

# NEW STRUCTURE FOR ORTHOGONAL QUANTUM GROUP INVARIANTS

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ABSTRACT. Based on the orthogonal Labastida-Mariño-Ooguri-Vafa conjecture made by L. Chen & Q. Chen [5], we derive an infinite product formula for Chern-Simons partition functions, which generalizes the Liu-Peng's [19] recent results to the orthogonal case. Symmetry property of this new infinite product structure is also discussed.

## 1. INTRODUCTION

In 1984 Jones [9, 10] discovered a polynomial invariant of oriented knots. Later, along this clue, the two-variable HOMFLY-PT [7, 22] and one-variable Kauffman polynomials [11] of oriented and unoriented links have also been discovered in 1985 and 1987 respectively. HOMFLY-PT polynomial generalizes both Alexander and Jones polynomial. In 1990, two-variable Kauffman polynomial [12] was also introduced, which only generalizes the Jones polynomial.

By using the Chern-Simons path integral method Witten [29] gave a quantum field theory interpretation of the Jones polynomial. Witten [29] also predicted the existence of 3-manifold quantum invariants. Following this track, Reshetikhin and Turaev [26, 27] gave a construction of 3-manifold invariants by using quantum universal enveloping algebra (quantum group)  $U_q(sl_2)$  at roots of unity, which leads to the colored version of classical HOMFLY-PT and Kauffman polynomial invariants. These work actually give a unified understanding of the quantum group invariants of links. Here color means the representation of quantum groups. It turns out that colored HOMFLY-PT polynomial is a special linear quantum group invariants, i.e. the quantum group of  $A_n$  type; while colored Kauffman is a quantum group invariants of  $B_n$ ,  $C_n$  and  $D_n$  type.

In a series of papers, Labastida, Mariño, Ooguri and Vafa [13, 14, 21] proposed a conjectural description of deep relationship between reformulated invariants of colored HOMFLY-PT link in late 1990s and early 2000s. This conjecture was proved by K. Liu and P. Peng in [17, 18].

In some sense the Labastida-Mariño-Ooguri-Vafa (LMOV) conjecture can be expressed purely by using mathematical language, i.e. irreducible representation of quantum groups. The physics background of this conjecture can be dated back to 't Hooft's seminal work on large  $N$  expansion of  $U(N)$  gauge field theories in 1974. Gopakumar and Vafa [8] described the exact theory that closed topological string theory on the resolved conifold is dual to the  $U(N)$  Chern-Simons theory on  $S^3$ . The Gromov-Witten theory of the resolved conifold actually corresponds to the Chern-Simons theory of an unknot. The LMOV conjecture considers the more general case when the link or knot is nontrivial and the corresponding Wilson loop expectation values, i.e. colored HOMFLY-PT polynomial of the link. So the LMOV conjecture could be viewed as a counterpart of Gopakumar-Vafa conjecture.

Previously people thought that only HOMFLY-PT can be expressed as a series in  $q - q^{-1}$  and  $t$  with integer coefficients; while the colored HOMFLY-PT polynomial is just a polynomial of  $q^{\pm 1}$  and  $t^{\pm 1}$  with rational coefficients. The LMOV conjecture predicts an intrinsic symmetry of  $q - q^{-1}$  about reformulated invariants of the colored HOMFLY-PT polynomial and hidden integrality encoded in the colored HOMFLY-PT polynomial.

In the late 2010s, the orthogonal LMOV conjecture was also formulated by Lin Chen & Qingtao Chen [4, 5, 6]. and Marcos Mariño [20]. Chen-Chen's formulation put an emphasis on the colored Kauffman solely; while Mariño's put an emphasis on the relation between the colored HOMFLY-PT polynomials and the colored Kauffman polynomials.

More recently, Kefeng Liu and Pan Peng [19] obtained a new structure of the colored HOMFLY-PT polynomial that the Chern-Simons partition function appeared in the LMOV conjecture can be expressed as an infinite product, which indicates some potential modularity of the Chern-Simons partition function [19].

In this paper, an infinite product expression for the orthogonal Chern-Simons partition function appeared in the orthogonal LMOV type conjecture is established and the case of unknot is presented in explicit formula.

This paper is organized as follows. In Section 2, we introduce the basic setups for the quantum group invariants of links. In Section 3, we describe the original and the orthogonal LMOV conjecture and notations. In Section 4, we derive the orthogonal Chern-Simons partition function as an infinite product and illustrate an example of the unknot. In Section 5, we discuss the symmetric properties associated to this infinite product structure.

## 2. QUANTUM INVARIANTS OF LINKS

Let  $\mathfrak{g}$  be a finite dimensional complex semi-simple Lie algebra of rank  $N$  with Cartan matrix  $(C_{ij})$ . Let  $U_q(\mathfrak{g})$  be the quantum enveloping algebra of  $\mathfrak{g}$ . Let  $V$  be a vector space over a field  $k$ . A linear automorphism  $c$  of  $V \otimes V$  is said to be an  $R$ -matrix if it is a solution of the following Yang-Baxter equation

$$(2.1) \quad (c \otimes id_V)(id_V \otimes c)(c \otimes id_V) = (id_V \otimes c)(c \otimes id_V)(id_V \otimes c)$$

that holds in the automorphism group of  $V \otimes V \otimes V$ .

It is well known that the solution of the Yang-Baxter equation provides a representation of the braid group. The solution we used is the following so-called universal  $R$ -matrix.

$$(2.2) \quad R = q^{\sum_{i,j} C_{ij}^{-1} H_i \otimes H_j} \prod_{\beta} exp_q[(1 - q^{-2})X_{\beta}^+ \otimes X_{\beta}^-],$$

where  $\beta$  runs over positive roots of  $sl(N, \mathbf{C})$ ,  $(C_{ij})$  is the Cartan matrix, and  $q$ -exponential is given by

$$(2.3) \quad exp_q[x] = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k+1)} \frac{x^k}{[k]_q!},$$

where  $[k]_q! = [k]_q \cdot [k-1]_q \cdots [1]_q$ ,  $[k]_q = \frac{[k]}{[1]}$  and  $[n] = q^n - q^{-n}$ .

Given a link  $\mathcal{L}$  with  $L$  components. It is well-known that  $\mathcal{L}$  can be represented by an element in some braid group  $B_m$  with  $m$  strands. For each component, we associate to it with an irreducible representation  $A^\alpha$  of quantized universal enveloping algebra  $U_q(sl(N, \mathbb{C}))$ .  $A^\alpha$  is labeled by the highest weight  $\Lambda_\alpha$ . As usual, we associate to them with the Young diagrams. Without loss of generality, one can assume the first  $m_1$  strands correspond to the first component, the second  $m_2$  strands correspond to the second component, and so on. Let

$$(2.4) \quad \widehat{V} = \bigotimes_{\alpha=1}^L V_{\Lambda_\alpha}^{m_\alpha}.$$

And write the braiding  $\check{R} = P_{12}R: V \otimes W \rightarrow W \otimes V$ , where  $P_{12}(s \otimes t) = t \otimes s$ .

For a generator of the braid group  $B_m$ ,  $\sigma_i$ , define

$$(2.5) \quad \pi(\sigma_i^{\pm 1}) = Id_{V_1} \otimes Id_{V_2} \otimes \cdots \otimes Id_{V_{i-1}} \otimes \check{R}^{\pm 1} \otimes \cdots \otimes Id_{V_m}.$$

The quantum group invariants of the link  $\mathcal{L}$  is defined as follows

$$(2.6) \quad W_{A^1, \dots, A^L}^{\mathfrak{g}}(\mathcal{L}) = q^{d(\mathcal{L})} Tr_{\widehat{V}}(\mu^m \cdot \pi(\mathcal{L})),$$

where  $\mu = q^{\rho^*}$ ,  $\rho^*$  is the element in  $\mathfrak{h} \subset U_q(\mathfrak{h})$  corresponding to Weyl vector (i.e. the sum of fundamental weights) under the natural isomorphism  $\mathfrak{h} \simeq \mathfrak{h}^*$  and  $d(\mathcal{L})$  is given by the following formula

$$(2.7) \quad d(\mathcal{L}) = -\sum_{\alpha=1}^L \omega(\mathcal{K}_\alpha)(\Lambda_\alpha, \Lambda_\alpha + 2\rho) + \frac{2}{N} \sum_{\alpha < \beta}^L lk(\mathcal{K}_\alpha, \mathcal{K}_\beta) l_\alpha l_\beta.$$

Special case 1: For the unknot  $\bigcirc$ ,  $W_A(\bigcirc)$  is the quantum dimension  $\dim_q(V_A)$  of the corresponding representation space  $V_A$ .

Special case 2: If  $\mathfrak{g} = sl_N$  and  $A^1 = A^2 = \cdots = A^L = (1)$ , the quantum group invariant of links equal to the HOMFLY polynomial at  $t = q^N$  up to a universal factor  $\frac{t - t^{-1}}{q - q^{-1}}$ .

Special case 3: If  $\mathfrak{g} = so_{2N+1}$  and  $A^1 = A^2 = \cdots = A^L = (1)$ , quantum group invariant of links equal to Kauffman polynomial at  $t = q^{2N}$  up to a universal factor  $1 + \frac{t - t^{-1}}{q - q^{-1}}$  and some  $t$  power of the linking numbers.

Thus the quantum group invariant associated to  $\mathfrak{g} = sl_N$  and  $\mathfrak{g} = so_{2N+1}$  are called the colored HOMFLY and the colored Kauffman polynomials respectively.

Actually the irreducible representation of the quantum groups of special linear and orthogonal cases can be labeled by the Young Tableau. Now we would like to introduce some basic notation of the partition and the corresponding Young Tableau.

A partition of  $n$  is a tuple of positive integers  $\mu = (\mu_1, \mu_2, \dots, \mu_k)$  such that  $|\mu| \triangleq \sum_{i=1}^k \mu_i = n$  and  $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$ , where  $|\mu|$  is called the degree of  $\mu$  and  $k$  is called

the length of  $\mu$ , denoted by  $\ell(\mu)$ . A partition can be represented by a Young diagram, for example, partition  $(5, 4, 2, 1)$  can be identified as the following Young diagram.

Denote by  $\mathcal{P}$  the set of all Young diagrams. Let  $\chi_A$  be the character of irreducible representation of symmetric group, labeled by partition  $A$ . Given a partition  $\mu$ , define  $m_j = \#(\mu_k = j; k \geq 1)$ . The order of the conjugate class of type  $\mu$  is given by:

$$(2.8) \quad \mathfrak{z}_\mu = \prod_{j \geq 1} j^{m_j} m_j!.$$

The theory of symmetric functions has a close relationship with the representations of symmetric group. The symmetric power functions of a given set of variables  $x = \{x_j\}_{j \geq 1}$  are defined as the direct limit of the Newton polynomials:

$$(2.9) \quad p_n(x) = \sum_{j=1}^{\infty} x_j^n, \quad p_\mu(x) = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}(x).$$

We will consistently denote by  $\mathcal{L}$  a link and by  $L$  the number of components in  $\mathcal{L}$ .

The irreducible  $U_q(\mathfrak{g})$  modules associated to  $\mathcal{L}$  will be labeled by their highest weights, thus by Young diagrams. We usually denote it by a vector form  $\vec{A} = (A^1, \dots, A^L)$ .

Let  $\vec{x} = (x^1, \dots, x^L)$  is  $L$  sets of variables, each of which is associated to a component of  $\mathcal{L}$  and  $\vec{\mu} = (\mu^1, \dots, \mu^L) \in \mathcal{P}^L$  be a tuple of  $L$  partitions. Define:

$$\begin{aligned} [\mu] &= \prod_{i=1}^{\ell(\mu)} [\mu_i], & [\vec{\mu}] &= \prod_{\alpha=1}^L [\mu^\alpha], & \mathfrak{z}_{\vec{\mu}} &= \prod_{\alpha=1}^L \mathfrak{z}_{\mu^\alpha}, \\ \chi_{\vec{A}}(C_{\vec{\mu}}) &= \prod_{\alpha=1}^L \chi_{A^\alpha}(C_{\mu^\alpha}), & s_{\vec{A}}(\vec{x}) &= \prod_{\alpha=1}^L s_{A^\alpha}(x^\alpha), & p_{\vec{\mu}}(\vec{x}) &= \prod_{\alpha=1}^L p_{\mu^\alpha}(x^\alpha). \end{aligned}$$

When we consider the orthogonal quantum group invariants, we need to study the Brauer algebra  $Br_n$  which contains the group algebra  $\mathbb{C}[S_n]$  as a direct summand. Thus all the irreducible representations of  $S_n$  are also irreducible representations of  $Br_n$ , labeled by partitions of the integer  $n$ . Indeed, the set of irreducible representations of  $Br_n$  are bijective to the set of partitions of the integers  $n-2k$ , where  $k = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$  [24, 28]. Thus the semi-simple algebra  $Br_n$  can be decomposed into the direct sum of simple algebras

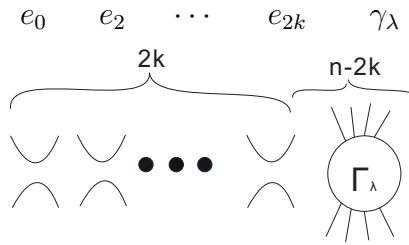
$$(2.10) \quad Br_n \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} \bigoplus_{\lambda \vdash n-2k} M_{d_\lambda \times d_\lambda}(\mathbb{C}).$$

The work of Beliakova and Blanchet [1] constructed an explicit basis of the above decomposition. An up and down tableau  $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  is a tube of  $n$  Young diagrams such that  $\lambda_1 = (1)$  and each  $\lambda_i$  is obtained by adding or removing one box from  $\lambda_{i-1}$ . Let  $\lambda$  be a partition of  $n-2k$ . Denote by  $|\Lambda| = \lambda$  if  $\lambda_n = \lambda$ , and we say an up and down tableau  $\Lambda$  is of shape  $\lambda$ . There is a minimal path idempotent  $p_\Lambda \in Br_n$  associated to each  $\Lambda$ . Then the minimal central idempotent  $\pi_\lambda$  of  $Br_n$  correspond to the irreducible representation labeled by  $\lambda$  is given by

$$(2.11) \quad \pi_\lambda = \sum_{|\Lambda|=\lambda} p_\Lambda.$$

In particular, the dimension of the irreducible representations  $d_\lambda$  is the number of up and down tableau of shape  $\lambda$ . More detail can be found in [1, 28].

The table of the characters and orthogonal relations can be found in [23, 24, 25]. The values of a character of  $Br_n$  is completely determined by its values on the set of elements  $e^k \otimes \gamma_\lambda$ , where  $e$  is the conjugacy class of  $e_1, \dots, e_{n-1}$  and  $\gamma_\lambda$  is the conjugacy class in  $S_{n-2k}$  labeled by the partition  $\lambda$  of  $n - 2k$ . The notion  $e^k \otimes \gamma_\lambda$  stands for the tangle in the following diagram.



where  $\Gamma_\lambda$  is a diagram in the conjugacy class of  $S_{n-2k}$  labeled by a partition  $\lambda$  of  $n - 2k$ .

Denote by  $\chi_A$  the character of the irreducible representation of  $Br_n$  labeled by a partition  $A \vdash n - 2k$  for some  $k$ , and denote by  $\chi_B^{S_n}$  the character of the irreducible representation of  $S_n$  labeled by a partition  $B \vdash n$ . It is known that when  $A$  is a partition of  $n$ , then  $\chi_A(e^m \otimes \gamma_\lambda) = 0$  for all  $m > 0$  and partition  $\lambda \vdash n - 2m$ , and  $\chi_A(\gamma_\mu) = \chi_A^{S_n}(\gamma_\mu)$  for partition  $\mu \vdash n$  coincide with the characters of the permutation group  $S_n$  [24].

### 3. LABASTIDA-MARIÑO-OOGURI-VAFA TYPE CONJECTURE FOR COLORED KAUFFMAN POLYNOMIAL

Let's quickly review the original LMOV conjecture first.

For each link  $\mathcal{L}$ , the type- $A$  Chern-Simons partition function of  $\mathcal{L}$  is defined by

$$(3.1) \quad Z_{CS}^{SL}(\mathcal{L}; q, t; \vec{x}) = \sum_{\vec{A} \in \mathcal{P}^L} W_{\vec{A}}^{SL}(\mathcal{L}; q, t) s_{\vec{A}}(\vec{x}) = 1 + \sum_{\vec{\mu} \neq \vec{0}} Z_{\vec{\mu}}^{SL} p_{\vec{\mu}}(\vec{x}),$$

where  $s_{\vec{A}}(\vec{x})$  are the Schur polynomials.

The original LMOV conjecture describes a very subtle structure of  $Z_{CS}^{SL}(\mathcal{L}; q, t; \vec{x})$ , which was proved by Kefeng Liu and Pan Peng [17, 18], based on the cabling technique and a careful degree analysis of the cut-join equation. As an application, the LMOV conjecture gives highly non-trivial relations between colored HOMFLY polynomials. The first such relation is the classical Lichorish-Millett theorem [15].

The study of the colored Kauffman polynomials are more difficult. For instance, the definition of the Chern-Simons partition function for the orthogonal quantum groups involves the representations of Brauer centralizer algebras, which admit a more complicated orthogonal relations [23, 24, 25]. In a joint work with Lin Chen, we [5] rigorously formulate the orthogonal quantum group version of LMOV conjecture by using the representation of the Brauer centralizer algebra.

Now we set  $\mathfrak{g} = \mathfrak{so}_{2N+1}$ .

Let  $\mathcal{L}$  be a link with  $L$  components and

$$(3.2) \quad pb_n(z) = \sum_{j=-\infty}^{\infty} z_j^n, \quad pb_{\mu}(z) = \prod_{i=1}^{\ell(\mu)} pb_{\mu_i}(z), \quad pb_{\vec{\mu}}(\vec{z}) = \prod_{\alpha=1}^L pb_{\mu^{\alpha}}(z^{\alpha}).$$

Let  $Z_{CS}^{SO}(\mathcal{L}, q, t)$  be the orthogonal Chern-Simons partition function defined by

$$(3.3) \quad Z_{CS}^{SO}(\mathcal{L}; q, t; \vec{z}) = \sum_{\vec{\mu} \in \mathcal{P}^L} \frac{pb_{\vec{\mu}}(\vec{z})}{\mathfrak{z}_{\vec{\mu}}} \sum_{\vec{A} \in \widehat{Br}_{|\vec{\mu}|}} \chi_{\vec{A}}(\gamma_{\vec{\mu}}) W_{\vec{A}}^{SO}(\mathcal{L}; q, t),$$

where  $\mathfrak{z}_{\vec{\mu}} = \frac{|\vec{\mu}|!}{|C_{\vec{\mu}}|}$ ,  $|\vec{\mu}| = (d^1, \dots, d^L)$ ,  $\widehat{Br}_{|\vec{\mu}|}$  denotes the set  $\widehat{Br}_{d^1} \times \dots \times \widehat{Br}_{d^L}$  (every element is a representation of the Brauer algebra),  $\vec{\mu} = (\mu^1, \dots, \mu^L)$  for partitions  $\mu^i$  of  $d^i \in \mathbb{Z}$  and  $\chi_{\vec{A}}(\gamma_{\vec{\mu}}) = \prod_{i=1}^L \chi_{A^i}(\gamma_{\mu^i})$  for the character  $\chi_{A^i}$  of  $Br_{d^i}$  labeled by  $A^i$ .

Extend the free energy

$$(3.4) \quad F_{CS}^{SO}(\mathcal{L}; q, t; \vec{z}) = \log Z_{CS}^{SO}(\mathcal{L}; q, t; \vec{z}) = \sum_{\vec{\mu} \neq \vec{0}} F_{\vec{\mu}}^{SO}(\mathcal{L}; q, t) pb_{\vec{\mu}}(\vec{z}),$$

Then the reformulated invariants are defined by

$$(3.5) \quad g_{\vec{\mu}}(\mathcal{L}; q, t) = \sum_{k|\vec{\mu}} \frac{\mu(k)}{k} F_{\vec{\mu}/k}(\mathcal{L}; q^k, t^k).$$

The orthogonal LMOV conjecture was formulated by L. Chen and Q. Chen [5] as follows

**Conjecture 3.1** (Orthogonal LMOV, Chen-Chen [5]).

$$\frac{\mathfrak{z}_{\vec{\mu}}[1]^2 \cdot [g_{\vec{\mu}}(\mathcal{L}; q, t) - g_{\vec{\mu}}(\mathcal{L}; q, -t)]}{2[\vec{\mu}]} = \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} N_{\vec{\mu}, g, \beta} (q - q^{-1})^g t^{\beta},$$

where  $N_{\vec{\mu}, g, \beta}$  are the integer coefficients and vanish for sufficiently large  $g$  and  $|\beta|$ ,

This conjecture is a rigorous mathematical formulation of the LMOV type conjecture about the colored Kauffman polynomial; while in [2, 20], their conjecture emphasizes on the relationship between colored HOMFLY-PT and colored Kauffman. The integer coefficients  $N_{\vec{\mu}, g, \beta}$  are closely related to the BPS numbers.

#### 4. INFINITE PRODUCT FORMULA FOR ORTHOGONAL CHERN-SIMONS PARTITION FUNCTIONS

To derive an infinite product formula, we will state the result for a knot first, since the notations in the computation for a knot are relatively simpler.

4.1. **The case of a knot.** Given  $z = \{z_i\}_{-\infty < i < \infty}$ ,  $x = \{x_j\}_{j \geq 1}$ , define

$$x * y = \{x_i \cdot y_j\}_{-\infty < i < \infty, j \geq 1}.$$

We also define  $z^d = \{z_i^d\}_{-\infty < i < \infty}$ . The  $d$ -th Adam operation of a type-B Schur function is given by  $sb_A(z^d)$ .

Introduce variables  $q^\rho = \{-q^{2j-1}\}_{j \geq 1}$ . we have

$$(4.1) \quad p_n(q^\rho) = \frac{1}{[n]},$$

where we assume  $|q| < 1$ .

Set  $w = z * q^\rho$ , then we have

$$(4.2) \quad pb_n(w) = \frac{pb_n(z)}{[n]}$$

and

$$(4.3) \quad pb_\mu(w) = \frac{pb_\mu(z)}{[\mu]}.$$

Consider the free energy weighted by the  $pb_\mu(w)$ , orthogonal LMOV conjecture implies the following reformulation of the free energy:

$$(4.4) \quad F^{SO}(\mathcal{K}; q, t; w) = \sum_{d=1}^{\infty} \sum_{\mu \neq 0} \frac{1}{d} g_\mu(\mathcal{K}; q^d, t^d) pb_\mu(w^d)$$

and

$$(4.5) \quad g_\mu(\mathcal{K}; q, t) - g_\mu(\mathcal{K}; q, -t) = \frac{2[\mu]}{\mathfrak{z}_\mu[1]^2} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} N_{\vec{\mu}, g, \beta} (q - q^{-1})^g t^\beta.$$

There exists integers  $n_{B; g, \beta}$  such that

$$(4.6) \quad \sum_{g=0}^{\infty} N_{B, g, \beta} (q - q^{-1})^g = \sum_{g=0}^{\infty} n_{B, g, \beta} \sum_{k=0}^g q^{g-2k}.$$

By the orthogonal LMOV conjecture,  $N_{B; g, \beta}$  vanish for sufficiently large  $g$  and  $|\beta|$ , thus  $n_{B; g, \beta}$  vanish for sufficiently large  $g$  and  $|\beta|$ .

We have

$$\begin{aligned}
& F^{SO}(\mathcal{K}; q, t; w) - F^{SO}(\mathcal{K}; q, -t; w) \\
&= \sum_{d=1}^{\infty} \sum_{\mu \neq 0} \frac{1}{d} g_{\mu}(\mathcal{K}; q^d, t^d) pb_{\mu}(w^d) - \sum_{d=1}^{\infty} \sum_{\mu \neq 0} \frac{1}{d} g_{\mu}(\mathcal{K}; q^d, (-1)^d t^d) pb_{\mu}(w^d) \\
&= \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} (g_{\mu}(\mathcal{K}; q^d, t^d) - g_{\mu}(\mathcal{K}; q^d, -t^d)) pb_{d\mu}(w) \\
&= \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} \frac{2[d\mu]}{\mathfrak{z}_{\mu}[d]^2} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} N_{\mu, g, \beta} (q^d - q^{-d})^g t^{d\beta} pb_{d\mu}(w) \\
(4.7) \quad &= \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} \frac{2}{\mathfrak{z}_{\mu}[d]^2} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} N_{\mu, g, \beta} (q^d - q^{-d})^g t^{d\beta} pb_{d\mu}(z) \\
&= \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} \frac{2}{\mathfrak{z}_{\mu}[d]^2} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{k=0}^g n_{\mu, g, \beta} q^{(g-2k)d} t^{d\beta} pb_{d\mu}(z) \\
&= \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} \frac{2}{\mathfrak{z}_{\mu}} \sum_{m=1}^{\infty} m q^{2md} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{k=0}^g n_{\mu, g, \beta} q^{(g-2k)d} t^{d\beta} pb_{d\mu}(z) \\
&= \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} \frac{2}{\mathfrak{z}_{\mu}} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{m=1}^{\infty} \sum_{k=0}^g m n_{\mu, g, \beta} q^{(g-2k+2m)d} t^{d\beta} pb_{d\mu}(z) \\
(4.8) \quad &= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{m=1}^{\infty} \sum_{k=0}^g \frac{m n_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} q^{(g-2k+2m)d} t^{d\beta} pb_{d\mu}(z),
\end{aligned}$$

where  $O\mathbb{Z}_+ = \{1, 3, 5, \dots\}$  denotes the set of all positive odd integers. Now we analyze the following computation in detail,

$$\begin{aligned}
& \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} q^{(g-2k+2m)d} t^{d\beta} pb_{d\mu}(z) \\
&= \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} q^{(g-2k+2m)d} t^{d\beta} \prod_{j=1}^{\ell(\mu)} \left( \sum_{i=-\infty}^{\infty} (z_i)^{d\mu_j} \right) \\
&= \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} q^{(g-2k+2m)d} t^{d\beta} \sum_{i_1, \dots, i_{\ell(\mu)}} (z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}})^d \\
&= \sum_{i_1, \dots, i_{\ell(\mu)}} \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} (q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}})^d
\end{aligned}$$

Now we compute the series  $\sum_{d \in O\mathbb{Z}_+} \frac{2}{d} x^d$  as follows



$$\begin{aligned}
& \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} x^d \\
&= 2 \left( \sum_{d=1}^{\infty} \frac{1}{d} x^d - \sum_{d \in 2\mathbb{Z}_+} \frac{1}{d} x^d \right) \\
&= 2 \left( \sum_{d=1}^{\infty} \frac{1}{d} x^d - \sum_{d=1}^{\infty} \frac{1}{2d} x^{2d} \right) \\
&= 2 \left( -\log(1-x) + \frac{1}{2} \log(1-x^2) \right) \\
&= \log \frac{1+x}{1-x},
\end{aligned}$$

where  $2\mathbb{Z}_+ = \{2, 4, 6, \dots\}$  denotes the set of all positive even integers.

Thus we obtain

$$\begin{aligned}
& F^{SO}(\mathcal{K}; q, t; w) - F^{SO}(\mathcal{K}; q, -t; w) \\
&= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{m=1}^{\infty} \sum_{k=0}^g \frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} q^{(g-2k+2m)d} t^{d\beta} p b_{d\mu}(z) \\
&= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{m=1}^{\infty} \sum_{k=0}^g \frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \sum_{i_1, \dots, i_{l(\mu)}} \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} (q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{l(\mu)}}^{\mu_{l(\mu)}})^d \\
&= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{m=1}^{\infty} \sum_{k=0}^g \frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \sum_{i_1, \dots, i_{l(\mu)}} \log \frac{1 + q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{l(\mu)}}^{\mu_{l(\mu)}}}{1 - q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{l(\mu)}}^{\mu_{l(\mu)}}} \\
&= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{m=1}^{\infty} \sum_{k=0}^g \frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \log \prod_{i_1, \dots, i_{l(\mu)}} \frac{1 + q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{l(\mu)}}^{\mu_{l(\mu)}}}{1 - q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{l(\mu)}}^{\mu_{l(\mu)}}}
\end{aligned}$$

Define the symmetric product as shown in the following formula:

$$\langle 1 \pm \omega z^{\mu} \rangle = \prod_{i_1, \dots, i_{l(\mu)}} \left( 1 \pm \omega z_{i_1}^{\mu_1} \dots z_{i_{l(\mu)}}^{\mu_{l(\mu)}} \right).$$

Thus we have

$$\begin{aligned}
& F^{SO}(\mathcal{K}; q, t; w) - F^{SO}(\mathcal{K}; q, -t; w) \\
&= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{m=1}^{\infty} \sum_{k=0}^g \frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \log \frac{\langle 1 + q^{g-2k+2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{g-2k+2m} t^{\beta} z^{\mu} \rangle} \\
&= \log \prod_{\mu \neq 0} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}} \prod_{m=1}^{\infty} \prod_{k=0}^g \left( \frac{\langle 1 + q^{g-2k+2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{g-2k+2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}}}
\end{aligned}$$

Now we obtain the infinite product formula for the orthogonal Chern-Simons partition function of knots.

**Theorem 4.1** (Orthogonal Infinite Product Formula for Knots). *The Chern-Simons partition function for orthogonal quantum group invariants can be expressed as the following infinite product formula*

$$(4.9) \quad \frac{Z_{CS}^{SO}(\mathcal{K}; q, t; w)}{Z_{CS}^{SO}(\mathcal{K}; q, -t; w)} = \prod_{\mu \neq 0} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}} \prod_{m=1}^{\infty} \prod_{k=0}^g \left( \frac{\langle 1 + q^{g-2k+2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{g-2k+2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu, g, \beta}}{3\mu}}$$

**4.2. The case of a link.** Now we consider the case of link.

Given a link  $\mathcal{L}$  of  $L$  components, let  $\vec{w} = (w^1, \dots, w^L)$  and  $\vec{z} = (z^1, \dots, z^L)$  satisfying  $w^i = z^i * q^{\rho}$ , for  $i = 1, \dots, L$ .

We generalize the symmetric product to the case of link as follows:

$$\langle 1 \pm \omega (z^1)^{\mu^1} \dots (z^L)^{\mu^L} \rangle = \prod_{i_{1,1}, \dots, i_{1, \ell(\mu^1)}, \dots, i_{L,1}, \dots, i_{L, \ell(\mu^L)}} \left( 1 \pm \omega \prod_{\alpha=1}^L \left( (z_{i_{\alpha,1}}^{\alpha})^{\mu_{\alpha}^1} \dots (z_{i_{\alpha, \ell(\mu^{\alpha})}}^{\alpha})^{\mu_{\alpha}^{\ell(\mu^{\alpha})}} \right) \right),$$

There exist integers  $n_{\vec{B}; g, \beta}$  such that

$$\sum_{g=0}^{\infty} N_{\vec{B}; g, \beta} (q - q^{-1})^g = \sum_{g=0}^{\infty} n_{\vec{B}; g, \beta} \sum_{k=0}^g q^{g-2k}.$$

In a similar way, the infinite product formula for orthogonal Chern-Simons partition function of link can be obtained as follows

**Theorem 4.2** (Orthogonal Infinite Product Formula for Links). *The Chern-Simons partition function for orthogonal quantum group invariants can be expressed as the following infinite product formula*

$$(4.10) \quad \frac{Z_{CS}^{SO}(\mathcal{L}; q, t; \vec{w})}{Z_{CS}^{SO}(\mathcal{L}; q, -t; \vec{w})} = \prod_{\vec{\mu} \neq \vec{0}} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}} \prod_{m=1}^{\infty} \prod_{k=0}^g \left( \frac{\langle 1 + q^{g-2k+2m} t^{\beta} (z^1)^{\mu^1} \dots (z^L)^{\mu^L} \rangle}{\langle 1 - q^{g-2k+2m} t^{\beta} (z^1)^{\mu^1} \dots (z^L)^{\mu^L} \rangle} \right)^{\frac{mn_{\vec{\mu}, g, \beta}}{3\vec{\mu}}}$$

**4.3. The case of the unknot.** In Proposition 10.2 of [5], we have computed the free energy associated to orthogonal Chern-Simons partition function of the knot.

$$F^{SO}(\bigcirc; q, t; z) = \sum_{k=1}^{\infty} \frac{1}{k} \left( 1 + \frac{t^k - t^{-k}}{q^k - q^{-k}} \right) pb_k(z).$$

Thus we have

$$\begin{aligned} & F^{SO}(\bigcirc; q, t; w) - F^{SO}(\bigcirc; q, -t; w) \\ &= \sum_{k \in \mathbb{O}\mathbb{Z}_+} \frac{2}{k} \frac{t^k - t^{-k}}{q^k - q^{-k}} pb_k(w) \\ &= \sum_{k \in \mathbb{O}\mathbb{Z}_+} \frac{2}{k} \frac{t^k - t^{-k}}{[k]^2} pb_k(z) \end{aligned}$$

Compared with (4.7), we obtain

$$(4.11) \quad N_{(1),0,1} = -N_{(1),0,-1} = 1.$$

All other coefficients  $N_{B;g,Q}$  are zero.

Thus we have

$$(4.12) \quad n_{\mu,g,\beta} = \delta_{\mu,(1)} \delta_{g,0} \text{sign}(\beta)$$

and

$$(4.13) \quad \frac{Z_{CS}^{SO}(\mathbb{O}; q, t; w)}{Z_{CS}^{SO}(\mathbb{O}; q, -t; w)} = \prod_{m=1}^{\infty} \prod_{i=-\infty}^{\infty} \left( \frac{(1 + q^{2m} t z_i)(1 - q^{2m} t^{-1} z_i)}{(1 - q^{2m} t z_i)(1 + q^{2m} t^{-1} z_i)} \right)^m$$

**4.4. Symmetry Property of  $q \rightarrow q^{-1}$  in Infinite Product Structure.** In this subsection, we discuss a basic symmetric property of this infinite of product structure obtained from the orthogonal LMOV partition function. Here we focus on the knot case only, while the case of links exactly follows from the same analysis. In the derivation of the infinite product formula, we assume  $|q| < 1$  for the Taylor expansion of  $\frac{1}{[d]^2}$ . In the case of  $|q| > 1$ , the Taylor expansion is given by

$$(4.14) \quad \frac{1}{[d]^2} = \sum_{m=1}^{\infty} m q^{-2md}$$

Therefore, the infinite product formula will be read as

$$(4.15) \quad \begin{aligned} & \frac{Z_{CS}^{SO}(\mathcal{K}; q, t; w)}{Z_{CS}^{SO}(\mathcal{K}; q, -t; w)} \\ &= \prod_{\mu \neq 0}^{\infty} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}}^{\infty} \prod_{m=1}^g \prod_{k=0}^g \left( \frac{\langle 1 + q^{g-2k-2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{g-2k-2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu,g,\beta}}{\delta_{\mu}}} \\ &= \prod_{\mu \neq 0}^{\infty} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}}^{\infty} \prod_{m=1}^g \prod_{k=0}^g \left( \frac{\langle 1 + q^{-g+2(g-k)-2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{-g+2(g-k)-2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu,g,\beta}}{\delta_{\mu}}} \\ &= \prod_{\mu \neq 0}^{\infty} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}}^{\infty} \prod_{m=1}^g \prod_{k=0}^g \left( \frac{\langle 1 + q^{-g+2k-2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{-g+2k-2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu,g,\beta}}{\delta_{\mu}}} \\ &= \prod_{\mu \neq 0}^{\infty} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}}^{\infty} \prod_{m=1}^g \prod_{k=0}^g \left( \frac{\langle 1 + (q^{-1})^{g-2k+2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - (q^{-1})^{g-2k+2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu,g,\beta}}{\delta_{\mu}}} \end{aligned}$$

This is the symmetry of  $q \rightarrow q^{-1}$  for the infinite product formula.

## 5. ACKNOWLEDGMENTS

We thank Pan Peng for many valuable discussions and Shengmao Zhu for giving many helpful suggestions and for proof reading the paper.

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