Compressible Viscous Flows in a Symmetric Domain with Complete Slip Boundary

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Abstract

This work is devoted to study the global behavior of viscous flows contained in a symmetric domain with complete slip boundary. In such scenario the boundary no longer provides friction and therefore the perturbation of angular velocity lacks decaying structure. In fact, we show the existence of uniformly rotating solutions as steady states for the compressible Navier-Stokes equations. By manipulating the conservation law of angular momentum, we establish a suitable Korn's type inequality to control the perturbation and show the asymptotic stability of the uniformly rotating solutions with small angular velocity. In particular, the initial perturbation which preserves the angular momentum would decay exponentially in time and the solution to the Navier-Stokes equations converges to the steady state as time grows up.

Contents

1 Introduction

1.1 Description and Related Works

In this work, we consider the isentropic compressible Navier-Stokes system in the following, which models the motion of viscous gases (or fluids) in a bounded domain $\Omega \subset \mathbb{R}^3$,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{in } \Omega, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div} \mathbb{S}(u) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\rho, u, P, \mathbb{S}(u)$ represent the density, the velocity, the pressure potential, and the viscous tensor respectively. Moreover, the flow is assumed to be Newtonian. For simplicity, the pressure potential and the viscous tensor are taken in the following forms,

$$P = P(\rho) = \rho^\gamma, \ \mathbb{S}(u) = \mu\left(\nabla u + \nabla u^\top\right) + \lambda \operatorname{div} u \mathbb{I}_3, \ \gamma > 1, \ \mu, \lambda > 0,$$

where $\mathbb{I}_3$ is the 3×3 identity matrix and μ, λ are the viscosity coefficients. (1.1) can be complemented with various boundary conditions. In this work, the associated boundary condition is taken as the complete slip boundary condition, i.e.,

$$u \cdot \vec{n} = 0, \ \vec{\tau} \cdot \mathbb{S}(u)\vec{n} = 0, \quad \text{on } \Gamma = \partial\Omega, \tag{1.2}$$

where $\vec{\tau}, \vec{n}$ denote the tangential and normal vectors on the boundary Γ. (1.1) and (1.2) are given with the initial data

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \qquad x \in \Omega. \tag{1.3}$$

Moreover, we take $\Omega = B_1$ being the ball centred at the origin with radius 1. Indeed, the geometry of Ω would be an important factor in our problem. On one hand, as it is illustrated in the study of the stationary problem [10], the shape of the boundary Γ is an important factor in determining the steady states of (1.1) with the boundary condition (1.2). On the other hand, as pointed out in [7], the Korn's inequality

$$C_\Omega \left\| \nabla V + \nabla V^\top \right\|_{L^2} \geq \| \nabla V \|_{L^2}, \tag{1.4}$$

for a vector field V with the tangency boundary condition ($u \cdot \vec{n} = 0$ on $\Gamma = \partial\Omega$) only holds when Ω is a non-axisymmetric domain. Therefore the Korn's inequality (1.4), which gives the dissipation and plays an important role in the global analysis of (1.1), no longer applies directly in our setting. We shall resolve such issue in this work (see (2.16)).

As it will be pointed out, (1.1) with (1.2) admits a class of steady states with non-trivial velocity, which satisfy the equations (1.6) with $\bar{\rho} \neq$ constant. In short, instead of external forces, such non-trivial steady states are consequences of the self-rotation and the geometry of occupied domain. The main goal in this article is to investigate the stability of these steady states.

It should be emphasised that, most of the available stability theories are subject to the no-slip boundary condition ($u|_{\partial\Omega} = 0$). However, the Navier-slip boundary condition is actually more appropriate when studying many phenomena such as hurricanes and tornadoes (see, [2]). The general Navier-slip boundary condition is of the form

$$u \cdot \vec{n} = 0, \; \vec{\tau} \cdot \mathbb{S}(u)\vec{n} = B(u \cdot \vec{\tau}), \quad \text{on } \Gamma = \partial\Omega. \tag{1.5}$$

Such kinds of boundary conditions were first introduced by Navier [25]. The first work on the mathematical rigorous analysis was due to Solonnikov and Ščadilov [27] on the linearized stationary equations. A simplified form of (1.5) is by taking $B(u \cdot \vec{\tau}) = -\kappa^{-1} u \cdot \vec{\tau}$ with κ being a constant. The most studied case in the literatures is when $\kappa > 0$, corresponding to slip with friction. In this situation, κ is called the slip length. The case $\kappa < 0$ corresponds to the case in which the boundary wall accelerates the fluids (see [8]). Our focus is on the case when the boundary does not provide friction or acceleration to the fluids, i.e. $B(u \cdot \vec{\tau}) = 0$. Such boundary is called the complete slip boundary (see [5]).

For a homogeneous incompressible flow ($\operatorname{div} u = 0, \rho = \text{constant}$), Chen and Qian in [5] demonstrated that when $\kappa \to 0^+$, the weak solutions to the incompressible Navier-Stokes equations converge to the solution for the problem subject to the no-slip boundary condition ($u|_{\partial\Omega} = 0$) for almost all time. Also, as $\kappa \to +\infty$, the solutions converge to a solution for the problem subject to the complete slip boundary condition. Recently, Ding, Li, Xin [8] show that some instability may occur when $\kappa < 0$ in two spatial dimensional setting. The instability is in the following sense. $\exists \epsilon > 0$ such that $\forall \delta > 0$, there exists an initial perturbation of the steady state ($u_s = 0$) that will grow larger than the size ϵ in a suitable function space even thought such perturbation is smaller than the size δ initially. Indeed, a critical value of the viscosity coefficient depending on κ serves as the threshold of stability and instability, and the instability would occur if and only if the viscosity coefficient is smaller than the critical value. The problem with complete slip boundary in an axisymmetric domain was studied by Watanabe in [31]. It is shown that the global weak solution would converge to the projection of initial velocity on the rigid body motion (1.12) in L^2 sense.

For a compressible flow, Hoff [14] proved the local-in-time existence of smooth solutions to the Navier-Stokes equations with the boundary condition (1.5). In [16], Huang, Li, Xin established the global dynamic property to the Stokes approximation of Navier-Stokes equations with the no-stick boundary condition in two dimensional setting. It is shown the solution (ρ, u) would converge the the equilibria $(\rho_s, 0)$ in $L^\alpha \times W^{1,\beta}$ space as time grows up. Recently, H. Li and X. Zhang in [21] establish the nonlinear stability of Couette flows with the moving condition on the top and the Navier-slip boundary condition (1.5) in which $B(u \cdot \vec{\tau}) = -\kappa^{-1} u \cdot \tau, \kappa > 0$ on the bottom. The asymptotic stability of the trivial steady state ($\rho_s = \text{constant}, u_s = 0$) to the problem with frictional boundary ($\kappa > 0$) was also demonstrated by Zajączkowski in [35].

As for the Cauchy problems and initial boundary valued problems with no-slip boundary for the compressible Navier-Stokes equations, there are rich literatures available . We shall only mention a few. In the absence of vacuum ($\rho \geq \underline{\rho} > 0$), the local and global well-posedness of classical solutions have been widely discussed. The uniqueness of both viscous and inviscid compressible flows was studied by Serrin [26]. Itaya [18] and Tani [28] showed the local-in-time existence of classical solutions. Matsumura and Nishida [23, 24] first established the global well-posedness of classical solutions with a small pertuerbation of a uniform non-vacuum state. In the present of vacuum, as pointed out by Xin in [32] (also Xin, Yan in [33]), some singular behaviour may occur. With small initial energy, Huang, Li, Xin [17] constructed the global smooth solutions for isentropic compressible viscous flows in $\mathbb{R}^3$. For more stability and instability problems, see [12, 13, 30, 19].

To study (1.1) with (1.2), we start with the stability theory as the first step. In this work, we will show the existence of rigid motions as steady solutions which rotate with uniform angular velocity. Unfortunately, such rotating profile may contain vacuum when the angular velocity is large. Indeed, the vacuum would appear around the symmetric axis of the domain. Thus it is supposed to be a vacuum interface problem. Moreover, the density profile admits physical vacuum across the vacuum interface, which contains singularities and is not suited in our functional framework (see [22]). We will study such vacuum interface problem in the future and focus on the non-vacuum problem here. To study the stability of the rigid motion, we would make use of the conservation of angular momentum (Lemma 3) to establish a Korn's type inequality (2.16), which would play important roles in global analysis. This is inspired by the study of free boundary problems in [34]. Moreover, it is introduced a spherical frame which matches the geometry of Ω. With such structure, we would be able to define some differential operators (2.21), which separate the normal derivatives and tangential derivatives. The benefit of doing so is that it avoids applying the partition of unity of the domain and works without introducing local charts. Therefore, we can calculate in the entire domain at once. The differential operators in this work are natural to the spherical domain and it is convenient to separate $\operatorname{div} \mathbb{S}(u)$ on the right of the momentum equation $(1.1)_2$ into tangential and normal directions (see (3.17)).

The rest of this work would be organised as follows. In the next section, we shall construct the steady state which we are going to study. In Section 2, we present the main tools that will be used in this work, including the Korn's type inequality, the differential operators and the classical elliptic estimates on the Stokes' problem. The equations and the main theorem in the perturbation variables would be given in Section 2.5. The main energy estimates are listed in Section 3. To illustrate the program, the estimates on the temporal derivatives and lower order spatial derivatives will be recorded in Section 3.1. Then we move onto the higher order spatial derivatives and interior estimates. With such blocks in hands, we chain them together in Section 4 and show the asymptotic stability. In particular, we demonstrate the nonlinearities can be indeed controlled by the energy in Section 4.2.

1.2 Steady States and Main Results

Here we search for non-vacuum steady states of (1.1). That is, to find $(\bar\rho, \bar u)$ with $\bar\rho > 0$ satisfying

$$\begin{cases} \operatorname{div}(\bar\rho\bar u) = 0 & \text{in } \Omega, \\ \operatorname{div}(\bar\rho\bar u \otimes \bar u) + \nabla \bar P = \operatorname{div}\mathbb{S}(\bar u) & \text{in } \Omega, \\ \bar u \cdot \vec n, \ \vec\tau \cdot \mathbb{S}(\bar u)\vec n = 0 & \text{on } \Gamma, \end{cases} \tag{1.6}$$

with $\bar P = \bar\rho^\gamma, \mathbb{S}(\bar u) = \mu(\nabla \bar u + \nabla \bar u^\top) + \lambda \operatorname{div}\bar u \mathbb{I}_3$. After taking inner product of $(1.6)_2$ with $\bar u$ and integration by parts, record the resulting equation,

$$\begin{aligned} &-\frac{1}{2}\int_\Omega \operatorname{div}(\bar\rho\bar u)|\bar u|^2\,dx + \frac{1}{2}\int_\Gamma \rho|\bar u|^2 \bar u \cdot \vec n\,dS - \frac{\gamma}{\gamma-1}\int_\Omega \operatorname{div}(\bar\rho\bar u)\cdot\bar\rho^{\gamma-1}\,dx \\ &+ \frac{\gamma}{\gamma-1}\int_\Gamma \bar\rho^\gamma \bar u\cdot\vec n\,dS = -\frac{\mu}{2}\int_\Omega \left|\nabla\bar u + \nabla\bar u^\top\right|^2\,dx - \lambda\int_\Omega |\operatorname{div}\bar u|^2\,dx \\ &+ \int_\Gamma \bar u\cdot\mathbb{S}(\bar u)\vec n\,dS. \end{aligned} \tag{1.7}$$

Since $\bar u \cdot \vec n = 0$ on Γ, $\bar u$ is in the tangential direction of Γ. Therefore, $(1.6)_3$ implies $\bar u\cdot\mathbb{S}(\bar u)\vec n = 0$ on Γ. Together with $(1.6)_1$, $(1.6)_3$, (1.7) can be rewritten as

$$\frac{\mu}{2}\int_\Omega \left|\nabla\bar u + \nabla\bar u^\top\right|^2\,dx + \lambda\int_\Omega |\operatorname{div}\bar u|^2\,dx = 0,$$

and hence

$$\nabla\bar u + \nabla\bar u^\top = 0, \ \text{in } \Omega. \tag{1.8}$$

It follows, via similar arguments as in [33] (also [6])

$$\bar u = \bar A x + \bar b, \quad x \in \Omega,$$

for some anti-symmetric matrix $\bar{A}$ and constant vector $\bar{b}$. Moreover, on Γ, x is paralleled to $\vec{n}$ as the consequence of the geometry of Ω. Hence $0 = \bar{u}\cdot\vec{n} = \bar{A}x\cdot\vec{n} + \bar{b}\cdot\vec{n} = \bar{b}\cdot\vec{n}$ on Γ. It follows $\bar{b} = 0$. Therefore, there is a vector $\bar{a}$ such that

$$\bar{u} = x \times \bar{a}, \quad \text{in } \Omega. \tag{1.9}$$

Without lost of generality, by assuming $\bar{a} = \bar{\omega}e_3$ for $\bar{\omega} \geq 0$, (1.6) is reduced to the following ordinary differential equation(ODE)

$$-\bar{\omega}^2\bar{\rho}r + \partial_r\bar{\rho}^\gamma = 0, \tag{1.10}$$

with $\bar{\rho} = \bar{\rho}(r)$ where $r = \sqrt{(x_1)^2 + (x_2)^2}$. Therefore,

$$\begin{cases} \bar{\rho} = \left(\bar{\rho}^{\gamma-1}(0) + \dfrac{\gamma-1}{2\gamma}\bar{\omega}^2 r^2\right)^{1/(\gamma-1)}, \\ \bar{u} = \bar{\omega}\,(x_2, -x_1, 0)^\top, \end{cases} \tag{1.11}$$

where $\bar{\rho}(0) > 0, \bar{\omega} \geq 0$.

Furthermore, we shall denote the rigid body motions

$$\begin{aligned} S &= \left\{V \in H^1(\Omega, \mathbb{R}^3); \nabla V + \nabla V^\top = 0, V\cdot\vec{n} = 0 \text{ on } \partial\Omega\right\} \\ &= \left\{V(x) = x \times \zeta; V\cdot\vec{n} = 0 \text{ on } \partial\Omega, \zeta \text{ is a constant vector}\right\}. \end{aligned} \tag{1.12}$$

Also, let P_S be the orthogonal projection from $H^1(\Omega, \mathbb{R}^3)$ onto S. Notice, $\forall V \in S$, since the domain is symmetric, it holds $\int_\Omega V\,dx = 0$. In particular, $\forall V \in H^1(\Omega, \mathbb{R}^3)$,

$$\int_\Omega P_S V\,dx = 0.$$

Remark We use the assumption $\mu, \lambda > 0$ in the deviation of (1.8). However, the Lamé relation $(\mu > 0, 3\lambda + 2\mu > 0)$ is sufficient to show this.

Remark It is assumed $\inf_{x\in\Omega}\bar{\rho}(x) > 0$. From (1.10), the minimum of $\bar{\rho}$ is achieved on the axis $\{x_1 = x_2 = 0\}$. For a fixed $\bar{\omega} > 0$ and the fixed domain Ω, it follows that the total mass of the fluid inside Ω has a lower bound $\underline{M_{\bar{\omega}}} > 0$ in order to avoid vacuum. In reality, when the total mass drops below $\underline{M_{\bar{\omega}}}$, there would be a vacuum area inside Ω and a vacuum interface.

Indeed, from (1.10), the density profile across the vacuum interface would admit physical vacuum [22]. Similar phenomena would occur when the total mass is fixed and the angular velocity increases. We leave such vacuum problems as future works.

In this work, it is devoted to study the stability of the steady state (1.11). In fact, we have the following informal statement of our theorem.

Theorem 1.1 (Informal Statement) *Provided the angular velocity* $|\bar{\omega}|$ *is small though, the steady state* $(\bar{\rho}, \bar{u})$ *given by* (1.11) *to the compressible Navier-Stokes equations* (1.1) *with the complete slip boundary condition* (1.2) *is asymptotically stable in the following sense. For an initial perturbation with the size less than* ϵ_1 *for some* $\epsilon_1 > 0$ *in some appropriate function space (defined in* (4.16) *and* (4.20)*), there is a globally defined classical solution to* (1.1) *and the solution converges to the steady state as time grows up exponentially. The perturbation is taken such that it preserves the angular momentum (see* (2.35)*).*

We shall state the theorem in perturbation variables later in Section 2.5.

2 Preliminaries

2.1 Notations

Through out this work, conventionally, for any quantities A, B,

$$\text{by } \ A \lesssim B, \ \text{ it is to say } \ A \leq CB,$$

where the constant $C > 0$ may depend on $\sup_{x\in\Omega} \bar{\rho}, \inf_{x\in\Omega} \bar{\rho}, \Omega, \mu, \lambda$ but is independent of $\bar{\omega}, \rho, u$. Similarly,

$$A \simeq B \ \text{ is equal to say } \ \frac{1}{C} A \leq B \leq CA.$$

For a constant $0 < \omega < 1$, the corresponding value C_ω would denote a positive value satisfying $1 \leq C_\omega \leq 1/\omega$. In the meantime, for a vector $\vec{w} = (w^1, w^2, w^3)^\top$, the associated differential operator is defined as,

$$\nabla_{\vec{w}} = \vec{w} \cdot \nabla = w^1 \partial_1 + w^2 \partial_2 + w^3 \partial_3,$$

where $\partial_i = \partial_{x_i}$ represents the spatial derivative for $i = 1, 2, 3$. Also, the commutator operator is defined by

$$[A, B] = AB - BA,$$

where A, B may stand for functions or differential operators. Notice $[\cdot, \cdot]$ is bilinear. The Sobolev norm in Ω and on the boundary $\Gamma = \partial\Omega$ is denoted as

$$\|f\|_{L^2(\Omega)}^2 = \int_\Omega f^2 \, dx, \qquad \|f\|_{H^k(\Omega)} = \sum_{i=0}^{k} \left\| \nabla^i f \right\|_{L^2(\Omega)},$$

$$\|f\|_{L^2(\Gamma)}^2 = \int_\Gamma f^2 \, dS, \qquad \|f\|_{H^k(\Gamma)} = \sum_{i=0}^{k} \left\| \bar{\nabla}^i f \right\|_{L^2(\Gamma)},$$

where $\nabla, \bar{\nabla}$ denote the differential operators in Ω and on the boundary Γ respectively.

2.2 Korn's Inequality

The following form of Korn's inequality is from [34, Lemma 5.1].

Lemma 1 (Korn's Inequality) *For $V = (V^1, V^2, V^3)^\top$ defined in Ω, it holds*

$$\int_\Omega |\nabla V|^2 \, dx \lesssim \int_\Omega \left|\nabla V + \nabla V^\top\right|^2 \, dx + \int_\Omega |V|^2 \, dx \tag{2.1}$$

provided the right hand side is finite.

For the sake of completeness, we show the proof here.

Proof Denote the energy of the symmetric part of any vector $U = (U^1, U^2, U^3)^\top$ as ,

$$E_\Omega(U) = \int_\Omega \left|\nabla U + \nabla U^\top\right|^2 \, dx.$$

Introduce a decomposition of V,

$$V = \sum_{i=1}^{3} b^i \varphi_i(x) + \bar{V}, \tag{2.2}$$

where

$$\varphi_i(x) = (x - \bar{x}) \times e_i, \tag{2.3}$$

with $x = (x_1, x_2, x_3)^\top$, $\bar{x} = \frac{1}{|\Omega|}(\int_\Omega x_1\,dx, \int_\Omega x_2\,dx, \int_\Omega x_3, dx)^\top$ and $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})^\top$, $i = 1, 2, 3$. In particular, if Ω is a ball, $\bar{x} = 0$. Define $b = (b^1, b^2, b^3)^\top$ by

$$b = -\frac{1}{2|\Omega|}\int_\Omega \operatorname{curl} V\,dx. \tag{2.4}$$

Then direct calculation yields

$$\int_\Omega V\,dx = \int_\Omega \bar{V}\,dx,\ \ \int_\Omega \operatorname{curl}\bar{V}\,dx = 0,\ \ E_\Omega(V) = E_\Omega(\bar{V}). \tag{2.5}$$

It follows from the original Korn's inequality (see [11, 15]),

$$\int_\Omega \left|\nabla\bar{V}\right|^2\,dx \lesssim E_\Omega(\bar{V}) = E_\Omega(V). \tag{2.6}$$

From the decomposition (2.2),

$$\int_\Omega |\nabla V|^2\,dx \lesssim |b|^2 + \int_\Omega \left|\nabla\bar{V}\right|^2\,dx \lesssim |b|^2 + E_\Omega(V). \tag{2.7}$$

It remains to estimate $|b|^2$. From (2.2), we consider the equations

$$\sum_{i=1}^{3} b^i \int_\Omega \varphi_i(x)\cdot\varphi_j(x)\,dx = \int_\Omega (V - \bar{V})\cdot\varphi_j(x)\,dx, \qquad j = 1, 2, 3.$$

The non-degeneracy of the coefficient matrix $\left\{\int_G \varphi_i(x)\cdot\varphi_j(x)\,dx\right\}$ implies

$$|b|^2 \lesssim \int_\Omega |V|^2\,dx + \int_\Omega |\bar{V}|^2\,dx. \tag{2.8}$$

Therefore, applying the Poincaré inequality together with (2.5) and (2.6), it holds,

$$\begin{aligned}\int_\Omega \left|\bar{V}\right|^2\,dx &\lesssim \int_\Omega \left|\bar{V} - \frac{1}{|\Omega|}\int_\Omega \bar{V}\,dx\right|^2\,dx + \int_\Omega \left|\frac{1}{|\Omega|}\int_\Omega \bar{V}\,dx\right|^2\,dx \\ &\lesssim \int_\Omega \left|\nabla\bar{V}\right|^2\,dx + \left(\int_\Omega V\,dx\right)^2 \lesssim E_\Omega(V) + \int_\Omega |V|^2\,dx.\end{aligned} \tag{2.9}$$

(2.1) follows from (2.7), (2.8) and (2.9). □

In addition, the following lemma consists of the L^2 estimate of the orthogonal complement of V with respect to the rigid motions (1.12).

Lemma 2 (Poincaré-Morrey Inequality) *For* $V \in H^1(\Omega, \mathbb{R}^3)$ *with* $V \cdot \vec{n} = 0$ *on* $\partial\Omega$*, we have*

$$\|V - P_S V\|^2_{L^2(\Omega)} \lesssim \int_\Omega \left|\nabla V + \nabla V^\top\right|^2 dx. \tag{2.10}$$

The proof contains a compactness argument. We refer the proof to [31, Lemma 4.2]. See also [10]. As a corollary, one can derive the following Poincaré inequality.

Corollary 1 *The same vector* V *as in Lemma 2 would satisfy the following*

$$\|V\|_{L^2(\Omega)} \lesssim \|\nabla V\|_{L^2(\Omega)}. \tag{2.11}$$

Proof This form of Poincaré inequalities can be found in [3]. However, a new proof is provided here. One can rewrite

$$V = V - \frac{1}{|\Omega|}\int_\Omega V\,dx + \frac{1}{|\Omega|}\int_\Omega (V - P_S V)\,dx,$$

where it has been used the fact $\int_\Omega P_S V\,dx = 0$. Therefore, the Poincaré inequality and (2.10) then yield

$$\|V\|^2_{L^2(\Omega)} \lesssim \|\nabla V\|^2_{L^2(\Omega)} + \left\|\nabla V + \nabla V^\top\right\|^2_{L^2(\Omega)} \lesssim \|\nabla V\|^2_{L^2(\Omega)}.$$

□

An interesting property of (1.1) comes from the complete slip boundary (1.2). More precisely, in the symmetric domain $\Omega = B_1$, the flow admits conservation of angular momentum.

Lemma 3 (Conservation of Angular Momentum) *With the complete slip boundary condition* (1.2)*, for any smooth solution to* (1.1) *with the initial data* (1.3)*, it holds*

$$\int_\Omega \rho u \cdot \varphi_i\,dx = \int_\Omega \rho_0 u_0 \cdot \varphi_i\,dx, \quad i = 1, 2, 3, \tag{2.12}$$

where φ_i *are defined in* (2.3) *with* $\bar{x} = 0$*.*

Proof Take inner product of $(1.1)_2$ with φ_i $(i = 1, 2, 3)$, and record the resulting after integration by parts,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \rho u \cdot \varphi_i \, dx - \int_\Omega \rho u \cdot (u \cdot \nabla)\varphi_i \, dx + \int_\Gamma (\rho u \cdot \varphi_i) \cdot (u \cdot \vec{n}) \, dS \\
&- \int_\Omega P \cdot \operatorname{div} \varphi_i \, dx + \int_\Gamma P \varphi_i \cdot \vec{n} \, dS = - \int_\Omega \mathbb{S}(u) : \nabla \varphi_i \, dx + \int_\Gamma \varphi_i \cdot (\mathbb{S}(u)\vec{n}) \, dS.
\end{aligned} \tag{2.13}$$

Notice $\varphi_i \cdot \vec{n} = (x \times e_i) \cdot \frac{x}{|x|} = 0, u \cdot \vec{n} = 0$ on Γ, which means both φ_i and u are in the tangential direction on Γ. Therefore all the boundary integrations above vanish. In the meantime, $\nabla \varphi_i$ is an anti-symmetric matrix, while $\mathbb{S}(u)$ is symmetric. Hence $\operatorname{div} \varphi_i = \operatorname{tr}(\nabla \varphi_i) = 0, u \cdot (u \cdot \nabla)\varphi_i = u \cdot (\nabla \varphi) u = 0, \mathbb{S}(u) : \nabla \varphi_i = 0$. Therefore, from (2.13)

$$\frac{d}{dt}\int_\Omega \rho u \cdot \varphi_i \, dx = 0. \tag{2.14}$$

□

Now we would able to derive the Korn's type inequality which is our first important block in this work.

Lemma 4 (Korn's Type Inequality) *For any smooth solution to* (1.1) *with*

$$\int_\Omega \rho u \cdot \varphi_i \, dx = \int_\Omega \bar{\rho} \bar{u} \cdot \varphi_i \, dx, \quad i = 1, 2, 3, \quad \text{and } \bar{\rho} \geq \inf_{x \in \Omega} \bar{\rho} > 0 \text{ in } \Omega, \tag{2.15}$$

it holds

$$\begin{aligned}
\int_\Omega |\nabla(u - \bar{u})|^2 \, dx \lesssim & \int_\Omega \left|\nabla u + \nabla u^\top\right|^2 \, dx + \int |\bar{u}|^2 |\rho - \bar{\rho}|^2 \, dx \\
& + \int_\Omega |\rho - \bar{\rho}|^2 |u - \bar{u}|^2 \, dx + \left(\int_\Omega u \, dx\right)^2.
\end{aligned} \tag{2.16}$$

Moreover, the last term $\int_\Omega u \, dx$ *on the right can be dropped in* (2.16).

Proof Decompose $u - \bar{u} = \sum_{i=1}^3 b^i \varphi_i + v$, where $b = (b^1, b^2, b^3)^\top$ are defined similarly as in (2.4) with $V = u - \bar{u}$. Similarly, it holds

$$\begin{aligned}
\int_\Omega v \, dx = \int_\Omega u \, dx, \quad \int_\Omega |\nabla v|^2 \, dx \lesssim \int_\Omega \left|\nabla u + \nabla u^\top\right|^2 \, dx, \\
\int_\Omega |\nabla(u - \bar{u})|^2 \, dx \lesssim |b|^2 + \int_\Omega \left|\nabla u + \nabla u^\top\right|^2 \, dx.
\end{aligned} \tag{2.17}$$

Notice

$$0 = \int_\Omega \rho u \cdot \varphi_i \, dx - \int_\Omega \bar{\rho}\bar{u} \cdot \varphi_i \, dx = \int_\Omega (\rho - \bar{\rho})(u - \bar{u}) \cdot \varphi_i \, dx$$
$$+ \int_\Omega (\rho - \bar{\rho})\bar{u} \cdot \varphi_i \, dx + \int_\Omega \bar{\rho}(u - \bar{u}) \cdot \varphi_i \, dx.$$

Therefore, after plugging in the decomposition of $u - \bar{u}$, we have the following system of equations ($i = 1, 2, 3$),

$$\sum_{j=1}^{3} b^j \int_\Omega \bar{\rho}\varphi_j \cdot \varphi_i \, dx = -\int_\Omega \bar{\rho} v \cdot \varphi_i \, dx$$
$$- \int_\Omega (\rho - \bar{\rho})(u - \bar{u}) \cdot \varphi_i \, dx - \int_\Omega (\rho - \bar{\rho})\bar{u} \cdot \varphi_i \, dx.$$

Notice $\left(\int_\Omega \bar{\rho}\varphi_i \cdot \varphi_j \, dx\right)$ is a non-degenerate matrix. It follows

$$|b|^2 \lesssim \int_\Omega |\bar{\rho}|^2 |v|^2 \, dx + \int_\Omega |\rho - \bar{\rho}|^2 |u - \bar{u}|^2 \, dx + \int_\Omega |\bar{u}|^2 |\rho - \bar{\rho}|^2 \, dx.$$

By using the Poincaré inequality and (2.17),

$$\int_\Omega |\bar{\rho}|^2 |v|^2 \, dx \lesssim \int_\Omega \left| v - \frac{1}{|\Omega|} \int_\Omega v \, dx \right|^2 dx + \int_\Omega \left| \frac{1}{|\Omega|} \int_\Omega v \, dx \right|^2 dx$$
$$\lesssim \int_\Omega |\nabla v|^2 \, dx + \left(\int_\Omega v \, dx \right)^2 \lesssim \int_\Omega \left| \nabla u + \nabla u^\top \right|^2 dx + \left(\int_\Omega u \, dx \right)^2.$$

(2.16) follows by chaining the above inequalities. In addition, the identity

$$\int_\Omega u \, dx = \int_\Omega (u - P_S u) \, dx$$

yields

$$\left(\int_\Omega u \, dx \right)^2 \lesssim \|u - P_S u\|_{L^2(\Omega)}^2 \lesssim \int_\Omega \left| \nabla u + \nabla u^\top \right|^2 dx$$

by applying (2.10). □

Similarly, we have the following

Lemma 5 *Under the same assumptions as in Lemma 4,*

$$\begin{aligned}\int_\Omega |\nabla\partial_t u|^2\,dx &\lesssim \int_\Omega \left|\nabla\partial_t u + \nabla\partial_t u^\top\right|^2\,dx + \int_\Omega |\partial_t\rho|^2|u-\bar{u}|^2\,dx \\ &+ \int_\Omega |\rho-\bar{\rho}|^2|\partial_t u|^2\,dx + \int_\Omega |\bar{u}|^2|\partial_t\rho|^2\,dx + \left(\int_\Omega \partial_t u\,dt\right)^2.\end{aligned} \tag{2.18}$$

Generally, for any integer $k \geq 1$,

$$\begin{aligned}\int_\Omega \left|\nabla\partial_t^k u\right|^2\,dx &\lesssim \int_\Omega \left|\nabla\partial_t^k u + \nabla\partial_t^k u^T\right|^2\,dx + \int_\Omega |\bar{u}|^2|\partial_t^k\rho|^2\,dx \\ &+ \sum_{j=0}^{k}\int_\Omega |\partial_t^j(\rho-\bar{\rho})|^2|\partial_t^{k-j}(u-\bar{u})|^2\,dx + \left(\int_\Omega \partial_t^k u\,dx\right)^2.\end{aligned} \tag{2.19}$$

In particular,the last terms on the right of (2.18) *and* (2.19) *can be dropped.*

The proofs of (2.18), (2.19) are similar to that of (2.16).

2.3 Spherical Differential Frame

Here, we introduce the decomposition of differential operators $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ near the boundary Γ into two kinds of operators, corresponding to tangential and normal derivatives respectively.

To begin with, define the following cut-off function,

$$\psi : C_c^\infty(B_{3/4}) \longmapsto [0,1]$$

satisfying $|\nabla\psi(x)| \leq 8$ and

$$\psi(x)\begin{cases} = 1 & |x| \leq 1/2, \\ = 0 & |x| \geq 3/4. \end{cases} \tag{2.20}$$

Our differential operators are defined as

$$\begin{aligned}&\nabla_T = \varphi_i(x)\cdot\nabla, \text{ for } i = 1,2,3, \\ &\nabla_N = N(x)\cdot\nabla,\end{aligned} \tag{2.21}$$

where φ_i is defined in (2.3) with $\bar{x} = 0$, $N(x) = (x_1, x_2, x_3)^\top$. In the following, we study the properties of these differential operators. First, the commutator of two differential operators is also a differential operator. More precisely,

Lemma 6 (Commutator) *For any function $f : \mathbb{R}^3 \longmapsto \mathbb{R}$ and any vector fields $\alpha = (\alpha^1, \alpha^2, \alpha^3)^\top, \beta = (\beta^1, \beta^2, \beta^3)^\top$,*

$$[\nabla_\alpha, \nabla_\beta] f(x) = \phi_{\alpha,\beta} \cdot \nabla f(x), \tag{2.22}$$

for some $\phi_{\alpha,\beta}$ satisfying

$$\|\phi_{\alpha,\beta}\|_{L^\infty(\Omega)} \lesssim \|\beta\|_{L^\infty(\Omega)} \|\nabla \alpha\|_{L^\infty(\Omega)} + \|\alpha\|_{L^\infty(\Omega)} \|\nabla \beta\|_{L^\infty(\Omega)}.$$

Moreover, $\phi_{\alpha,\beta} \in C^\infty$ provided $\alpha, \beta \in C^\infty$.

Proof For $i, j \in \{1, 2, 3\}$, direct calculation yields

$$\left[\alpha^i \partial_i, \beta^j \partial_j\right] f = \alpha^i \partial_i (\beta^j \partial_j f) - \beta^j \partial_j (\alpha^i \partial_i f) = \alpha^i \partial_i \beta^j \cdot \partial_j f - \beta^j \partial_j \alpha^i \cdot \partial_i f.$$

This finishes the proof. □

In the meantime, the following lemma shows that ∇ can be indeed decomposed into ∇_T and ∇_N in the boundary subdomain, in the sense that the estimates of L^2 norms of $\nabla_T f, \nabla_N f$ would be sufficient to obtain the corresponding estimate of ∇f.

Lemma 7 *For any smooth function $f : \mathbb{R}^3 \longmapsto \mathbb{R}$,*

$$\|\nabla f\|_{L^2(\Omega \backslash B_{1/2})} \lesssim \|\nabla_T f\|_{L^2(\Omega \backslash B_{1/2})} + \|\nabla_N f\|_{L^2(\Omega \backslash B_{1/2})}, \tag{2.23}$$

$$\|\nabla f\|_{L^2(\Omega)} \lesssim \|\nabla(\psi f)\|_{L^2(\Omega)} + \|\nabla_T f\|_{L^2(\Omega)} + \|\nabla_N f\|_{L^2(\Omega)} \lesssim \|f\|_{H^1(\Omega)}, \tag{2.24}$$

where $\|\nabla(\psi f)\|_{L^2(\Omega)}$ can be replaced by $\|\nabla f\|_{L^2(B_{1/2})}$. Here and in the following, $\|\nabla_T f\|_{(\cdot)}$ stands for the sum of

$$\|\nabla_{\varphi_1} f\|_{(\cdot)}, \|\nabla_{\varphi_2} f\|_{(\cdot)}, \|\nabla_{\varphi_3} f\|_{(\cdot)}.$$

We adopt this convention through the rest of this work.

Proof First we separate Ω into the interior and boundary subdomains, $\Omega = B_{1/2} \cup \left(\Omega \backslash B_{1/2}\right)$. Then

$$\|\nabla f\|^2_{L^2(\Omega)} = \|\nabla f\|^2_{L^2(B_{1/2})} + \|\nabla f\|^2_{L^2(\Omega \backslash B_{1/2})}.$$

Then (2.24) follows from (2.23) and the property of ψ in (2.20). Thus it remains to show (2.23). It is sufficient to show that in $\Omega \backslash B_{1/2}$, the rank of $\{\varphi_1, \varphi_2, \varphi_3, N\}$ is equal to three. Notice, in the boundary subdomain, $x_1^2 + x_2^2 + x_3^2 \geq 1/4$ and thus at least one of $|x_1|, |x_2|, |x_3|$ is no less than $1/4$. Without loss of generality, we only consider the case when $x_1 \geq 1/4$. Direct calculation yields

$$\det(\varphi_2, \varphi_3, N) = \begin{vmatrix} -x_3 & x_2 & x_1 \\ 0 & -x_1 & x_2 \\ x_1 & 0 & x_3 \end{vmatrix} = x_1 \left(x_1^2 + x_2^2 + x_3^2 \right) \geq (1/4)^2 > 0.$$

This finishes the proof. □

For higher order derivatives, it also admits the following lemma.

Lemma 8

$$\begin{aligned} \left\| \nabla^2 f \right\|_{L^2(\Omega)} \lesssim & \left\| \nabla^2 (\psi f) \right\|_{L^2(\Omega)} + \left\| \nabla_T^2 f \right\|_{L^2(\Omega)} + \| \nabla_T \nabla_N f \|_{L^2(\Omega)} \\ & + \left\| \nabla_N^2 f \right\|_{L^2(\Omega)} + \| \nabla f \|_{L^2(\Omega)} , \end{aligned} \tag{2.25}$$

$$\begin{aligned} \left\| \nabla^k f \right\|_{L^2(\Omega)} \lesssim & \left\| \nabla^k (\psi f) \right\|_{L^2(\Omega)} + \sum_{i+j=k} \left\| \nabla_T^i \nabla_N^j f \right\|_{L^2(\Omega)} \\ & + \sum_{i \leq k-1} \left\| \nabla^i f \right\|_{L^2(\Omega)} , \end{aligned} \tag{2.26}$$

where $\|\nabla^2 (\psi f)\|_{L^2(\Omega)}$, $\left\| \nabla^k (\psi f) \right\|_{L^2(\Omega)}$ *can be replaced by* $\|\nabla^2 f\|_{L^2(B_{1/2})}$ *and* $\left\| \nabla^k f \right\|_{L^2(B_{1/2})}$ *respectively. In these inequalities,* ∇_N *and* ∇_T *can be interchanged.*

Proof We show (2.25) only. (2.26) can be proved via a similar argument. Notice,

$$\left\| \nabla^2 f \right\|_{L^2} = \left\| \nabla^2 f \right\|_{L^2(B_{1/2})} + \left\| \nabla^2 f \right\|_{L^2(\Omega \backslash B_{1/2})} .$$

Then, apply (2.23) repeatly,

$$\begin{aligned} \left\| \nabla^2 f \right\|_{L^2(\Omega \backslash B_{1/2})} \lesssim & \| \nabla_T \nabla f \|_{L^2(\Omega \backslash B_{1/2})} + \| \nabla_N \nabla f \|_{L^2(\Omega \backslash B_{1/2})} \\ \lesssim & \| \nabla \nabla_T f \|_{L^2(\Omega \backslash B_{1/2})} + \| \nabla \nabla_N f \|_{L^2(\Omega \backslash B_{1/2})} + \| \nabla f \|_{L^2(\Omega \backslash B_{1/2})} \\ \lesssim & \| \nabla_T \nabla_T f \|_{L^2(\Omega \backslash B_{1/2})} + \| \nabla_N \nabla_T f \|_{L^2(\Omega \backslash B_{1/2})} \\ & + \| \nabla_T \nabla_N f \|_{L^2(\Omega \backslash B_{1/2})} + \| \nabla_N \nabla_N f \|_{L^2(\Omega \backslash B_{1/2})} + \| \nabla f \|_{L^2(\Omega \backslash B_{1/2})} \end{aligned}$$

where in the second inequality (2.22) has been applied. Again, after applying (2.22) again, it holds

$$\|\nabla_N \nabla_T f\|_{L^2(\Omega\setminus B_{1/2})} \lesssim \|\nabla_T \nabla_N f\|_{L^2(\Omega\setminus B_{1/2})} + \|\nabla f\|_{L^2(\Omega\setminus B_{1/2})} .$$

Thus this finishes the proof of (2.25). $\square$

The next lemma concerns the calculus on the boundary Γ. In fact, $\nabla_T|_\Gamma$ is a differential operator on the boundary.

Lemma 9 *For any smooth function $g : \Gamma \longmapsto \mathbb{R}$, it holds,*

$$\int_\Gamma \nabla_T g \, dS = 0. \tag{2.27}$$

As a corollary, for $g_1, g_2 : \Gamma \longmapsto \mathbb{R}$,

$$\int_\Gamma \nabla_T g_1 \cdot g_2 \, dS = -\int_\Gamma g_1 \cdot \nabla_T g_2 \, dS. \tag{2.28}$$

Similarly,

$$\int_\Omega \nabla_T f \, dx = 0, \tag{2.29}$$

$$\int_\Omega \nabla_T f_1 \cdot f_2 \, dx = -\int_\Omega f_1 \cdot \nabla_T f_2 \, dx. \tag{2.30}$$

where $f, f_1, f_2 : \Omega \longmapsto \mathbb{R}$.

Proof Without lost of generality, we show the lemma for $\nabla_T = \varphi_1 \cdot \nabla = x_3 \partial_{x_2} - x_2 \partial_{x_3}$. To do so, we introduce the following coordinate representation of Γ,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r \\ \sqrt{1-r^2}\cos\theta \\ \sqrt{1-r^2}\sin\theta \end{pmatrix} \in \Gamma,$$

with $\theta \in [0, 2\pi)$, $-1 \le r \le 1$. Then it holds

$$\nabla_T = -\partial_\theta, \text{ and } \det(D_{r,\theta} X^\top D_{r,\theta} X) = 1,$$

Therefore,

$$\int_\Gamma \nabla_T g \, dS = \int_{-1}^1 \int_0^{2\pi} (-\partial_\theta g) \sqrt{\det(D_{r,\theta} X^\top D_{r,\theta} X)} \, d\theta \, dr = 0.$$

This finishes the proof. $\square$

2.4 Embedding Theories and the Stokes Problem

We start with the trace theorem. The following is from [1].

Lemma 10 (Trace Theory) *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary Γ. Suppose that $kp < 3$ and $p \leq q \leq p^* = 2p/(3-kp)$. The trace operator $Tr : W^{k,p}(\Omega) \to L^q(\Gamma)$ is bounded. Moreover, for $u \in W^{k,p}(\Omega)$,*

$$\|u\|_{L^q(\Gamma)} \lesssim \|u\|_{W^{k,p}(\Omega)} .$$

If $kp = 3$, then the above relation holds for $p \leq q < \infty$. In particular,

$$\|u\|_{L^2(\Gamma)} \lesssim \|u\|_{H^{1/2}(\Omega)} \lesssim \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2} . \tag{2.31}$$

Proof See [1, Theorem 5.36] and [4]. □

Meanwhile, we shall record the following form of trace theorem.

Lemma 11 (Trace Theory: Fractional Sobolev space) *For $n \geq 1$, and $u \in H^n(\Omega)$, it holds,*

$$\|u\|_{H^{n-1/2}(\Gamma)} \lesssim \sum_{j=0}^{n-1} \left\| \nabla_T^j u \right\|_{H^1(\Omega)} . \tag{2.32}$$

Proof We shall apply the following fact about the trace operator (see [9, Theorem 1]): the trace operator $Tr : H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ is bounded for $1/2 < s < 3/2$. Then (2.32) is a consequence of the fact the rank of $\{\varphi_1, \varphi_2, \varphi_3\}$ is equal to two and therefore

$$\|u\|_{H^{n-1/2}(\Gamma)} \lesssim \sum_{j=0}^{n-1} \left\| \nabla_T^j u \right\|_{H^{1/2}(\Gamma)} .$$

□

In the meantime, we shall record the classical regularity theory for the Stokes problem. Consider the Stokes problem,

$$\begin{cases} -\mu\Delta u + \nabla P = f & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \\ u = h & \text{on } \Gamma. \end{cases} \tag{2.33}$$

Ω is a domain with a smooth boundary Γ. The following is from [30, 19],

Lemma 12 *Let $n \geq 2$. If $f \in H^{n-2}(\Omega), g \in H^{n-1}(\Omega), h \in H^{n-1/2}(\Gamma)$ be given such that*

$$\int_\Omega g\,dx = \int_\Gamma h \cdot \vec{n}\,dS,$$

then there exists unique $u \in H^n(\Omega), P \in H^{n-1}(\Omega)$(up to constants) solving (2.33). *Moreover,*

$$\|u\|_{H^n(\Omega)} + \|\nabla P\|_{H^{n-2}(\Omega)} \lesssim \|f\|_{H^{n-2}(\Omega)} + \|g\|_{H^{n-1}(\Omega)} + \|h\|_{H^{n-1/2}(\Gamma)}. \quad (2.34)$$

Proof See [20, 29]. □

2.5 Perturbed Formulation and Poincaré Inequality

We aim to study the stability of (1.11). It is assumed the initial data (ρ_0, u_0) in (1.3) satisfies,

$$\int_\Omega \rho_0\,dx = \int_\Omega \bar\rho\,dx, \int_\Omega \rho_0 u_0 \cdot \varphi_i\,dx = \int_\Omega \bar\rho\bar u \cdot \varphi_i\,dx \quad i = 1, 2, 3, \quad (2.35)$$

where $(\bar\rho, \bar u)$ is given in (1.11) with some $\bar\omega > 0$. In particular, it is assumed $\bar\rho(0) > 0$ so that $\bar\rho$ admits uniform lower and upper bounds. Moreover, $\left|\nabla^k \bar u\right| \lesssim \bar\omega$ for any $k \geq 0$.

Now it is time to introduce the perturbed formulation of (1.1) around the steady solution (1.11). Define $q := \rho - \bar\rho, v := u - \bar u$. Also, write

$$\rho^\gamma - \bar\rho^\gamma = \gamma\bar\rho^{\gamma-1}(\rho - \bar\rho) + \gamma(\gamma - 1)\int_{\bar\rho}^{\rho} (\rho - y)y^{\gamma-2}\,dy := \gamma\bar\rho^{\gamma-1}q + \mathcal{R}. \quad (2.36)$$

Notice, from the definition of $\mathcal{R}$,

$$\begin{aligned}
\nabla\mathcal{R} &= \gamma(\rho^{\gamma-1} - \bar\rho^{\gamma-1})\nabla\rho - \gamma(\gamma-1)\bar\rho^{\gamma-2}(\rho - \bar\rho)\nabla\bar\rho \\
&= \gamma((\gamma-1)\bar\rho^{\gamma-2}q + \mathcal{O}(2))(\nabla q + \nabla\bar\rho) - \gamma(\gamma-1)\bar\rho^{\gamma-2}q\nabla\bar\rho \\
&= \gamma(\gamma-1)\bar\rho^{\gamma-2}q\nabla q + (\nabla q + \nabla\bar\rho)\mathcal{O}(2),
\end{aligned}$$

is at least quadratic in $q, \nabla q$. From (1.1) and (1.6), we easily derive the system of the perturbation variables (q, v),

$$\begin{cases}
\partial_t q + \operatorname{div}(\bar\rho v) + (v + \bar u)\cdot\nabla q = G_1 & \text{in } \Omega, \\
(\bar\rho + q)\partial_t v + \bar\rho v \cdot \nabla v + \gamma\bar\rho\nabla(\bar\rho^{\gamma-2}q) - \operatorname{div}\mathbb{S}(v) = F_2 + G_2 & \text{in } \Omega, \\
v\cdot\vec{n},\ \vec{\tau}\cdot\mathbb{S}(v)\vec{n} = 0 & \text{on } \Gamma,
\end{cases} \quad (2.37)$$

where

$$
\begin{aligned}
&G_1 = -q\text{div}\, v, \\
&G_2 = -\nabla\mathcal{R} - \left(q\bar{u}\cdot\nabla v + qv\cdot\nabla\bar{u} + qv\cdot\nabla v\right), \\
&F_2 = -\frac{\gamma}{\gamma-1}q\nabla\bar{\rho}^{\gamma-1} - \left(\bar{\rho}\bar{u}\cdot\nabla v + \bar{\rho}v\cdot\nabla\bar{u} + q\bar{u}\cdot\nabla\bar{u}\right) \\
&\quad\;\, = -\bar{\rho}\bar{u}\cdot\nabla v - \bar{\rho}v\cdot\nabla\bar{u}.
\end{aligned}
\tag{2.38}
$$

It should be noticed in the definition of F_2, it has been used the fact

$$
\bar{u}\cdot\nabla\bar{u} + \frac{\gamma}{\gamma-1}\nabla\bar{\rho}^{\gamma-1} = 0.
$$

In addition, we have the following Poincaré inequality for q and v.

Lemma 13 (Poincaré Inequality for q, v) *For $i, j \geq 0$,*

$$
\int_\Omega \left|\nabla_T^i\partial_t^j q\right|^2 dx \lesssim \int_\Omega \left|\nabla\nabla_T^i\partial_t^j q\right|^2 dx, \tag{2.39}
$$

$$
\int_\Omega \left|\nabla_T^i\partial_t^j v\right|^2 dx \lesssim \int_\Omega \left|\nabla\nabla_T^i\partial_t^j v\right|^2 dx. \tag{2.40}
$$

Proof For $i = 0$, it is a direct consequence of (2.35) that

$$
\int_\Omega \partial_t^j q\, dx = 0.
$$

For $i \geq 1$, from (2.29),

$$
\int_\Omega \nabla_T^i\partial_t^j q\, dx = \int_\Omega \nabla_T^i\partial_t^j v\, dx = 0.
$$

Then (2.39), (2.40) follow from the standard Poincaré inequality as well as (2.11). □

In terms of (q, v), our stability theorem can be stated as follows.

Theorem 2.1 (Main Theorem in Perturbed Variables) *The steady rigid motion* (1.11) *with* $|\bar{\omega}| < \epsilon_0$ *for some* $\epsilon_0 > 0$ *(given in Lemma 28) is nonlinearly stable.*

In particular, there is a constants $\epsilon_1 > 0$ *(given in Lemma 29) such that the classical solution* (q, v) *to* (2.37) *exists globally with the given initial* *data (q_0, v_0) satisfying (2.35) and $\bar{\mathfrak{E}}_L(0), \bar{\mathcal{E}}_L(0) < \epsilon_1$ (defined in (4.16) and (4.20)) for $L \geq 3$. Moreover, the following inequalities hold for the energy functionals,*

$$e^{\sigma t}\bar{\mathcal{E}}_L(t) \leq \epsilon_0, \; e^{\sigma t}\bar{\mathfrak{E}}_L(t) \leq \epsilon_1, \tag{2.41}$$

for some positive constant $\sigma > 0$. Consequently, the initial perturbation would decay to zero as time grows up and therefore $(\rho, u)(x,t) \to (\bar{\rho}, \bar{u})(x)$ as $t \to +\infty$.

Remark Roughly speaking, the energy functional $\bar{\mathcal{E}}_L$ consists of the $H^s(\Omega)$ norms of $\partial_t^l q, \partial_t^l v$ for some $s, l \geq 0$. On the other hand, the functional $\bar{\mathfrak{E}}_L$ consists of the anisotropic $H^s(\Omega)$ norms. In particular, $\bar{\mathfrak{E}}_L$ mainly includes $L^2(\Omega)$ norms of the tangential and interior derivatives. However, the relation $\bar{\mathfrak{E}}_L \leq \bar{\mathcal{E}}_L$ does not hold as one can easily check.

3 Energy Estimates

To investigate the stability theory, we shall build some blocks in order to describe the propagation of the initial regularities. Thought out this section, the nonnegative integers l, m, n are not specified, and would be addressed with appropriate values in the next section.

3.1 On Temporal and Lower Order Spatial Derivatives

To begin with, it is shown the estimate of the temporal derivatives of (q, v) in this section. Moreover, we present the estimates of the first tangential derivatives and the estimates of normal derivatives through a Stokes problem in terms of the differential frames introduced in Section 2.3.

Applying the temporal derivative ∂_t^l for $l = 0, 1, \cdots$ to (2.37), it results in the following system,

$$\begin{cases} \partial_t^{l+1} q + \operatorname{div}(\bar{\rho}\partial_t^l v) + (v + \bar{u}) \cdot \nabla \partial_t^l q = G_1^l & \text{in } \Omega, \\ (\bar{\rho} + q)\partial_t^{l+1} v + \bar{\rho} v \cdot \nabla \partial_t^l v + \gamma \bar{\rho} \nabla (\bar{\rho}^{\gamma-2} \partial_t^l q) - \operatorname{div} \mathbb{S}(\partial_t^l v) = F_2^l + G_2^l & \text{in } \Omega, \\ \partial_t^l v \cdot \vec{n}, \; \vec{\tau} \cdot \mathbb{S}(\partial_t^l v)\vec{n} = 0 & \text{on } \Gamma, \end{cases} \tag{3.1}$$

where by using the Leibniz's rule,

$$
\begin{aligned}
G_1^l &= \partial_t^l G_1 - \sum_{j=0}^{l-1} C_{j,l} \partial_t^{l-j} v \cdot \nabla \partial_t^j q, \\
G_2^l &= \partial_t^l G_2 - \sum_{j=0}^{l-1} C_{j+1,l} \partial_t^{j+1} q \partial_t^{l-j} v - \bar{\rho} \sum_{j=0}^{l-1} C_{j+1,l} \partial_t^{j+1} v \cdot \nabla \partial_t^{l-j-1} v, \\
F_2^l &= \partial_t^l F_2.
\end{aligned}
\tag{3.2}
$$

We record the following energy identity.

Lemma 14 *For any smooth solution $(\partial_t^l q, \partial_t^l v)$ to (3.1), it holds the following energy identities, for any integer $l \geq 0$,*

$$
\begin{aligned}
&\frac{d}{dt} \left\{ \frac{1}{2} \int_\Omega (\bar{\rho} + q) \left| \partial_t^l v \right|^2 \, dx + \frac{\gamma}{2} \int_\Omega \bar{\rho}^{\gamma-2} \left| \partial_t^l q \right|^2 \, dx \right\} \\
&\quad + \int_\Omega \left(\frac{\mu}{2} \left| \nabla \partial_t^l v + \nabla \partial_t^l v^\top \right|^2 + \lambda \left| \operatorname{div} \partial_t^l v \right|^2 \right) dx \\
&= \frac{\gamma}{2} \int_\Omega \left| \partial_t^l q \right|^2 (v + \bar{u}) \cdot \nabla \bar{\rho}^{\gamma-2} \, dx + \frac{\gamma}{2} \int_\Omega \bar{\rho}^{\gamma-2} \left| \partial_t^l q \right|^2 \operatorname{div} (v + \bar{u}) \, dx \\
&\quad + \gamma \int_\Omega \bar{\rho}^{\gamma-2} \partial_t^l q G_1^l \, dx + \frac{1}{2} \int_\Omega (\partial_t q + \operatorname{div} (\bar{\rho} v)) \left| \partial_t^l v \right|^2 \, dx \\
&\quad + \int_\Omega (F_2^l + G_2^l) \cdot \partial_t^l v \, dx.
\end{aligned}
\tag{3.3}
$$

Proof Take inner product of $(3.1)_2$ with $\partial_t^l v$ and then record the resulting after integration by parts,

$$
\begin{aligned}
&\frac{d}{dt} \frac{1}{2} \int_\Omega (\bar{\rho} + q) \left| \partial_t^l v \right|^2 \, dx - \gamma \int_\Omega \operatorname{div} (\bar{\rho} \partial_t^l v) \bar{\rho}^{\gamma-2} \partial_t^l q \, dx \\
&\quad + \int_\Omega \left(\frac{\mu}{2} \left| \nabla \partial_t^l v + \nabla \partial_t^l v^\top \right|^2 + \lambda \left| \operatorname{div} \partial_t^l v \right|^2 \right) dx \\
&\quad + \gamma \int_\Gamma \bar{\rho}^{\gamma-1} \partial_t^l q \partial_t^l v \cdot \vec{n} \, dS + \frac{1}{2} \int_\Gamma \bar{\rho} \left| \partial_t^l v \right|^2 v \cdot \vec{n} \, dS - \int_\Gamma \partial_t^l v \cdot \mathbb{S}(\partial_t^l v) \vec{n} \, dS \\
&= \frac{1}{2} \int_\Omega \partial_t q \left| \partial_t^l v \right|^2 \, dx + \frac{1}{2} \int_\Omega \operatorname{div} (\bar{\rho} v) \left| \partial_t^l v \right|^2 \, dx + \int_\Omega (F_2^l + G_2^l) \cdot \partial_t^l v \, dx.
\end{aligned}
$$

(3.4)

Similar as before, from $(3.1)_3$, $\partial_t^k v$ is in the tangential direction of Γ. Therefore, from $(3.1)_3$ and $(2.37)_3$, the boundary integrals in (3.4) vanish. Meanwhile, using $(3.1)_1$, it holds

$$
\begin{aligned}
&-\gamma\int_\Omega \operatorname{div}(\bar\rho\partial_t^l v)\bar\rho^{\gamma-2}\partial_t^l q\,dx = \gamma\int_\Omega \left(\partial_t^{l+1}q + (v+\bar u)\cdot\nabla\partial_t^l q - G_1^l\right)\bar\rho^{\gamma-2}\partial_t^l q\,dx\\
&= \frac{d}{dt}\frac{\gamma}{2}\int_\Omega \bar\rho^{\gamma-2}\left|\partial_t^l q\right|^2 dx - \gamma\int_\Omega \bar\rho^{\gamma-2}\partial_t^l q G_1^l\,dx - \frac{\gamma}{2}\int_\Omega \left|\partial_t^l q\right|^2 (v+\bar u)\cdot\nabla\bar\rho^{\gamma-2}\,dx\\
&\quad - \frac{\gamma}{2}\int_\Omega \bar\rho^{\gamma-2}\left|\partial_t^l q\right|^2 \operatorname{div}(v+\bar u)\,dx,
\end{aligned}
$$

where we have applied integration by parts and the boundary condition $(2.37)_3$. Thus (3.3) follows after chaining the above identity. □

Lemma 15 *For any smooth solution $(\partial_t^l q, \partial_t^l v)$ to (3.1), any $0<\omega<1$, the following estimate holds,*

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac12\int_\Omega (\bar\rho+q)\left|\partial_t^l v\right|^2 dx + \frac{\gamma}{2}\int_\Omega \bar\rho^{\gamma-2}\left|\partial_t^l q\right|^2 dx\right\} + \frac{\mu}{2}\int_\Omega \left|\nabla\partial_t^l v\right|^2 dx\\
&\quad + \lambda\int_\Omega \left|\operatorname{div}\partial_t^l v\right|^2 dx \lesssim (\omega + a_{l,1})\int_\Omega \left|\nabla\partial_t^l q\right|^2 dx + a_{l,2}\int_\Omega \left|\nabla\partial_t^l v\right|^2 dx\\
&\quad + \sum_{j=0}^{[l/2]} b_{l,j}\left(\int_\Omega \left|\nabla\partial_t^{l-1-j}q\right|^2 dx + \int_\Omega \left|\nabla\partial_t^{l-j}v\right|^2 dx\right) \qquad (3.5)\\
&\quad + C_\omega\int_\Omega \left|G_1^l\right|^2 dx + \int_\Omega \left|G_2^l\right|^2 dx,
\end{aligned}
$$

where, for some integer $a>0$,

$$
\begin{aligned}
a_{l,1} &= \bar\omega^a + \|v\|^2_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)},\\
a_{l,2} &= \bar\omega^a + \|\partial_t q\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|v\|^2_{L^\infty(\Omega)},\\
b_{l,j} &= \left\|\partial_t^{j+1}v\right\|^2_{L^\infty(\Omega)} + \left\|\partial_t^j q\right\|^2_{L^\infty(\Omega)}, \quad 0\le j\le [l/2].
\end{aligned}
$$

Proof After chaining (2.19), (3.3) and applying Cauchy's inequality, it holds for any $0<\delta<1$ and $0<\omega<1$,

$$
\frac{d}{dt}\left\{\frac12\int_\Omega (\bar\rho+q)\left|\partial_t^l v\right|^2 dx + \frac{\gamma}{2}\int_\Omega \bar\rho^{\gamma-2}\left|\partial_t^l q\right|^2 dx\right\}
$$

$$
\begin{aligned}
&+\frac{\mu}{2}\int_\Omega \left|\nabla\partial_t^l v\right|^2\, dx+\lambda\int_\Omega \left|\operatorname{div}\partial_t^l v\right|^2\, dx\\
&\lesssim \left(\omega+\bar{\omega}^a+\|v\|^2_{L^\infty(\Omega)}+\|\nabla v\|_{L^\infty(\Omega)}\right)\int_\Omega \left|\partial_t^l q\right|^2\, dx\\
&+\left(\delta+\|\partial_t q\|_{L^\infty(\Omega)}+\|\nabla v\|_{L^\infty(\Omega)}+\bar{\omega}^a\|v\|_{L^\infty(\Omega)}\right)\int_\Omega \left|\partial_t^l v\right|^2\, dx\\
&+\sum_{j=0}^{l-1}\int_\Omega \left|\partial_t^j q\right|^2\left|\partial_t^{l-j} v\right|^2\, dx+C_\delta\int_\Omega \left|F_2^l\right|^2\, dx\\
&+C_\omega\int_\Omega \left|G_1^l\right|^2\, dx+C_\delta\int_\Omega \left|G_2^l\right|^2\, dx,
\end{aligned}
$$

where it has been used the fact $\|\bar{u}\|_{L^\infty(\Omega)}, \|\nabla\bar{u}\|_{L^\infty(\Omega)}, \|\nabla\bar{\rho}\|_{L^\infty(\Omega)} \lesssim \bar{\omega}^a$ for some integer $a>0$. Meanwhile

$$
\begin{aligned}
\sum_{j=0}^{l-1}\int_\Omega \left|\partial_t^j q\right|^2\left|\partial_t^{l-j} v\right|^2\, dx &\lesssim \sum_{j=0}^{[l/2]}\left(\left\|\partial_t^{j+1} v\right\|^2_{L^\infty(\Omega)}+\left\|\partial_t^j q\right\|^2_{L^\infty(\Omega)}\right)\\
&\times\left(\int_\Omega \left|\partial_t^{l-1-j} q\right|^2\, dx+\int_\Omega \left|\partial_t^{l-j} v\right|^2\, dx\right).
\end{aligned}
$$

On the other hand, from the definition of F_2^l,

$$
\int_\Omega \left|F_2^l\right|^2\, dx \lesssim \bar{\omega}^a\left(\int_\Omega \left|\nabla\partial_t^l v\right|^2\, dx+\int_\Omega \left|\partial_t^l v\right|^2\, dx\right).
$$

Then by chaining these estimates, together with (2.40), (2.39) and choosing an appropriately small $\delta>0$, (3.5) holds. □

Here and after, $a>0$ would denote an integer which might be different from line to line. Next lemma is concerning the estimate of the spatial derivative.

Lemma 16 *Under the same assumptions as in Lemma 14,*

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{\mu}{4}\int_\Omega \left|\nabla\partial_t^l v+\nabla\partial_t^l v^T\right|^2\, dx+\frac{\lambda}{2}\int_\Omega \left|\operatorname{div}\partial_t^l v\right|^2\, dx\right.\\
&\left.-\gamma\int_\Omega \bar{\rho}^{\gamma-2}\operatorname{div}(\bar{\rho}\partial_t^l v)\partial_t^l q\, dx\right\}+\int_\Omega (1+q)\left|\partial_t^{l+1} v\right|^2\, dx \lesssim (\bar{\omega}^a \qquad (3.6)\\
&+\|v\|^2_{L^\infty(\Omega)})\int_\Omega \left|\nabla\partial_t^l q\right|^2\, dx+(1+\bar{\omega}^a+\|v\|^2_{L^\infty(\Omega)})\int_\Omega \left|\nabla\partial_t^l v\right|^2\, dx
\end{aligned}
$$

$$+\int_\Omega \left|G_1^l\right|^2\,dx+\int_\Omega \left|G_2^l\right|^2\,dx.$$

Proof Take inner product of $(3.1)_2$ with $\partial_t^{l+1}v$ and then record the resulting after integration by parts. Similar arguments as in the previous lemma yield,

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{\mu}{4}\int_\Omega \left|\nabla\partial_t^l v+\nabla\partial_t^l v^\top\right|^2\,dx+\frac{\lambda}{2}\int_\Omega \left|\mathrm{div}\,\partial_t^l v\right|^2\,dx\right\}+\int_\Omega(\bar\rho+q)\left|\partial_t^{l+1}v\right|^2\,dx\\
&\quad\underbrace{-\gamma\int_\Omega \mathrm{div}\,(\bar\rho\partial_t^{l+1}v)\bar\rho^{\gamma-2}\partial_t^l q\,dx}_{(i)}=\underbrace{-\int_\Omega \bar\rho v\cdot\nabla\partial_t^l v\partial_t^{l+1}v\,dx}_{(ii)}\\
&\quad+\int_\Omega (F_2^l+G_2^l)\cdot\partial_t^{l+1}v\,dx.
\end{aligned}$$

Meanwhile, together with (2.39) and (2.40),

$$(i)=\frac{d}{dt}\left\{-\gamma\int_\Omega \bar\rho^{\gamma-2}\mathrm{div}\,(\bar\rho\partial_t^l v)\partial_t^l q\,dx\right\}+\underbrace{\gamma\int_\Omega \bar\rho^{\gamma-2}\mathrm{div}\,(\bar\rho\partial_t^l v)\partial_t^{l+1}q\,dx}_{(iii)},$$

$$\begin{aligned}
(iii)&\lesssim(1+\bar\omega^a)\int_\Omega\left|\nabla\partial_t^l v\right|^2\,dx+\int_\Omega\left|\partial_t^{l+1}q\right|^2\,dx\\
&\lesssim(1+\bar\omega^a)\int_\Omega\left|\nabla\partial_t^l v\right|^2\,dx+(\|v\|_{L^\infty(\Omega)}^2+\bar\omega^a)\int_\Omega\left|\nabla\partial_t^l q\right|^2\,dx+\int_\Omega\left|G_1^l\right|^2\,dx,\\
(ii)&\lesssim\delta\int_\Omega\left|\partial_t^{l+1}v\right|^2\,dx+C_\delta\|v\|_{L^\infty(\Omega)}^2\int_\Omega\left|\nabla\partial_t^l v\right|^2\,dx,
\end{aligned}$$

where it has been making use of the fact, from $(3.1)_1$ and (2.40)

$$\begin{aligned}
\int_\Omega\left|\partial_t^{l+1}q\right|^2\,dx&\lesssim(1+\bar\omega^a)\int_\Omega\left|\nabla\partial_t^l v\right|^2\,dx\\
&\quad+(\|v\|_{L^\infty(\Omega)}^2+\|\bar u\|_{L^\infty(\Omega)}^2)\int_\Omega\left|\nabla\partial_t^l q\right|^2\,dx+\int_\Omega\left|G_1^l\right|^2\,dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Omega F_2^l\cdot\partial_t^{l+1}v\,dx\lesssim\delta\int_\Omega\left|\partial_t^{l+1}v\right|^2\,dx+C_\delta\bar\omega^a\int_\Omega\left|\nabla\partial_t^l v\right|^2\,dx,\\
&\int_\Omega G_2^l\cdot\partial_t^{l+1}v\,dx\lesssim\delta\int_\Omega\left|\partial_t^{l+1}v\right|^2\,dx+C_\delta\int_\Omega\left|G_2^l\right|^2\,dx.
\end{aligned}$$

Thus chaining these estimates with an appropriately small $\delta > 0$ leads to (3.6). $\square$

As the consequence of (3.5) and (3.6), we have the following estimates on the temporal derivatives.

Proposition 1 (Temporal Derivatives) *Denote $\Lambda_l = \Lambda_l(\cdot)$ as a polynomial of the following quantities*

$$\bar{\omega}, \|v\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)}, \|\partial_t q\|_{L^\infty(\Omega)}, \sum_{j=0}^{[l/2]}(||\partial_t^{j+1} v||_{L^\infty(\Omega)} + ||\partial_t^j q||_{L^\infty(\Omega)}),$$

with the property $\Lambda_l(0) = 0$. Then it shall hold the following estimate. For any $0 < \omega < 1$,

$$\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega (\bar{\rho} + q) \left|\partial_t^l v\right|^2 dx + c\int_\Omega \left|\nabla\partial_t^l v + \nabla\partial_t^l v^T\right|^2 dx + c\int_\Omega \left|\operatorname{div}\partial_t^l v\right|^2 dx \\
&+ \int_\Omega \bar{\rho}^{\gamma-2}\left|\partial_t^l q\right|^2 dx - c\int_\Omega \bar{\rho}^{\gamma-2}\operatorname{div}(\bar{\rho}\partial_t^l v)\partial_t^l q\, dx\Big\} \\
&+ (1-c)\int_\Omega \left|\nabla\partial_t^l v\right|^2 dx + \int_\Omega \left|\operatorname{div}\partial_t^l v\right|^2 dx + c\int_\Omega (1+q)\left|\partial_t^{l+1} v\right|^2 dx \\
&\lesssim (\omega + \Lambda_l)\int_\Omega \left|\nabla\partial_t^l q\right|^2 dx + \Lambda_l\int_\Omega \left|\nabla\partial_t^l v\right|^2 dx + \Lambda_l\sum_{j=0}^{[l/2]}\Big(\int_\Omega \left|\nabla\partial_t^{l-1-j} q\right|^2 dx \\
&+ \int_\Omega \left|\nabla\partial_t^{l-j} v\right|^2 dx\Big) + (1 + C_\omega)\int_\Omega \left|G_1^l\right|^2 dx + \int_\Omega \left|G_2^l\right|^2 dx,
\end{aligned} \tag{3.7}$$

where $0 < c < 1$ is a positive constant such that

$$\begin{aligned}
&\int_\Omega \left|\partial_t^l v\right|^2 dx + \int_\Omega \left|\nabla\partial_t^l v\right|^2 dx + \int_\Omega \left|\partial_t^l q\right|^2 dx \\
&\lesssim \int_\Omega (\bar{\rho} + q)\left|\partial_t^l v\right|^2 dx + c\int_\Omega \left|\nabla\partial_t^l v + \nabla\partial_t^l v^T\right|^2 dx + c\int_\Omega |\operatorname{div} v|^2 dx \\
&+ \int_\Omega \bar{\rho}^{\gamma-2}\left|\partial_t^l q\right|^2 dx - c\int_\Omega \bar{\rho}^{\gamma-2}\operatorname{div}(\bar{\rho}\partial_t^l v)\partial_t^l q\, dx.
\end{aligned} \tag{3.8}$$

Proof This is a direct consequence of the linear combination $c\times(3.6)+(3.5)$. We only show the the choice of c can be justified. By applying the Cauchy's inequality and (2.1), (2.40),

$$
\begin{aligned}
&\int_\Omega \left|\nabla \partial_t^l v\right|^2 \, dx - \int_\Omega \left|\partial_t^l v\right|^2 \, dx - \int_\Omega \left|\partial_t^l q\right|^2 \, dx \\
&\lesssim \int_\Omega \left|\nabla \partial_t^l v + \nabla \partial_t^l v^T\right|^2 \, dx - \int_\Omega \bar{\rho}^{\gamma-2} \mathrm{div}\, (\bar{\rho} \partial_t^l v) \partial_t^l q \, dx.
\end{aligned}
$$

Together with the fact $\inf_{x\in\Omega} \bar{\rho} > 0$, (3.8) holds after choosing $c > 0$ sufficiently small. □

Notice, in Proposition 1, the estimate (3.7) contains the term $\int_\Omega \left|\nabla \partial_t^l q\right|^2 \, dx$ on the right hand side. To derive an estimate in a consistent form, it is desirable to perform the estimate on the spatial derivatives in the rest of this section.

Here we establish the estimates on the tangential derivatives. Starting with the first order tangential derivative, ∇_T(3.1) can be written as,

$$
\begin{cases}
\partial_t^{l+1} \nabla_T q + \mathrm{div}\, \left(\bar{\rho} \partial_t^l \nabla_T v\right) + (v + \bar{u}) \cdot \nabla \partial_t^l \nabla_T q = F_1^{l,1} + G_1^{l,1} & \text{in } \Omega, \\
(\bar{\rho} + q) \partial_t^{l+1} \nabla_T v + \bar{\rho} v \cdot \nabla \partial_t^l \nabla_T v + \gamma \bar{\rho} \nabla \left(\bar{\rho}^{\gamma-2} \partial_t^l \nabla_T q\right) & \\
\qquad - \mathrm{div}\, \mathbb{S}(\partial_t^l \nabla_T v) = F_2^{l,1} + \nabla_T G_2^l + G_2^{l,1} & \text{in } \Omega, \\
\partial_t^l \nabla_T v \cdot \vec{n} = -\partial_t^l v \cdot \nabla_T \vec{n}, \ \nabla_T \left(\vec{\tau} \cdot \mathbb{S}(\partial_t^l v) \vec{n}\right) = 0 & \text{on } \Gamma,
\end{cases}
\tag{3.9}
$$

with

$$
\begin{aligned}
G_1^{l,1} &= \nabla_T G_1^l - \nabla_T v \cdot \nabla \partial_t^l q - v \cdot \left[\nabla_T, \nabla\right] \partial_t^l q, \\
G_2^{l,1} &= -\nabla_T q \partial_t^{l+1} v - \bar{\rho} v \cdot \left[\nabla_T, \nabla\right] \partial_t^l v - \nabla_T(\bar{\rho} v) \cdot \nabla \partial_t^l v, \\
F_1^{l,1} &= -\left[\nabla_T, \mathrm{div}\,\right] (\bar{\rho} \partial_t^l v) - \mathrm{div}\, (\nabla_T \bar{\rho} \partial_t^l v) - \nabla_T \bar{u} \cdot \nabla \partial_t^l q - \bar{u} \cdot \left[\nabla_T, \nabla\right] \partial_t^l q, \\
F_2^{l,1} &= \nabla_T F_2^l - \nabla_T \bar{\rho} \partial_t^{l+1} v - \gamma \nabla_T \bar{\rho} \nabla (\bar{\rho}^{\gamma-2} \partial_t^l q) - \gamma \bar{\rho} \left[\nabla_T, \nabla\right] (\bar{\rho}^{\gamma-2} \partial_t^l q) \\
&\quad - \gamma \bar{\rho} \nabla (\nabla_T \bar{\rho}^{\gamma-2} \partial_t^l q) + \left[\nabla_T, \mathrm{div}\,\right] \mathbb{S}(\partial_t^l v) + \mathrm{div}\, (\left[\nabla_T, \mathbb{S}\right] (\partial_t^l v)).
\end{aligned}
\tag{3.10}
$$

Take inner product of $(3.9)_2$ with $\partial_t^l \nabla_T v$ and then record the resulting after integration by parts,

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Omega (\bar\rho+q)\left|\partial_t^l\nabla_T v\right|^2\,dx \underbrace{-\gamma\int_\Omega \mathrm{div}\,(\bar\rho\partial_t^l\nabla_T v)\bar\rho^{\gamma-2}\partial_t^l\nabla_T q\,dx}_{(i)}\\
&\quad+\int_\Omega\left(\frac{\mu}{2}\left|\nabla\partial_t^l\nabla_T v+\nabla\partial_t^l\nabla_T v^T\right|^2+\lambda\left|\mathrm{div}\,(\partial_t^l\nabla_T v)\right|^2\right)\,dx\\
&=\underbrace{\frac{1}{2}\int_\Omega \partial_t q\left|\partial_t^l\nabla_T v\right|^2\,dx+\frac{1}{2}\int_\Omega \mathrm{div}\,(\bar\rho v)\left|\partial_t^l\nabla_T v\right|^2\,dx}_{(ii)}\\
&\quad-\gamma\underbrace{\int_\Gamma \bar\rho^{\gamma-1}\partial_t^l\nabla_T q\partial_t^l\nabla_T v\cdot\vec n\,dS}_{(iii)}+\underbrace{\int_\Gamma \partial_t^l\nabla_T v\cdot\mathbb{S}(\partial_t^l\nabla_T v)\vec n\,dS}_{(iv)}\\
&\quad+\underbrace{\int_\Omega (F_2^{l,1}+G_2^{l,1})\cdot\partial_t^l\nabla_T v\,dx.}_{(v)}+\underbrace{\int_\Omega \nabla_T G_2^l\cdot\partial_t^l\nabla_T v\,dx,}_{(vi)}
\end{aligned}
\tag{3.11}
$$

where we have used the boundary condition $(2.37)_3$. Meanwhile, from $(3.9)_1$, $(2.37)_3$,

$$
\begin{aligned}
(i)&=\gamma\int_\Omega \bar\rho^{\gamma-2}\partial_t^l\nabla_T q\cdot(\partial_t^{l+1}\nabla_T q+(v+\bar u)\cdot\nabla\partial_t^l\nabla_T q-F_1^{l,1}-G_1^{l,1})\,dx\\
&=\frac{d}{dt}\frac{\gamma}{2}\int_\Omega \bar\rho^{\gamma-2}\left|\partial_t^l\nabla_T q\right|^2\,dx-\underbrace{\frac{\gamma}{2}\int_\Omega \mathrm{div}\,(\bar\rho^{\gamma-2}(v+\bar u))\left|\partial_t^l\nabla_T q\right|^2\,dx}_{(vii)}\\
&\quad-\underbrace{\gamma\int_\Omega (F_1^{l,1}+G_1^{l,1})\cdot\bar\rho^{\gamma-2}\partial_t^l\nabla_T q\,dx.}_{(viii)}
\end{aligned}
$$

Applying Cauchyś inequality and Poincaré inequality as follows,

$$
\begin{aligned}
&(vii)\lesssim\left(\bar\omega^a+\|v\|_{L^\infty(\Omega)}^2+\|\nabla v\|_{L^\infty(\Omega)}\right)\int_\Omega\left|\nabla_T\partial_t^l q\right|^2\,dx,\\
&(viii)\lesssim\omega\int_\Omega\left|\nabla_T\partial_t^l q\right|^2\,dx+C_\omega\left(\int_\Omega\left|F_1^{l,1}\right|^2\,dx+\int_\Omega\left|G_1^{l,1}\right|^2\,dx\right),\\
&(ii)\lesssim\left(\bar\omega^a+\|\partial_t q\|_{L^\infty(\Omega)}+\|v\|_{L^\infty(\Omega)}^2+\|\nabla v\|_{L^\infty(\Omega)}\right)\int_\Omega\left|\nabla_T\partial_t^l v\right|^2\,dx,
\end{aligned}
$$

$$(v) \lesssim \omega \int_\Omega \left|F_2^{l,1}\right|^2 \, dx + C_\omega \left(\int_\Omega \left|\nabla_T \partial_t^l v\right|^2 \, dx + \int_\Omega \left|G_2^{l,1}\right|^2 \, dx \right).$$

To estimate the boundary terms, from $(3.9)_3$, as consequences of the trace theorem (2.31), Hölder inequality and (2.28), (2.39), (2.40), it holds

$$\begin{aligned}
(iii) &= -\int_\Gamma \bar{\rho}^{\gamma-1} \partial_t^l \nabla_T q \partial_t^l v \cdot \nabla_T \vec{n} \, dS = \int_\Gamma \partial_t^l q \nabla_T (\bar{\rho}^{\gamma-1} \partial_t^l v \cdot \nabla_T \vec{n}) \, dS \\
&\lesssim \left\| \partial_t^l q \right\|_{L^2(\Gamma)} \left((1+\bar{\omega}) \left\| \partial_t^l v \right\|_{L^2(\Gamma)} + \left\| \nabla_T \partial_t^l v \right\|_{L^2(\Gamma)} \right) \\
&\lesssim \left\| \partial_t^l q \right\|_{L^2(\Omega)}^{1/2} \left\| \partial_t^l q \right\|_{H^1(\Omega)}^{1/2} \Big((1+\bar{\omega}) \left\| \partial_t^l v \right\|_{L^2(\Omega)}^{1/2} \left\| \partial_t^l v \right\|_{H^1(\Omega)}^{1/2} \\
&\qquad + \left\| \nabla_T \partial_t^l v \right\|_{L^2(\Omega)}^{1/2} \left\| \nabla_T \partial_t^l v \right\|_{H^1(\Omega)}^{1/2} \Big) \\
&\lesssim \omega \left(\int_\Omega \left|\nabla \partial_t^l q\right|^2 \, dx + \int_\Omega \left|\nabla \nabla_T \partial_t^l v\right|^2 \, dx \right) \\
&\qquad + C_\omega (1+\bar{\omega}^a) \int_\Omega \left|\nabla \partial_t^l v\right|^2 \, dx,
\end{aligned}$$

On the other hand, on Γ, we have the following identities from $(3.9)_3$

$$\begin{aligned}
0 &= \nabla_T (\vec{\tau} \cdot \mathbb{S}(\partial_t^l v) \vec{n}) = \vec{\tau} \cdot \mathbb{S}(\partial_t^l \nabla_T v) \vec{n} + \nabla_T \vec{\tau} \cdot \mathbb{S}(\partial_t^l v) \vec{n} \\
&\qquad + \vec{\tau} \cdot [\nabla_T, \mathbb{S}] (\partial_t^l v) \vec{n} + \vec{\tau} \cdot \mathbb{S}(\partial_t^l v) \nabla_T \vec{n}, \\
\partial_t^l \nabla_T v &= (\partial_t^l \nabla_T v \cdot \vec{n}) \vec{n} + v_{l,1} \vec{\tau} = -(\partial_t^l v \cdot \nabla_T \vec{n}) \vec{n} + v_{l,1} \vec{\tau},
\end{aligned} \tag{3.12}$$

with $|v_{l,1}| \lesssim \left|\partial_t^l \nabla_T v\right|$. Therefore, the calculus on the boundary (2.28) then yields

$$\begin{aligned}
(iv) &= -\int_\Gamma (\partial_t^l v \cdot \nabla_T \vec{n}) \vec{n} \cdot \mathbb{S}(\partial_t^l \nabla_T v) \vec{n} \, dS + \int_\Gamma v_{l,1} \vec{\tau} \cdot \mathbb{S}(\partial_t^l \nabla_T v) \vec{n} \, dS \\
&= -\int_\Gamma (\partial_t^l v \cdot \nabla_T \vec{n}) \vec{n} \cdot [\mathbb{S}, \nabla_T] (\partial_t^l v) \vec{n} \, dS + \int_\Gamma \nabla_T ((\partial_t^l v \cdot \nabla_T \vec{n}) \vec{n}) \cdot \mathbb{S}(\partial_t^l v) \vec{n} \, dS \\
&\quad + \int_\Gamma (\partial_t^l v \cdot \nabla_T \vec{n}) \vec{n} \cdot \mathbb{S}(\partial_t^l v) \nabla_T \vec{n} \, dS - \int_\Gamma v_{l,1} \nabla_T \vec{\tau} \cdot \mathbb{S}(\partial_t^l v) \vec{n} \, dS \\
&\quad - \int_\Gamma v_{l,1} \vec{\tau} \cdot [\nabla_T, \mathbb{S}] (\partial_t^l v) \vec{n} \, dS - \int_\Gamma v_{l,1} \vec{\tau} \cdot \mathbb{S}(\partial_t^l v) \nabla_T \vec{n} \, dS \\
&\lesssim \int_\Gamma \left(\left|\partial_t^l v\right| + \left|\partial_t^l \nabla_T v\right| \right) \cdot \left|\nabla \partial_t^l v\right| \, dS \lesssim \left\| \partial_t^l v \right\|_{L^2(\Gamma)}^2 + \left\| \nabla \partial_t^l v \right\|_{L^2(\Gamma)}^2 \\
&\quad + \left\| \nabla_T \partial_t^l v \right\|_{L^2(\Gamma)}^2 \lesssim \left\| \partial_t^l v \right\|_{L^2(\Omega)} \left\| \partial_t^l v \right\|_{H^1(\Omega)}
\end{aligned} \tag{3.13}$$

$$+\left\|\nabla\partial_t^l v\right\|_{L^2(\Omega)}\left\|\nabla\partial_t^l v\right\|_{H^1(\Omega)} \lesssim \omega\int_\Omega\left|\nabla^2\partial_t^l v\right|^2 dx + C_\omega\int_\Omega\left|\nabla\partial_t^l v\right|^2 dx.$$

Indeed, the above calculation indicates the next lemma.

Lemma 17 (Tangential Direction Estimate) *For any smooth solution* $(\nabla_T\partial_t^l q, \nabla_T\partial_t^l v)$ *to* (3.9)*, the following inequality holds,*

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega(\bar\rho+q)\left|\nabla_T\partial_t^l v\right|^2 dx + \frac{\gamma}{2}\int_\Omega\bar\rho^{\gamma-2}\left|\nabla_T\partial_t^l q\right|^2 dx\right\}\\
&\quad+\int_\Omega\left|\nabla\nabla_T\partial_t^l v\right|^2 dx + \int_\Omega\left|\operatorname{div}(\nabla_T\partial_t^l v)\right|^2 dx\\
&\lesssim(\omega+\Lambda_l)\int_\Omega\left|\nabla^2\partial_t^l v\right|^2 dx + (\omega+C_\omega\bar\omega^a+\Lambda_l)\int_\Omega\left|\nabla\partial_t^l q\right|^2 dx\\
&\quad+(1+\omega+C_\omega+\Lambda_l)\int_\Omega\left|\nabla\partial_t^l v\right|^2 dx + \omega\bar\omega^2\int_\Omega\left|\partial_t^{l+1}v\right|^2 dx\\
&\quad+C_\omega\left(\int_\Omega\left|G_1^{l,1}\right|^2 dx + \int_\Omega\left|G_2^{l,1}\right|^2 dx\right) + \int_\Omega\left|G_2^l\right|^2 dx,
\end{aligned}\tag{3.14}$$

for any $0<\omega<1$.

Proof By summing up the estimates between (3.11) and (3.13) , and (2.1), we only need to estimate $F_1^{l,1}, F_2^{l,1}$ and (vi). Direct calculations show

$$\begin{aligned}
&\int_\Omega\left|F_1^{l,1}\right|^2 dx \lesssim (1+\bar\omega^a)\int_\Omega\left|\nabla\partial_t^l v\right|^2 dx + \bar\omega^a\int_\Omega\left|\nabla\partial_t^l q\right|^2 dx,\\
&\int_\Omega\left|F_2^{l,1}\right|^2 dx \lesssim \int_\Omega\left|\nabla_T F_2^l\right|^2 dx + \bar\omega^a\int_\Omega\left|\partial_t^{l+1}v\right|^2 dx\\
&\qquad+(1+\bar\omega^a)\int_\Omega\left|\nabla\partial_t^l q\right|^2 dx + \int_\Omega\left|\nabla^2\partial_t^l v\right|^2 dx\\
&\quad\lesssim \bar\omega^a\int_\Omega\left|\nabla\partial_t^l v\right|^2 dx + \bar\omega^a\int_\Omega\left|\partial_t^{l+1}v\right|^2 dx\\
&\qquad+(1+\bar\omega^a)\int_\Omega\left|\nabla\partial_t^l q\right|^2 dx + (1+\bar\omega^a)\int_\Omega\left|\nabla^2\partial_t^l v\right|^2 dx,
\end{aligned}$$

where $a>0$ and we have applied (2.39), (2.40). Also, as the consequence of (2.30),

$$(vi) = -\int_\Omega G_2^l\cdot\nabla_T\partial_t^l\nabla_T v\,dx \lesssim \delta\int_\Omega\left|\nabla\nabla_T\partial_t^l v\right|^2 dx + C_\delta\int_\Omega\left|G_2^l\right|^2 dx.$$

Then by choosing an appropriately small $\delta>0$, (3.14) is proved. □

Remark The identities (3.12) on the boundary and the boundary condition $(3.9)_3$ should be understood as follows. τ stands for one of the tangential vector fields $\varphi_1, \varphi_2, \varphi_3$ (defined in (2.3)), which are smooth and defined globally on Γ. Therefore, $\nabla_T \tau$ is non-singular and smooth. Moreover, the rank of $\{\varphi_1, \varphi_2, \varphi_3\}$ is equal to two and hence any tangental vector on Γ can be represented by them. To show that $v_{l,1}\tau$ makes sense and $|v_{l,1}| \lesssim \left|\partial_t^l \nabla_T v\right|$, we adopt the following representation of tangential vector fields. Indeed, we claim that any tangential vector fields V on Γ can be denoted as

$$V = V_1\varphi_1 + V_2\varphi_2 + V_3\varphi_3, \tag{3.15}$$

with $|V_1|, |V_2|, |V_3|$ bounded by $|V|$. To show this is possible, consider a point $p = (x_1, x_2, x_3)$ on Γ. Without loss of generality, we assume $x_1 > 1/4$. Then for a neighbourhood W_p of p, $\{\varphi_2, \varphi_3\}$ forms a non-degenerate basis of the tangential space and $|\varphi_2|, |\varphi_3| > 1/4$. Then V can be written as

$$V = V_{1,p}\varphi_1 + V_{2,p}\varphi_2 + V_{3,p}\varphi_3$$

with $V_{1,p} = 0$ and $|V_{2,p}|, |V_{3,p}| \lesssim |V|$ inside W_p. Since Γ is compact, one can construct a finite cover $\{W_p\}$ of Γ and corresponding $\{p; V_{1,p}, V_{2,p}, V_{3,p}\}$. Then a partition of unity argument would yield (3.15). Notice, V_1, V_2, V_3 are not necessarily continuous.

Next, we show how to develop the estimates of the normal derivatives in the spherical differential frame. Just as it is classically done (see, for example, [19]), we shall derive the ordinary differential equation(ODE) satisfied by $\partial_t^l \nabla_N q$. In order to do so, taking inner product of $(3.1)_2$ with N yields,

$$\begin{aligned}(\bar\rho + q)\partial_t^{l+1} v \cdot N + \bar\rho v \cdot \nabla \partial_t^l v \cdot N - (F_2^l + G_2^l) \cdot N \\ = -\gamma\bar\rho\nabla\left(\bar\rho^{\gamma-2}\partial_t^l q\right) \cdot N + \operatorname{div}\mathbb{S}(\partial_t^l v) \cdot N.\end{aligned} \tag{3.16}$$

Notice, $N = (x_1, x_2, x_3)^\top$, and

$$\begin{aligned}\operatorname{div}\mathbb{S}(\partial_t^l v) \cdot N &= \sum_{1\le i,j\le 3} x_i\partial_j\left(\mu\left(\partial_i\partial_t^l v^j + \partial_j\partial_t^l v^i\right) + \lambda\delta_{ij}\operatorname{div}\partial_t^l v\right) \\ &= (\mu+\lambda)\nabla_N \operatorname{div}\partial_t^l v + \mu\sum_{1\le i,j\le 3} x_i\partial_j\partial_j\partial_t^l v^i = (2\mu+\lambda)\nabla_N\operatorname{div}\partial_t^l v \\ &\quad + \mu\sum_{1\le s,m,n\le 3}\varphi_s \cdot \nabla(\epsilon_{smn}\partial_m\partial_t^l v^n),\end{aligned} \tag{3.17}$$

where in the last equality we have substituted the following identity,

$$\begin{aligned}
&\sum_{1\le i,j\le 3} x_i\partial_j\partial_j\partial_t^l v^i = \sum_{1\le i,j,m,n\le 3} x_i(\delta_{jm}\delta_{in})\partial_m\partial_j\partial_t^l v^n \\
&= \sum_{1\le i,j,m,n\le 3} x_i(\delta_{jm}\delta_{in}-\delta_{jn}\delta_{im})\partial_m\partial_j\partial_t^l v^n + \sum_{1\le i,j,m,n\le 3} x_i(\delta_{jn}\delta_{im})\partial_m\partial_j\partial_t^l v^n \\
&= \sum_{1\le i,j,m,n,s\le 3} x_i\epsilon_{jis}\epsilon_{smn}\partial_m\partial_j\partial_t^l v^n + \sum_{1\le i,j\le 3} x_i\partial_i\partial_j\partial_t^l v^j \\
&= \sum_{1\le s,m,n\le 3} \varphi_s\cdot\nabla(\epsilon_{smn}\partial_m\partial_t^l v^n) + \nabla_N \mathrm{div}\,\partial_t^l v.
\end{aligned}$$

In the meantime, from $(3.1)_1$

$$\begin{aligned}
\bar\rho\nabla_N \mathrm{div}\,\partial_t^l v = \nabla_N G_1^l - \big(&\partial_t^{l+1}\nabla_N q + (v+\bar u)\cdot\nabla\partial_t^l\nabla_N q + \nabla_N(v+\bar u)\cdot\nabla\partial_t^l q \\
&+(v+\bar u)\cdot[\nabla_N,\nabla]\,\partial_t^l q + \nabla_N\partial_t^l v\cdot\nabla\bar\rho + \partial_t^l v\cdot\nabla_N\nabla\bar\rho + \nabla_N\bar\rho\mathrm{div}\,\partial_t^l v\big)\,.
\end{aligned} \tag{3.18}$$

Combining (3.16), (3.17), (3.18), we can derive the following equations

$$\begin{cases}
(2\mu+\lambda)\partial_t^{l+1}\nabla_N q + (2\mu+\lambda)(v+\bar u)\cdot\nabla\partial_t^l\nabla_N q + \gamma\bar\rho^2\nabla_N\left(\bar\rho^{\gamma-2}\partial_t^l q\right) \\
\qquad\qquad = F_{1,N}^l + G_{1,N}^l, \\
(2\mu+\lambda)\nabla_N \mathrm{div}\,\partial_t^l v = F_{2,N}^l + G_{2,N}^l,
\end{cases} \tag{3.19}$$

with

$$\begin{aligned}
F_{1,N}^l &= -(2\mu+\lambda)\left(\bar u\cdot[\nabla_N,\nabla]\,\partial_t^l q + \nabla_N\bar u\cdot\nabla\partial_t^l q\right. \\
&\quad \left.+\nabla_N\partial_t^l v\cdot\nabla\bar\rho + \partial_t^l v\cdot\nabla_N\nabla\bar\rho + \nabla_N\bar\rho\mathrm{div}\,\partial_t^l v\right) \\
&\quad + \mu\bar\rho\sum_{1\le s,m,n\le 3}\varphi_s\cdot\nabla(\epsilon_{smn}\partial_m\partial_t^l v^n) - \bar\rho\left(\bar\rho\partial_t^{l+1}v\cdot N - F_2^l\cdot N\right), \\
F_{2,N}^l &= -\mu\sum_{1\le s,m,n\le 3}\varphi_s\cdot\nabla(\epsilon_{smn}\partial_m\partial_t^l v^n) + \left(\bar\rho\partial_t^{l+1}v\cdot N\right. \\
&\quad \left.-F_2^l\cdot N + \gamma\bar\rho\nabla\left(\bar\rho^{\gamma-2}\partial_t^l q\right)\cdot N\right), \\
G_{1,N}^l &= (2\mu+\lambda)\left(\nabla_N G_1^l - (\nabla_N v\cdot\nabla\partial_t^l q + v\cdot[\nabla_N,\nabla]\,\partial_t^l q)\right) \\
&\quad - \bar\rho\left(q\partial_t^{l+1}v\cdot N + \bar\rho v\cdot\nabla\partial_t^l v\cdot N - G_2^l\cdot N\right), \\
G_{2,N}^l &= q\partial_t^{l+1}v\cdot N + \bar\rho v\cdot\nabla\partial_t^l v\cdot N - G_2^l\cdot N.
\end{aligned} \tag{3.20}$$

Lemma 18 (Normal Direction Estimate) *The following estimate on* $\nabla_N \partial_t^l q$ *holds,*

$$\frac{d}{dt}\left\{\frac{2\mu+\lambda}{2}\int_\Omega \left|\nabla_N \partial_t^l q\right|^2\, dx\right\} + \int_\Omega \left|\nabla_N \partial_t^l q\right|^2\, dx \lesssim \Lambda_l \int_\Omega \left|\nabla \partial_t^l v\right|^2\, dx$$
$$+ \Lambda_l \int_\Omega \left|\nabla \partial_t^l q\right|^2\, dx + \int_\Omega \left|\partial_t^{l+1} v\right|^2\, dx + \int_\Omega \left|\nabla \nabla_T \partial_t^l v\right|^2\, dx + \int_\Omega \left|G_{1,N}^l\right|^2\, dx. \tag{3.21}$$

Also, $\operatorname{div} \partial_t^l v$ *admits the estimate,*

$$\int_\Omega \left|\nabla \operatorname{div} \partial_t^l v\right|^2\, dx + \int_\Omega \left|\nabla_N \operatorname{div} \partial_t^l v\right|^2\, dx \lesssim \Lambda_l \int_\Omega \left|\nabla \partial_t^l q\right|^2\, dx$$
$$+ (1+\Lambda_l) \int_\Omega \left|\nabla \partial_t^l v\right|^2\, dx + \int_\Omega \left|\partial_t^{l+1} v\right|^2\, dx + \int_\Omega \left|\nabla \nabla_T \partial_t^l v\right|^2\, dx \tag{3.22}$$
$$+ \int_\Omega \left|\nabla_N \partial_t^l q\right|^2\, dx + \int_\Omega \left|G_{2,N}^l\right|^2\, dx + \int_{B_{1/2}} \left|\nabla^2 \partial_t^l v\right|^2\, dx.$$

Proof After taking inner product of $(3.19)_1$ with $\partial_t^l \nabla_N q$ and recording the resulting after integration by parts, it holds,

$$\frac{d}{dt}\left\{\frac{2\mu+\lambda}{2}\int_\Omega \left|\partial_t^l \nabla_N q\right|^2\, dx\right\} + \gamma \int_\Omega \bar\rho^\gamma \left|\partial_t^l \nabla_N q\right|^2\, dx$$
$$= -(\gamma-2)\int_\Omega \partial_t^l q \partial_t^l \nabla_N q \nabla_N \bar\rho^\gamma\, dx + \frac{2\mu+\lambda}{2}\int_\Omega \operatorname{div}(v+\bar u)\left|\partial_t^l \nabla_N q\right|^2\, dx$$
$$+ \int_\Omega F_{1,N}^l \partial_t^l \nabla_N q\, dx + \int_\Omega G_{1,N}^l \partial_t^l \nabla_N q\, dx.$$

Meanwhile,

$$\int_\Omega \left|F_{1,N}^l\right|^2\, dx \lesssim \bar\omega^a \left(\int_\Omega \left|\nabla \partial_t^l v\right|^2\, dx + \int_\Omega \left|\nabla \partial_t^l q\right|^2\, dx\right) + \int_\Omega \left|\partial_t^{l+1} v\right|^2\, dx$$
$$+ \int_\Omega \left|\nabla \nabla_T \partial_t^l v\right|^2\, dx + \int_\Omega \left|F_2^l\right|^2\, dx \lesssim \bar\omega^a \left(\int_\Omega \left|\nabla \partial_t^l v\right|^2\, dx + \int_\Omega \left|\nabla \partial_t^l q\right|^2\, dx\right)$$
$$+ \int_\Omega \left|\partial_t^{l+1} v\right|^2\, dx + \int_\Omega \left|\nabla \nabla_T \partial_t^l v\right|^2\, dx,$$

for some $a > 0$. (3.21) then follows from Hölder's inequality, (2.39), (2.40) and the fact $\inf_{x\in\Omega}\bar{\rho} > 0$. On the other hand,

$$\int_\Omega \left|\nabla_N \mathrm{div}\,\partial_t^l v\right|^2\,dx \lesssim \int_\Omega \left|F_{2,N}^l\right|^2\,dx + \int_\Omega \left|G_{2,N}^l\right|^2\,dx.$$

Similarly,

$$\begin{aligned}\int_\Omega \left|F_{2,N}^l\right|^2\,dx &\lesssim \bar{\omega}^2 \int_\Omega \left|\nabla\partial_t^l v\right|^2\,dx + \int_\Omega \left|\partial_t^{l+1} v\right|^2\,dx + \int_\Omega \left|\nabla\nabla_T\partial_t^l v\right|^2\,dx \\ &+ \int_\Omega \left|\nabla_N\partial_t^l q\right|^2\,dx + \bar{\omega}^a \int_\Omega \left|\nabla\partial_t^l q\right|^2\,dx.\end{aligned}$$

Then together with the fact

$$\begin{aligned}\int_\Omega \left|\nabla \mathrm{div}\,\partial_t^l v\right|^2\,dx &\lesssim \int_{B_{1/2}} \left|\nabla \mathrm{div}\,\partial_t^l v\right|^2\,dx + \int_\Omega \left|\nabla_T \mathrm{div}\,\partial_t^l v\right|^2\,dx \\ &+ \int_\Omega \left|\nabla_N \mathrm{div}\,\partial_t^l v\right|^2\,dx \lesssim \int_{B_{1/2}} \left|\nabla^2\partial_t^l v\right|^2\,dx + \int_\Omega \left|\nabla\nabla_T\partial_t^l v\right|^2\,dx \\ &+ \int_\Omega \left|\nabla\partial_t^l v\right|^2\,dx + \int_\Omega \left|\nabla_N \mathrm{div}\,\partial_t^l v\right|^2\,dx,\end{aligned}$$

as the consequence of (2.24) and (2.22), (3.22) follows after chaining the these inequalities. □

Now it is time to introduce the associated Stokes problem. From $(3.1)_2$, $(\partial_t^l v, \gamma\bar{\rho}^{\gamma-1}\partial_t^l q)$ satisfies the following Stokes system.

$$\begin{cases} -\mathrm{div}\,\mathbb{S}(\partial_t^l v) + \nabla(\gamma\bar{\rho}^{\gamma-1}\partial_t^l q) = F_s^l + G_s^l & \text{in } \Omega, \\ \mathrm{div}\,\partial_t^l v = \mathrm{div}\,\partial_t^l v & \text{in } \Omega, \\ \partial_t^l v = \partial_t^l v & \text{on } \Gamma, \end{cases} \tag{3.23}$$

with

$$\begin{aligned} F_s^l &= F_2^l - \bar{\rho}\partial_t^{l+1} v + \gamma\bar{\rho}^{\gamma-2}\partial_t^l q\nabla\bar{\rho}, \\ G_s^l &= G_2^l - q\partial_t^{l+1} v - \bar{\rho} v\cdot\nabla\partial_t^l v. \end{aligned} \tag{3.24}$$

Lemma 19 *By applying the Stokes estimate in Lemma 12 to* (3.23)*, we shall obtain,*

$$\begin{aligned} \left\| \partial_t^l v \right\|_{H^2(\Omega)}^2 + \left\| \nabla \partial_t^l q \right\|_{L^2(\Omega)}^2 \lesssim \left\| \nabla \mathrm{div}\, \partial_t^l v \right\|_{L^2(\Omega)}^2 + \left\| \nabla \nabla_T \partial_t^l v \right\|_{L^2(\Omega)}^2 \\ + \Lambda_l \int_\Omega \left| \nabla \partial_t^l q \right|^2 dx + (1+\Lambda_l) \int_\Omega \left| \nabla \partial_t^l v \right|^2 dx \\ + \int_\Omega \left| \partial_t^{l+1} v \right|^2 dx + \left\| G_s^l \right\|_{L^2(\Omega)}^2 . \end{aligned} \tag{3.25}$$

Proof By noticing

$$\mathrm{div}\, \mathbb{S} = \mu \Delta + (\mu + \lambda) \nabla \mathrm{div}\,, \tag{3.26}$$

the Stokes estimate in Lemma 12 then yields

$$\begin{aligned} \left\| \partial_t^l v \right\|_{H^2(\Omega)}^2 + \left\| \nabla (\gamma \bar{\rho}^{\gamma-1} \partial_t^l q) \right\|_{L^2(\Omega)}^2 \leq \left\| F_s^l \right\|_{L^2(\Omega)}^2 + \left\| G_s^l \right\|_{L^2(\Omega)}^2 \\ + \left\| \mathrm{div}\, \partial_t^l v \right\|_{H^1(\Omega)}^2 + \left\| \partial_t^l v \right\|_{H^{3/2}(\Gamma)}^2 . \end{aligned}$$

Meanwhile, the trace embedding inequality (2.32) implies

$$\left\| \partial_t^l v \right\|_{H^{3/2}(\Gamma)}^2 \lesssim \sum_{j=0}^{1} \left\| \nabla_T^j \partial_t^l v \right\|_{H^1(\Omega)}^2 \lesssim \left\| \nabla \partial_t^l v \right\|_{L^2(\Omega)}^2 + \left\| \nabla \nabla_T \partial_t^l v \right\|_{L^2(\Omega)}^2 .$$

On the other hand, direct calculation gives the following,

$$\begin{aligned} &\left\| F_s^l \right\|_{L^2}^2 \lesssim \bar{\omega}^a \int_\Omega \left| \nabla \partial_t^l q \right|^2 dx + \bar{\omega}^a \int_\Omega \left| \nabla \partial_t^l v \right|^2 dx + \int_\Omega \left| \partial_t^{l+1} v \right|^2 dx, \\ &\left\| \nabla \partial_t^l q \right\|_{L^2(\Omega)}^2 \lesssim \left\| \nabla (\gamma \bar{\rho}^{\gamma-1} \partial_t^l q) \right\|_{L^2(\Omega)}^2 + \bar{\omega}^a \left\| \nabla \partial_t^l q \right\|_{L^2(\Omega)}^2 . \end{aligned}$$

Hence (3.25) holds. □

The following lemma is a consequence of (3.14), (3.21), (3.22) and (3.25).

Proposition 2 *We have obtained the estimate concerning the spatial derivatives as follows,*

$$\frac{d}{dt} \left\{ \int_\Omega (\bar{\rho} + q) \left| \nabla_T \partial_t^l v \right|^2 dx + \int_\Omega \bar{\rho}^{\gamma-2} \left| \nabla_T \partial_t^l q \right|^2 dx + \int_\Omega \left| \nabla_N \partial_t^l q \right|^2 dx \right\}$$

$$
\begin{aligned}
&+ \int_\Omega \left|\nabla^2 \partial_t^l v\right|^2 dx + \int_\Omega \left|\nabla \mathrm{div}\, \partial_t^l v\right|^2 dx + \int_\Omega \left|\nabla \partial_t^l q\right|^2 dx \\
&\lesssim \Lambda_l \left(\int_\Omega \left|\nabla^2 \partial_t^l v\right|^2 dx + \int_\Omega \left|\nabla \partial_t^l q\right|^2 dx \right) + (1+\Lambda_l) \left(\int_\Omega \left|\nabla \partial_t^l v\right|^2 dx \right. \\
&\left. + \int_\Omega \left|\partial_t^{l+1} v\right|^2 dx \right) + \int_\Omega \left|G_1^{l,1}\right|^2 dx + \int_\Omega \left|G_2^{l,1}\right|^2 dx + \int_\Omega \left|G_2^{l}\right|^2 dx \\
&+ \int_\Omega \left|G_{1,N}^{l}\right|^2 dx + \int_\Omega \left|G_{2,N}^{l}\right|^2 dx + \int_\Omega \left|G_s^{l}\right|^2 dx + \int_{B_{1/2}} \left|\nabla^2 \partial_t^l v\right|^2 dx.
\end{aligned} \tag{3.27}
$$

Proof From (3.21), (3.22), (3.25), it holds

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega \left|\nabla_N \partial_t^l q\right|^2 dx + \int_\Omega \left|\nabla_N \partial_t^l q\right|^2 dx + \int_\Omega \left|\nabla \mathrm{div}\, \partial_t^l v\right|^2 dx \\
&+ \int_\Omega \left|\nabla^2 \partial_t^l v\right|^2 dx + \int_\Omega \left|\nabla \partial_t^l q\right|^2 dx \lesssim \Lambda_l \int_\Omega \left|\nabla \partial_t^l q\right|^2 dx \\
&+ (1+\Lambda_l) \left(\int_\Omega \left|\nabla \partial_t^l v\right|^2 dx + \int_\Omega \left|\partial_t^{l+1} v\right|^2 dx \right) + \int_\Omega \left|\nabla \nabla_T \partial_t^l v\right|^2 dx \\
&+ \int_\Omega \left|G_{1,N}^{l}\right|^2 dx + \int_\Omega \left|G_{2,N}^{l}\right|^2 dx + \int_\Omega \left|G_s^{l}\right|^2 dx + \int_{B_{1/2}} \left|\nabla^2 \partial_t^l v\right|^2 dx.
\end{aligned}
$$

Together with (3.14) and an appropriate choice of $\omega > 0$, (3.27) follows easily. □

3.2 On Higher Order Spatial Derivatives

Through the following arguments similar to those in the last section, the estimates involving higher order spatial derivatives would be shown. In particular, the estimates of tangential derivatives are obtained through a high-order version of (3.9). Also, by taking mixed derivatives to (3.19), we shall track the regularities of q and more importantly, the regularities of $\mathrm{div}\, v$. Then a high-order version of the Stokes problem (3.23) would eventually yield the estimates of v and q. These estimates would play important roles in the global analysis.

Apply ∇_T^m to (3.1) and record the resulting system as follows,

$$\begin{cases} \partial_t^{l+1}\nabla_T^m q + \mathrm{div}\,(\bar\rho\partial_t^l\nabla_T^m v) + (v+\bar u)\cdot\nabla\partial_t^l\nabla_T^m q = F_1^{l,m} + G_1^{l,m} & \text{in } \Omega, \\ (\bar\rho + q)\partial_t^{l+1}\nabla_T^m v + \bar\rho v\cdot\nabla\partial_t^l\nabla_T^m v + \gamma\bar\rho\nabla(\bar\rho^{\gamma-2}\partial_t^l\nabla_T^m q) & \\ \qquad - \mathrm{div}\,\nabla_T^m\mathbb{S}(\partial_t^l v) = F_2^{l,m} + \nabla_T^m G_2^l + G_2^{l,m} & \text{in } \Omega, \\ \partial_t^l\nabla_T^m v\cdot\vec n = b_1^m & \text{on } \Gamma, \\ \vec\tau\cdot\nabla_T^m\mathbb{S}(\partial_t^l v)\vec n = b_2^m & \text{on } \Gamma, \end{cases} \tag{3.28}$$

with

$$\begin{aligned} F_1^{l,m} &= -\Big\{\bar u\cdot[\nabla_T^m,\nabla]\,\partial_t^l q + \sum_{j=0}^{m-1} C_{j,m}\nabla_T^{m-j}\bar u\cdot\nabla_T^j\nabla\partial_t^l q \\ &\quad + [\nabla_T^m,\mathrm{div}\,]\,(\bar\rho\partial_t^l v) + \sum_{j=0}^{m-1} C_{j,m}\mathrm{div}\,(\nabla_T^{m-j}\bar\rho\partial_t^l\nabla_T^j v)\Big\}, \\ F_2^{l,m} &= \nabla_T^m F_2^l - \Big\{\sum_{j=0}^{m-1} C_{j,m}\nabla_T^{m-j}\bar\rho\partial_t^{l+1}\nabla_T^j v + \gamma\bar\rho\,[\nabla_T^m,\nabla]\,(\bar\rho^{\gamma-2}\partial_t^l q) \\ &\quad + \sum_{j=0}^{m-1} C_{j,m}\{\gamma\bar\rho\nabla(\nabla_T^{m-j}\bar\rho^{\gamma-2}\nabla_T^j\partial_t^l q) + \gamma\nabla_T^{m-j}\bar\rho\nabla_T^j\nabla(\bar\rho^{\gamma-2}\partial_t^l q)\}\Big\} \\ &\quad + [\nabla_T^m,\mathrm{div}\,]\,\mathbb{S}(\partial_t^l v), \\ G_1^{l,m} &= \nabla_T^m G_1^l - \Big\{v\cdot[\nabla_T^m,\nabla]\,\partial_t^l q + \sum_{j=0}^{m-1} C_{j,m}\nabla_T^{m-j}v\cdot\nabla_T^j\nabla\partial_t^l q\Big\}, \\ G_2^{l,m} &= -\Big\{\sum_{j=0}^{m-1} C_{j,m}\nabla_T^{m-j}q\partial_t^{l+1}\nabla_T^j v + \bar\rho v\cdot[\nabla_T^m,\nabla]\,\partial_t^l v \\ &\quad + \sum_{j=0}^{m-1} C_{j,m}\nabla_T^{m-j}(\bar\rho v)\cdot\nabla_T^j\nabla\partial_t^l v\Big\}, \end{aligned}$$

and

$$b_1^m = -\sum_{j=0}^{m-1} C_{j,m}\partial_t^l\nabla_T^j v\cdot\nabla_T^{m-j}\vec n,$$

$$
b_2^m = -\sum_{j=0}^{m-1} \sum_{a+b=m-j} C_{j,m} C_{a,m-j} \nabla_T^a \vec{\tau} \cdot \nabla_T^j \mathbb{S}(\partial_t^l v) \nabla_T^b \vec{n}.
$$

Applying standard energy estimate to (3.28) then will yield the following lemma.

Lemma 20 *For $m \geq 1$, the following higher version of* (3.14) *holds,*

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega (\bar{\rho}+q)\left|\partial_t^l \nabla_T^m v\right|^2 \, dx + \frac{\gamma}{2}\int_\Omega \bar{\rho}^{\gamma-2}\left|\partial_t^l \nabla_T^m q\right|^2 \, dx\right\} \\
&\quad + \frac{\mu}{4}\int_\Omega \left|\nabla \nabla_T^m \partial_t^l v\right|^2 \, dx + \lambda \int_\Omega \left|\operatorname{div}\left(\nabla_T^m \partial_t^l v\right)\right|^2 \, dx \\
&\lesssim (\omega + \Lambda_l)\int_\Omega \left|\nabla^{m+1}\partial_t^l v\right|^2 \, dx + (\omega + C_\omega \bar{\omega}^a + \Lambda_l)\int_\Omega \left|\nabla^m \partial_t^l q\right|^2 \, dx \\
&+ (\omega + C_\omega + \Lambda_l)\left(\sum_{j=1}^{m}\int_\Omega \left|\nabla^j \partial_t^l v\right|^2 \, dx + \sum_{j=1}^{m-1}\int_\Omega \left|\nabla^j \partial_t^l q\right|^2 \, dx\right) \\
&+ \omega \bar{\omega}^a \sum_{j=0}^{m-1}\int_\Omega \left|\nabla^j \partial_t^{l+1} v\right|^2 \, dx + C_\omega \int_\Omega \left|G_1^{l,m}\right|^2 \, dx + C_\omega \int_\Omega \left|G_2^{l,m}\right|^2 \, dx \\
&+ \int_\Omega \left|\nabla_T^{m-1} G_2^l\right|^2 \, dx.
\end{aligned} \tag{3.29}
$$

with any $0 < \omega < 1$.

Proof Taking inner product of $(3.28)_2$ with $\partial_t^l \nabla_T^m v$ and recording the resulting after integration by parts,

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega (\bar{\rho}+q)\left|\partial_t^l \nabla_T^m v\right|^2 \, dx\right\} \underbrace{-\gamma \int_\Omega \bar{\rho}^{\gamma-2}\partial_t^l \nabla_T^m q \operatorname{div}\left(\bar{\rho}\partial_t^l \nabla_T^m v\right) dx}_{(i)} \\
&+ \frac{\mu}{2}\int_\Omega \left|\nabla \partial_t^l \nabla_T^m v + \nabla \partial_t^l \nabla_T^m v^T\right|^2 \, dx + \lambda \int_\Omega \operatorname{div}\left(\partial_t^l \nabla_T^m v\right) dx \\
&+ \underbrace{\int_\Omega \left[\nabla_T^m, \mathbb{S}\right]\left(\partial_t^l v\right) : \nabla \partial_t^l \nabla_T^m v \, dx}_{(ii)} = \underbrace{\int_\Omega \left(F_2^{l,m} + G_2^{l,m}\right) \cdot \partial_t^l \nabla_T^m v \, dx}_{(iii)}
\end{aligned}
$$

$$
+\underbrace{\int_\Omega \nabla_T^m G_2^l \cdot \partial_t^l \nabla_T^m v \, dx}_{(iv)} + \underbrace{\frac{1}{2}\int_\Omega \partial_t q \left|\partial_t^l \nabla_T^m v\right|^2 \, dx + \frac{1}{2}\int_\Omega \operatorname{div}(\bar\rho v) \left|\partial_t^l \nabla_T^m v\right|^2 \, dx}_{(v)}
$$

$$
-\gamma \underbrace{\int_\Gamma \bar\rho^{\gamma-1} \partial_t^l \nabla_T^m q \partial_t^l \nabla_T^m v \cdot \vec n \, dS}_{(vi)} + \underbrace{\int_\Gamma \partial_t^l \nabla_T^m v \cdot \nabla_T^m \mathbb{S}(\partial_t^l v) \vec n \, dS}_{(vii)} .
$$

From $(3.28)_1$, the following identity holds,

$$
\begin{aligned}
(i) &= \gamma \int_\Omega \bar\rho^{\gamma-2} \partial_t^l \nabla_T^m q \cdot (\partial_t^{l+1} \nabla_T^m q + (v+\bar u) \cdot \nabla \partial_t^l \nabla_T^m q - F_1^{l,m} - G_1^{l,m}) \, dx \\
&= \frac{d}{dt}\frac{\gamma}{2}\int_\Omega \bar\rho^{\gamma-2} \left|\partial_t^l \nabla_T^m q\right|^2 \, dx - \frac{\gamma}{2} \underbrace{\int_\Omega \operatorname{div}(\bar\rho^{\gamma-2}(v+\bar u)) \left|\partial_t^l \nabla_T^m q\right|^2 \, dx}_{(viii)} \\
&\quad - \gamma \underbrace{\int_\Omega (F_1^{l,m} + G_1^{l,m}) \cdot \bar\rho^{\gamma-2} \partial_t^l \nabla_T^m q \, dx}_{(ix)} .
\end{aligned}
$$

By applying Cauchy's inequality, Poincaré inequality and (2.30),

$$
\begin{aligned}
&(viii) \lesssim (\bar\omega^a + \bar\omega \|v\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)}) \int_\Omega \left|\partial_t^l \nabla_T^m q\right|^2 \, dx, \\
&(ix) \lesssim \omega \int_\Omega \left|\nabla_T^m \partial_t^l q\right|^2 \, dx + C_\omega \left(\int_\Omega \left|F_1^{l,m}\right|^2 \, dx + \int_\Omega \left|G_1^{l,m}\right|^2 \, dx \right), \\
&(ii) \lesssim \delta \int_\Omega \left|\nabla \nabla_T^m \partial_t^l v\right|^2 \, dx + C_\delta \sum_{j=0}^m \int_\Omega \left|\nabla^j \partial_t^l v\right|^2 \, dx, \\
&(iii) \lesssim \omega \int_\Omega \left|F_2^{l,m}\right|^2 \, dx + C_\omega \left(\int_\Omega \left|G_2^{l,m}\right|^2 \, dx + \int_\Omega \left|\partial_t^l \nabla_T^m v\right|^2 \, dx \right), \\
&(iv) = -\int_\Omega \nabla_T^{m-1} G_2^l \cdot \nabla_T \partial_t^l \nabla_T^m v \, dx \lesssim \delta \int_\Omega \left|\nabla \nabla_T^m \partial_t^l v\right|^2 \, dx \\
&\qquad + C_\delta \int_\Omega \left|\nabla_T^{m-1} G_2^l\right|^2 \, dx, \\
&(v) \lesssim (\|\partial_t q\|_{L^\infty(\Omega)} + \bar\omega \|v\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)}) \int_\Omega \left|\partial_t^l \nabla_T^m v\right|^2 \, dx.
\end{aligned}
$$

Also, followed from the definition of $F_1^{l,m}, F_2^{l,m}$, it holds

$$\int_\Omega \left|F_1^{l,m}\right|^2 \, dx \lesssim (1+\bar\omega^a)\sum_{j=0}^m \int_\Omega \left|\nabla^j \partial_t^l v\right|^2 \, dx + \bar\omega^a \sum_{j=0}^m \int_\Omega \left|\nabla^j \partial_t^l q\right|^2 \, dx,$$

$$\begin{aligned}\int_\Omega \left|F_2^{l,m}\right|^2 \, dx &\lesssim (1+\bar\omega^a)\int_\Omega \left|\nabla^{m+1} \partial_t^l v\right|^2 \, dx + (1+\bar\omega^a)\int_\Omega \left|\nabla^m \partial_t^l q\right|^2 \, dx \\ &\quad + \bar\omega^a \sum_{j=0}^{m-1}\left(\int_\Omega \left|\nabla^j \partial_t^l q\right|^2 \, dx + \int_\Omega \left|\nabla^j \partial_t^{l+1} v\right|^2 \, dx\right) \\ &\quad + (1+\bar\omega^a)\sum_{j=0}^m \int_\Omega \left|\nabla^j \partial_t^l v\right|^2 \, dx.\end{aligned}$$

Meanwhile, to handle the boundary integration (vi), from $(3.28)_3$ and the calculus on the boundary (2.28),

$$\begin{aligned}(vi) &= \int_\Gamma \bar\rho^{\gamma-1}\partial_t^l \nabla_T^m q b_1^m \, dS = -\int_\Gamma \partial_t^l \nabla_T^{m-1} q \nabla_T(\bar\rho^{\gamma-1} b_1^m)\, dS \\ &\lesssim \left\| \partial_t^l \nabla_T^{m-1} q \right\|_{L^2(\Gamma)} \left(\bar\omega^a \left\|b_1^m\right\|_{L^2(\Gamma)} + \left\|\nabla_T b_1^m\right\|_{L^2(\Gamma)}\right) \\ &\lesssim (1+\bar\omega^a)\left\| \partial_t^l \nabla_T^{m-1} q \right\|_{L^2(\Gamma)} \times \sum_{j=0}^m \left\| \partial_t^l \nabla_T^j v \right\|_{L^2(\Gamma)},\end{aligned}$$

where it has been used from the definition of b_1^m,

$$|b_1^m| \lesssim \sum_{j=0}^{m-1}\left|\partial_t^l \nabla_T^j v\right|, \quad |\nabla_T b_1^m| \lesssim \sum_{j=0}^m \left|\partial_t^l \nabla_T^j v\right|.$$

Then the trace embedding inequality (2.31) together with (2.39) and (2.40) implies

$$\begin{aligned}(vi) &\lesssim \omega\left(\int_\Omega \left|\nabla^m \partial_t^l q\right|^2 \, dx + \int_\Omega \left|\nabla^{m+1} \partial_t^l v\right|^2 \, dx\right) \\ &\quad + C_\omega (1+\bar\omega^a)\left(\int_\Omega \left|\nabla^{m-1} \partial_t^l q\right|^2 \, dx + \sum_{j=0}^m \int_\Omega \left|\nabla^j \partial_t^l v\right|^2 \, dx\right).\end{aligned}$$

On the other hand, $(3.28)_3$ implies the following decomposition on Γ,

$$\partial_t^l \nabla_T^m v = b_1^m \vec n + v_{l,m} \vec\tau$$

with $|v_{l,m}| \leq \left|\partial_t^l \nabla_T^m v\right|$. Then together with $(3.28)_4$

$$
\begin{aligned}
(vii) &= \int_\Gamma b_1^m \vec{n} \cdot \nabla_T^m \mathbb{S}(\partial_t^l v) \vec{n}\, dS + \int_\Gamma v_{l,m} \vec{\tau} \cdot \nabla_T^m \mathbb{S}(\partial_t^l v) \vec{n}\, dS \\
&= - \int_\Gamma \nabla_T (b_1^m \vec{n}) \cdot \nabla_T^{m-1} \mathbb{S}(\partial_t^l v) \vec{n}\, dS - \int_\Gamma b_1^m \vec{n} \cdot \nabla_T^{m-1} \mathbb{S}(\partial_t^l v) \nabla_T \vec{n}\, dS \\
&\quad + \int_\Gamma v_{l,m} \cdot b_2^m \, dS \lesssim \left\| \nabla_T^{m-1} \mathbb{S}(\partial_t^l v) \right\|_{L^2(\Gamma)} \left(\|b_1^m\|_{L^2(\Gamma)} + \|\nabla_T b_1^m\|_{L^2(\Gamma)} \right) \\
&\quad + \left\| \partial_t^l \nabla_T^m v \right\|_{L^2(\Gamma)} \|b_2^m\|_{L^2(\Gamma)} .
\end{aligned}
$$

From the definition,

$$
|b_2^m| \lesssim \sum_{j=0}^{m-1} \left| \nabla_T^j \mathbb{S}(\partial_t^l v) \right| .
$$

Therefore, by applying trace embedding inequality (2.31), Cauchy's inequality and Poincaré inequality,

$$
(vii) \lesssim \omega \int_\Omega \left| \nabla^{m+1} \partial_t^l v \right|^2 \, dx + C_\omega \sum_{j=0}^{m} \int_\Omega \left| \nabla^j \partial_t^l v \right|^2 \, dx.
$$

Summing up these estimates with an appropriately small $\delta > 0$ and (2.1), it holds,

$$
\begin{aligned}
&\frac{d}{dt} \left\{ \frac{1}{2} \int_\Omega (\bar\rho + q) \left| \partial_t^l \nabla_T^m v \right|^2 \, dx + \frac{\gamma}{2} \int_\Omega \bar\rho^{\gamma-2} \left| \partial_t^l \nabla_T^m q \right|^2 \, dx \right\} \\
&\qquad + \frac{\mu}{4} \int_\Omega \left| \nabla \nabla_T^m \partial_t^l v \right|^2 \, dx + \lambda \int_\Omega \left| \operatorname{div} (\nabla_T^m \partial_t^l v) \right|^2 \, dx \\
&\lesssim (\omega + \Lambda_l) \int_\Omega \left| \nabla^{m+1} \partial_t^l v \right|^2 \, dx + (\omega + C_\omega \bar\omega^a + \Lambda_l) \int_\Omega \left| \nabla^m \partial_t^l q \right|^2 \, dx \\
&+ (\omega + C_\omega + \Lambda_l) \left(\sum_{j=0}^{m} \int_\Omega \left| \nabla^j \partial_t^l v \right|^2 \, dx + \sum_{j=0}^{m-1} \int_\Omega \left| \nabla^j \partial_t^l q \right|^2 \, dx \right) \\
&+ \omega \bar\omega^a \sum_{j=0}^{m-1} \int_\Omega \left| \nabla^j \partial_t^{l+1} v \right|^2 \, dx + C_\omega \int_\Omega \left| G_1^{l,m} \right|^2 \, dx + C_\omega \int_\Omega \left| G_2^{l,m} \right|^2 \, dx \\
&+ \int_\Omega \left| \nabla_T^{m-1} G_2^l \right|^2 \, dx.
\end{aligned}
$$

Therefore, (3.29) follows after applying (2.39) and (2.40).

□

Next it is to derive the estimates of mixed derivatives of q. Applying $\nabla_T^m \nabla_N^{n-1}$ to (3.19), it holds

$$
\begin{cases}
(2\mu+\lambda)\partial_t^{l+1}\nabla_T^m\nabla_N^n q + (2\mu+\lambda)(v+\bar{u})\cdot\nabla\partial_t^l\nabla_T^m\nabla_N^n q + \gamma\bar{\rho}^\gamma\partial_t^l\nabla_T^m\nabla_N^n q \\
\qquad\qquad = F_{1,N}^{l,m,n} + G_{1,N}^{l,m,n}, \\
(2\mu+\lambda)\nabla_T^m\nabla_N^n \mathrm{div}\,\partial_t^l v = F_{2,N}^{l,m,n} + G_{2,N}^{l,m,n},
\end{cases}
\tag{3.30}
$$

with

$$
\begin{aligned}
F_{1,N}^{l,m,n} &= \nabla_T^m\nabla_N^{n-1}F_{1,N}^l - (2\mu+\lambda)\bar{u}\cdot\left[\nabla_T^m\nabla_N^{n-1},\nabla\right]\partial_t^l\nabla_N q \\
&\quad - (2\mu+\lambda)\sum_{j=0}^{m+n-2}\sum_{a+b=j} C_{a,m}C_{b,n-1}\nabla_T^{m-a}\nabla_N^{n-1-b}\bar{u}\cdot\nabla_T^a\nabla_N^b\nabla\partial_t^l\nabla_N q \\
&\quad - (\gamma-2)\sum_{j=0}^{m+n-1}\sum_{a+b=j} C_{a,m}C_{b,n-1}\nabla_T^{m-a}\nabla_N^{n-b}\bar{\rho}^\gamma\partial_t^l\nabla_T^a\nabla_N^b q \\
&\quad - \gamma\sum_{j=0}^{m+n-2}\sum_{a+b=j} C_{a,m}C_{b,n-1}\nabla_T^{m-a}\nabla_N^{n-1-b}\bar{\rho}^\gamma\partial_t^l\nabla_T^a\nabla_N^{b+1} q, \\
F_{2,N}^{l,m,n} &= \nabla_T^m\nabla_N^{n-1}F_{2,N}^l, \\
G_{1,N}^{l,m,n} &= \nabla_T^m\nabla_N^{n-1}G_{1,N}^l - (2\mu+\lambda)v\cdot\left[\nabla_T^m\nabla_N^{n-1},\nabla\right]\partial_t^l\nabla_N q \\
&\quad - (2\mu+\lambda)\sum_{j=0}^{m+n-2}\sum_{a+b=j} C_{a,m}C_{b,n-1}\nabla_T^{m-a}\nabla_N^{n-1-b}v\cdot\nabla_T^a\nabla_N^b\nabla\partial_t^l\nabla_N q, \\
G_{2,N}^{l,m,n} &= \nabla_T^m\nabla_N^{n-1}G_{2,N}^l.
\end{aligned}
$$

Then the following estimate holds,

Lemma 21 *For $n \geq 1, m+n \geq 2$,*

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{2\mu+\lambda}{2}\int_\Omega \left|\nabla_T^m\nabla_N^n\partial_t^l q\right|^2\,dx\right\}+\gamma\int_\Omega \bar{\rho}^\gamma\left|\nabla_T^m\nabla_N^n\partial_t^l q\right|^2\,dx\\
&\lesssim \int_\Omega\left|\nabla\nabla_T^{m+1}\nabla_N^{n-1}\partial_t^l v\right|^2\,dx+\int_\Omega\left|\nabla_T^m\nabla_N^{n-1}\partial_t^{l+1}v\right|^2\,dx\\
&\quad+(1+\Lambda_l)\sum_{j=1}^{m+n}\int_\Omega\left|\nabla^j\partial_t^l v\right|^2\,dx+\Lambda_l\sum_{j=1}^{m+n}\int_\Omega\left|\nabla^j\partial_t^l q\right|^2\,dx \qquad (3.31)\\
&\quad+(1+\Lambda_l)\sum_{j=0}^{m+n-2}\int_\Omega\left|\nabla^j\partial_t^{l+1}v\right|^2\,dx+\int_\Omega\left|G_{1,N}^{l,m,n}\right|^2\,dx.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\int_\Omega\left|\nabla_T^m\nabla_N^n\mathrm{div}\,\partial_t^l v\right|^2\,dx\lesssim\int_\Omega\left|\nabla_T^m\nabla_N^n\partial_t^l q\right|^2\,dx+\int_\Omega\left|\nabla\nabla_T^{m+1}\nabla_N^{n-1}\partial_t^l v\right|^2\,dx\\
&\quad+\int_\Omega\left|\nabla_T^m\nabla_N^{n-1}\partial_t^{l+1}v\right|^2\,dx+(1+\Lambda_l)\sum_{j=1}^{m+n}\int_\Omega\left|\nabla^j\partial_t^l v\right|^2\,dx \qquad (3.32)\\
&\quad+\Lambda_l\sum_{j=1}^{m+n-1}\int_\Omega\left|\nabla^j\partial_t^l q\right|^2\,dx+(1+\Lambda_l)\sum_{j=0}^{m+n-2}\int_\Omega\left|\nabla^j\partial_t^{l+1}v\right|^2\,dx\\
&\quad+\int_\Omega\left|G_{2,N}^{l,m,n}\right|^2\,dx.
\end{aligned}
$$

Proof After taking inner product of $(3.30)_1$ with $\partial_t^l\nabla_T^m\nabla_N^n q$ and recording the resulting after integration, it admits,

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{2\mu+\lambda}{2}\int_\Omega\left|\nabla_T^m\nabla_N^n\partial_t^l q\right|^2\,dx\right\}+\gamma\int_\Omega\bar{\rho}^\gamma\left|\nabla_T^m\nabla_N^n\partial_t^l q\right|^2\,dx\\
&=\frac{2\mu+\lambda}{2}\int_\Omega\mathrm{div}\,(v+\bar{u})\left|\nabla_T^m\nabla_N^n\partial_t^l q\right|^2\,dx+\int_\Omega(F_{1,N}^{l,m,n}+G_{1,N}^{l,m,n})\nabla_T^m\nabla_N^n\partial_t^l q\,dx\\
&\lesssim(\delta+\bar{\omega}+\|\nabla v\|_{L^\infty(\Omega)})\int_\Omega\left|\nabla_T^m\nabla_N^n\partial_t^l q\right|^2\,dx+C_\delta\int_\Omega\left|F_{1,N}^{l,m,n}\right|^2+\left|G_{1,N}^{l,m,n}\right|^2\,dx.
\end{aligned}
\qquad (3.33)
$$

Moreover repeatedly applying the formula (2.22) would imply

$$
\begin{aligned}
\int_\Omega \left|F_{1,N}^{l,m,n}\right|^2 dx &\lesssim \int_\Omega \left|\nabla_T^m \nabla_N^{n-1} F_{1,N}^l\right|^2 dx + \bar{\omega}^a \sum_{j=0}^{m+n} \int_\Omega \left|\nabla^j \partial_t^l q\right|^2 dx \\
&\lesssim \int_\Omega \left|\nabla \nabla_T^{m+1} \nabla_N^{n-1} \partial_t^l v\right|^2 dx + \int_\Omega \left|\nabla_T^m \nabla_N^{n-1} \partial_t^{l+1} v\right|^2 dx \\
&\quad + (1+\bar{\omega}^a) \sum_{j=0}^{m+n} \int_\Omega \left|\nabla^j \partial_t^l v\right|^2 dx + \bar{\omega}^a \sum_{j=0}^{m+n} \int_\Omega \left|\nabla^j \partial_t^l q\right|^2 dx \\
&\quad + (1+\bar{\omega}^a) \sum_{j=0}^{m+n-2} \int_\Omega \left|\nabla^j \partial_t^{l+1} v\right|^2 dx,
\end{aligned} \tag{3.34}
$$

from which (3.31) follows together with (2.39), (2.40) and an appropriate $\delta > 0$. On the other hand, (3.32) is the consequence of $(3.30)_2$, (2.39), (2.40) and

$$
\begin{aligned}
\int_\Omega \left|F_{2,N}^{l,m,n}\right|^2 dx &\lesssim \int_\Omega \left|\nabla_T^m \nabla_N^n \partial_t^l q\right|^2 dx + \int_\Omega \left|\nabla \nabla_T^{m+1} \nabla_N^{n-1} \partial_t^l v\right|^2 dx \\
&\quad + \int_\Omega \left|\nabla_T^m \nabla_N^{n-1} \partial_t^{l+1} v\right|^2 dx + (1+\bar{\omega}^a) \sum_{j=0}^{m+n} \int_\Omega \left|\nabla^j \partial_t^l v\right|^2 dx \\
&\quad + \bar{\omega}^a \sum_{j=0}^{m+n-1} \int_\Omega \left|\nabla^j \partial_t^l q\right|^2 dx + (1+\bar{\omega}^a) \sum_{j=0}^{m+n-2} \int_\Omega \left|\nabla^j \partial_t^{l+1} v\right|^2 dx.
\end{aligned} \tag{3.35}
$$

□

The last estimate we shall obtain in this section is from the following Stokes problem. After applying ∇_T^m to (3.23), record the resulting system as follows.

$$
\begin{cases}
-\mathrm{div}\, \mathbb{S}(\partial_t^l \nabla_T^m v) + \nabla(\gamma \bar{\rho}^{\gamma-1} \partial_t^l \nabla_T^m q) = F_s^{l,m} + G_s^{l,m} & \text{in } \Omega, \\
\mathrm{div}\, \partial_t^l \nabla_T^m v = \mathrm{div}\, \partial_t^l \nabla_T^m v & \text{in } \Omega, \\
\partial_t^l \nabla_T^m v = \partial_t^l \nabla_T^m v & \text{on } \Gamma,
\end{cases} \tag{3.36}
$$

with

$$
\begin{aligned}
F_s^{l,m} &= \nabla_T^m F_s^l + [\nabla_T^m, \mathrm{div}\, \mathbb{S}]\,(\partial_t^l v) - \gamma\, [\nabla_T^m, \nabla]\,(\bar{\rho}^{\gamma-1} \partial_t^l q) \\
&\qquad - \gamma \sum_{j=0}^{m-1} C_{j,m} \nabla(\nabla_T^{m-j} \bar{\rho}^{\gamma-1} \partial_t^l \nabla_T^j q), \\
G_s^{l,m} &= \nabla_T^m G_s^l.
\end{aligned} \tag{3.37}
$$

As consequences of the Stokes estimate in Lemma 12, the following lemma indicates the estimates of normal derivatives.

Lemma 22 *For $n \geq 2, m+n > 2$,*

$$
\begin{aligned}
&\left\| \nabla^n \nabla_T^m \partial_t^l v \right\|_{L^2(\Omega)}^2 + \left\| \nabla^{n-1} \nabla_T^m \partial_t^l q \right\|_{L^2(\Omega)}^2 \lesssim \left\| \nabla_T^m \nabla^{n-1} \operatorname{div} \partial_t^l v \right\|_{L^2(\Omega)}^2 \\
&\quad + \left\| \nabla \nabla_T^{n+m-1} \partial_t^l v \right\|_{L^2(\Omega)}^2 + \left\| \nabla^{m+n-2} \partial_t^{l+1} v \right\|_{L^2(\Omega)}^2 \\
&\quad + (1+\Lambda_l) \left\{ \sum_{j=0}^{n+m-3} \left\| \nabla^j \partial_t^{l+1} v \right\|_{L^2(\Omega)}^2 + \sum_{j=1}^{n+m-1} \left\| \nabla^j \partial_t^l v \right\|_{L^2(\Omega)}^2 \right. \qquad (3.38) \\
&\quad \left. + \sum_{j=1}^{n+m-2} \left\| \nabla^j \partial_t^l q \right\|_{L^2(\Omega)}^2 \right\} + \left\| G_s^{l,m} \right\|_{H^{n-2}(\Omega)}^2 .
\end{aligned}
$$

Proof Similarly as before, by noticing (3.26), the Stokes estimate in Lemma 12 then yields,

$$
\begin{aligned}
&\left\| \partial_t^l \nabla_T^m v \right\|_{H^n(\Omega)}^2 + \left\| \nabla (\bar{\rho}^{\gamma-1} \partial_t^l \nabla_T^m q) \right\|_{H^{n-2}(\Omega)}^2 \leq \left\| F_s^{l,m} \right\|_{H^{n-2}(\Omega)}^2 \\
&\quad + \left\| G_s^{l,m} \right\|_{H^{n-2}(\Omega)}^2 + \left\| \operatorname{div} \partial_t^l \nabla_T^m v \right\|_{H^{n-1}(\Omega)}^2 + \left\| \partial_t^l \nabla_T^m v \right\|_{H^{n-1/2}(\Gamma)}^2 .
\end{aligned}
$$

On one hand, triangle inequality implies

$$
\begin{aligned}
&\left\| \nabla^{n-1} \nabla_T^m \partial_t^l q \right\|_{L^2(\Omega)}^2 \lesssim \left\| \nabla (\bar{\rho}^{\gamma-1} \partial_t^l \nabla_T^m q) \right\|_{H^{n-2}(\Omega)}^2 \\
&\qquad + \bar{\omega}^a \sum_{j=0}^{n-2} \left\| \nabla^j \nabla_T^m \partial_t^l q \right\|_{L^2(\Omega)}^2 , \\
&\left\| \nabla^n \nabla_T^m \partial_t^l v \right\|_{L^2(\Omega)}^2 \lesssim \left\| \partial_t^l \nabla_T^m v \right\|_{H^n(\Omega)}^2 + \sum_{j=0}^{n-1} \left\| \nabla^j \nabla_T^m \partial_t^l v \right\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

Again, the trace embedding inequality (2.32), together with (2.40), yields

$$
\left\| \partial_t^l \nabla_T^m v \right\|_{H^{n-1/2}(\Gamma)}^2 \lesssim \sum_{j=0}^{n-1} \left\| \partial_t^l \nabla_T^{m+j} v \right\|_{H^1(\Omega)}^2 \lesssim \sum_{j=0}^{n-1} \left\| \nabla \nabla_T^{m+j} \partial_t^l v \right\|_{L^2(\Omega)}^2 .
$$

Meanwhile, from the definition of $F_s^{l,m}$, it admits

$$
\left\| F_s^{l,m} \right\|_{H^{n-2}(\Omega)}^2 \lesssim \left\| \nabla_T^m F_s^l \right\|_{H^{n-2}(\Omega)}^2 + \sum_{j=0}^{m+1} \left\| \nabla^j \partial_t^l v \right\|_{H^{n-2}(\Omega)}^2
$$

$$+ (1+\bar{\omega}^a)\left\|\nabla^m\partial_t^l q\right\|^2_{H^{n-2}(\Omega)} + \bar{\omega}^a\sum_{j=0}^{m-1}\left\|\nabla^j\partial_t^l q\right\|^2_{H^{n-2}(\Omega)}$$

$$\lesssim \left\|\nabla^{m+n-2}\partial_t^{l+1}v\right\|^2_{L^2(\Omega)} + (1+\bar{\omega}^a)\Bigg\{\sum_{j=0}^{n+m-3}\left\|\nabla^j\partial_t^{l+1}v\right\|^2_{L^2(\Omega)}$$

$$+ \sum_{j=0}^{n+m-1}\left\|\nabla^j\partial_t^l v\right\|^2_{L^2(\Omega)} + \sum_{j=0}^{n+m-2}\left\|\nabla^j\partial_t^l q\right\|^2_{L^2(\Omega)}\Bigg\},$$

where we have employed the following estimate of F_s^l,

$$\left\|\nabla_T^m F_s^l\right\|^2_{H^{n-2}(\Omega)} \lesssim \left\|\nabla^m\partial_t^{l+1}v\right\|^2_{H^{n-2}(\Omega)} + \bar{\omega}^a\Bigg(\sum_{j=0}^{m-1}\left\|\nabla^j\partial_t^{l+1}v\right\|^2_{H^{n-2}(\Omega)}$$

$$+ \sum_{j=0}^{m+1}\left\|\nabla^j\partial_t^l v\right\|^2_{H^{n-2}(\Omega)} + \sum_{j=0}^{m}\left\|\nabla^j\partial_t^l q\right\|^2_{H^{n-2}(\Omega)}\Bigg).$$

Moreover, the commutator formula (2.22) implies

$$\left\|\operatorname{div}\partial_t^l\nabla_T^m v\right\|^2_{H^{n-1}(\Omega)} \lesssim \left\|\nabla_T^m\nabla^{n-1}\operatorname{div}\partial_t^l v\right\|^2_{L^2(\Omega)} + \sum_{j=0}^{n+m-1}\left\|\nabla^j\partial_t^l v\right\|^2_{L^2(\Omega)}. \tag{3.39}$$

Therefore, after summing up these estimates, together with (2.39), (2.40) (3.38) follows. □

3.3 Interior Estimates

We will need one last block to establish the stability theory. To track the propagation of regularity away from the boundary, we shall write down the corresponding system in the interior subdomain. Let $\tilde{v} = \psi\cdot v,\ \tilde{q} = \psi\cdot q$. Then from (2.37), the system satisfied by $\tilde{v},\tilde{q}$ admits the form

$$\begin{cases} \partial_t\tilde{q} + \operatorname{div}(\bar{\rho}\tilde{v}) + (v+\bar{u})\cdot\nabla\tilde{q} = \tilde{F}_1 + \tilde{G}_1 & \text{in } \Omega, \\ (\bar{\rho}+\rho)\partial_t\tilde{v} + \bar{\rho}v\cdot\nabla\tilde{v} + \gamma\bar{\rho}\nabla(\bar{\rho}^{\gamma-2}\tilde{q}) - \operatorname{div}\mathbb{S}(\tilde{v}) = \tilde{F}_2 + \tilde{G}_2 & \text{in } \Omega, \\ (\tilde{q},\tilde{v}) = (0,0) & \text{in } \Omega\backslash B_{1/2}, \end{cases} \tag{3.40}$$

where

$$
\begin{aligned}
\tilde{F}_1 &= \bar{\rho} v \cdot \nabla \psi + q \bar{u} \cdot \nabla \psi, \\
\tilde{F}_2 &= \psi F_2 + \gamma \bar{\rho}^{\gamma-1} q \nabla \psi - (\mu+\lambda) \mathrm{div}\, v \nabla \psi - 2\mu (\nabla \psi \cdot \nabla) v \\
&\quad - \mu \Delta \psi v - (\mu+\lambda) \nabla (v \cdot \nabla \psi), \\
\tilde{G}_1 &= \psi G_1 + q v \cdot \nabla \psi, \\
\tilde{G}_2 &= \psi G_2 + \bar{\rho} (v \cdot \nabla \psi) v.
\end{aligned}
$$

After applying the differential operator $\partial^m \partial_t^l$ to (3.40), record the resulting system in the following, where ∂ represents the spatial derivative in any direction (i.e., $\partial = \partial_i$ for $i = 1, 2, 3$),

$$
\begin{cases}
\partial_t \partial^m \partial_t^l \tilde{q} + \mathrm{div}\, (\bar{\rho} \partial^m \partial_t^l \tilde{v}) + (v + \bar{u}) \cdot \nabla \partial^m \partial_t^l \tilde{q} = \tilde{F}_1^{l,m} + \tilde{G}_1^{l,m} & \text{in } \Omega, \\
(\bar{\rho} + q) \partial_t \partial^m \partial_t^l \tilde{v} + \bar{\rho} v \cdot \nabla \partial^m \partial_t^l \tilde{v} + \gamma \bar{\rho} \nabla (\bar{\rho}^{\gamma-2} \partial^m \partial_t^l \tilde{q}) & \\
\qquad - \mathrm{div}\, \mathbb{S}(\partial^m \partial_t^l \tilde{v}) = \tilde{F}_2^{l,m} + \partial^m \partial_t^l \tilde{G}_2 + \tilde{G}_2^{l,m} & \text{in } \Omega, \\
(\partial^m \partial_t^l \tilde{q}, \partial^m \partial_t^l \tilde{v}) = (0, 0) & \text{in } \Omega \backslash B_{3/4},
\end{cases}
\tag{3.41}
$$

where

$$
\begin{aligned}
\tilde{F}_1^{l,m} &= \partial^m \partial_t^l \tilde{F}_1 - [\partial^m, \mathrm{div}\,] (\bar{\rho} \partial_t^l \tilde{v}) - \sum_{j=0}^{m-1} C_{j,m} \mathrm{div}\, (\partial^{m-j} \bar{\rho} \partial^j \partial_t^l \tilde{v}) \\
&\quad - \sum_{j=0}^{m-1} C_{j,m} \partial^{m-j} \bar{u} \cdot \nabla \partial^j \partial_t^l \tilde{q}, \\
\tilde{F}_2^{l,m} &= \partial^m \partial_t^l \tilde{F}_2 - \sum_{j=0}^{m-1} C_{j,m} \partial^{m-j} \bar{\rho} \partial^j \partial_t^{l+1} \tilde{v} \\
&\quad - \gamma \sum_{j=0}^{m-1} \sum_{a=0}^{m-j} C_{j,m} C_{a,m-j} \partial^a \bar{\rho} \nabla (\partial^{m-j-a} \bar{\rho}^{\gamma-2} \partial^j \partial_t^l \tilde{q}), \\
\tilde{G}_1^{l,m} &= \partial^m \partial_t^l \tilde{G}_1 - \sum_{j=0}^{m+l-1} \sum_{a+b=j} C_{a,m} C_{b,l} \partial^{m-a} \partial_t^{l-b} v \cdot \nabla \partial^a \partial_t^b \tilde{q}, \\
\tilde{G}_2^{l,m} &= - \sum_{j=0}^{m+l-1} \sum_{a+b=j} C_{a,m} C_{b,l} \partial^{m-a} \partial_t^{l-b} q \partial^a \partial_t^{b+1} \tilde{v}
\end{aligned}
$$

$$
-\sum_{j=0}^{m+l-1}\sum_{a+b=j}\sum_{c=0}^{m-a}C_{a,m}C_{b,l}C_{c,m-a}\partial^c\bar\rho\partial^{m-a-c}\partial_t^{l-b}v\cdot\nabla\partial^a\partial_t^b\tilde v.
$$

Next lemma is concerning the regularity of (q,v) in the interior subdomain.

Lemma 23 *For $m\geq 1$,*

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\frac12\int_\Omega(\bar\rho+q)\left|\nabla^m\partial_t^l\tilde v\right|^2\,dx+\frac{\gamma}{2}\int_\Omega\bar\rho^{\gamma-2}\left|\nabla^m\partial_t^l\tilde q\right|^2\,dx\Big\}\\
&\qquad+\mu\int_\Omega\left|\nabla^{m+1}\partial_t^l\tilde v\right|^2\,dx\\
&\lesssim(\omega+C_\omega\bar\omega^a+\Lambda_l)\int_\Omega\left|\nabla^m\partial_t^l q\right|^2\,dx+(\omega+\Lambda_l)\int_\Omega\left|\nabla^{m+1}\partial_t^l v\right|^2\,dx\\
&\quad+(C_\omega+\Lambda_l)\Big(\sum_{j=1}^m\int_\Omega\left|\nabla^j\partial_t^l v\right|^2\,dx+\sum_{j=1}^{m-1}\int_\Omega\left|\nabla^j\partial_t^l q\right|^2\,dx\Big)\\
&\quad+\omega\Lambda_l\sum_{j=0}^{m-1}\int_\Omega\left|\nabla^j\partial_t^{l+1}v\right|^2\,dx\\
&\quad+C_\omega\int_\Omega\left|\tilde G_1^{l,m}\right|^2\,dx+\int_\Omega\left|\tilde G_2^{l,m}\right|^2\,dx+\int_\Omega\left|\nabla^{m-1}\partial_t^l\tilde G_2\right|^2\,dx,
\end{aligned}
\tag{3.42}
$$

where $0<\omega<1$ would be addressed later.

Proof Take inner product of $(3.41)_2$ with $\partial^m\partial_t^l\tilde v$ and then record the resulting after integration by parts,

$$
\begin{aligned}
&\frac{d}{dt}\frac12\int_\Omega(\bar\rho+q)\left|\partial^m\partial_t^l\tilde v\right|^2\,dx\underbrace{-\gamma\int_\Omega\operatorname{div}(\bar\rho\partial^m\partial_t^l\tilde v)\bar\rho^{\gamma-2}\partial^m\partial_t^l\tilde q\,dx}_{(i)}\\
&\quad+\mu\int_\Omega\left|\nabla\partial^m\partial_t^l\tilde v\right|^2\,dx+(\mu+\lambda)\int_\Omega\left|\operatorname{div}\partial^m\partial_t^l\tilde v\right|^2\,dx\\
&=\underbrace{\frac12\int_\Omega\partial_t q\left|\partial^m\partial_t^l\tilde v\right|^2\,dx+\frac12\int_\Omega\operatorname{div}(\bar\rho v)\left|\partial^m\partial_t^l\tilde v\right|^2\,dx}_{(ii)}
\end{aligned}
$$

$$+\underbrace{\int_\Omega \tilde F_2^{l,m}\cdot \partial^m\partial_t^l\tilde v\,dx}_{(iii)}+\underbrace{\int_\Omega \tilde G_2^{l,m}\cdot \partial^m\partial_t^l\tilde v\,dx}_{(iv)}+\underbrace{\int_\Omega \partial^m\partial_t^l\tilde G_2\cdot \partial^m\partial_t^l\tilde v\,dx}_{(v)}.$$

Using $(3.41)_1$

$$\begin{aligned}(i)&=\gamma\int_\Omega(\partial_t\partial^m\partial_t^l\tilde q+(v+\bar u)\cdot\nabla\partial^m\partial_t^l\tilde q-\tilde F_1^{l,m}-\tilde G_1^{l,m})\bar\rho^{\gamma-2}\partial^m\partial_t^l\tilde q\,dx\\ &=\frac{d}{dt}\frac{\gamma}{2}\int_\Omega\bar\rho^{\gamma-2}\left|\partial^m\partial_t^l\tilde q\right|^2\,dx-\frac{\gamma}{2}\underbrace{\int_\Omega \operatorname{div}\left(\bar\rho^{\gamma-2}(v+\bar u)\right)\left|\partial^m\partial_t^l\tilde q\right|^2\,dx}_{(vi)}\\ &\quad-\gamma\underbrace{\int_\Omega\tilde F_1^{l,m}\bar\rho^{\gamma-2}\partial^m\partial_t^l\tilde q\,dx}_{(vii)}-\gamma\underbrace{\int_\Omega\tilde G_1^{l,m}\bar\rho^{\gamma-2}\partial^m\partial_t^l\tilde q\,dx}_{(viii)}.\end{aligned}$$

Now, we shall analyse $\tilde F_1^{l,m},\tilde F_2^{l,m}$. From the definition, it holds

$$\begin{aligned}&\left\|\tilde F_1^{l,m}\right\|_{L^2(\Omega)}^2\lesssim(1+\bar\omega^a)\sum_{j=0}^m\int_\Omega\left|\nabla^j\partial_t^l v\right|^2\,dx+\bar\omega^a\sum_{j=0}^m\int_\Omega\left|\nabla^j\partial_t^l q\right|^2\,dx,\\ &\left\|\tilde F_2^{l,m}\right\|_{L^2(\Omega)}^2\lesssim\left\|\partial^m\partial_t^l\tilde F_2\right\|_{L^2(\Omega)}^2+\bar\omega^a\sum_{j=0}^{m-1}\int_\Omega\left|\partial^j\partial_t^{l+1}v\right|^2\,dx\\ &\quad+\bar\omega^a\sum_{j=0}^m\int_\Omega\left|\partial^j\partial_t^l q\right|^2\,dx\lesssim(1+\bar\omega^a)\left(\int_\Omega\left|\nabla^{m+1}\partial_t^l v\right|^2\,dx+\int_\Omega\left|\nabla^m\partial_t^l q\right|^2\,dx\right)\\ &\quad+\bar\omega^a\left(\sum_{j=0}^{m-1}\int_\Omega\left|\nabla^j\partial_t^{l+1}v\right|^2\,dx+\sum_{j=0}^{m-1}\int_\Omega\left|\nabla^j\partial_t^l q\right|^2\,dx+\sum_{j=0}^m\int_\Omega\left|\nabla^j\partial_t^l v\right|^2\,dx\right).\end{aligned}$$

Therefore, Cauchy's inequality then yields, for any $\omega>0,\delta>0$,

$$\begin{aligned}(ii),(vi)&\lesssim(\|\partial_t q\|_{L^\infty(\Omega)}+\bar\omega\,\|v\|_{L^\infty(\Omega)}+\|\nabla v\|_{L^\infty(\Omega)}+\bar\omega^a)\\ &\quad\times\sum_{j=0}^m\left(\int_\Omega\left|\nabla^j\partial_t^l v\right|^2\,dx+\int_\Omega\left|\nabla^j\partial_t^l q\right|^2\,dx\right),\\ (vii)&\lesssim\omega\int_\Omega\left|\nabla^m\partial_t^l q\right|^2\,dx+C_\omega(1+\bar\omega^a)\sum_{j=0}^m\int_\Omega\left|\nabla^j\partial_t^l v\right|^2\,dx\end{aligned}$$

$$+ C_\omega \bar{\omega}^a \sum_{j=0}^{m} \int_\Omega \left|\nabla^j \partial_t^l q\right|^2 \, dx,$$

$$(iii) \lesssim \omega \left\| \tilde{F}_2^{l,m} \right\|_{L^2(\Omega)}^2 + C_\omega \sum_{j=0}^{m} \int_\Omega \left|\nabla^j \partial_t^l v\right|^2 \, dx,$$

$$(v) = - \int_\Omega \partial^{m-1} \partial_t^l \tilde{G}_2 \cdot \partial^{m+1} \partial_t^l \tilde{v} \, dx \lesssim \delta \int_\Omega \left|\nabla^{m+1} \partial_t^l \tilde{v}\right|^2 \, dx$$

$$+ C_\delta \int_\Omega \left|\nabla^{m-1} \partial_t^l \tilde{G}_2\right|^2 \, dx,$$

$$(iv), (viii) \lesssim \omega \sum_{j=1}^{m} \int_\Omega \left|\nabla^j \partial_t^l q\right|^2 \, dx + \sum_{j=0}^{m} \int_\Omega \left|\nabla^j \partial_t^l v\right|^2 \, dx$$

$$+ C_\omega \int_\Omega \left|\tilde{G}_1^{l,m}\right|^2 \, dx + \int_\Omega \left|\tilde{G}_2^{l,m}\right|^2 \, dx.$$

After summing up these estimates with an appropriately small $\delta > 0$, (3.42) follows. □

4 Synthesis and Asymptotic Stability

In this section, we shall chain the energy estimates obtained in the previous sections. In particular, the full energy dissipation and evolution would be achieved, which would capture the propagation of initial regularities as time grows up. Moreover, the decay of the corresponding energy functionals implies that the solution with small initial perturbations would converge to the stationary solution given in (1.11).

4.1 Synthesis

To chain the energy estimates, we first introduce some notations. For $l \geq 0$, denote the energy and dissipation functionals

$$\mathfrak{E}_l = \sum_{i=0}^{l} \left\{ \int_\Omega (\bar{\rho} + q) \left|\partial_t^i v\right|^2 \, dx + c \int_\Omega \left|\nabla \partial_t^i v + \nabla \partial_t^i v^T\right|^2 \, dx \right.$$

$$\left. + c \int_\Omega \left|\mathrm{div}\, \partial_t^i v\right|^2 \, dx + \int_\Omega \bar{\rho}^{\gamma-2} \left|\partial_t^i q\right|^2 \, dx - c \int_\Omega \bar{\rho}^{\gamma-2} \mathrm{div}\,(\bar{\rho} \partial_t^i v) \partial_t^i q \, dx \right\},$$

$$\mathfrak{D}_l = \sum_{i=0}^{l} \Big\{ \int_\Omega \left|\nabla \partial_t^i v\right|^2 \, dx + \int_\Omega \left|\operatorname{div} \partial_t^i v\right|^2 \, dx + \int_\Omega (1+q) \left|\partial_t^{i+1} v\right|^2 \, dx \Big\},$$

where $c > 0$ is determined by (3.8). Also, the tangential and interior energy and dissipation functionals are denoted as,

$$\begin{aligned}
\mathfrak{E}_{l,m}^\tau = & \sum_{i=0}^{l} \sum_{j=0}^{m} \Big\{ \int_\Omega ((\bar\rho + q) \left|\nabla_T^j \partial_t^i v\right|^2 \, dx + \int_\Omega \bar\rho^{\gamma-2} \left|\nabla_T^j \partial_t^i q\right|^2 \, dx \\
& + \int_\Omega (\bar\rho + q) \left|\nabla^j \partial_t^i \tilde v\right|^2 \, dx + \int_\Omega \bar\rho^{\gamma-2} \left|\nabla^j \partial_t^i \tilde q\right|^2 \, dx \Big\}, \\
\mathfrak{D}_{l,m}^\tau = & \sum_{i=0}^{l} \sum_{j=0}^{m} \Big\{ \int_\Omega \left|\nabla \nabla_T^j \partial_t^i v\right|^2 \, dx + \int_\Omega \left|\operatorname{div} \nabla_T^j \partial_t^i v\right|^2 \, dx \\
& + \int_\Omega \left|\nabla^{j+1} \partial_t^i \tilde v\right|^2 \, dx \Big\},
\end{aligned}$$

for $m \geq 0$. The normal direction energy and dissipation functionals are represented by, for $n \geq 1$,

$$\begin{aligned}
\mathfrak{E}_{l,m,n} = & \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=1}^{n} \Big\{ \int_\Omega \left|\nabla_T^j \nabla_N^k \partial_t^i q\right|^2 \, dx \Big\}, \\
\mathfrak{D}_{l,m,n} = & \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=1}^{n} \Big\{ \int_\Omega \left|\nabla_T^j \nabla_N^k \partial_t^i q\right|^2 \, dx + \int_\Omega \left|\nabla_T^j \nabla_N^k \operatorname{div} \partial_t^i v\right|^2 \, dx \\
& + \int_\Omega \left|\nabla^{k+1} \nabla_T^j \partial_t^i v\right|^2 \, dx + \int_\Omega \left|\nabla^k \nabla_T^j \partial_t^i q\right|^2 \, dx \Big\}.
\end{aligned}$$

For $n = 0$, denote

$$\mathfrak{D}_{l,m,0} = \mathfrak{D}_{l,m}^\tau.$$

Additionally, let $\mathfrak{E}_{l,m}, \mathfrak{D}_{l,m}$ be defined as

$$\begin{aligned}
\mathfrak{E}_{l,m} &= \mathfrak{E}_{l,m}^\tau + \sum_{j=1}^{m} \mathfrak{E}_{l,m-j,j}, \\
\mathfrak{D}_{l,m} &= \mathfrak{D}_{l,m}^\tau + \sum_{j=1}^{m} \mathfrak{D}_{l,m-j,j},
\end{aligned}$$

with $m \geq 1$. Moreover, a comparable quantity of $\mathfrak{D}_{l,m}$ can be written as

$$\bar{\mathfrak{D}}_{l,m} = \sum_{i=0}^{l} \sum_{j=1}^{m} \left(\int_\Omega \left| \nabla^{j+1} \partial_t^i v \right|^2 dx + \int_\Omega \left| \nabla^j \partial_t^i q \right|^2 dx \right).$$

Then it holds,

$$\bar{\mathfrak{D}}_{l,m} \lesssim \mathfrak{D}_{l,m}, \; m \geq 1. \tag{4.1}$$

Also, denote

$$\bar{\mathfrak{D}}_{l,0} = \mathfrak{D}_l, \; \bar{\mathfrak{D}}_{l+1,-1} = \mathfrak{D}_l.$$

Notice, the above quantities are monotone increasing in each index l, m, n.

Lemma 24

$$\frac{d}{dt} \{ \mathfrak{E}_l + \mathfrak{E}_{l,1} \} + \mathfrak{D}_l + \mathfrak{D}_{l,1} \lesssim \Lambda_l \left(\mathfrak{D}_l + \mathfrak{D}_{l,1} \right) + \mathfrak{G}_l, \tag{4.2}$$

where $\mathfrak{G}_l$ *is defined as*

$$\begin{aligned} \mathfrak{G}_l = \sum_{i=0}^{l} \Bigg(& \int_\Omega \left| G_1^i \right|^2 dx + \int_\Omega \left| G_2^i \right|^2 dx + \int_\Omega \left| G_1^{i,1} \right|^2 dx + \int_\Omega \left| G_2^{i,1} \right|^2 dx \\ & + \int_\Omega \left| \tilde{G}_1^{i,1} \right|^2 dx + \int_\Omega \left| \tilde{G}_2^{i,1} \right|^2 dx + \int_\Omega \left| \partial_t^i \tilde{G}_2 \right|^2 dx \\ & + \int_\Omega \left| G_{1,N}^i \right|^2 dx + \int_\Omega \left| G_{2,N}^i \right|^2 dx + \int_\Omega \left| G_s^i \right|^2 dx \Bigg). \end{aligned} \tag{4.3}$$

Proof From (3.7),

$$\begin{aligned} \frac{d}{dt} \mathfrak{E}_l + \mathfrak{D}_l \lesssim & (\omega + \Lambda_l) \mathfrak{D}_{l,1} + \Lambda_l \mathfrak{D}_l \\ & + (1 + C_\omega) \sum_{i=0}^{l} \int_\Omega \left| G_1^i \right|^2 dx + \sum_{i=0}^{l} \int_\Omega \left| G_2^i \right|^2 dx. \end{aligned} \tag{4.4}$$

Meanwhile, as the consequences of (3.27) and (3.42) with $m = 1$, by choosing an appropriately small $\omega > 0$,

$$\frac{d}{dt} \mathfrak{E}_{l,1} + \mathfrak{D}_{l,1} \lesssim (1 + \Lambda_l) \mathfrak{D}_l + \Lambda_l \mathfrak{D}_{l,1} + \sum_{i=0}^{l} \Bigg(\int_\Omega \left| G_1^{i,1} \right|^2 dx + \int_\Omega \left| G_2^{i,1} \right|^2 dx$$

$$+\int_\Omega \left|G_2^i\right|^2\,dx+\int_\Omega \left|\tilde{G}_1^{i,1}\right|^2\,dx+\int_\Omega \left|\tilde{G}_2^{i,1}\right|^2\,dx+\int_\Omega \left|\partial_t^i\tilde{G}_2\right|^2\,dx \quad (4.5)$$

$$+\int_\Omega \left|G_{1,N}^i\right|^2\,dx+\int_\Omega \left|G_{2,N}^i\right|^2\,dx+\int_\Omega \left|G_s^i\right|^2\,dx\Bigg).$$

Then a sum of (4.5) and (4.4) with an appropriately small ω in (4.4) yields (4.2). □

From (3.31), (3.32) and (3.38), the following estimates holds with $m+n\geq 2, n\geq 1$,

$$\frac{d}{dt}\int_\Omega \left|\nabla_T^m\nabla_N^n\partial_t^l q\right|^2\,dx+\int_\Omega \bar{\rho}^\gamma\left|\nabla_T^m\nabla_N^n\partial_t^l q\right|^2\,dx\lesssim \Lambda_l\bar{\mathfrak{D}}_{l,m+n}$$

$$+\mathfrak{D}_{l,m+1,n-1}+\bar{\mathfrak{D}}_{l+1,m+n-2}+(1+\Lambda_l)\left(\bar{\mathfrak{D}}_{l,m+n-1}+\bar{\mathfrak{D}}_{l+1,m+n-3}\right) \quad (4.6)$$

$$+\int_\Omega \left|G_{1,N}^{l,m,n}\right|^2\,dx,$$

$$\int_\Omega \left|\nabla_T^m\nabla_N^n\mathrm{div}\,\partial_t^l v\right|^2\,dx\lesssim\int_\Omega \left|\nabla_T^m\nabla_N^n\partial_t^l q\right|^2\,dx+\mathfrak{D}_{l,m+1,n-1}+\bar{\mathfrak{D}}_{l+1,m+n-2}$$

$$+(1+\Lambda_l)\left(\bar{\mathfrak{D}}_{l,m+n-1}+\bar{\mathfrak{D}}_{l+1,m+n-3}\right)+\int_\Omega \left|G_{2,N}^{l,m,n}\right|^2\,dx, \quad (4.7)$$

$$\int_\Omega \left|\nabla^{n+1}\nabla_T^m\partial_t^l v\right|^2\,dx+\int_\Omega \left|\nabla^n\nabla_T^m\partial_t^l q\right|^2\,dx\lesssim\int_\Omega \left|\nabla_T^m\nabla^n\mathrm{div}\,\partial_t^l v\right|^2\,dx$$

$$+\mathfrak{D}_{l,m+n}^\tau+\bar{\mathfrak{D}}_{l+1,m+n-2}+(1+\Lambda_l)\left(\bar{\mathfrak{D}}_{l+1,n+m-3}+\bar{\mathfrak{D}}_{l,m+n-1}\right) \quad (4.8)$$

$$+\left\|G_s^{l,m}\right\|_{H^{n-1}(\Omega)}^2.$$

Additionally, by using (2.26) and the commutator property (2.22) repeatedly,

$$\int_\Omega \left|\nabla_T^m\nabla^n\mathrm{div}\,\partial_t^l v\right|^2\,dx\lesssim\int_\Omega \left|\nabla_T^m\nabla_N^n\mathrm{div}\,\partial_t^l v\right|^2\,dx+\int_\Omega \left|\nabla^n\nabla_T^{m+1}\partial_t^l v\right|^2\,dx$$

$$+\int_{B_{1/2}} \left|\nabla^{m+n+1}\partial_t^l v\right|^2\,dx+\sum_{j=1}^{m+n}\int_\Omega \left|\nabla^j\partial_t^l v\right|^2\,dx\lesssim\int_\Omega \left|\nabla_T^m\nabla_N^n\mathrm{div}\,\partial_t^l v\right|^2\,dx$$

$$+\mathfrak{D}_{l,m+1,n-1}+\mathfrak{D}_{l,m+n}^\tau+\bar{\mathfrak{D}}_{l,m+n-1}.$$

Therefore, chaining these inequalities together with (3.21), (3.22) and (3.25) then yields, for $m+n\geq 2, n\geq 1$,

$$\frac{d}{dt}\mathfrak{E}_{l,m,n}+\mathfrak{D}_{l,m,n}\lesssim\Lambda_l\bar{\mathfrak{D}}_{l,m+n}+\mathfrak{D}_{l,m+1,n-1}+\mathfrak{D}_{l,m+n}^\tau$$

$$
+ \bar{\mathfrak{D}}_{l+1,m+n-2} + (1+\Lambda_l)\Big(\bar{\mathfrak{D}}_{l,m+n-1} + \bar{\mathfrak{D}}_{l+1,m+n-3}\Big) \tag{4.9}
$$

$$
+ \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=1}^{n}\Big(\int_\Omega \left|G_{1,N}^{i,j,k}\right|^2 \, dx + \int_\Omega \left|G_{2,N}^{i,j,k}\right|^2 \, dx + \left\| G_s^{i,j} \right\|_{H^{k-1}(\Omega)}^2\Big).
$$

On the other hand, as the consequence of (3.29) and (3.42) together with (3.5) for $m \geq 1$,

$$
\begin{aligned}
\frac{d}{dt}\mathfrak{E}_{l,m}^\tau + \mathfrak{D}_{l,m}^\tau &\lesssim (\omega + C_\omega \bar{\omega}^a + \Lambda_l)\bar{\mathfrak{D}}_{l,m} + (\omega + C_\omega + \Lambda_l)\bar{\mathfrak{D}}_{l,m-1} \\
&+ \omega\Lambda_l \bar{\mathfrak{D}}_{l+1,m-2} + C_\omega \sum_{i=0}^{l}\sum_{j=1}^{m}\Big(\int_\Omega \left|G_1^{i,j}\right|^2 \, dx + \int_\Omega \left|G_2^{i,j}\right|^2 \, dx \qquad (4.10) \\
&+ \int_\Omega \left|\nabla_T^{j-1} G_2^i\right|^2 \, dx + \int_\Omega \left|\tilde{G}_1^{i,j}\right|^2 \, dx + \int_\Omega \left|\tilde{G}_2^{i,j}\right|^2 \, dx + \int_\Omega \left|\nabla^{j-1}\partial_t^i \tilde{G}_2\right|^2 \, dx\Big).
\end{aligned}
$$

Now we would be able to chain the total energy estimates.

Lemma 25 *For $k \geq 2$,*

$$
\begin{aligned}
\frac{d}{dt}\mathfrak{E}_{l,k} + \mathfrak{D}_{l,k} &\lesssim \Lambda_l \mathfrak{D}_{l,k} + (1+\Lambda_l)\bar{\mathfrak{D}}_{l+1,k-2} + (1+\Lambda_l)\bar{\mathfrak{D}}_{l,k-1} \\
&+ (1+\Lambda_l)\bar{\mathfrak{D}}_{l+1,k-3} + \mathfrak{G}_{l,k},
\end{aligned} \tag{4.11}
$$

where $\mathfrak{G}_{l,k}$ is defined in (4.14)*. In particular,*

$$
\frac{d}{dt}\mathfrak{E}_{l,2} + \mathfrak{D}_{l,2} \lesssim \Lambda_l \mathfrak{D}_{l,2} + (1+\Lambda_l)\left(\mathfrak{D}_{l+1} + \mathfrak{D}_{l,1}\right) + \mathfrak{G}_{l,2}, \tag{4.12}
$$

$$
\frac{d}{dt}\mathfrak{E}_{l,3} + \mathfrak{D}_{l,3} \lesssim \Lambda_l \mathfrak{D}_{l,3} + (1+\Lambda_l)\left(\mathfrak{D}_{l+1,1} + \mathfrak{D}_{l+1} + \mathfrak{D}_{l,2}\right) + \mathfrak{G}_{l,3}. \tag{4.13}
$$

Proof Consider (4.9) with $m = k-j, n = j$ for $j \geq 1$,

$$
\begin{aligned}
\frac{d}{dt}\mathfrak{E}_{l,k-j,j} + \mathfrak{D}_{l,k-j,j} &\lesssim \Lambda_l \bar{\mathfrak{D}}_{l,k} + \mathfrak{D}_{l,k-j+1,j-1} \\
&+ \mathfrak{D}_{l,k}^\tau + \bar{\mathfrak{D}}_{l+1,k-2} + (1+\Lambda_l)\left(\bar{\mathfrak{D}}_{l,k-1} + \bar{\mathfrak{D}}_{l+1,k-3}\right) + \sum_{a=0}^{l}\sum_{b=0}^{k-j}\sum_{c=1}^{j}\mathfrak{G}_{a,b,c},
\end{aligned}
$$

where

$$
\mathfrak{G}_{a,b,c} = \int_\Omega \left|G_{1,N}^{a,b,c}\right|^2 \, dx + \int_\Omega \left|G_{2,N}^{a,b,c}\right|^2 \, dx + \left\| G_s^{a,b} \right\|_{H^{c-1}(\Omega)}^2 .
$$

Thus, after summing from $j=1$ to $j=k$, together with (4.10) with $m=k$ and an appropriately small $\omega>0$, it holds

$$\frac{d}{dt}\mathfrak{E}_{l,k}+\mathfrak{D}_{l,k}\lesssim\Lambda_l\mathfrak{D}_{l,k}+(1+\Lambda_l)\bar{\mathfrak{D}}_{l+1,k-2}+(1+\Lambda_l)\bar{\mathfrak{D}}_{l,k-1}$$
$$+(1+\Lambda_l)\bar{\mathfrak{D}}_{l+1,k-3}+\mathfrak{G}_{l,k},$$

where

$$\mathfrak{G}_{l,k}=\sum_{j=1}^{k}\sum_{a=0}^{l}\sum_{b=0}^{k-j}\sum_{c=1}^{j}\mathfrak{G}_{a,b,c}+\sum_{a=0}^{l}\sum_{b=1}^{k}\Bigg(\int_\Omega\left|G_1^{a,b}\right|^2dx+\int_\Omega\left|G_2^{a,b}\right|^2dx$$
$$+\int_\Omega\left|\nabla_T^{b-1}G_2^a\right|^2dx+\int_\Omega\left|\tilde{G}_1^{a,b}\right|^2dx+\int_\Omega\left|\tilde{G}_2^{a,b}\right|^2dx+\int_\Omega\left|\nabla^{b-1}\partial_t^a\tilde{G}_2\right|^2dx\Bigg).\tag{4.14}$$

□

Thus we have shown the following

Proposition 3 *For $L\geq 2$,*

$$\frac{d}{dt}\bar{\mathfrak{E}}_L(t)+(1-\Lambda_L)\bar{\mathfrak{D}}_L(t)\lesssim\mathfrak{G}_L(t)+\sum_{i=1}^{L}\mathfrak{G}_{L-i,2i+1}(t),\tag{4.15}$$

where

$$\bar{\mathfrak{E}}_L(t)=\mathfrak{E}_L(t)+\mathfrak{E}_{L,1}(t)+\sum_{i=1}^{L}\left(\mathfrak{E}_{L-i,2i}(t)+\mathfrak{E}_{L-i,2i+1}(t)\right),\tag{4.16}$$

$$\bar{\mathfrak{D}}_L(t)=\mathfrak{D}_L(t)+\mathfrak{D}_{L,1}(t)+\sum_{i=1}^{L}\left(\mathfrak{D}_{L-i,2i}(t)+\mathfrak{D}_{L-i,2i+1}(t)\right).\tag{4.17}$$

Proof For $i=1$, as the consequence of (4.12) and (4.13),

$$\begin{aligned}\frac{d}{dt}\left(\mathfrak{E}_{L-1,2}+\mathfrak{E}_{L-1,3}\right)+\mathfrak{D}_{L-1,2}+\mathfrak{D}_{L-1,3}&\lesssim\Lambda_L\left(\mathfrak{D}_{L-1,2}+\mathfrak{D}_{L-1,3}\right)\\&\quad+(1+\Lambda_L)\left(\mathfrak{D}_L+\mathfrak{D}_{L,1}\right)+\mathfrak{G}_{L-1,3}.\end{aligned}\tag{4.18}$$

Similarly, for $i\geq 2, 1\leq j\leq i$, from (4.11),

$$\frac{d}{dt}\mathfrak{E}_{L-i,2j}+\mathfrak{D}_{L-i,2j}\lesssim\Lambda_L\mathfrak{D}_{L-i,2j}$$

$$
\begin{aligned}
&\quad + (1+\Lambda_L)\left(\mathfrak{D}_{L-i+1,2j-2} + \mathfrak{D}_{L-i,2j-1}\right) + \mathfrak{G}_{L-i,2j},\\
&\frac{d}{dt}\mathfrak{E}_{L-i,2j+1} + \mathfrak{D}_{L-i,2j+1} \lesssim \Lambda_L \mathfrak{D}_{L-i,2j+1}\\
&\quad + (1+\Lambda_L)\left(\mathfrak{D}_{L-i+1,2j-1} + \mathfrak{D}_{L-i,2j}\right) + \mathfrak{G}_{L-i,2j+1}.
\end{aligned}
$$

Hence, after summing over j, it holds

$$
\begin{aligned}
&\frac{d}{dt}\left(\mathfrak{E}_{L-i,2i} + \mathfrak{E}_{L-i,2i+1}\right) + \mathfrak{D}_{L-i,2i} + \mathfrak{D}_{L-i,2i+1} \lesssim \Lambda_L(\mathfrak{D}_{L-i,2i} + \mathfrak{D}_{L-i,2i+1})\\
&\quad + (1+\Lambda_L)(\mathfrak{D}_{L-i,1} + \mathfrak{D}_{L-i+1,2i-1}) + \mathfrak{G}_{L-i,2i+1} \lesssim \Lambda_L(\mathfrak{D}_{L-i,2i}\\
&\quad + \mathfrak{D}_{L-i,2i+1}) + (1+\Lambda_L)(\mathfrak{D}_{L,1} + \mathfrak{D}_{L-i+1,2i-1}) + \mathfrak{G}_{L-i,2i+1}.
\end{aligned} \tag{4.19}
$$

Then (4.15) is the consequence of (4.2) with $l = L$, (4.18) and summing (4.19) with i from 2 to L. □

4.2 Estimates on Nonlinearities

In this section, we shall perform the estimates on the nonlinearities. To do so, denote the intermediate energy and dissipation $\mathcal{E}_L, \mathcal{D}_L$ as

$$
\bar{\mathcal{E}}_L = \left\| \partial_t^L v \right\|^2_{H^1(\Omega)} + \sum_{i=0}^{L-1} \left\| \partial_t^i v \right\|^2_{H^{2L-2i}(\Omega)} + \sum_{i=0}^{L} \left\| \partial_t^i q \right\|^2_{H^{2L-2i+1}(\Omega)}, \tag{4.20}
$$

$$
\bar{\mathcal{D}}_L = \sum_{i=0}^{L} \left(\left\| \partial_t^i v \right\|^2_{H^{2L-2i+2}(\Omega)} + \left\| \partial_t^i q \right\|^2_{H^{2L-2i+1}(\Omega)} \right) + \left\| \partial_t^{L+1} v \right\|^2_{L^2(\Omega)}. \tag{4.21}
$$

Then it holds,

$$
\sum_{2a+b\le 2L-2} \left\| \nabla^b \partial_t^a v \right\|^2_{L^\infty(\Omega)} + \sum_{2a+b\le 2L-1} \left\| \nabla^b \partial_t^a q \right\|^2_{L^\infty(\Omega)} \lesssim \bar{\mathcal{E}}_L, \tag{4.22}
$$

$$
\sum_{2a+b\le 2L+2} \left\| \nabla^b \partial_t^a v \right\|^2_{L^2(\Omega)} + \sum_{2a+b\le 2L+1} \left\| \nabla^b \partial_t^a q \right\|^2_{L^2(\Omega)} \lesssim \bar{\mathcal{D}}_L. \tag{4.23}
$$

We shall use (4.22) and (4.23) to manipulate the nonlinear terms on the right hand side of (4.15).

Lemma 26 *For $L \geq 3$,*

$$\mathfrak{G}_L + \sum_{i=1}^{L} \mathfrak{G}_{L-i,2i+1} \lesssim \mathfrak{P}(\bar{\mathcal{E}}_L)\bar{\mathcal{D}}_L. \tag{4.24}$$

In the meantime,

$$\Lambda_L \lesssim \mathfrak{P}(\bar{\omega}, \bar{\mathcal{E}}_L). \tag{4.25}$$

where $\mathfrak{P}(\cdot)$ is a polynomial with the property $\mathfrak{P}(0) = 0$.

Proof We only show $\mathfrak{G}_L$, and others can be handled in a similar way. Recall,

$$\begin{aligned}\mathfrak{G}_L = \sum_{i=0}^{L} \Bigg(& \underbrace{\int_\Omega \left|G_1^i\right|^2 dx + \int_\Omega \left|G_2^i\right|^2 dx}_{(i)} + \underbrace{\int_\Omega \left|G_1^{i,1}\right|^2 dx + \int_\Omega \left|G_2^{i,1}\right|^2 dx}_{(ii)} \\ & + \underbrace{\int_\Omega \left|\tilde{G}_1^{i,1}\right|^2 dx + \int_\Omega \left|\tilde{G}_2^{i,2}\right|^2 dx + \int_\Omega \left|\partial_t^i \tilde{G}_2\right|^2 dx}_{(iii)} \\ & + \underbrace{\int_\Omega \left|G_{1,N}^i\right|^2 dx + \int_\Omega \left|G_{2,N}^i\right|^2 dx + \int_\Omega \left|G_s^i\right|^2 dx}_{(iv)} \Bigg).\end{aligned}$$

We shall consider the highest order terms only. In the following, let $l.o.t$ denote the terms involving relatively lower order derivatives than others, which might be different from line by line. To evaluate (i), from (2.38), (3.2),

$$\begin{aligned}\left|G_1^L\right| &\lesssim \left|\partial_t^L G_1\right| + \sum_{j=0}^{L-1} \left|\partial_t^{L-j} v\right| \left|\nabla \partial_t^j q\right| \lesssim \sum_{j=0}^{L} \left|\partial_t^j q\right| \left|\nabla \partial_t^{L-j} v\right| \\ &+ \sum_{j=0}^{L-1} \left|\partial_t^{L-j} v\right| \left|\nabla \partial_t^j q\right| \lesssim \sum_{j=0}^{[L/2]} \left(\left|\partial_t^j q\right| + \left|\nabla \partial_t^j v\right|\right) \left(\left|\nabla \partial_t^{L-j} v\right| + \left|\partial_t^{L-j} q\right|\right) \\ &+ \sum_{j=0}^{[L/2]} \left(\left|\partial_t^{j+1} v\right| + \left|\nabla \partial_t^j q\right|\right) \left(\left|\nabla \partial_t^{L-j-1} q\right| + \left|\partial_t^{L-j} v\right|\right).\end{aligned}$$

Similarly,

$$
\begin{aligned}
\left|G_2^L\right| &\lesssim \left|\partial_t^L G_2\right| + \sum_{j=0}^{L-1}\left(\left|\partial_t^{j+1}q\right|\left|\partial_t^{L-j}v\right| + \left|\partial_t^{j+1}v\right|\left|\nabla\partial_t^{L-j-1}v\right|\right)\\
&\lesssim \sum_{j=0}^{[L/2]}\left(\left|\partial_t^j q\right| + \left|\nabla\partial_t^j q\right| + \left|\partial_t^j v\right| + \left|\nabla\partial_t^j v\right|\right)\\
&\qquad\qquad \times\left(\left|\partial_t^{L-j}q\right| + \left|\nabla\partial_t^{L-j}q\right| + \left|\partial_t^{L-j}v\right| + \left|\nabla\partial_t^{L-j}v\right|\right)\\
&\quad + \sum_{j=0}^{[L/2]}\left(\left|\partial_t^{j+1}q\right| + \left|\partial_t^{j+1}v\right|\right)\left(\left|\partial_t^{L-j}v\right| + \left|\partial_t^{L-j}q\right|\right)\\
&\quad + \sum_{j=0}^{[L/2]}\left(\left|\partial_t^{j+1}v\right| + \left|\nabla\partial_t^j v\right|\right)\left(\left|\nabla\partial_t^{L-j-1}v\right| + \left|\partial_t^{L-j}v\right|\right) + \mathfrak{C},
\end{aligned}
$$

where $\mathfrak{C}$ denotes cubic terms. Therefore,

$$
\begin{aligned}
(i) \lesssim \mathcal{A}\cdot\sum_{j=0}^{[L/2]}&\left(\int_\Omega\left|\partial_t^{L-j}q\right|^2 dx + \int_\Omega\left|\nabla\partial_t^{L-j}q\right|^2 dx + \int_\Omega\left|\nabla\partial_t^{L-j-1}q\right|^2 dx\right.\\
&\left. + \int_\Omega\left|\partial_t^{L-j}v\right|^2 dx + \int_\Omega\left|\nabla\partial_t^{L-j}v\right|^2 dx + \int_\Omega\left|\nabla\partial_t^{L-j-1}v\right|^2 dx\right),
\end{aligned}
$$

where $\mathcal{A} = \mathcal{A}(\cdot)$ is a polynomial of

$$
\begin{aligned}
\Big\{&\left\|\partial_t^j q\right\|_{L^\infty(\Omega)}, \left\|\partial_t^{j+1}q\right\|_{L^\infty(\Omega)}, \left\|\nabla\partial_t^j q\right\|_{L^\infty(\Omega)}\\
&\left\|\partial_t^j v\right\|_{L^\infty(\Omega)}, \left\|\partial_t^{j+1}v\right\|_{L^\infty(\Omega)}, \left\|\nabla\partial_t^j v\right\|_{L^\infty(\Omega)}\Big\}_{0\le j\le[L/2]},
\end{aligned}
$$

with the property $\mathcal{A}(0) = 0$. Thus from (4.22) and (4.23), $(i) \lesssim \mathfrak{P}(\bar{\mathcal{E}}_L)\bar{\mathcal{D}}_L$ provided

$$
2([L/2]+1) \le 2L-2, \text{ or equivalently } L \ge 3.
$$

Similarly, from (3.10)

$$
\left|G_1^{L,1}\right| \lesssim \left|\nabla G_1^L\right| + |\nabla v|\left|\nabla\partial_t^L q\right| + |v|\left|\nabla\partial_t^L q\right|
$$

$$
\begin{aligned}
&\lesssim \sum_{j=0}^{L}\left(\left|\nabla\partial_t^j q\right|\left|\nabla\partial_t^{L-j} v\right|+\left|\partial_t^j q\right|\left|\nabla^2\partial_t^{L-j} v\right|\right)\\
&\quad+\sum_{j=0}^{L-1}\left(\left|\nabla\partial_t^{L-j} v\right|\left|\nabla\partial_t^j q\right|+\left|\partial_t^{L-j} v\right|\left|\nabla^2\partial_t^j q\right|\right)+\underbrace{|\nabla v|}_{L^\infty}\underbrace{\left|\nabla\partial_t^L q\right|}_{L^2}\\
&\quad+\underbrace{|v|}_{L^\infty}\underbrace{\left|\nabla\partial_t^L q\right|}_{L^2}+l.o.t,\\
\left|G_2^{L,1}\right| &\lesssim \underbrace{|\nabla q|}_{L^\infty}\underbrace{\left|\partial_t^{L+1} v\right|}_{L^2}+\underbrace{|v|}_{L^\infty}\underbrace{\left|\nabla\partial_t^L v\right|}_{L^2}+\underbrace{(|\nabla v|+|v|)}_{L^\infty}\underbrace{\left|\nabla\partial_t^L v\right|}_{L^2}+l.o.t,
\end{aligned}
$$

where the subscripts represent the corresponding norms to bound the terms. The same notations would be adopted in the following. Meanwhile, the rests in $\left|G_1^{L,1}\right|$ are bounded in a similar way.

$$
\begin{aligned}
\sum_{j=0}^{L}\left|\nabla\partial_t^j q\right|\left|\nabla\partial_t^{L-j} v\right| &\lesssim \sum_{\substack{2j+1\le 2L-1,\\ 2(L-j)+1\le 2L+2}}\left|\nabla\partial_t^j q\right|\left|\nabla\partial_t^{L-j} v\right|\\
&\quad+\sum_{\substack{2(L-j)+1\le 2L-2,\\ 2j+1\le 2L+1}}\left|\nabla\partial_t^{L-j} v\right|\left|\nabla\partial_t^j q\right| = \sum_{0\le j\le L-1}\underbrace{\left|\nabla\partial_t^j q\right|}_{L^\infty}\underbrace{\left|\nabla\partial_t^{L-j} v\right|}_{L^2}\\
&\quad+\sum_{3/2\le j\le L}\underbrace{\left|\nabla\partial_t^{L-j} v\right|}_{L^\infty}\underbrace{\left|\nabla\partial_t^j q\right|}_{L^2},
\end{aligned}
$$

provided

$$3/2 \le L-1, \text{ or equivalently } L \ge 3.$$

Similar arguments then yield,

$$
\begin{aligned}
\sum_{j=0}^{L}\left|\partial_t^j q\right|\left|\nabla^2\partial_t^{L-j} v\right| &\lesssim \sum_{0\le j\le L-1}\underbrace{\left|\partial_t^j q\right|}_{L^\infty}\underbrace{\left|\nabla^2\partial_t^{L-j} v\right|}_{L^2}+\sum_{2\le j\le L}\underbrace{\left|\nabla^2\partial_t^{L-j} v\right|}_{L^\infty}\underbrace{\left|\partial_t^j q\right|}_{L^2},\\
\sum_{j=0}^{L-1}\left|\nabla\partial_t^{L-j} v\right|\left|\nabla\partial_t^j q\right| &\lesssim \sum_{2\le j\le L}\underbrace{\left|\nabla\partial_t^{L-j} v\right|}_{L^\infty}\underbrace{\left|\nabla\partial_t^j q\right|}_{L^2}+\sum_{0\le j\le L-1}\underbrace{\left|\nabla\partial_t^j q\right|}_{L^\infty}\underbrace{\left|\nabla\partial_t^{L-j} v\right|}_{L^2},
\end{aligned}
$$

$$\sum_{j=0}^{L-1} \left|\partial_t^{L-j} v\right| \left|\nabla^2 \partial_t^j q\right| \lesssim \sum_{1\le j\le L-1} \underbrace{\left|\partial_t^{L-j} v\right|}_{L^\infty} \underbrace{\left|\nabla^2 \partial_t^j q\right|}_{L^2} + \sum_{0\le j\le L-2} \underbrace{\left|\nabla^2 \partial_t^j q\right|}_{L^\infty} \underbrace{\left|\partial_t^{L-j} v\right|}_{L^2},$$

provided

$$L \geq 3.$$

Thus, $(ii) \lesssim \mathfrak{P}(\bar{\mathcal{E}}_L)\bar{\mathcal{D}}_L$. Similarly $(iii) \lesssim \mathfrak{P}(\bar{\mathcal{E}}_L)\bar{\mathcal{D}}_L$. To handle (iv), from (3.20), (3.24)

$$\begin{aligned} &\left|G_{1,N}^L\right|, \left|G_{2,N}^L\right|, \left|G_s^L\right| \lesssim \left|\nabla G_1^L\right| + |\nabla v| \left|\nabla \partial_t^L q\right| + |v| \left|\nabla \partial_t^L q\right| \\ &\quad + |q| \left|\partial_t^{L+1} v\right| + |v| \left|\nabla \partial_t^L v\right| + \left|G_2^L\right|, \end{aligned}$$

in which the terms on the right hand side have already appeared before. Hence, $(iv) \lesssim \mathfrak{P}(\bar{\mathcal{E}}_L)\bar{\mathcal{D}}_L$. We have shown $\mathfrak{G}_L \lesssim \mathfrak{P}(\bar{\mathcal{E}}_L)\bar{\mathcal{D}}_L$. The estimate on Λ_L is the direct consequence of (4.22). □

Next lemma concerns the quantity analysis of $\bar{\mathcal{E}}_L$ and $\bar{\mathcal{D}}_L$.

Lemma 27 *It holds the equivalent relation*

$$\bar{\mathcal{D}}_L \simeq \bar{\mathfrak{D}}_L. \tag{4.26}$$

In the meantime,

$$\left\| \partial_t^L v \right\|_{H^1(\Omega)}^2 + \sum_{i=0}^{L} \left\| \partial_t^i q \right\|_{H^{2L-2i+1}(\Omega)}^2 \lesssim \bar{\mathfrak{E}}_L(t) \lesssim \bar{\mathcal{D}}_L(t), \tag{4.27}$$

$$\bar{\mathcal{E}}_L \lesssim \bar{\mathcal{D}}_L. \tag{4.28}$$

Also, a direct calculation yields,

$$\frac{d}{dt} \sum_{i=0}^{L-1} \left\| \partial_t^i v \right\|_{H^{2L-2i}(\Omega)}^2 \lesssim \sum_{i=0}^{L} \left\| \partial_t^i v \right\|_{H^{2L-2i+2}(\Omega)}^2 \lesssim \bar{\mathcal{D}}_L. \tag{4.29}$$

Proof As a consequence of the definitions, it is easy to show,

$$\mathfrak{D}_L + \mathfrak{D}_{L,1} + \sum_{i=1}^{L} \left(\mathfrak{D}_{L-i,2i} + \mathfrak{D}_{L-i,2i+1} \right) \lesssim \bar{\mathcal{D}}_L.$$

Meanwhile, from the estimate (2.26), the commutator property (2.22), and the Poincaré inequality (2.39), (2.40), it can be shown without difficulty

$$\bar{\mathcal{D}}_L \lesssim \mathfrak{D}_L + \mathfrak{D}_{L,1} + \sum_{i=1}^{L} \left(\mathfrak{D}_{L-i,2i} + \mathfrak{D}_{L-i,2i+1}\right).$$

(4.27), (4.28) can be derived in a similar way. □

4.3 Global Prior Estimate and Asymptotic Stability

Let $\mathfrak{L}(t)$ be defined as

$$\mathfrak{L}(t) = \max_{0<s<t} \bar{\mathcal{E}}_L(t) + \int_0^t \bar{\mathcal{D}}_L(s)\, ds.$$

Lemma 28 *There is a $\epsilon_0 > 0$, such that $\bar{\omega} \leq \epsilon_0, \mathfrak{L}(t) \leq \epsilon_0$ would imply*

$$e^{\sigma t}\bar{\mathfrak{E}}_L(t) + \int_0^t e^{\sigma s}\bar{\mathfrak{D}}_L(s)\, ds \leq \bar{\mathfrak{E}}_L(0), \tag{4.30}$$

for some $\sigma > 0$.

Proof From (4.15), and (4.24), (4.25) and the equivalence (4.26), it holds

$$\frac{d}{dt}\bar{\mathfrak{E}}_L(t) + \bar{\mathfrak{D}}_L(t) \lesssim \mathfrak{P}(\bar{\omega}, \bar{\mathcal{E}}_L(t)) \left(\bar{\mathcal{D}}_L(t) + \bar{\mathfrak{D}}_L(t)\right) \lesssim \mathfrak{P}(\bar{\omega}, \mathfrak{L}(t))\bar{\mathfrak{D}}_L(t),$$

where $\mathfrak{P} = \mathfrak{P}(\cdot)$ is a polynomial with the property $\mathfrak{P}(0) = 0$. Thus, $\exists\epsilon_0 > 0$ such that $\mathfrak{L} \leq \epsilon_0, \bar{\omega} \leq \epsilon_0$ would yield $\mathfrak{P} \leq \bar{\epsilon}$ with $\bar{\epsilon} > 0$ small enough such that the following holds

$$\frac{d}{dt}\bar{\mathfrak{E}}_L(t) + \bar{\mathfrak{D}}_L(t) \lesssim 0.$$

Multiply this inequality with $e^{\sigma t}$, together with (4.27),

$$\frac{d}{dt}\left\{e^{\sigma t}\bar{\mathfrak{E}}_L(t)\right\} + e^{\sigma t}\bar{\mathfrak{D}}_L(t) \lesssim \sigma e^{\sigma t}\bar{\mathfrak{E}}_L(t) \lesssim \sigma e^{\sigma t}\bar{\mathfrak{D}}_L(t).$$

Thus (4.30) follows by integrating this inequality over temporal variable with an appropriately small $\sigma > 0$. □

Meanwhile, we shall demonstrate that the estimate (4.30) would in turn yield the boundedness of $\mathfrak{L}(t)$. More precisely,

Lemma 29 *There is a* $0 < \epsilon_1 \leq \epsilon_0$*, such that under the assumption that* $\bar{\mathfrak{E}}_L(0) \leq \epsilon_1$ *and* $\bar{\mathcal{E}}_L(0) \leq \epsilon_1$*, (4.30) would imply*

$$e^{\sigma t}\bar{\mathcal{E}}_L(t) + \int_0^t e^{\sigma s}\bar{\mathcal{D}}_L(s)\,ds \leq \epsilon_0. \tag{4.31}$$

In particular,

$$\mathfrak{L}(t) \leq \epsilon_0. \tag{4.32}$$

Proof From (4.27) and (4.30)

$$e^{\sigma t}\left(\left\| \partial_t^L v \right\|^2_{H^1(\Omega)} + \sum_{i=0}^{L} \left\| \partial_t^i q \right\|^2_{H^{2L-2i+1}(\Omega)} \right) \lesssim e^{\sigma t}\bar{\mathfrak{E}}(t) \leq \epsilon_1. \tag{4.33}$$

Similarly, from (4.29), (4.28), (4.26).

$$\begin{aligned}\frac{d}{dt}\left(e^{\sigma t}\sum_{i=0}^{L-1} \left\| \partial_t^i v \right\|^2_{H^{2L-2i}(\Omega)} \right) &\lesssim \sigma e^{\sigma t}\sum_{i=0}^{L-1} \left\| \partial_t^i v \right\|^2_{H^{2L-2i}(\Omega)} + e^{\sigma t}\bar{\mathcal{D}}_L \\ &\lesssim \sigma e^{\sigma t}\bar{\mathcal{E}}_L + e^{\sigma t}\bar{\mathcal{D}}_L \lesssim (\sigma+1)e^{\sigma t}\bar{\mathcal{D}}_L \lesssim (\sigma+1)e^{\sigma t}\bar{\mathfrak{D}}_L.\end{aligned}$$

Integration in the temporal variable then yields,

$$e^{\sigma t}\sum_{i=0}^{L-1} \left\| \partial_t^i v \right\|^2_{H^{2L-2i}(\Omega)} \lesssim \epsilon_1 + \bar{\mathcal{E}}_L(0) \lesssim \epsilon_1, \tag{4.34}$$

as the consequence of (4.30). Together with (4.33), (4.26) and (4.30), this then yields (4.31) with a sufficiently small $\epsilon_1 > 0$. (4.32) is a direct consequence of (4.31). □

Proof of Main Theorem Let $\epsilon_0, \epsilon_1 > 0$ be defined in Lemma 28 and Lemma 29. Then for $\bar{\omega} < \epsilon_0$, $\bar{\mathfrak{E}}_L(0), \bar{\mathcal{E}}_L(0) < \epsilon_1$ with $L \geq 3$, through continuous arguments, the following estimates hold

$$\begin{aligned} &e^{\sigma t}\bar{\mathcal{E}}_L(t) + \int_0^t e^{\sigma s}\bar{\mathcal{D}}_L(s)\,ds \leq \epsilon_0, \\ &e^{\sigma t}\bar{\mathfrak{E}}_L(t) + \int_0^t e^{\sigma s}\bar{\mathfrak{D}}_L(s)\,ds \leq \epsilon_1. \end{aligned} \tag{4.35}$$

In particular, the energy functional $\bar{\mathfrak{E}}_L(t) + \bar{\mathcal{E}}_L(t)$ admits exponential decay as time grows up. This finishes the proof.

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