

INTERSECTION NUMBERS AND AUTOMORPHISMS OF STABLE CURVES

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1. INTRODUCTION

Denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n -pointed genus g complex algebraic curves. We have the morphism that forgets the last marked point

$$\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Denote by $\sigma_1, \dots, \sigma_n$ the canonical sections of π , and by D_1, \dots, D_n the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf, we have the following tautological classes on moduli spaces of curves.

$$\begin{aligned} \psi_i &= c_1(\sigma_i^*(\omega_\pi)) \\ \kappa_i &= \pi_* \left(c_1 \left(\omega_\pi \left(\sum D_i \right) \right)^{i+1} \right) \\ \lambda_l &= c_l(\pi_*(\omega_\pi)), \quad 1 \leq l \leq g. \end{aligned}$$

The classes κ_i were first introduced by Mumford [22] on $\overline{\mathcal{M}}_g$, their generalization to $\overline{\mathcal{M}}_{g,n}$ here is due to Arbarello-Cornalba [1].

For background materials about the intersection theory of moduli spaces of curves, we refer to the book [19] and the survey paper [25].

Hodge integrals are intersection numbers of the form

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \mid \lambda_1^{k_1} \cdots \lambda_g^{k_g} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g},$$

which are rational numbers because the moduli space of curves are orbifolds. They are nonzero only when $\sum_{i=1}^n d_i + \sum_{i=1}^m a_i + \sum_{i=1}^g i k_i = 3g - 3 + n$.

Hodge integrals arise naturally in the localization computation of Gromov-Witten invariants. They are extensively studied by mathematicians and physicists. Hodge integrals involving only ψ classes can be computed recursively by the celebrated Witten-Kontsevich theorem [26, 18], which can be equivalently formulated by the following DVV recursion relation [5]

$$\begin{aligned} (1) \quad \langle \tau_{k+1} \tau_{d_1} \cdots \tau_{d_n} \rangle_g &= \frac{1}{(2k+3)!!} \left[\sum_{j=1}^n \frac{(2k+2d_j+1)!!}{(2d_j-1)!!} \langle \tau_{d_1} \cdots \tau_{d_j+k} \cdots \tau_{d_n} \rangle_g \right. \\ &\quad + \frac{1}{2} \sum_{r+s=k-1} (2r+1)!!(2s+1)!! \langle \tau_r \tau_s \tau_{d_1} \cdots \tau_{d_n} \rangle_{g-1} \\ &\quad \left. + \frac{1}{2} \sum_{r+s=k-1} (2r+1)!!(2s+1)!! \sum_{\underline{n}=I \amalg J} \langle \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \right] \end{aligned}$$

where $\underline{n} = \{1, 2, \dots, n\}$.

Now there are several new proofs of Witten's conjecture [3, 13, 14, 15, 21, 23].

Let $\text{denom}(r)$ denote the denominator of a rational number r in reduced form (coprime numerator and denominator, positive denominator). For $2g - 2 + n \geq 1$, we define

$$D_{g,n} = \text{lcm} \left\{ \text{denom} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \mid \sum_{i=1}^n d_i = 3g - 3 + n \right\},$$

and for $g \geq 2$,

$$\mathcal{D}_g = \text{lcm} \left\{ \text{denom} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \mid \sum_{i=1}^n d_i = 3g - 3 + n, d_i \geq 2, n \geq 1 \right\},$$

$$\tilde{\mathcal{D}}_g = \text{lcm} \left\{ \text{denom} \left(\int_{\overline{\mathcal{M}}_g} \kappa_{a_1} \cdots \kappa_{a_m} \right) \mid \sum_{i=1}^m a_i = 3g - 3 \right\},$$

where lcm denotes the *least common multiple*.

Note that \mathcal{D}_g was previously defined by Itzykson and Zuber [12].

We know that a neighborhood of $\Sigma \in \overline{\mathcal{M}}_{g,n}$ is of the form $U/\text{Aut}(\Sigma)$, where U is an open subset of \mathbb{C}^{3g-3+n} . This gives the orbifold structure for $\overline{\mathcal{M}}_{g,n}$. Since denominators of intersection numbers on $\overline{\mathcal{M}}_{g,n}$ all come from these orbifold quotient singularities, the divisibility properties of $D_{g,n}$ and \mathcal{D}_g should reflect the overall behavior of singularities.

In Section 2, we study basic relations between $D_{g,n}$, \mathcal{D}_g and $\tilde{\mathcal{D}}_g$. In Section 3, we discuss briefly automorphism groups of Riemann surfaces and stable curves. In Section 4, we study prime factors of \mathcal{D}_g and prove a strong form of a conjecture of Itzykson and Zuber [12] concerning denominators of intersection numbers. In Section 5, we present a conjectural multinomial type property for intersection numbers and verify it in low genera.

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2. BASIC PROPERTIES OF \mathcal{D}_g

If we take $k = -1$ and $k = 0$ respectively in DVV formula (1), we get the string equation

$$\langle \tau_0 \prod_{i=1}^n \tau_{k_i} \rangle_g = \sum_{j=1}^n \langle \tau_{k_j-1} \prod_{i \neq j} \tau_{k_i} \rangle_g$$

and the dilaton equation

$$\langle \tau_1 \prod_{i=1}^n \tau_{k_i} \rangle_g = (2g - 2 + n) \langle \prod_{i=1}^n \tau_{k_i} \rangle_g$$

Their proof may be found in the book [19].

Lemma 2.1. *If $n \geq 1$, then*

- i) $D_{0,n} = 1$,
- ii) $D_{1,n} = 24$,
- iii) $D_{g,1} = 24^g \cdot g!$.

Proof. The lemma follows from the string equation, the dilaton equation and the following well known formulae

$$\begin{aligned}\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_0 &= \binom{n-3}{d_1 \cdots d_n} = \frac{(n-3)!}{d_1! \cdots d_n!}, \\ \langle \tau_1 \rangle_1 &= \frac{1}{24}, \quad \langle \tau_{3g-2} \rangle_g = \frac{1}{24^g g!}.\end{aligned}$$

Their proofs can be found in [19, 26]. \square

Note that $D_{0,n} = 1$ is expected since $\overline{\mathcal{M}}_{0,n}$ is a smooth manifold.

Theorem 2.2. *We have*

$$D_{g,n} \mid D_{g,n+1}.$$

Proof. Let $q^s \mid D_{g,n}$, where q is a prime number and $q^{s+1} \nmid D_{g,n}$.

We sort $\{\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \mid \sum_{i=1}^n d_i = 3g - 3 + n, 0 \leq d_1 \leq \cdots \leq d_n\}$ in lexicographical order, we say $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g \prec \langle \tau_{m_1} \cdots \tau_{m_n} \rangle_g$, if there is some i , such that $k_j = m_j, j < i$ and $k_i < m_i$.

Let $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$ be the minimal element with respect to the lexicographical order such that its denominator is divisible by q^s .

There exist integers c, d, a_i, b_i where $i = 1, \dots, n-1$ such that

$$\begin{aligned}\langle \tau_0 \tau_{k_1} \cdots \tau_{k_{n+1}} \rangle_g &= \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g + \sum_{i=1}^{n-1} \langle \tau_{k_1} \cdots \tau_{k_{i-1}} \cdots \tau_{k_{n-1}} \tau_{k_{n+1}} \rangle_g \\ &= \frac{c}{q^s d} + \sum_{i=1}^{n-1} \frac{b_i}{a_i},\end{aligned}$$

we require $q \nmid c, q \nmid d$ and $(a_i, b_i) = 1$.

Since for $i = 1, \dots, n-1$, we have $\langle \tau_{k_1} \cdots \tau_{k_{i-1}} \cdots \tau_{k_{n-1}} \tau_{k_{n+1}} \rangle_g \prec \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$, so $a_i = q^{s_i} e_i$, where $s_i < l$ and $q \nmid e_i$. We now have

$$\langle \tau_0 \tau_{k_1} \cdots \tau_{k_{n+1}} \rangle_g = \frac{c \prod_{i=1}^{n-1} e_i + qd (\sum_{j=1}^{n-1} q^{s-s_j-1} \prod_{i \neq j} e_i)}{q^s d \prod_{i=1}^{n-1} e_i}$$

we see that q can not divide the numerator, so we have proved $q^s \mid D_{g,n+1}$. Since q is arbitrary, we proved the theorem. \square

Theorem 2.3. *We have $D_{g,n} \mid \tilde{\mathcal{D}}_g$ for all $g \geq 2, n \geq 1$. Moreover $\tilde{\mathcal{D}}_g = D_{g,3g-3}$.*

Proof. Let

$$\pi_n : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n-1},$$

be the morphism that forgets the last marked point, then we have [1],

$$(2) \quad (\pi_1 \cdots \pi_n)_* (\psi_1^{a_1+1} \cdots \psi_n^{a_n+1}) = \sum_{\sigma \in S_n} \kappa_\sigma,$$

where κ_σ is defined as follows. Write the permutation σ as a product of $\nu(\sigma)$ disjoint cycles, including 1-cycles: $\sigma = \beta_1 \cdots \beta_{\nu(\sigma)}$, where we think of the symmetric group S_n as acting on the n -tuple (a_1, \dots, a_n) . Denote by $|\beta|$ the sum of the elements of a cycle β . Then

$$\kappa_\sigma = \kappa_{|\beta_1|} \kappa_{|\beta_2|} \cdots \kappa_{|\beta_{\nu(\sigma)}|}.$$

From the formula (2), we get

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1+1} \cdots \psi_n^{a_n+1} = \sum_{\sigma \in S_n} \int_{\overline{\mathcal{M}}_g} \kappa_\sigma,$$

so we proved $D_{g,n} \mid \tilde{\mathcal{D}}_g$.

On the other hand, any $\int_{\overline{\mathcal{M}}_g} \kappa_{a_1} \cdots \kappa_{a_m}$ can be written as a sum of $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$'s. This can be seen by induction on the number of kappa classes, for integrals with only one kappa class, we have $\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1} \psi_1^{d_1} \cdots \psi_n^{d_n} = \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a_1+1} \psi_1^{d_1} \cdots \psi_n^{d_n}$. We also have

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1} \cdots \kappa_{a_m} \psi_1^{d_1} \cdots \psi_n^{d_n} = \int_{\overline{\mathcal{M}}_{g,n+m}} \psi_{n+1}^{a_1+1} \cdots \psi_{n+m}^{a_m+1} \psi_1^{d_1} \cdots \psi_n^{d_n} - \{\text{integrals with at most } m-1 \text{ } \kappa \text{ classes}\}.$$

thus finishing the induction argument. So we proved $\tilde{\mathcal{D}}_g = D_{g,3g-3}$. \square

Corollary 2.4. *For $g \geq 2$, we have $\mathcal{D}_g = \tilde{\mathcal{D}}_g$.*

We have computed \mathcal{D}_g for $g \leq 20$ using the DVV formula (1) and observed the following conjectural exact values of \mathcal{D}_g (see also [20]).

Conjecture 2.5. Let p be a prime number and $g \geq 2$. Let $\text{ord}(p, n)$ denote the maximum integer such that $p^{\text{ord}(p, n)} \mid n$, then

- i) $\text{ord}(2, \mathcal{D}_g) = 3g + \text{ord}(2, g!)$,
- ii) $\text{ord}(3, \mathcal{D}_g) = g + \text{ord}(3, g!)$,
- iii) $\text{ord}(p, \mathcal{D}_g) = \lfloor \frac{2g}{p-1} \rfloor$ for $p \geq 5$, where $\lfloor x \rfloor$ denotes the maximum integer that is not larger than x .

On the other hand, we may get explicit expressions for multiples of \mathcal{D}_g by applying either Kazarian-Lando's formula [14] expressing intersection indices by Hurwitz numbers or Proposition 4.4.

3. AUTOMORPHISM GROUPS OF STABLE CURVES

First we recall some facts about automorphisms of compact Riemann surfaces following [8].

Let X be a compact Riemann surface of genus g and $\text{Aut}(X)$ the group of conformal automorphisms of X . It's a classical theorem of Hurwitz that if $g \geq 2$, then $|\text{Aut}(X)| \leq 84(g-1)$.

Let $G \subset \text{Aut}(X)$ be a group of automorphisms of X , consider the natural map

$$\pi : X \rightarrow X/G$$

we know that π has degree $|G|$ and X/G is a compact Riemann surface of genus g_0 .

The mapping π is branched only at the fixed points of G and the branching order

$$b(P) = \text{ord}G_P - 1$$

where G_P is the isotropy group at $P \in X$ which is known to be cyclic.

Let P_1, \dots, P_r be a maximal set of inequivalent fixed points of elements of $G \setminus \{1\}$. (that is, $P_i \neq h(P_j)$ for all $h \in G$ and all $i \neq j$.)

Let $n_i = \text{ord}G_{P_i}$, then the total branch number of π is given by

$$B = \sum_{i=1}^r \frac{|G|}{n_i} (n_i - 1) = |G| \sum_{i=1}^r \left(1 - \frac{1}{n_i}\right)$$

the Riemann-Hurwitz formula now reads

$$2g - 2 = |G| \left[2g_0 - 2 + \sum_{i=1}^r \left(1 - \frac{1}{n_i}\right) \right]$$

so we have

$$(3) \quad |G| \mid (2g - 2) \cdot \text{lcm}(n_1, \dots, n_r),$$

this fact is crucial in the study of automorphism groups of compact Riemann surfaces.

The following is a special case of a theorem due to W. Harvey (Theorem 6 in [10]).

Proposition 3.1. [10] *The minimum genus g of a compact Riemann surface which admits an automorphism of order p^r (p is prime) is given by*

$$g = \max \left\{ 2, \frac{p-1}{2} p^{r-1} \right\}.$$

In (3), we have $n_i = \text{ord}G_{P_i}$ and G_{P_i} is cyclic, so Proposition 3.1 implies the following

Corollary 3.2. *Let X be a compact Riemann surface of genus $g \geq 2$ and $G = |\text{Aut}(X)|$. Then*

$$\text{ord}(p, |G|) \leq \lfloor \log_p \frac{2pg}{p-1} \rfloor + \text{ord}(p, 2(g-1)).$$

In particular, $p \nmid |G|$ if $p > 2g + 1$.

Definition 3.3. A *node* on a curve is a point that is locally analytically isomorphic to a neighborhood of the origin of $xy = 0$ in the complex plane \mathbb{C}^2 .

If Σ is a nodal curve, define its *normalization* $\tilde{\Sigma}$ to be the Riemann surface obtained by “ungluing” its nodes. Let $p : \tilde{\Sigma} \rightarrow \Sigma$ denote the canonical normalization map. The preimages in $\tilde{\Sigma}$ of the nodes of Σ are called *node-branches*.

A *stable curve* is a connected and compact nodal curve, which means that its singular points are nodes and satisfy the stability conditions: (i) each genus 0 component has at least three node-branches; (ii) each genus 1 component has at least one node-branch.

Stability is equivalent to the finiteness of the automorphism group. Suppose Σ is a stable curve of arithmetic genus g such that its normalization has m components $\Sigma_1, \dots, \Sigma_m$ of genus g_1, \dots, g_m .

Definition 3.4. An automorphism φ of the dual graph Γ of Σ will be called *geometric*, if it is induced by an automorphism of the corresponding stable curve Σ . All geometric automorphisms of Γ form a group $GAut(\Gamma)$, which is a subgroup of $Aut(\Gamma)$.

The notion of geometric automorphism is introduced by Opstall and Veliche [24] in their study of sharp bounds for the automorphism group of stable curves of a given genus.

Theorem 3.5. *Let $\widetilde{Aut}(\Sigma_i)$ be the group of automorphisms of Σ_i fixing node-branches on Σ_i . Then we have*

$$|Aut(\Sigma)| = |GAut(\Gamma)| \cdot \prod_{i=1}^m |\widetilde{Aut}(\Sigma_i)|$$

Proof. First note the following fact, if $f(x)$ and $g(y)$ are two holomorphic functions defined near the origin of \mathbb{C}^1 and satisfy $f(0) = g(0)$, then $F(x, y) = f(x) + g(y) - f(0)$ is a holomorphic function near the origin of \mathbb{C}^2 satisfying $F(x, 0) = f(x)$ and $F(0, y) = g(y)$. So to check whether a function on a nodal curve is analytic, we need only check it is analytic restricting to each connected component.

There is a natural map $p : Aut(\Sigma) \rightarrow GAut(\Gamma)$ mapping an automorphism of Σ to the induced automorphism on its dual graph Γ .

For each $b \in GAut(\Gamma)$, fix a $T_b \in Aut(\Sigma)$ such that $p(T_b) = b$. If $f_i \in \widetilde{Aut}(\Sigma_i), i = 1 \cdots m$, we denote by $(f_1, \cdots, f_m) \in Aut(\Sigma)$ the gluing morphism. We define the following map

$$\begin{aligned} GAut(\Gamma) \times \prod_{i=1}^m \widetilde{Aut}(\Sigma_i) &\longrightarrow Aut(\Sigma) \\ (b, f_1, \cdots, f_m) &\longmapsto T_b \circ (f_1, \cdots, f_m). \end{aligned}$$

It's not difficult to see that this map is in fact bijective. Its converse is

$$\begin{aligned} Aut(\Sigma) &\longrightarrow GAut(\Gamma) \times \prod_{i=1}^m \widetilde{Aut}(\Sigma_i) \\ T &\longmapsto (p(T), (T_{p(T)}^{-1} \circ T)|_{\Sigma_1}, \cdots, (T_{p(T)}^{-1} \circ T)|_{\Sigma_m}). \end{aligned}$$

So we proved the theorem. \square

Proposition 3.6. *Let Σ be a stable curve of arithmetic genus $g \geq 2$, if a prime number p divides $|Aut(\Sigma)|$, then $p \leq 2g + 1$.*

Proof. Let's assume that there are δ nodes on Σ and δ_i node-branches on each Σ_i . Then we have the following relations,

$$(4) \quad g = \sum_{i=1}^m (g_i - 1) + \delta + 1,$$

$$(5) \quad 2g_i + \delta_i - 2 \geq 1,$$

$$(6) \quad 2\delta = \sum_{i=1}^m \delta_i.$$

Sum up (5) for $i = 1$ to n and substitute (4) and (6) into (5), we get

$$m \leq 2g - 2.$$

Let e_{ij} denote the number of edges between Σ_i and Σ_j in the dual graph of Σ , then it's obvious that $e_{ij} \leq g + 1$.

Since $|Aut(\Gamma)|$ divides $m! \prod_{(i,j)} (e_{ij}!)$ which is not divisible by prime numbers greater than $2g + 1$, and $g_i \leq g$, so the proposition follows from Theorem 3.5 and Corollary 3.2. \square

We remark that for non-stable nodal curves, Proposition 3.6 may not hold.

4. PRIME FACTORS OF \mathcal{D}_g

Definition 4.1. In [7], the following generating function

$$F(x_1, \cdots, x_n) = \sum_{g=0}^{\infty} \sum_{\sum d_i = 3g - 3 + n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^n x_i^{d_i}$$

is called the n -point function.

In particular, 2-point function has a simple explicit form due to Dijkgraaf (see [7])

$$F(x_1, x_2) = \frac{1}{x_1 + x_2} \exp\left(\frac{x_1^3}{24} + \frac{x_2^3}{24}\right) \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!} \left(\frac{1}{2}x_1x_2(x_1+x_2)\right)^k.$$

Lemma 4.2. *Let p be a prime number and $g \geq 2$, then*

- i) If $p > 2g + 1$, then $p \nmid D_{g,2}$,
- ii) If $g + 1 \leq p \leq 2g + 1$, then

$$p \mid \text{denom} \langle \tau_{\frac{p-1}{2}} \tau_{3g-1-\frac{p-1}{2}} \rangle_g,$$

- iii) If $2g + 1$ is prime, then $(2g + 1) \mid \text{denom} \langle \tau_d \tau_{3g-1-d} \rangle_g$ if and only if $g \leq d \leq 2g - 1$.
- iv) If $2g + 1$ is prime, then $\text{ord}(2g + 1, D_{g,2}) = 1$.

Proof. From the 2-point function, we get

$$\begin{aligned} \langle \tau_d \tau_{3g-1-d} \rangle_g &= \sum_{i=0}^g \sum_k \binom{g-k}{i} \binom{k-1}{d-3i-k} \frac{k!}{(g-k)! 24^{g-k} (2k+1)! 2^k} \\ &\quad + \frac{(-1)^{d \bmod 3}}{g! 24^g} \binom{g-1}{\lfloor \frac{d}{3} \rfloor}, \end{aligned}$$

where the summation range of k is $\max(\frac{d_1-3i+1}{2}, 1) \leq k \leq \min(g-i, d_1-3i)$. Then the lemma follows easily. \square

Theorem 4.3. *Let p be a prime number, $g \geq 2$ and let $\text{ord}(p, q)$ denote the maximum integer such that $p^{\text{ord}(p,q)} \mid q$, then*

- i) If $p > 2g + 1$, then $p \nmid \mathcal{D}_g$,
- ii) For any prime $p \leq 2g + 1$, we have $p \mid \mathcal{D}_g$,
- iii) If $2g + 1$ is prime, then $\text{ord}(2g + 1, \mathcal{D}_g) = 1$,
- iv) $\text{ord}(2, \mathcal{D}_g) = 3g + \text{ord}(2, g!)$.

Proof. For part (i), we use induction on the pair of genus and the number of marked points (g, n) to prove that denominators of all ψ class intersection numbers $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$ are not divisible by prime numbers greater than $2g + 1$. If $p > 2g + 1$, then $p \nmid D_{g,2}$ has been proved in Lemma 4.2(i). Also $\mathcal{D}_2 = 2^7 \cdot 3^2 \cdot 5$ is not divisible by $p > 5$. So we may assume $g \geq 3, n \geq 3$. We rewrite the DVV formula here,

$$\begin{aligned} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g &= \frac{1}{(2d_1 + 1)!!} \left[\sum_{j=2}^n \frac{(2d_1 + 2d_j - 1)!!}{(2d_j - 1)!!} \langle \tau_{d_2} \cdots \tau_{d_j+d_1-1} \cdots \tau_{d_n} \rangle_g \right. \\ &\quad + \frac{1}{2} \sum_{r+s=d_1-1} (2r+1)!! (2s+1)!! \langle \tau_r \tau_s \tau_{d_2} \cdots \tau_{d_n} \rangle_{g-1} \\ &\quad \left. + \frac{1}{2} \sum_{r+s=d_1-1} (2r+1)!! (2s+1)!! \sum_{\{2, \dots, n\} = I \amalg J} \langle \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \right] \end{aligned}$$

For $n \geq 3$ marked points, we may take $d_1 \leq g$, then by induction on (g, n) it's easy to see that the denominator of the right hand side is not divisible by prime numbers greater than $2g + 1$.

For part (ii), it follows from Lemma 2.1(iii), Theorem 2.3 and Lemma 4.2(ii).

For part (iii), we again use induction on (g, n) as in the proof of part (i), we may assume $n \geq 3$. In view of Lemma 4.2(iii)-(iv), we need only prove $\text{ord}(2g + 1, D_{g,n}) \leq 1$. If $n > 3$, then we may take $d_1 < g$ in $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$, whose denominator is not divisible by $(2g + 1)^2$. This is easily seen by induction on the right hand side of the DVV formula. So we are only left to prove that the denominator of $\langle \tau_g \tau_g \tau_g \rangle_g$ is not divisible by $(2g + 1)^2$.

$$\langle \tau_g \tau_g \tau_g \rangle_g = \frac{1}{(2g+1)!!} \left[\frac{2(4g-1)!!}{(2g-1)!!} \langle \tau_g \tau_{2g-1} \rangle_g + \{\text{lower genus terms}\} \right]$$

Since the factor $2g+1$ in the denominator of $\langle \tau_g \tau_{2g-1} \rangle$ will be cancelled by $(4g-1)!!$, by induction we proved (iii).

For part (iv), since $\langle \tau_{3g-2} \rangle_g = \frac{1}{24^g g!}$, we have $\text{ord}(2, D_g) \geq 3g + \text{ord}(2, g!)$, the reverse inequality can be seen from the DVV formula by induction on (g, n) and note the following,

$$\begin{aligned} & \frac{1}{2} \sum_{r+s=k-1} (2r+1)!!(2s+1)!! \sum_{\underline{n}=I \amalg J} \langle \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \\ = & \sum_{r+s=k-1} (2r+1)!!(2s+1)!! \sum_{\{2, \dots, n\}=I \amalg J} \langle \tau_r \tau_{d_1} \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}. \end{aligned}$$

□

Lemma 4.4. *If $2 \leq p \leq g+1$ is a prime number, then $\text{ord}(p, D_{g,3}) \geq 2$.*

Proof. From Lemma 2.1(3), we have $24^g \mid D_{g,3}$, so the lemma is obvious for $p = 2$ or 3 . We assume $p \geq 5$ below.

The following formula of the special three-point function is due to Faber [7].

$$\begin{aligned} F_g(x, y, -y) &= \sum_{b \geq 0} \sum_{j=0}^{2b} (-1)^j \langle \tau_{3g-2b} \tau_j \tau_{2b-j} \rangle_g x^{3g-2b} y^{2b} \\ &= \sum_{\substack{a+b+c=g \\ b \geq a}} \frac{(a+b)!}{4^{a+b} 24^c (2a+2b+1)!! (b-a)! (2a+1)! c!} x^{3a+3c+b} y^{2b}. \end{aligned}$$

If $p > \frac{2g+1}{3}$, then consider the coefficient of $x^{3g-p+1} y^{p-1}$ in $F_g(x, y, -y)$

$$[F_g(x, y, -y)]_{x^{3g-p+1} y^{p-1}} = \sum_{\substack{a+b+c=g \\ a \leq \frac{p-1}{2}}} \frac{(a+b)!}{4^{a+b} 24^c (2a+2b+1)!! (b-a)! (2a+1)! c!}$$

where $b = \frac{p-1}{2}$. We must have $c < p$, so it's not difficult to see that only the term with $a = b = \frac{p-1}{2}$ can contain factor p^2 in the denominator.

If $p \leq \frac{2g+1}{3}$, we have

$$[F_g(x, y, -y)]_{x^g y^{2g}} = \frac{1}{4^g (2g+1)!!},$$

and $\text{ord}(p, (2g+1)!!) \geq 2$.

So we proved the lemma. □

Theorem 4.5. *Let X be a compact Riemann surface of genus $g' \geq 2$ and $g \geq g'$, then $|Aut(X)|$ divides $D_{g,3}$.*

Proof. We first prove the case $g' = g$.

Let p denote a prime number. By Corollary 3.2, it is sufficient to prove

$$(7) \quad \left\lfloor \log_p \frac{2pg}{p-1} \right\rfloor + \text{ord}(p, 2(g-1)) \leq \text{ord}(p, D_{g,3})$$

for all prime $p \leq 2g+1$.

If $\max(g, 5) \leq p \leq 2g+1$, then we have $\left\lfloor \log_p \frac{2pg}{p-1} \right\rfloor \leq 1$ and $\text{ord}(p, 2(g-1)) = 0$, so from Theorem 4.3(2), the above inequality (8) holds in this case.

Now we assume $5 \leq p \leq g-1$, the cases $p = 2$ and $p = 3$ will be treated at last. We still need to divide into three finer cases.

Case i) If $p = g - 1 \geq 5$ is prime, then we have $(g - 1)(g - 2) > 2g$. By Lemma 4.7, we have

$$\lfloor \log_{g-1} \frac{2g(g-1)}{g-2} \rfloor + 1 \leq 2 \leq \text{ord}(g-1, D_{g,3}).$$

Case ii) Otherwise if $p \nmid (g - 1)$, since $\text{ord}(p, 2(g - 1)) = 0$, $g! \mid D_{g,3}$ and $\text{ord}(p, g!) \geq \lfloor \frac{g}{p} \rfloor$, so in order to check (8), it's sufficient to prove

$$\lfloor \log_p \frac{2pg}{p-1} \rfloor \leq \lfloor \frac{g}{p} \rfloor.$$

Let $g = kp + r$, where $-p \leq r < 0$. Then $\lfloor \frac{g}{p} \rfloor = k - 1$. Since for fixed k , the left hand side takes its maximum value when $g = kp - 1$, we need only prove the above identity for $g = kp - 1$, which is equivalent to for all $k \geq 2$, $p \geq 5$,

$$p^k > \frac{2p(kp-1)}{p-1}, \quad \text{i.e. } p^k - p^{k-1} - 2kp + 2 > 0,$$

which is not difficult to check.

Case iii) If $p \mid (g - 1)$ and $5 \leq p < g - 1$. Let $\text{ord}(p, 2(g - 1)) = r$. Then $p^r \mid (g - 1)$, we have

$$\begin{aligned} \text{ord}(p, D_{g,3}) \geq \text{ord}(p, g!) &= \lfloor \frac{g}{p} \rfloor + \lfloor \frac{g}{p^2} \rfloor + \lfloor \frac{g}{p^3} \rfloor + \dots \\ &\geq \lfloor \frac{g}{p} \rfloor + r - 1. \end{aligned}$$

So it's sufficient to prove

$$\lfloor \log_p \frac{2pg}{p-1} \rfloor + 1 \leq \lfloor \frac{g}{p} \rfloor.$$

Let $g = kp + 1$, $k \geq 2$, we need to prove

$$p^k > \frac{2p(kp+1)}{p-1}, \quad \text{i.e. } p^k - p^{k-1} - 2kp - 2 > 0.$$

The above inequality holds except in the case $p = 5$, $k = 2$ and $g = 11$, which should be treated separately. We have

$$\text{ord}(5, |G|) \leq \lfloor \log_5 \frac{110}{4} \rfloor + 1 = 3$$

and $\text{ord}(5, D_{11,3}) = 3$, in fact

$$D_{11,3} = 2^{41} \cdot 3^{15} \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23.$$

We finished checking in this case.

Now we consider the remaining two cases, $p = 2$ and $p = 3$. Note that $24^g g! \mid D_{g,3}$.

If $p = 2$, it's sufficient to prove $\log_2 4g \leq 3g - 1$.

If $p = 3$, it's sufficient to prove $\log_3 3g \leq g$.

Both cases are easy to check. So we conclude the proof of the theorem when $g' = g$.

The proof of the cases $g' < g$ can be proved by exactly the same argument and using Lemma 4.7. \square

We remark that there exists a compact Riemann surface X of genus 6 with $|Aut(X)| = 150$ (see Table 13 in [2]). While the power of 5 in $D_{6,2} = 2^{22} \cdot 3^8 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ is only 1, so $|Aut(X)| \nmid D_{6,2}$. In this sense, we may say that Theorem 4.8 is optimal.

The following immediate corollary of Theorem 4.8 is a conjecture of Itzykson and Zuber, stated at the end of Section 5 of [12].

Corollary 4.6. *For $1 < g' \leq g$, the order of automorphism group of any compact Riemann surface of genus g' divides \mathcal{D}_g .*

We remark that the statement of Corollary 4.9 doesn't hold for stable curves, namely there exists some stable curve of genus g , the order of whose automorphism group does not divide \mathcal{D}_g . A counterexample can be constructed as follows. Let $n = \lfloor \frac{2g}{p-1} \rfloor$ Riemann surfaces of genus $\frac{p-1}{2}$ attached to a sphere at $e^{\frac{2\pi i}{n}}$ for $0 \leq i \leq n-1$. When $n \geq p$, the order of automorphism group of such a stable curve will have the power of p larger than $\lfloor \frac{2g}{p-1} \rfloor$ (see conjecture 2.5).

5. A CONJECTURAL NUMERICAL PROPERTY OF INTERSECTION NUMBERS

During our work on intersection numbers, we noticed a multinomial type property for intersection numbers. Although still conjectural, we feel they are interesting constraints of intersection numbers on moduli spaces, so we briefly present them here.

From

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_0 = \binom{n-3}{d_1 \cdots d_n} = \frac{(n-3)!}{d_1! \cdots d_n!},$$

we see that if $d_1 < d_2$, we have

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_0 \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_0.$$

Now we prove that the same inequality holds in genus 1.

Proposition 5.1. *For $\sum_{i=1}^n d_i = n$ and $d_1 < d_2$, we have*

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1.$$

Proof. We prove the inequality by induction on n . If $n = 2$, we have

$$\langle \tau_0 \tau_2 \rangle_1 = \langle \tau_1 \tau_1 \rangle_1 = \frac{1}{24}.$$

Now assume that the proposition has been proved for $n-1$. We may also assume $d_2 - d_1 \geq 2$, otherwise it is trivial. So by the symmetry property of intersection numbers, we may assume without loss of generality that $d_n = 0$ or $d_n = 1$.

If $d_n = 1$ then by dilaton equation

$$\begin{aligned} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 &= (n-1) \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_{n-1}} \rangle_1 \\ \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1 &= (n-1) \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_{n-1}} \rangle_1. \end{aligned}$$

So $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1$ holds in this case by induction.

If $d_n = 0$ then by string equation

$$\begin{aligned} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 &= \langle \tau_{d_1-1} \tau_{d_2} \cdots \tau_{d_{n-1}} \rangle_1 + \langle \tau_{d_1} \tau_{d_2-1} \cdots \tau_{d_{n-1}} \rangle_1 \\ &\quad + \sum_{i=3}^{n-1} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_{i-1}} \cdots \tau_{d_{n-1}} \rangle_1 \\ \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1 &= \langle \tau_{d_1} \tau_{d_2-1} \cdots \tau_{d_{n-1}} \rangle_1 + \langle \tau_{d_1+1} \tau_{d_2-2} \cdots \tau_{d_{n-1}} \rangle_1 \\ &\quad + \sum_{i=3}^{n-1} \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_{i-1}} \cdots \tau_{d_{n-1}} \rangle_1. \end{aligned}$$

So $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1$ holds again by induction. \square

Now we formulate the following conjecture

Conjecture 5.2. For $\sum_{i=1}^n d_i = 3g - 3 + n$ and $d_1 < d_2$, we have

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_g \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_g.$$

Namely the more evenly $3g - 3 + n$ be distributed among indices, the larger the intersection numbers.

By the same argument of Proposition 5.1, we can see that for each g , it's enough to check only those intersection numbers with $n \leq 3g - 1$ and $d_3 \geq 2, \dots, d_n \geq 2$.

We checked Conjecture 5.2 for $g \leq 16$ with the help of Faber's Maple program. Moreover, for $n = 2$, we checked all $g \leq 300$ using Dijkgraaf's 2-point function; for $n = 3$, we checked all $g \leq 50$ using Zagier's 3-point function.

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