



Modular invariance and anomaly cancellation formulas in odd dimension



Kefeng Liu^{a,b}, Yong Wang^{c,*}

^a Center of Mathematical Sciences, Zhejiang University, Hangzhou Zhejiang 310027, China

^b Department of Mathematics, University of California at Los Angeles, Los Angeles CA 90095-1555, USA

^c School of Mathematics and Statistics, Northeast Normal University, Changchun Jilin, 130024, China

ARTICLE INFO

Article history:

Received 2 May 2015

Accepted 14 October 2015

Available online 24 October 2015

MSC:

58C20

57R20

53C80

Keywords:

Modular invariance

Cancellation formulas in odd dimension

Divisibilities

ABSTRACT

By studying modular invariance properties of some characteristic forms, we get some new anomaly cancellation formulas on $(4r - 1)$ dimensional manifolds. As an application, we derive some results on divisibilities of the index of Toeplitz operators on $(4r - 1)$ dimensional spin manifolds and some congruent formulas on characteristic number for $(4r - 1)$ dimensional spin^c manifolds.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

In 1983, the physicists Alvarez-Gaumé and Witten [1] discovered the “miraculous cancellation” formula for gravitational anomaly which reveals a beautiful relation between the top components of the Hirzebruch \hat{L} -form and A -form of a 12-dimensional smooth Riemannian manifold. Kefeng Liu [2] established higher dimensional “miraculous cancellation” formulas for $(8k + 4)$ -dimensional Riemannian manifolds by developing modular invariance properties of characteristic forms. These formulas could be used to deduce some divisibility results. In [3,4], for each $(8k + 4)$ -dimensional smooth Riemannian manifold, a more general cancellation formula that involves a complex line bundle was established. This formula was applied to spin^c manifolds, then an analytic Ochanine congruence formula was derived. In [5], Qingtao Chen and Fei Han obtained more twisted cancellation formulas for $8k$ and $8k + 4$ dimensional manifolds and they also applied their cancellation formulas to study divisibilities on spin manifolds and congruences on spin^c manifolds.

Another important application of modular invariance properties of characteristic forms is to prove the rigidity theorem on elliptic genera. For example, see [2,6–10]. For odd dimensional manifolds, we proved the similar rigidity theorem for elliptic genera under the condition that fixed point submanifolds are 1-dimensional in [11]. In [12], Han and Yu dropped off our condition and proved more general odd dimensional rigidity theorem for elliptic genera. In [12], in order to prove the rigidity theorem, they constructed some interesting modular forms under the condition that the 3th de-Rham cohomology of manifolds vanishes.

* Corresponding author.

E-mail addresses: liu@ucla.edu.cn, liu@cms.zju.edu.cn (K. Liu), wangy581@nenu.edu.cn (Y. Wang).

In parallel, a natural question is whether we can get some interesting cancellation formulas in odd dimension by modular forms constructed in [12]. In this paper, we will give the confirmative answer of this question. That is, by studying modular invariance properties of some characteristic forms, we get some new anomaly cancellation formulas on $(4r - 1)$ dimensional manifolds. As an application, we derive some results on divisibilities on $(4r - 1)$ dimensional spin manifolds and congruences on $(4r - 1)$ dimensional spin^c manifolds. In [13], a cancellation formula on 11-dimensional manifolds was derived. To the authors' best knowledge, our cancellation formulas appear for the first time for general odd dimensional manifolds.

This paper is organized as follows: In Section 2, we review some knowledge on characteristic forms and modular forms that we are going to use. In Section 3, we prove some odd dimensional cancellation formulas and we apply them to get some results on divisibilities on the index of Toeplitz operators on spin manifolds. In Section 4, we prove some odd cancellation formulas involving a complex line bundle. By these formulas, we get some congruent formulas on characteristic number for odd spin^c manifolds.

2. Characteristic forms and modular forms

The purpose of this section is to review the necessary knowledge on characteristic forms and modular forms that we are going to use.

2.1. Characteristic forms

Let M be a Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection on TM and $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM} . Let $\widehat{A}(TM, \nabla^{TM})$ and $\widehat{L}(TM, \nabla^{TM})$ be the Hirzebruch characteristic forms defined respectively by (cf. [14])

$$\begin{aligned} \widehat{A}(TM, \nabla^{TM}) &= \det^{\frac{1}{2}} \left(\frac{\frac{\sqrt{-1}}{4\pi} R^{TM}}{\sinh(\frac{\sqrt{-1}}{4\pi} R^{TM})} \right), \\ \widehat{L}(TM, \nabla^{TM}) &= \det^{\frac{1}{2}} \left(\frac{\frac{\sqrt{-1}}{2\pi} R^{TM}}{\tanh(\frac{\sqrt{-1}}{4\pi} R^{TM})} \right). \end{aligned} \tag{2.1}$$

Let F, F' be two Hermitian vector bundles over M carrying Hermitian connection $\nabla^F, \nabla^{F'}$ respectively. Let $R^F = (\nabla^F)^2$ (resp. $R^{F'} = (\nabla^{F'})^2$) be the curvature of ∇^F (resp. $\nabla^{F'}$). If we set the formal difference $G = F - F'$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as

$$\text{ch}(G, \nabla^G) = \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^F \right) \right] - \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^{F'} \right) \right]. \tag{2.2}$$

For any complex number t , let

$$\wedge_t(F) = \mathbf{C}|_M + tF + t^2 \wedge^2(F) + \dots, \quad S_t(F) = \mathbf{C}|_M + tF + t^2 S^2(F) + \dots$$

denote respectively the total exterior and symmetric powers of F , which live in $K(M)[[t]]$. The following relations between these operations hold,

$$S_t(F) = \frac{1}{\wedge_{-t}(F)}, \quad \wedge_t(F - F') = \frac{\wedge_t(F)}{\wedge_t(F')}. \tag{2.3}$$

Moreover, if $\{\omega_i\}, \{\omega'_j\}$ are formal Chern roots for Hermitian vector bundles F, F' respectively, then

$$\text{ch}(\wedge_t(F)) = \prod_i (1 + e^{\omega_i t}). \tag{2.4}$$

Then we have the following formulas for Chern character forms,

$$\text{ch}(S_t(F)) = \frac{1}{\prod_i (1 - e^{\omega_i t})}, \quad \text{ch}(\wedge_t(F - F')) = \frac{\prod_i (1 + e^{\omega_i t})}{\prod_j (1 + e^{\omega'_j t})}. \tag{2.5}$$

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W , then its complexification $W_{\mathbf{C}} = W \otimes \mathbf{C}$ is a complex vector bundle over M carrying a canonical induced Hermitian metric from that of W , as well as a Hermitian connection $\nabla^{W_{\mathbf{C}}}$ induced from ∇^W . If F is a vector bundle (complex or real) over M , set $F = F - \dim F$ in $K(M)$ or $KO(M)$.

2.2. Some properties about the Jacobi theta functions and modular forms

We first recall the four Jacobi theta functions are defined as follows (cf. [15]):

$$\theta(v, \tau) = 2q^{\frac{1}{8}} \sin(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi\sqrt{-1}v}q^j)(1 - e^{-2\pi\sqrt{-1}v}q^j)], \tag{2.6}$$

$$\theta_1(v, \tau) = 2q^{\frac{1}{8}} \cos(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi\sqrt{-1}v}q^j)(1 + e^{-2\pi\sqrt{-1}v}q^j)], \tag{2.7}$$

$$\theta_2(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi\sqrt{-1}v}q^{j-\frac{1}{2}})(1 - e^{-2\pi\sqrt{-1}v}q^{j-\frac{1}{2}})], \tag{2.8}$$

$$\theta_3(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi\sqrt{-1}v}q^{j-\frac{1}{2}})(1 + e^{-2\pi\sqrt{-1}v}q^{j-\frac{1}{2}})], \tag{2.9}$$

where $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbf{H}$, the upper half complex plane. Let

$$\theta'(0, \tau) = \frac{\partial\theta(v, \tau)}{\partial v} \Big|_{v=0}. \tag{2.10}$$

Then the following Jacobi identity (cf. [15]) holds,

$$\theta'(0, \tau) = \pi\theta_1(0, \tau)\theta_2(0, \tau)\theta_3(0, \tau). \tag{2.11}$$

Denote $SL_2(\mathbf{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbf{Z}, ad - bc = 1 \right\}$ the modular group. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be the two generators of $SL_2(\mathbf{Z})$. They act on \mathbf{H} by $S\tau = -\frac{1}{\tau}$, $T\tau = \tau + 1$. One has the following transformation laws of theta functions under the actions of S and T (cf. [15]):

$$\theta(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta(v, \tau), \quad \theta\left(v, -\frac{1}{\tau}\right) = \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} e^{\pi\sqrt{-1}\tau v^2} \theta(\tau v, \tau); \tag{2.12}$$

$$\theta_1(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta_1(v, \tau), \quad \theta_1\left(v, -\frac{1}{\tau}\right) = \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} e^{\pi\sqrt{-1}\tau v^2} \theta_2(\tau v, \tau); \tag{2.13}$$

$$\theta_2(v, \tau + 1) = \theta_3(v, \tau), \quad \theta_2\left(v, -\frac{1}{\tau}\right) = \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} e^{\pi\sqrt{-1}\tau v^2} \theta_1(\tau v, \tau); \tag{2.14}$$

$$\theta_3(v, \tau + 1) = \theta_2(v, \tau), \quad \theta_3\left(v, -\frac{1}{\tau}\right) = \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} e^{\pi\sqrt{-1}\tau v^2} \theta_3(\tau v, \tau), \tag{2.15}$$

$$\theta'(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta'(v, \tau), \quad \theta'\left(0, -\frac{1}{\tau}\right) = \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} \tau \theta'(0, \tau). \tag{2.16}$$

Definition 2.1. A modular form over Γ , a subgroup of $SL_2(\mathbf{Z})$, is a holomorphic function $f(\tau)$ on \mathbf{H} such that

$$f(g\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(g)(c\tau + d)^k f(\tau), \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{2.17}$$

where $\chi : \Gamma \rightarrow \mathbf{C}^*$ is a character of Γ . k is called the weight of f .

Let

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \pmod{2} \right\},$$

$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid b \equiv 0 \pmod{2} \right\},$$

be the two modular subgroups of $SL_2(\mathbf{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST , the generators of $\Gamma^0(2)$ are STS, T^2STS (cf. [15]).

If Γ is a modular subgroup, let $\mathcal{M}_{\mathbf{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients. Writing $\theta_j = \theta_j(0, \tau)$, $1 \leq j \leq 3$, we introduce four explicit modular forms (cf. [2]),

$$\begin{aligned} \delta_1(\tau) &= \frac{1}{8}(\theta_2^4 + \theta_3^4), & \varepsilon_1(\tau) &= \frac{1}{16}\theta_2^4\theta_3^4, \\ \delta_2(\tau) &= -\frac{1}{8}(\theta_1^4 + \theta_3^4), & \varepsilon_2(\tau) &= \frac{1}{16}\theta_1^4\theta_3^4. \end{aligned} \tag{2.18}$$

They have the following Fourier expansions in $q^{\frac{1}{2}}$:

$$\begin{aligned} \delta_1(\tau) &= \frac{1}{4} + 6q + 6q^2 \dots, & \varepsilon_1(\tau) &= \frac{1}{16} - q + 7q^2 + \dots, \\ \delta_2(\tau) &= -\frac{1}{8} - 3q^{\frac{1}{2}} - 3q + \dots, & \varepsilon_2(\tau) &= q^{\frac{1}{2}} + 8q + \dots, \end{aligned} \tag{2.19}$$

where the “...” terms are the higher degree terms, all of which have integral coefficients. They also satisfy the transformation laws,

$$\delta_2\left(-\frac{1}{\tau}\right) = \tau^2 \delta_1(\tau), \quad \varepsilon_2\left(-\frac{1}{\tau}\right) = \tau^4 \varepsilon_1(\tau). \tag{2.20}$$

Lemma 2.2 ([2]). $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$ and moreover $\mathcal{M}_{\mathbf{R}}(\Gamma^0(2)) = \mathbf{R}[\delta_2(\tau), \varepsilon_2(\tau)]$.

3. Some cancellation formulas in odd dimension

Let M be a $(4r - 1)$ dimensional Riemannian manifold. Set

$$\begin{aligned} \Theta_1(T_C M) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_C M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{q^m}(\widetilde{T_C M}), \\ \Theta_2(T_C M) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_C M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{-q^{m-\frac{1}{2}}}(\widetilde{T_C M}), \\ \Theta_3(T_C M) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_C M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{q^{m-\frac{1}{2}}}(\widetilde{T_C M}). \end{aligned} \tag{3.1}$$

We recall the odd Chern character of a smooth map g from M to the general linear group $GL(N, \mathbf{C})$ with N a positive integer (see [14]). Let d denote a trivial connection on $\mathbf{C}^N|_M$. We will denote by $c_n(M, [g])$ the cohomology class associated to the closed n -form

$$c_n(\mathbf{C}^N|_M, g, d) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{(n+1)}{2}} \text{Tr}[(g^{-1}dg)^n]. \tag{3.2}$$

The odd Chern character form $\text{ch}(\mathbf{C}^N|_M, g, d)$ associated to g and d by definition is

$$\text{ch}(\mathbf{C}^N|_M, g, d) = \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!} c_{2n+1}(\mathbf{C}^N|_M, g, d). \tag{3.3}$$

Let the connection ∇_u on the trivial bundle $\mathbf{C}^N|_M$ defined by

$$\nabla_u = (1 - u)d + ug^{-1} \cdot d \cdot g, \quad u \in [0, 1]. \tag{3.4}$$

Then we have

$$d\text{ch}(\mathbf{C}^N|_M, g, d) = \text{ch}(\mathbf{C}^N|_M, d) - \text{ch}(\mathbf{C}^N|_M, g^{-1} \cdot d \cdot g). \tag{3.5}$$

Now let $g : M \rightarrow SO(N)$ and we assume that N is even and large enough. Let E denote the trivial real vector bundle of rank N over M . We equip E with the canonical trivial metric and trivial connection d . Set

$$\nabla_u = d + ug^{-1}dg, \quad u \in [0, 1].$$

Let R_u be the curvature of ∇_u , then

$$R_u = (u^2 - u)(g^{-1}dg)^2. \tag{3.6}$$

We also consider the complexification of E and g extends to a unitary automorphism of $E_{\mathbf{C}}$. The connection ∇_u extends to a Hermitian connection on $E_{\mathbf{C}}$ with curvature still given by (3.6). Let $\Delta(E)$ be the spinor bundle of E , which is a trivial Hermitian bundle of rank $2^{\frac{N}{2}}$. We assume that g has a lift to the Spin group $\text{Spin}(N) : g^{\Delta} : M \rightarrow \text{Spin}(N)$. So g^{Δ} can be viewed as an automorphism of $\Delta(E)$ preserving the Hermitian metric. We lift d on E to be a trivial Hermitian connection d^{Δ} on $\Delta(E)$, then

$$\nabla_u^{\Delta} = (1 - u)d^{\Delta} + u(g^{\Delta})^{-1} \cdot d^{\Delta} \cdot g^{\Delta}, \quad u \in [0, 1] \tag{3.7}$$

lifts ∇^u on E to $\Delta(E)$. Let $Q_j(E), j = 1, 2, 3$ be the virtual bundles defined as follows:

$$Q_1(E) = \Delta(E) \otimes \bigotimes_{n=1}^{\infty} \wedge_{q^n}(\tilde{E}_C),$$

$$Q_2(E) = \bigotimes_{n=1}^{\infty} \wedge_{-q^{n-\frac{1}{2}}}(\tilde{E}_C); \quad Q_3(E) = \bigotimes_{n=1}^{\infty} \wedge_{q^{n-\frac{1}{2}}}(\tilde{E}_C). \tag{3.8}$$

Let g on E have a lift $g^{Q_j(E)}$ on $Q_j(E)$ and ∇_u have a lift $\nabla_u^{Q_j(E)}$ on $Q_j(E)$. Following [12], we defined $\text{ch}(Q_j(E), g^{Q_j(E)}, d, \tau)$ for $j = 1, 2, 3$ as follows:

$$\text{ch}(Q_j(E), \nabla_0^{Q_j(E)}, \tau) - \text{ch}(Q_j(E), \nabla_1^{Q_j(E)}, \tau) = d\text{ch}(Q_j(E), g^{Q_j(E)}, d, \tau), \tag{3.9}$$

where

$$\text{ch}(Q_1(E), g^{Q_1(E)}, d, \tau) = -\frac{2^{N/2}}{8\pi^2} \int_0^1 \text{Tr} \left[g^{-1} dg \frac{\theta'_1(R_u/(4\pi^2), \tau)}{\theta_1(R_u/(4\pi^2), \tau)} \right] du, \tag{3.10}$$

and for $j = 2, 3$

$$\text{ch}(Q_j(E), g^{Q_j(E)}, d, \tau) = -\frac{1}{8\pi^2} \int_0^1 \text{Tr} \left[g^{-1} dg \frac{\theta'_j(R_u/(4\pi^2), \tau)}{\theta_j(R_u/(4\pi^2), \tau)} \right] du. \tag{3.11}$$

By Proposition 2.2 in [12], we have if $c_3(E_C, g, d) = 0$, then for any integer $l \geq 1$ and $j = 1, 2, 3$, $\text{ch}(Q_j(E), g^{Q_j(E)}, d, \tau)^{(4l-1)}$ are modular forms of weight $2l$ over $\Gamma_0(2), \Gamma^0(2)$ and Γ_θ respectively. Let (see [12, Def. 2.3])

$$\Phi_L(\nabla^{TM}, g, d, \tau) = \widehat{L}(TM, \nabla^{TM}) \text{ch}(\Theta_1(TM), \nabla^{\Theta_1(TM)}, \tau) \text{ch}(Q_1(E), g^{Q_1(E)}, d, \tau); \tag{3.12}$$

$$\Phi_W(\nabla^{TM}, g, d, \tau) = \widehat{A}(TM, \nabla^{TM}) \text{ch}(\Theta_2(TM), \nabla^{\Theta_2(TM)}, \tau) \text{ch}(Q_2(E), g^{Q_2(E)}, d, \tau); \tag{3.13}$$

$$\Phi'_W(\nabla^{TM}, g, d, \tau) = \widehat{A}(TM, \nabla^{TM}) \text{ch}(\Theta_3(TM), \nabla^{\Theta_3(TM)}, \tau) \text{ch}(Q_3(E), g^{Q_3(E)}, d, \tau). \tag{3.14}$$

By Proposition 2.4 and Theorem 2.6 in [12], we have that if $c_3(E_C, g, d) = 0$, then for any integer $l \geq 1$ and $j = 1, 2, 3$, $\Phi_L(\nabla^{TM}, g, d, \tau)^{(4l-1)}$, $\Phi_W(\nabla^{TM}, g, d, \tau)^{(4l-1)}$ and $\Phi'_W(\nabla^{TM}, g, d, \tau)^{(4l-1)}$ are modular forms of weight $2l$ over $\Gamma_0(2), \Gamma^0(2)$ and Γ_θ respectively. We have

$$\begin{aligned} \langle \Phi_L(\nabla^{TM}, g, d, \tau), [M] \rangle &= -\text{Ind}(T \otimes \Delta(TM) \otimes \Theta_1(TM) \otimes (Q_1(E), g^{Q_1(E)})); \\ \langle \Phi_W(\nabla^{TM}, g, d, \tau), [M] \rangle &= -\text{Ind}(T \otimes \Theta_2(TM) \otimes (Q_2(E), g^{Q_2(E)})); \\ \langle \Phi'_W(\nabla^{TM}, g, d, \tau), [M] \rangle &= -\text{Ind}(T \otimes \Theta_3(TM) \otimes (Q_3(E), g^{Q_3(E)})), \end{aligned} \tag{3.15}$$

where $\text{Ind}(T \otimes \dots)$ denotes the index of the Toeplitz operator. Let $\{\pm 2\pi\sqrt{-1}x_j\}$ for $1 \leq j \leq 2r - 1$ be the Chern roots of $TM \otimes \mathbf{C}$. Similar to the computations in [2], we have

$$\Phi_L(\nabla^{TM}, g, d, \tau) = 2^{2r-1} \left(\prod_{j=1}^{2r-1} \frac{x_j \theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_1(x_j, \tau)}{\theta_1(0, \tau)} \right) \text{ch}(Q_1(E), g^{Q_1(E)}, d, \tau); \tag{3.16}$$

$$\Phi_W(\nabla^{TM}, g, d, \tau) = \left(\prod_{j=1}^{2r-1} \frac{x_j \theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_2(x_j, \tau)}{\theta_2(0, \tau)} \right) \text{ch}(Q_2(E), g^{Q_2(E)}, d, \tau); \tag{3.17}$$

$$\Phi'_W(\nabla^{TM}, g, d, \tau) = \left(\prod_{j=1}^{2r-1} \frac{x_j \theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_3(x_j, \tau)}{\theta_3(0, \tau)} \right) \text{ch}(Q_3(E), g^{Q_3(E)}, d, \tau). \tag{3.18}$$

Clearly, $\Theta_1(T_C M) \otimes Q_1(E)$ and $\Theta_2(T_C M) \otimes Q_2(E)$ admit formal Fourier expansion in $q^{\frac{1}{2}}$ as

$$\begin{aligned} \Theta_1(T_C M) \otimes Q_1(E) &= A_0(T_C M, E) + A_1(T_C M, E)q + \dots, \\ \Theta_2(T_C M) \otimes Q_2(E) &= B_0(T_C M, E) + B_1(T_C M, E)q^{\frac{1}{2}} + \dots, \end{aligned} \tag{3.19}$$

where the A_j and B_j are elements in the semi-group formally generated by Hermitian vector bundles over M . Moreover, they carry canonically induced Hermitian connections. If

$$B_j(T_C M, E) = B_{j,1}(T_C M) \otimes B_{j,2}(E),$$

we let

$$\widetilde{\text{ch}}(B_j(T_C M, E)) = \text{ch}(B_{j,1}(T_C M))\text{ch}(B_{j,2}(E), g^{B_{j,2}(E)}, d).$$

If ω is a differential form over M , we denote $\omega^{(4r-1)}$ its top degree component. Our main results include the following theorem.

Theorem 3.1. *If $c_3(E, g, d) = 0$, then*

$$\{\widehat{L}(TM, \nabla^{TM})\text{ch}(\Delta(E), g^{\Delta(E)}, d)\}^{(4r-1)} = 2^{3r-1+\frac{N}{2}} \sum_{l=1}^{\lfloor \frac{r}{2} \rfloor} 2^{-6l} h_l, \tag{3.20}$$

where each h_l , $1 \leq l \leq \lfloor \frac{r}{2} \rfloor$, is a canonical integral linear combination of

$$\{\widehat{A}(TM, \nabla^{TM})\widetilde{\text{ch}}(B_j(T_C M, E))\}^{(4r-1)},$$

$1 \leq j \leq l$ and h_1, h_2 are given by (3.25) and (3.30).

Proof. Similarly to the computations in [2, P. 35] and by (2.26) in [12] and the condition $c_3(E, g, d) = 0$, we have

$$\Phi_W \left(\nabla^{TM}, g, d, -\frac{1}{\tau} \right)^{(4r-1)} = \frac{\tau^{2r}}{2^{2r-1+\frac{N}{2}}} \Phi_L(\nabla^{TM}, g, d, \tau)^{(4r-1)}. \tag{3.21}$$

By $\Phi_W(\nabla^{TM}, g, d, \tau)^{(4r-1)}$ is a modular form of weight $2r$ over $\Gamma^0(2)$. By Lemma 2.2, we have

$$\Phi_W(\nabla^{TM}, g, d, \tau)^{(4r-1)} = h_0(8\delta_2)^r + h_1(8\delta_2)^{r-2}\varepsilon_2 + \dots + h_{\lfloor \frac{r}{2} \rfloor}(8\delta_2)^{r-2\lfloor \frac{r}{2} \rfloor}\varepsilon_2^{\lfloor \frac{r}{2} \rfloor}, \tag{3.22}$$

where each h_l , $0 \leq l \leq \lfloor \frac{r}{2} \rfloor$, is a real multiple of the volume form at x . By (2.20), (3.21) and (3.22), we get

$$\Phi_L(\nabla^{TM}, g, d, \tau)^{(4r-1)} = 2^{2r-1+\frac{N}{2}} \left[h_0(8\delta_1)^r + h_1(8\delta_1)^{r-2}\varepsilon_1 + \dots + h_{\lfloor \frac{r}{2} \rfloor}(8\delta_1)^{r-2\lfloor \frac{r}{2} \rfloor}\varepsilon_1^{\lfloor \frac{r}{2} \rfloor} \right]. \tag{3.23}$$

By comparing the constant term in (3.23), we get (3.20). By comparing the coefficients of $q^{\frac{j}{2}}, j \geq 0$ between the two sides of (3.22), we can use the induction method to prove that $h_0 = 0$ and each h_l $1 \leq l \leq \lfloor \frac{r}{2} \rfloor$, can be expressed through a canonical integral linear combination of

$$\{\widehat{A}(TM, \nabla^{TM})\widetilde{\text{ch}}(B_j(T_C M, E))\}^{(4r-1)},$$

$1 \leq j \leq l$. Direct computations show that

$$\Theta_2(T_C M) \otimes Q_2(E) = 1 - (\widetilde{T_C M} + \widetilde{E_C})q^{\frac{1}{2}} + (\widetilde{T_C M} + \wedge^2 \widetilde{T_C M} + \wedge^2 \widetilde{E_C} + \widetilde{T_C M} \otimes \widetilde{E_C})q + \dots \tag{3.24}$$

By (2.19) and comparing the coefficients of $q^{\frac{1}{2}}$ of (3.22), we get

$$\begin{aligned} h_1 &= (-1)^{r-2} \{\widehat{A}(TM, \nabla^{TM})\widetilde{\text{ch}}(B_1(T_C M, E))\}^{(4r-1)} \\ &= (-1)^{r-1} \{\widehat{A}(TM, \nabla^{TM})\text{ch}(E, g, d)\}^{(4r-1)}. \end{aligned} \tag{3.25}$$

By (2.19) and comparing the coefficients of q of (3.22), we get

$$h_2 = (-1)^{r-4} \{\widehat{A}(TM, \nabla^{TM})\widetilde{\text{ch}}(B_2(T_C M, E))\}^{(4r-1)} + [-8 + 24(-1)^r(r-2)]h_1. \tag{3.26}$$

By (2.3), then

$$\widetilde{\text{ch}}(B_2(T_C M, E)) = \text{ch}(\wedge^2 \widetilde{E_C}, g, d) + \text{ch}(\widetilde{T_C M})\text{ch}(E, g, d); \tag{3.27}$$

$$\wedge^2 \widetilde{E_C} = S^2(\mathbf{C}^N) + \wedge^2 E_C - E_C \otimes \mathbf{C}^N; \tag{3.28}$$

$$\text{ch}(\wedge^2 \widetilde{E_C}, g, d) = \text{ch}(\wedge^2 E_C, g, d) - N\text{ch}(E_C, g, d). \tag{3.29}$$

By (3.25)–(3.29), we have

$$\begin{aligned} h_2 &= (-1)^{r-4} \left[\widehat{A}(TM, \nabla^{TM})\text{ch}(T_C M)\text{ch}(E, g, d) + \widehat{A}(TM, \nabla^{TM})\text{ch}(\wedge^2 E, g, d) \right]^{(4r-1)} \\ &\quad + [(-1)^{r-4}(7-N-4r) - 24(r-2) + 8(-1)^r] \left[\widehat{A}(TM, \nabla^{TM})\text{ch}(E, g, d) \right]^{(4r-1)}. \quad \square \end{aligned} \tag{3.30}$$

Corollary 3.2. Let M be a $(4r - 1)$ -dimensional spin manifold and $c_3(E, g, d) = 0$, then

$$\text{Ind}(T \otimes \Delta TM \otimes (\Delta(E), g^{\Delta(E)})) = -2^{3r-1+\frac{N}{2}} \sum_{l=1}^{\lfloor \frac{r}{2} \rfloor} 2^{-6l} h_l, \tag{3.31}$$

where each h_l , $1 \leq l \leq \lfloor \frac{r}{2} \rfloor$, is a canonical integral linear combination of $\text{Ind}(T \otimes (B_j(T_C M, E)))$.

Corollary 3.3. Let M be a $(4r - 1)$ -dimensional spin manifold and $c_3(E, g, d) = 0$. If r is even, then

$$\text{Ind}(T \otimes \Delta TM \otimes (\Delta(E), g^{\Delta(E)})) \equiv 0 \pmod{2^{\frac{N}{2}-1}}.$$

If r is odd, then

$$\text{Ind}(T \otimes \Delta TM \otimes (\Delta(E), g^{\Delta(E)})) \equiv 0 \pmod{2^{\frac{N}{2}+2}}.$$

Corollary 3.4. Let M be a 11-dimensional manifold and $c_3(E, g, d) = 0$. Then we have

$$\{\widehat{L}(TM, \nabla^{TM})\text{ch}(\Delta(E), g^{\Delta(E)}, d)\}^{(11)} = 2^{\frac{N}{2}+2} \{\widehat{A}(TM, \nabla^{TM})\text{ch}(E, g, d)\}^{(11)}. \tag{3.32}$$

Corollary 3.5. Let M be a 15-dimensional manifold and $c_3(E, g, d) = 0$. Then we have

$$\begin{aligned} \{\widehat{L}(TM, \nabla^{TM})\text{ch}(\Delta(E), g^{\Delta(E)}, d)\}^{(15)} &= 2^{\frac{N}{2}-1} [-(113 + N)\widehat{A}(TM, \nabla^{TM})\text{ch}(E, g, d) \\ &\quad + \widehat{A}(TM, \nabla^{TM})\text{ch}(T_C M)\text{ch}(E, g, d) + \widehat{A}(TM, \nabla^{TM})\text{ch}(\wedge^2 E, g, d)]^{(15)}. \end{aligned} \tag{3.33}$$

By (5.21) in [5], we have

$$\begin{aligned} \Theta_1(TM) &= 2 + 2(T_C M - 4r + 1)q + 2[(-8r + 3)T_C M \\ &\quad + T_C M \otimes T_C M + (4r - 1)(4r - 2)]q^2 + \dots. \end{aligned} \tag{3.34}$$

$$\text{ch}(Q_1(E), g^{Q_1(E)}, d) = \text{ch}(\Delta(E) \otimes (E_C q + ((2 - 4r)E_C + \wedge^2 E_C)q^2 + \dots), g, d). \tag{3.35}$$

$$\begin{aligned} \Theta_1(TM) \otimes Q_1(E) &= 2\Delta(E) \otimes E_C q + [2\Delta(E) \otimes ((2 - 4r)E_C + \wedge^2 E_C) \\ &\quad + 2(T_C M - 4r + 1) \otimes \Delta(E) \otimes E_C]q^2 + O(q^3). \end{aligned} \tag{3.36}$$

By (5.22) in [5], we have for $1 \leq l \leq \lfloor \frac{r}{2} \rfloor$

$$(8\delta_1)^{r-2l} \varepsilon_1^l = 2^{r-6l} [1 + (24r - 64l)q + (288r^2 - 1536rl + 2048l^2 + 512l - 264r)q^2 + O(q^3)]. \tag{3.37}$$

By comparing the coefficients of q in (3.23), we get

Theorem 3.6. If $c_3(E, g, d) = 0$, then

$$\{\widehat{L}(TM, \nabla^{TM})\text{ch}(\Delta(E) \otimes E_C, g, d) - 12r\widehat{L}(TM, \nabla^{TM})\text{ch}(\Delta(E), g^{\Delta(E)}, d)\}^{(4r-1)} = -2^{3r+4+\frac{N}{2}} \sum_{l=1}^{\lfloor \frac{r}{2} \rfloor} 2^{-6l} l h_l. \tag{3.38}$$

Corollary 3.7. Let M be a $(4r - 1)$ -dimensional spin manifold and $c_3(E, g, d) = 0$.

If r is even, then

$$\text{Ind}(T \otimes \Delta(TM) \otimes (\Delta(E) \otimes E_C, g)) - 12r\text{Ind}(T \otimes \Delta(TM) \otimes (\Delta(E), g)) \equiv 0 \pmod{2^{\frac{N}{2}+4}}. \tag{3.39}$$

If r is odd, then

$$\text{Ind}(T \otimes \Delta(TM) \otimes (\Delta(E) \otimes E_C, g)) - 12r\text{Ind}(T \otimes \Delta(TM) \otimes (\Delta(E), g)) \equiv 0 \pmod{2^{\frac{N}{2}+7}}. \tag{3.40}$$

By comparing the coefficients of q^2 in (3.23), we get

Theorem 3.8. If $c_3(E, g, d) = 0$, then

$$\begin{aligned} & \left\{ \widehat{L}(TM, \nabla^{TM}) \left[(19 - 56r) \text{ch}(\Delta(E) \otimes E_C, g, d) - 144(r^2 - r) \text{ch}(\Delta(E), g, d) \right. \right. \\ & \quad \left. \left. + \text{ch}(T_C M) \text{ch}(\Delta(E) \otimes E_C, g, d) + \text{ch}(\Delta(E) \otimes \wedge^2 E_C, g, d) \right] \right\}^{(4r-1)} \\ & = 2^{3r+9+\frac{N}{2}} \sum_{l=1}^{\lfloor \frac{r}{2} \rfloor} 2^{-6l} l^2 h_l. \end{aligned} \tag{3.41}$$

Corollary 3.9. Let M be a $(4r - 1)$ -dimensional spin manifold and $c_3(E, g, d) = 0$. Write A for the index of the Toeplitz operator determined by the left hand of (3.38). If r is even, then $A \equiv 0 \pmod{2^{\frac{N}{2}+9}}$. If r is odd, then $A \equiv 0 \pmod{2^{\frac{N}{2}+12}}$.

Corollary 3.10. If $\dim M = 11$ and $c_3(E, g, d) = 0$, then

$$\left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(\Delta(E) \otimes E_C, g, d) - 36 \widehat{L}(TM, \nabla^{TM}) \text{ch}(\Delta(E), g, d) + 2^{7+\frac{N}{2}} \widehat{A}(TM, \nabla^{TM}) \text{ch}(E, g, d) \right\}^{(11)} = 0. \tag{3.42}$$

Corollary 3.11. If $\dim M = 11$ and $c_3(E, g, d) = 0$, then

$$\begin{aligned} & \left\{ \widehat{L}(TM, \nabla^{TM}) \left[-149 \text{ch}(\Delta(E) \otimes E_C, g, d) - 864 \text{ch}(\Delta(E), g, d) \right. \right. \\ & \quad \left. \left. + \text{ch}(T_C M) \text{ch}(\Delta(E) \otimes E_C, g, d) + \text{ch}(\Delta(E) \otimes \wedge^2 E_C, g, d) \right] \right\}^{(11)} \\ & = 2^{12+\frac{N}{2}} [\widehat{A}(TM, \nabla^{TM}) \text{ch}(E, g, d)]^{(11)}. \end{aligned} \tag{3.43}$$

4. Twisted cancellation formulas in odd dimension

Let M be a $4r - 1$ dimensional Riemannian manifold and ξ be a rank two real oriented Euclidean vector bundle over M carrying with a Euclidean connection ∇^ξ . Set

$$\begin{aligned} \Theta_1(T_C M, \xi_C) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_C M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{q^m}(\widetilde{T_C M} - 2\widetilde{\xi}_C) \otimes \bigotimes_{r=1}^{\infty} \wedge_{q^{r-\frac{1}{2}}}(\widetilde{\xi}_C) \otimes \bigotimes_{s=1}^{\infty} \wedge_{-q^{s-\frac{1}{2}}}(\widetilde{\xi}_C), \\ \Theta_2(T_C M, \xi_C) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_C M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{-q^{m-\frac{1}{2}}}(\widetilde{T_C M} - 2\widetilde{\xi}_C) \otimes \bigotimes_{r=1}^{\infty} \wedge_{q^{r-\frac{1}{2}}}(\widetilde{\xi}_C) \otimes \bigotimes_{s=1}^{\infty} \wedge_{q^s}(\widetilde{\xi}_C), \\ \Theta_3(T_C M, \xi_C) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_C M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{q^{m-\frac{1}{2}}}(\widetilde{T_C M} - 2\widetilde{\xi}_C) \otimes \bigotimes_{r=1}^{\infty} \wedge_{q^r}(\widetilde{\xi}_C) \otimes \bigotimes_{s=1}^{\infty} \wedge_{-q^{s-\frac{1}{2}}}(\widetilde{\xi}_C). \end{aligned} \tag{4.1}$$

Let $c = e(\xi, \nabla^\xi)$ be the Euler form of ξ canonically associated to ∇^ξ . Set

$$\begin{aligned} \Phi_L(\nabla^{TM}, \nabla^\xi, g, d, \tau) &= \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \text{ch}(\Theta_1(T_C M, \xi_C), \nabla^{\Theta_1(T_C M, \xi_C)}) \cdot \text{ch}(Q_1(E), g^{Q_1(E)}, d, \tau), \\ \Phi_W(\nabla^{TM}, \nabla^\xi, d, g, \tau) &= \widehat{A}(TM, \nabla^{TM}) \cosh\left(\frac{c}{2}\right) \text{ch}(\Theta_2(T_C M, \xi_C), \nabla^{\Theta_2(T_C M, \xi_C)}) \cdot \text{ch}(Q_2(E), g^{Q_1(E)}, d, \tau), \\ \Phi'_W(\nabla^{TM}, \nabla^\xi, d, g, \tau) &= \widehat{A}(TM, \nabla^{TM}) \cosh\left(\frac{c}{2}\right) \text{ch}(\Theta_3(T_C M, \xi_C), \nabla^{\Theta_3(T_C M, \xi_C)}) \cdot \text{ch}(Q_3(E), g^{Q_1(E)}, d, \tau). \end{aligned} \tag{4.2}$$

Let $\{\pm 2\pi \sqrt{-1}x_j \mid 1 \leq j \leq 2r - 1\}$ and $\{\pm 2\pi \sqrt{-1}u\}$ be the Chern roots of $T_C M$ and ξ_C respectively and $c = 2\pi \sqrt{-1}u$. Through direct computations, we get (cf. [4])

$$\Phi_L(\nabla^{TM}, \nabla^\xi, g, d, \tau) = 2^{2r-1} \left\{ \left(\prod_{j=1}^{2r-1} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_1(x_j, \tau)}{\theta_1(0, \tau)} \right) \frac{\theta_1^2(0, \tau)}{\theta_1^2(u, \tau)} \frac{\theta_3(u, \tau)}{\theta_3(0, \tau)} \frac{\theta_2(u, \tau)}{\theta_2(0, \tau)} \right\} \cdot \text{ch}(Q_1(E), g^{Q_1(E)}, d, \tau); \tag{4.3}$$

$$\Phi_W(\nabla^{TM}, \nabla^\xi, \tau) = \left(\prod_{j=1}^{2r-1} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_2(x_j, \tau)}{\theta_2(0, \tau)} \right) \frac{\theta_2^2(0, \tau)}{\theta_2^2(u, \tau)} \frac{\theta_3(u, \tau)}{\theta_3(0, \tau)} \frac{\theta_1(u, \tau)}{\theta_1(0, \tau)} \cdot \text{ch}(Q_1(E), g^{Q_1(E)}, d, \tau); \tag{4.4}$$

$$\Phi'_W(\nabla^{TM}, \nabla^\xi, \tau) = \left(\prod_{j=1}^{2r-1} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_3(x_j, \tau)}{\theta_3(0, \tau)} \right) \frac{\theta_3^2(0, \tau)}{\theta_3^2(u, \tau)} \frac{\theta_1(u, \tau)}{\theta_1(0, \tau)} \frac{\theta_2(u, \tau)}{\theta_2(0, \tau)} \cdot \text{ch}(Q_1(E), g^{Q_1(E)}, d, \tau). \tag{4.5}$$

Similarly to (3.21), we have

$$\Phi_W \left(\nabla^{TM}, \nabla^\xi, g, d, -\frac{1}{\tau} \right)^{(4r-1)} = \frac{\tau^{2r}}{2^{2r-1+\frac{N}{2}}} \Phi_L(\nabla^{TM}, \nabla^\xi, g, d, \tau)^{(4r-1)}. \tag{4.6}$$

Similarly to Theorem 2.6 in [12], we have

Proposition 4.1. *If $c_3(E_C, g, d) = 0$, then for any integer $l \geq 1$ and $j = 1, 2, 3$,*

$$\Phi_L(\nabla^{TM}, \nabla^\xi, g, d, \tau)^{(4l-1)}, \quad \Phi_W(\nabla^{TM}, \nabla^\xi, g, d, \tau)^{(4l-1)}$$

and $\Phi'_W(\nabla^{TM}, \nabla^\xi, g, d, \tau)^{(4l-1)}$ are modular forms of weight $2l$ over $\Gamma_0(2)$, $\Gamma^0(2)$ and Γ_θ respectively.

We know that (3.22) and (3.23) hold in the twisted case. We define $B_j(T_C M, \xi_C, E)$ similarly to $B_j(T_C M, E)$. Similarly to Theorem 3.1, we have

Theorem 4.2. *If $c_3(E, g, d) = 0$, then*

$$\left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{\xi}{2})} \text{ch}(\Delta(E), g^{\Delta(E)}, d) \right\}^{(4r-1)} = 2^{3r-1+\frac{N}{2}} \sum_{l=1}^{\lfloor \frac{r}{2} \rfloor} 2^{-6l} \widetilde{h}_l, \tag{4.7}$$

where each \widetilde{h}_l , $1 \leq l \leq \lfloor \frac{r}{2} \rfloor$, is a canonical integral linear combination of

$$\left\{ \widehat{A}(TM, \nabla^{TM}) \cosh\left(\frac{c}{2}\right) \widetilde{\text{ch}}(B_j(T_C M, \xi_C, E)) \right\}^{(4r-1)},$$

$1 \leq j \leq l$.

By the definition, we have

$$\begin{aligned} \Theta_2(TM, \xi_C) \otimes Q_2(E) &= 1 + (3\widetilde{\xi}_C - \widetilde{T}_C \widetilde{M} - \widetilde{E}_C)q^{\frac{1}{2}} + [-3\widetilde{\xi}_C \otimes (\widetilde{T}_C \widetilde{M} + \widetilde{E}_C) \\ &\quad + (\widetilde{T}_C \widetilde{M} + \wedge^2 \widetilde{T}_C \widetilde{M} + \wedge^2 \widetilde{E}_C + \widetilde{T}_C \widetilde{M} \otimes \widetilde{E}_C) \\ &\quad + (3\widetilde{\xi}_C \otimes \widetilde{\xi}_C + 2S^2(\widetilde{\xi}_C) + \wedge^2(\widetilde{\xi}_C) + \widetilde{\xi}_C)]q + \dots \end{aligned} \tag{4.8}$$

Comparing the coefficients of $q^{\frac{1}{2}}$ and q in (3.22), we get

$$\widetilde{h}_1 = (-1)^{r-1} \left\{ \widehat{A}(TM, \nabla^{TM}) \cosh\left(\frac{c}{2}\right) \text{ch}(E, g, d) \right\}^{(4r-1)}. \tag{4.9}$$

$$\begin{aligned} \widetilde{h}_2 &= (-1)^{r-4} \left[\widehat{A}(TM, \nabla^{TM}) \text{ch}(T_C M - 3\widetilde{\xi}_C) \text{ch}(E, g, d) \cosh\left(\frac{c}{2}\right) + \widehat{A}(TM, \nabla^{TM}) \text{ch}(\wedge^2 E, g, d) \cosh\left(\frac{c}{2}\right) \right]^{(4r-1)} \\ &\quad + [(-1)^{r-4}(7 - N - 4r) - 24(r - 2) + 8(-1)^r] \cdot \left[\widehat{A}(TM, \nabla^{TM}) \text{ch}(E, g, d) \cosh\left(\frac{c}{2}\right) \right]^{(4r-1)}. \end{aligned} \tag{4.10}$$

Corollary 4.3. *Let M be a $(4r - 1)$ -dimensional spin^c manifold and $c_3(E, g, d) = 0$. If r is even, then*

$$2^{1-\frac{N}{2}} \left\langle \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{\xi}{2})} \text{ch}(\Delta(E), g^{\Delta(E)}, d), [M] \right\rangle \equiv \widetilde{h}_{\frac{r}{2}} \pmod{64\mathbf{Z}}. \tag{4.11}$$

If r is odd, then

$$2^{-2-\frac{N}{2}} \left\langle \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{\xi}{2})} \text{ch}(\Delta(E), g^{\Delta(E)}, d), [M] \right\rangle \equiv \widetilde{h}_{\frac{r-1}{2}} \pmod{64\mathbf{Z}}. \tag{4.12}$$

Corollary 4.4. *Let M be a 11-dimensional manifold and $c_3(E, g, d) = 0$. Then we have*

$$\left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{\xi}{2})} \text{ch}(\Delta(E), g^{\Delta(E)}, d) \right\}^{(11)} = 2^{\frac{N}{2}+2} \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(E, g, d) \cosh\left(\frac{c}{2}\right) \right\}^{(11)}. \tag{4.13}$$

Corollary 4.5. Let M be a 15-dimensional manifold and $c_3(E, g, d) = 0$. Then we have

$$\left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch}(\Delta(E), g^{\Delta(E)}, d) \right\}^{(15)} = 2^{\frac{N}{2}-1} \left[- (113 + N) \widehat{A}(TM, \nabla^{TM}) \cosh \frac{c}{2} \text{ch}(E, g, d) \right. \\ \left. + \widehat{A}(TM, \nabla^{TM}) \text{ch}(T_C M - 3\xi_C) \text{ch}(E, g, d) \cosh \frac{c}{2} + \widehat{A}(TM, \nabla^{TM}) \text{ch}(\wedge^2 E, g, d) \cosh \frac{c}{2} \right]^{(15)}. \tag{4.14}$$

By the definition, we have

$$\widetilde{\text{ch}}(\Theta_1(TM) \otimes Q_1(E)) = 2\text{ch}(\Delta(E) \otimes E_C, g, d)q + O(q^{\frac{3}{2}}). \tag{4.15}$$

Comparing the coefficients of q in (3.23), we get

Theorem 4.6. If $c_3(E, g, d) = 0$, then

$$\left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch}(\Delta(E) \otimes E_C, g, d) - 12r \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch}(\Delta(E), g^{\Delta(E)}, d) \right\}^{(4r-1)} = -2^{3r+4+\frac{N}{2}} \sum_{l=1}^{\lfloor \frac{r}{2} \rfloor} 2^{-6l} \widetilde{h}_l. \tag{4.16}$$

Corollary 4.7. Let M be a $(4r - 1)$ -dimensional spin^c manifold and $c_3(E, g, d) = 0$. If r is even, then

$$2^{-4-\frac{N}{2}} \left\langle \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch}(\Delta(E) \otimes E_C, g, d) - 12r \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch}(\Delta(E), g^{\Delta(E)}, d), [M] \right\rangle \equiv -\frac{r}{2} \widetilde{h}_{\frac{r}{2}} \pmod{64\mathbf{Z}}. \tag{4.17}$$

If r is odd, then

$$2^{-6-\frac{N}{2}} \left\langle \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch}(\Delta(E) \otimes E_C, g, d) - 12r \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch}(\Delta(E), g^{\Delta(E)}, d), [M] \right\rangle \equiv -\frac{r-1}{2} \widetilde{h}_{\frac{r-1}{2}} \pmod{64\mathbf{Z}}. \tag{4.18}$$

Corollary 4.8. If $\dim M = 11$ and $c_3(E, g, d) = 0$, then

$$\left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch}(\Delta(E) \otimes E_C, g, d) - 36 \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch}(\Delta(E), g, d) \right. \\ \left. + 2^{7+\frac{N}{2}} \widehat{A}(TM, \nabla^{TM}) \text{ch}(E, g, d) \cosh \frac{c}{2} \right\}^{(11)} = 0. \tag{4.19}$$

Remark. 1. By the Dai-Zhang family index theorem associated to the Toeplitz operator in [16], similarly to [17], we can easily extend our anomaly cancellation formulas to the family case.

2. Similarly to [2,18] and [19], we also extend our anomaly cancellation formulas to these cases.

Acknowledgments

The work of the first author was supported by NSF. The work of the second author was supported by NSFC. 11271062 and NCET-13-0721. The authors are indebted to Dr. F. Han for helpful comments.

References

[1] L. Alvarez-Gaumé, E. Witten, Gravitational anomalies, Nuclear Phys. B 234 (1983) 269–330.
 [2] K. Liu, Modular invariance and characteristic numbers, Comm. Math. Phys. 174 (1995) 29–42.

- [3] F. Han, W. Zhang, Spin^c -manifold and elliptic genera, *C. R. Acad. Sci., Paris I* 336 (2003) 1011–1014.
- [4] F. Han, W. Zhang, Modular invariance, characteristic numbers and eta invariants, *J. Differential Geom.* 67 (2004) 257–288.
- [5] Q. Chen, F. Han, Modular invariance and twisted anomaly cancellations of characteristic numbers, *Trans. Amer. Math. Soc.* 361 (2009) 1463–1493.
- [6] K. Liu, On elliptic genera and theta-functions, *Topology* 35 (3) (1996) 617–640.
- [7] K. Liu, On modular invariance and rigidity theorems, *J. Differential Geom.* 41 (1995) 343–396.
- [8] K. Liu, X. Ma, On family rigidity theorems. I, *Duke Math. J.* 102 (2000) 451–474.
- [9] K. Liu, X. Ma, W. Zhang, Spin^c manifolds and rigidity theorems in K-theory, *Asian J. Math.* 4 (2000) 933–959.
- [10] K. Liu, X. Ma, W. Zhang, Rigidity and vanishing theorems in K-theory, *Comm. Anal. Geom.* 11 (2003) 121–180.
- [11] K. Liu, Y. Wang, Rigidity theorems on odd dimensional manifolds, *Pure Appl. Math. Q.* 5 (2009) 1139–1159.
- [12] F. Han, J. Yu, On the Witten rigidity theorem for odd dimensional manifolds, [arXiv:1504.03007](https://arxiv.org/abs/1504.03007).
- [13] Q. Chen, F. Han, Elliptic genera, transgression and loop space Chern–Simons forms, *Comm. Anal. Geom.* 17 (1) (2009) 73–106.
- [14] W. Zhang, Lectures on Chern–weil Theory and Witten Deformations, in: *Nankai Tracks in Mathematics*, vol. 4, World Scientific, Singapore, 2001.
- [15] K. Chandrasekharan, *Elliptic Functions*, Springer-Verlag, 1985.
- [16] X. Dai, W. Zhang, Higher spectral flow, *J. Funct. Anal.* 157 (1998) 432–469.
- [17] F. Han, K. Liu, Gravitational anomaly cancellation and modular invariance, *Algebr. Geom. Topol.* 14 (2014) 91–113.
- [18] F. Han, K. Liu, W. Zhang, Modular forms and generalized anomaly cancellation formulas, *J. Geom. Phys.* 62 (2012) 1038–1053.
- [19] K. Liu, Y. Wang, A note on modular forms and generalized anomaly cancellation formulas, *Sci. China Math.* 56 (2013) 55–65.