kappa - m\alpha)$ as given in Example 1 above, it is easy to see that in the limit $\lambda \to 0$, we have

$$HG[P](t) = e^{-Ht/\alpha} \sum_{d\ge 0} e^{dt} \frac{\prod_{m=0}^{ld}(lH - m\alpha)}{\prod_{m=1}^d (H - m\alpha)^{n+1}}$$

where H on the right hand side is the non-equivariant hyperplane class of $\mathbf{P}^n$. The coefficients of $(-H/h)^i$ for $i = 0, 1, .., n-1$, are exactly solutions to a hypergeometric differential equation (for $l = n+1$), hence the name hypergeometric series, discussed in the Introduction.

2.2. Mirror transformations

Let $\mathcal{G}$ be the group of lower triangular unipotent matrices $f = [f_{rs}]_{r,s=0,1,2,..}$ with entries in $\mathcal{R}^{-1}[\kappa]$ which are homogeneous functions of degree 0. We shall always *assume that the entries are regular along $\kappa = \lambda_i$, $\alpha = (\lambda_i - \lambda_j)/d$ for all d and all $i \ne j$.*

We now introduce three invertible operations $Q \mapsto Q^f$, $Q \mapsto {}^fQ$, and $Q \mapsto Q^\sigma$ on the set $\mathcal{Q}^\Omega$ defined as follows:

$$\begin{aligned}
Q^f: \ & Q^f_d(\kappa,\alpha,\lambda) = Q_d(\kappa,\alpha,\lambda) \\
& + \textstyle\sum_{r=0}^{d-1} f_{d,r} Q_r(\kappa,\alpha,\lambda) \prod_{k=0}^n \\
& \quad \textstyle\prod_{m=r+1}^d (\kappa - \lambda_k - m\alpha) \\
{}^fQ: \ & {}^fQ_d(\kappa,\alpha,\lambda) = Q_d(\kappa,\alpha,\lambda) \\
& + \textstyle\sum_{r=0}^{d-1} f_{d,r} Q_r(\kappa - (d-r)\alpha, \alpha, \lambda) \prod_{k=0}^n \\
& \quad \textstyle\prod_{m=0}^{d-r-1} (\kappa - \lambda_k - m\alpha) \\
Q^\sigma: \ & Q^\sigma_d(\kappa,\alpha,\lambda) = Q_d(\kappa - d\alpha, -\alpha, \lambda)
\end{aligned} \tag{2.1}$$

for $f \in \mathcal{G}$. One can check that these operations are well-defined on $\mathcal{Q}^\Omega$, ie. the right hand sides are independent of the choice of representative of $Q_d \in \mathcal{R}^{-1} H_{S^1\times T}(N_d)$. It is obvious that the operations $Q \mapsto Q^f$, $Q \mapsto {}^fQ$ preserve the property that $Q_0 = \Omega$. Since Ω is independent of α, by assumption, it follows that the operation $Q \mapsto Q^\sigma$ also preserves the property that $Q_0 = \Omega$. It is easy to verify that for $f, g \in \mathcal{G}$, we have

$$\begin{aligned}
(Q^f)^g &= Q^{gf} \\
{}^g({}^fQ) &= {}^{gf}Q \\
(Q^\sigma)^\sigma &= Q.
\end{aligned}$$

Definition 2.9 *Let $\mathcal{G}'$ be the group of transformations on $\mathcal{Q}^\Omega$ generated by all three types of transformations (2.1). An element of $\mathcal{G}'$ is called a mirror transformation if it sends an admissible data to an admissible data. We denote by $\mathcal{M}^\Omega$ the group of mirror transformations. Two admissible data lying in the same orbit of $\mathcal{M}^\Omega$ are called mirror transforms of one another.*

Example. Let f be a unipotent matrix of the form:

$$\begin{bmatrix}
1 & 0 & 0 & \cdots & & & \\
a_1 & 1 & 0 & 0 & \cdots & & \\
a_2 & a_1 & 1 & 0 & 0 & \cdots & \\
a_3 & a_2 & a_1 & 1 & 0 & 0 & \cdots \\
\cdots & & \cdots & & \cdots & & \cdots
\end{bmatrix}$$

with $a_i \in \mathbf{Q}$. Then it can be shown that

$$Q \mapsto \tilde{Q} = ((((Q^f)^\sigma)^f)^\sigma$$

defines a mirror transformation. One can also express this transformation in terms of the associated hypergeometric series HG. Namely,

$$HG[\tilde{Q}](t) = g(e^t) \cdot HG[Q](t)$$

where $g(q) = 1 + a_1 q + a_2 q^2 + \cdots$. (cf. the mirror transformation in the Introduction). For more details, see [29].

Theorem 2.10 *Admissible data in the same orbit of the mirror group are linked. In particular if P is linked to Q, then any mirror transform of P is linked to Q.*

3. THE MIRROR CONJECTURE

Throughout this section, we assume that $l = n + 1 = 5$, and we fix $\Omega(\kappa) = l\kappa$.

Consider the hypergeometric differential equation

$$\left((\frac{d}{dt})^n - le^t(l\frac{d}{dt} + 1) \cdots (l\frac{d}{dt} + n) \right) f = 0.$$

It is trivial to show that (cf. Introduction) that there exists unique solutions of the forms $u(t) = 1 + O(e^t)$ and $v(t) = u(t)t + O(e^t)$. Recall that

$$T(t) = \frac{v(t)}{u(t)} = t + O(e^t)$$

is the mirror map of Candelas et al. As in the Introduction, we have four solutions $f_0, .., f_3$ with $u = f_0$, $v = f_1$.

Theorem 3.1 *The linked admissible data $P:\ P_d = \prod_{m=0}^{ld}(l\kappa - m\alpha)$ and $Q:\ Q_d = \varphi_!(\chi_d)$ are mirror transforms of one another. Their respective hypergeometric series are related by*

$$HG[Q](T(t)) = \frac{e^{\frac{g}{\alpha}\sum_k \lambda_k}}{u} HG[P](t).$$

where g is some power series in e^t.

One constructs a mirror transformation by simply immitating the mirror transformation of Candelas et al described in the Introduction. The asserted equality can then be shown by applying the Uniqueness Theorem.

Theorem 3.2 *(Special Geometry) In the nonequivariant limit $\lambda \to 0$, we have*

$$\begin{aligned} HG[Q](T) &= 5H(1 + T\tfrac{H}{\alpha} + F'\tfrac{H^2}{5\alpha^2} \\ &\quad + (TF' - 2F)\tfrac{H^3}{5\alpha^3}) \\ F(T) &:= \tfrac{5T^3}{6} + \textstyle\sum_{d\geq 0} K_d e^{dT} \\ HG[P](t(T)) &= 5Hu(t)(1 + T\tfrac{H}{\alpha} + \mathcal{F}'\tfrac{H^2}{5\alpha^2} \\ &\quad + (T\mathcal{F}' - 2\mathcal{F})\tfrac{H^3}{5\alpha^3}) \\ \mathcal{F}(T) &:= \tfrac{5}{2}(\tfrac{f_1}{f_0}\tfrac{f_2}{f_0} - \tfrac{f_3}{f_0}). \end{aligned}$$

It is easy to compute the asymptotic behaviour of $\mathcal{F}$, namely

$$\mathcal{F}(T) = \frac{5T^3}{6} + O(e^T),$$

which is the same as the asymptotic behaviour of F.

The preceding theorems imply that

$$F' = \mathcal{F}'.$$

This shows that $F - \mathcal{F} = const$. But since $F, \mathcal{F}$ have the same asymptotic behaviour, the const. must be zero. Hence $F = \mathcal{F}$.

3.1. Concluding remarks

Much of the machinery we have introduced here can be generalized to a large class of counting problems. For example any convex bundle (see [29] for definition) $V \to \mathbf{P}^n$ induces a sequence of equivariant bundles $V_d \to \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$ on the stable moduli spaces, in the same way as lH does. The push-forwards $\varphi_!(\chi_d^V)$ of the equivariant Euler classes of V_d turn out to form an Ω-admissible data with Ω being the equivariant Euler class of V. The general theory of admissible data allows us to study the classes $\varphi_!(\chi_d^V)$ on the linear sigma model. Similarly any concave bundle (see [29] for definition), such as $-lH$, also induces an admissible data by considering the corresponding obstruction bundles on $\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$. One obtains the admissible data for the multiple cover formula of $\mathbf{P}^1$ and the local mirror symmetry formula for the canonical bundle on $\mathbf{P}^2$. See [29] for details.

As we have seen, the set of linked admissible data has an infinite dimensional transformation group – the mirror group. Two special linked admissible data: P arising from hypergeometric functions and the other from the classes

$Q: \ Q_d = \varphi_!(\chi_d)$ on the non-linear sigma model are related by a particular mirror transformation of Candelas et al. Since the mirror group is so big, there are many other admissible data which are linked to P. and can be obtained simply by acting on P by the mirror group. From the physical point of view, P arises from type IIB string theory while Q arises from type IIA string theory, and mirror symmetry is a duality between the two. This relationship manifests itself on the linear sigma model as a duality transformation. This suggests that other admissible data linked to P may arise from some other string theories which are dual to type IIA and IIB, via more general mirror transformations. From the point of view of moduli theory, P is associated to the linear sigma model compatification N_d of the moduli space M_d^0 we discussed in the Introduction. Whereas Q is associated to the non-linear sigma model M_d, which is the stable map compactification of M_d^0. This suggests that other admissible data linked to P may correspond to other compactifications of M_d^0. If true, we will have an association between string theories, linked admissible data, and compactifications of moduli space of maps, all in the same picture, whereby there is a duality in each kind which one sees in the linear sigma model. It would be interesting to understand this duality more precisely.

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