

Remarks on the Chern Classes of Calabi-Yau Moduli

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Abstract

We prove that the first Chern form of the moduli space of polarized Calabi-Yau manifolds, with the Hodge metric or the Weil-Petersson metric, represents the first Chern class of the canonical extensions of the tangent bundle to the compactification of the moduli space with normal crossing divisors.

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1 Introduction

A compact projective manifold X of complex dimension n with $n \geq 3$ is called a Calabi-Yau manifold in this paper, if it has a trivial canonical bundle and satisfies $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$. A polarized Calabi-Yau manifold is a pair (X, L) consisting of a Calabi-Yau manifold X , and an ample line bundle L over X . A basis of the quotient space $(H_n(X, \mathbb{Z})/\text{Tor})/m(H_n(X, \mathbb{Z})/\text{Tor})$ is called a level m structure with $m \geq 3$ on the polarized Calabi-Yau manifold.

We will consider the moduli space \mathcal{M}_m of polarized Calabi-Yau manifolds with level m structure with $m \geq 3$, which is called the Calabi-Yau moduli in this paper for simplicity. Over \mathcal{M}_m , we can construct various Hodge bundles. The holomorphic bundle H^n over \mathcal{M}_m whose fibers are the primitive cohomology group $PH^n(X_p, \mathbb{C}), p \in \mathcal{M}_m$, endowed with the Gauss-Manin connection, carry a polarized Hodge structure of weight n . Then the holomorphic bundle $(H^n)^* \otimes H^n \rightarrow \mathcal{M}_m$ defines a variation of polarized Hodge structure over \mathcal{M}_m , which is defined over \mathbb{Z} . Then, with the Hodge metric, we have the following useful observation,

Theorem 1.1. *Let \mathcal{M}_m be the moduli space of polarized Calabi-Yau manifolds with level m structure with $m \geq 3$. Then $(H^n)^* \otimes H^n$ defines a variation of polarized Hodge structure over \mathcal{M}_m , which is defined over \mathbb{Z} . Moreover, with the natural Hodge metric over the Calabi-Yau moduli \mathcal{M}_m , the tangent bundle*

$$T\mathcal{M}_m \hookrightarrow (H^n)^* \otimes H^n, \quad (1.1)$$

is a holomorphic subbundle of $(H^n)^ \otimes H^n$ over \mathcal{M}_m with the induced Hodge metric.*

Then, by the important results for the integrability of Chern forms of subbundles and quotient bundles of a variation of polarized Hodge structure over a quasi-projective manifold as given by [1] and [4], see Theorem 2.3 and Theorem 2.4, we can get that the first Chern form of the Calabi-Yau moduli are integrable with the induced Hodge metric. More precisely, let $\widetilde{T\mathcal{M}_m} \rightarrow \overline{\mathcal{M}_m}$ be the canonical extension of the tangent bundle of \mathcal{M}_m , then we have,

Theorem 1.2. *The first Chern form of the Calabi-Yau moduli \mathcal{M}_m with the induced Hodge metric define currents over the compactification $\overline{\mathcal{M}_m}$ with normal crossing boundary divisors. Moreover, let R_H be the curvature form of $T\mathcal{M}_m$ with the induced Hodge metric, then we have*

$$\left(\frac{-1}{2\pi i}\right)^N \int_{T\mathcal{M}_m} (\text{tr} R_H)^N = c_1(\widetilde{T\mathcal{M}_m})^N$$

where $N = \dim_{\mathbb{C}} \mathcal{M}_m$.

Another direct and easy consequence is that the Chern forms of the Hodge bundles on the Calabi-Yau moduli with their induced Hodge metrics are all integrable.

In this paper we focus on Calabi-Yau manifolds. Actually our method only needs the fact that the moduli space of the manifolds with certain structures are smooth and quasi-projective and the period map is locally injective (the local Torelli theorem). So our results can be easily extended to more general projective manifolds, including Calabi-Yau manifolds, Hyperkähler manifolds, many hypersurfaces and complete intersections in projective spaces. Here, we only summarize the results into the following theorem:

Theorem 1.3. *Let \mathcal{M} be the moduli space of polarized projective manifolds with certain structure. Assume that \mathcal{M} is smooth and quasi-projective. If the period map from \mathcal{M} into the period domain*

is locally injective, then the first Chern form of the moduli space \mathcal{M} with the induced Hodge metric defines currents over the compactification $\overline{\mathcal{M}}$ with normal crossing boundary divisors. Moreover, the first Chern form represents the first Chern class of the corresponding canonical extension $\widetilde{T\mathcal{M}} \rightarrow \overline{\mathcal{M}}$ of the tangent bundle.

By similar argument, one can show that the Chern forms of the moduli space \mathcal{M} with the Weil-Petersson metric define currents over the compactification $\overline{\mathcal{M}}$ of the moduli space \mathcal{M} , and the first Chern form also represent the first Chern class of the corresponding canonical extension of the tangent bundle.

This paper is organized as follows. In Section 2, firstly, we review the definition of the variation of Hodge structure. Then, some essential estimates for the degeneration of the Hodge metric of a variation of polarized Hodge structure near a normal crossing divisor was reviewed, which was used to derive the integrability of the Chern forms of subbundles and quotient bundles of the variation of polarized Hodge structure over a quasi-projective manifold. In Section 3, we review the definition of moduli space of polarized Calabi-Yau manifolds with level m structure with $m \geq 3$ and various Hodge bundles. Then, by a key observation that the tangent bundle of Calabi-Yau moduli is a subbundle of the variation of polarized Hodge structure $(H^n)^* \otimes H^n \rightarrow \mathcal{M}_m$, we prove that the first Chern form of the Calabi-Yau moduli \mathcal{M}_m are integrable, with the Hodge metric. In Section 4, the Weil-Petersson geometry was reviewed. And, by the isomorphism $T\mathcal{M}_m \cong (F^n)^* \otimes F^{n-1}/F^n$, we show that the Chern forms of the Calabi-Yau moduli \mathcal{M}_m are integrable, equipped with the Weil-Petersson metric.

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2 Chern Forms of the Hodge Bundles

In Section 2.1, we review the definition of variations of Hodge structure. In Section 2.2, some essential estimates for the degeneration of Hodge metric of a variation of polarized Hodge structure near a normal crossing divisor was reviewed, which was used to derive the integrability of the Chern forms of subbundles and quotient bundles of a variation of polarized Hodge structure over a quasi-projective manifold.

2.1 Variation of Hodge Structure and Period Map

Let $H_{\mathbb{R}}$ be a real vector space with a \mathbb{Z} -structure defined by a lattice $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$, and let $H_{\mathbb{C}}$ be the complexification of $H_{\mathbb{R}}$. A Hodge structure of weight n on $H_{\mathbb{C}}$ is a decomposition

$$H_{\mathbb{C}} = \bigoplus_{k=0}^n H^{k,n-k}, \quad \text{with } H^{n-k,k} = \overline{H^{k,n-k}}.$$

The integers $h^{k,n-k} = \dim_{\mathbb{C}} H^{k,n-k}$ are called the Hodge numbers. To each Hodge structure of weight n on $H_{\mathbb{C}}$, one assigns the Hodge filtration:

$$H_{\mathbb{C}} = F^0 \supset \cdots \supset F^n, \quad (2.1)$$

with $F^k = H^{n,0} \oplus \cdots \oplus H^{k,n-k}$ and $f^k = \dim_{\mathbb{C}} F^k = \sum_{i=k}^n h^{i,n-i}$. This filtration satisfies that

$$H_{\mathbb{C}} = F^k \oplus \overline{F^{n-k+1}}, \quad \text{for } 0 \leq k \leq n. \quad (2.2)$$

Conversely, every decreasing filtration (2.1), with the property (2.2) and fixed dimensions $\dim_{\mathbb{C}} F^k = f^k$, determines a Hodge structure $\{H^{k,n-k}\}_{k=0}^n$, with

$$H^{k,n-k} = F^k \cap \overline{F^{n-k}}.$$

A polarization for a Hodge structure of weight n consists of the data of a Hodge-Riemann bilinear form Q over \mathbb{Z} , which is symmetric for even n , skew symmetric for odd n , such that

$$Q(H^{k,n-k}, H^{r,n-r}) = 0 \quad \text{unless } k = n - r, \quad (2.3)$$

$$i^{2k-n} Q(v, \bar{v}) > 0 \quad \text{if } v \in H^{k,n-k}, v \neq 0. \quad (2.4)$$

In terms of the Hodge filtration $H_{\mathbb{C}} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^n$, the relations (2.3) and (2.4) can be written as

$$Q(F^k, F^{n-k+1}) = 0, \quad (2.5)$$

$$Q(Cv, \bar{v}) > 0 \quad \text{if } v \neq 0, \quad (2.6)$$

where C is the Weil operator given by $Cv = i^{2k-n}v$ when $v \in H^{k,n-k} = F^k \cap \overline{F^{n-k}}$.

Definition 2.1. Let S be a connected complex manifold, a variation of polarized Hodge structure of weight n over S consists of a polarized local system $H_{\mathbb{Z}}$ of \mathbb{Z} -modules and a filtration of the associated holomorphic vector bundle H :

$$\cdots \supseteq F^{k-1} \supseteq F^k \supseteq \cdots \quad (2.7)$$

by holomorphic subbundles F^k which satisfy:

1. $H = F^k \oplus \overline{F^{n-k+1}}$ as C^∞ bundles, where the conjugation is taking relative to the local system of real vectorspace $H_{\mathbb{R}} := H_{\mathbb{Z}} \otimes \mathbb{R}$.
2. $\nabla(F^k) \subseteq \Omega_S^1 \otimes F^{k-1}$, where ∇ denotes the flat connection on H .

We will refer to the holomorphic subbundles F^k as the Hodge bundles of the variation of polarized Hodge structure. And for each $s \in S$, we have the Hodge decomposition:

$$H_s = \bigoplus_{k=0}^n H_s^{k,n-k}; \quad H_s^{k,n-k} = \overline{H_s^{n-k,k}} \quad (2.8)$$

where $H^{k,n-k}$ is the C^∞ subbundle of H defined by:

$$H^{k,n-k} = F^k \cap \overline{F^{n-k}}.$$

Starting from a variation of polarized Hodge structure of weight n on a complex manifold S , fixing a point $s \in S$ as reference point, we can construct the period domain D and its dual \check{D} .

The classifying space or the period domain D for polarized Hodge structures with Hodge numbers $\{h^{k,n-k}\}_{k=0}^n$ is the space of all such Hodge filtrations

$$D = \{F^n \subset \cdots \subset F^0 = H_{\mathbb{C}} \mid \dim F^k = f^k, (2.5) \text{ and } (2.6) \text{ hold}\}.$$

The compact dual \check{D} of D is

$$\check{D} = \{F^n \subset \cdots \subset F^0 = H_{\mathbb{C}} \mid \dim F^k = f^k \text{ and } (2.5) \text{ hold}\}.$$

The classifying space or the period domain $D \subset \check{D}$ is an open subset.

Finally recall that a period map is given by a locally liftable, holomorphic mapping

$$\Phi : S \longrightarrow \Gamma \backslash D,$$

where D is the classifying space or the period domain for polarized Hodge structures with given Hodge numbers $h^{p,q}$, Γ is the monodromy group.

2.2 Degeneration of Hodge Structures

In this section, we will consider a variation of polarized Hodge structure over S , where S is a quasi-projective manifold with $\dim_{\mathbb{C}} S = k$. Let \bar{S} be its compactification such that $\bar{S} - S$ is a divisor of normal crossings.

Let $(\mathcal{U}, s) \subset \bar{S}$ be a special coordinate neighborhood, i.e., a coordinate neighborhood isomorphic to the polycylinder Δ^k such that

$$S \cap \mathcal{U} \cong \{s = (s_1, \cdots, s_l, \cdots, s_k) \in \Delta^k \mid \prod_{i=1}^l s_i \neq 0\} = (\Delta^*)^l \times \Delta^{k-l}.$$

where Δ , Δ^* are the unit disk and the punctured unit disk in the complex plane, respectively. Consider the period map

$$\Phi : (\Delta^*)^l \times \Delta^{k-l} \longrightarrow \Gamma \backslash D,$$

where Γ is the monodromy group. Let U be the upper half plane of \mathbb{C} . Then $U^l \times \Delta^{k-l}$ is the universal covering space of $(\Delta^*)^l \times \Delta^{k-l}$, and we can lift Φ to a mapping

$$\tilde{\Phi} : U^l \times \Delta^{k-l} \longrightarrow D.$$

Let $(z_1, \dots, z_l, s_{l+1}, \dots, s_k)$ be the coordinates of $U^l \times \Delta^{k-l}$ such that $s_i = e^{2\pi iz_i}$ for $1 \leq i \leq l$. Corresponding to each of the first l variables, we choose a monodromy transformation $\gamma_i \in \Gamma$, such that

$$\tilde{\Phi}(z_1, \dots, z_i + 1, \dots, z_l, s_{l+1}, \dots, s_k) = \gamma_i(\tilde{\Phi}(z_1, \dots, z_i, \dots, z_l, s_{l+1}, \dots, s_k))$$

holds identically in all variables. And the monodromy transformations γ_i 's commute with each other. By a theorem of Borel (see [8], Lemma 4.5 on p. 230), after passing to a finite cover if necessary, the monodromy transformation γ_i around the punctures $s_i = 0$ is unipotent, i.e.,

$$\begin{cases} (\gamma_i - I)^{m-1} = 0 \\ [\gamma_i, \gamma_j] = 0, \end{cases}$$

for some positive integer m . Therefore, we can define the monodromy logarithm $N_i = \log \gamma_i$ by the Taylor's expansion

$$N_i = \log \gamma_i := \sum_{j \geq 1} (-1)^{j+1} \frac{(\gamma_i - 1)^j}{j}, \quad \forall 1 \leq i \leq l,$$

then $N_i, 1 \leq i \leq l$ are nilpotent. Let $(v.)$ be a flat multivalued basis of H over $\mathcal{U} \cap S$. The formula

$$(\tilde{v}.)(s) := \exp \left(\frac{-1}{2\pi\sqrt{-1}} \sum_{i=1}^l \log s_i N_i \right) (v.)(s)$$

give us a single-valued basis of H . Deligne's canonical extension \tilde{H} of H over \mathcal{U} is generated by this basis $(\tilde{v}.)(\text{cf. [8]})$. And we have

Proposition 2.2. *If the local monodromy is unipotent, then the cononical extension is a vector bundle, otherwise it is a coherent sheaf.*

The construction of \tilde{H} is independent of the choice of the local coordinates s_i 's and the flat multivalued basis $(v.)$. For any holomorphic subbundle A of H , Deligne's canonical extension of A is defined to be $\tilde{A} := \tilde{H} \cap j_* A$ where $j : S \rightarrow \bar{S}$ is the inclusion map. Then we have the canonical extension of the Hodge filtration:

$$\tilde{H} = \tilde{F}^0 \supset \tilde{F}^1 \supset \dots \supset \tilde{F}^n \supset 0,$$

which is also a filtration of locally free sheaves.

Let N be a linear combination of $N_i, 1 \leq i \leq l$, then N defines a weight flat filtration $W_\bullet(N)$ of $H_{\mathbb{C}}$ (cf. [2], [8]) by

$$0 \subset \dots \subset W_{i-1}(N) \subset W_i(N) \subset W_{i+1}(N) \subset \dots \subset H_{\mathbb{C}}.$$

Denote by $W_{\bullet}^j := W_{\bullet}(\sum_{\alpha=1}^j N_{\alpha})$ for $j = 1, \dots, l$, we can choose a flat multigrading

$$H_{\mathbb{C}} = \sum_{\beta_1, \dots, \beta_l} H_{\beta_1, \dots, \beta_l},$$

such that

$$\bigcap_{j=1}^l W_{\beta_j}^j = \sum_{k_j \leq \beta_j} H_{k_1, \dots, k_l}.$$

Let h be the Hodge metric on the variation of polarized Hodge structure H . In the special neighborhood \mathcal{U} , let v be a nonzero local multivalued flat section of a multigrading component H_{k_1, \dots, k_l} , then $(\tilde{v})(s) := \exp(\frac{-\sum_{i=1}^l \log s_i N_i}{2\pi\sqrt{-1}})v(s)$ is a local single-valued section of \tilde{H} . And, there holds a norm estimate (Theorem 5.21 in [1])

$$\|\tilde{v}(s)\|_h \leq C_1 \left(\frac{-\log |s_1|}{-\log |s_2|}\right)^{k_1/2} \left(\frac{-\log |s_2|}{-\log |s_3|}\right)^{k_2/2} \dots (-\log |s_l|)^{k_l/2}, \quad (2.9)$$

on the region

$$\Xi(N_1, \dots, N_l) := \{(s_1, \dots, s_l, \dots, s_k) \in (\Delta^*)^l \times \Delta^{k-l} \mid |s_1| \leq \dots \leq |s_k| \leq \epsilon\}$$

for some small $\epsilon > 0$, where C_1 is a positive constant dependent on the ordering of $\{N_1, N_2, \dots, N_l\}$ and ϵ . Since the number of ordering of $\{N_1, N_2, \dots, N_l\}$ is finite, for any flat multivalued local section v of H , there exist positive constants C_2 and M_2 such that

$$\|\tilde{v}(s)\|_h \leq C_2 \left(\prod_{i=1}^l -\log |s_i|\right)^{M_2}, \quad (2.10)$$

in the domain $\{(s_1, \dots, s_l, \dots, s_k) \mid 0 < |s_i| < \epsilon (i = 1, \dots, l), |s_j| < \epsilon (j = l+1, \dots, k)\}$. Moreover, since the dual H^* is also a variation of polarized Hodge structure, we then know that, for any flat multivalued local section v of H , there holds

$$C' \left(\prod_{i=1}^l -\log |s_i|\right)^{-M} \leq \|\tilde{v}(s)\|_h \leq C'' \left(\prod_{i=1}^l -\log |s_i|\right)^M, \quad (2.11)$$

where C' and C'' both only depend on ϵ .

By using this norm estimate, E. Cattani, A. Kaplan and W. Schmid [1] get the following result for the Chern forms of Hodge bundles over the quasi-projective manifold S , which is Corollary 5.23 in [1].

Theorem 2.3. *Let S be a smooth variety, $\bar{S} \supset S$ be a smooth compactification such that $\bar{S} - S = D_{\infty}$ is a normal crossing divisor. If H is a variation of polarized Hodge structure over S with unipotent monodromies around D_{∞} , then the Chern forms of Hodge metric on various Hodge bundles F^p/F^q define currents on the compactification \bar{S} . Moreover, the first Chern form represents the first Chern class of the canonical extension $\widetilde{F^p/F^q} \rightarrow \bar{S}$.*

Base on this result, the proof of [4, Theorem 5.1] gives us the following result for any subbundle of the variation of Hodge structure H .

Theorem 2.4. *Let S be a smooth variety, $\bar{S} \supset S$ be a smooth compactification such that $\bar{S} - S = D_\infty$ is a normal crossing divisor. Let H be a variation of polarized Hodge structure over S with unipotent monodromies around D_∞ and A be a vector subbundle of H . Then the first Chern form of A with respect to the induced Hodge metric is integrable. Moreover, let R_H be the curvature form with the induced Hodge metric over A , then we have*

$$\left(\frac{-1}{2\pi i}\right)^n \int_S (\text{tr} R_H)^n = c_1(\tilde{A})^n,$$

where $n = \dim_{\mathbb{C}} S$.

3 Chern Forms of Calabi-Yau Moduli with the Hodge Metric

In Section 3.1, we review the definition of the moduli space of a polarized Calabi-Yau manifold with level m structure with $m \geq 3$ and various Hodge bundles over the moduli space. In Section 3.2 and 3.3, by a key observation that the tangent bundle of the Calabi-Yau moduli is a subbundle of the variation of polarized Hodge structure $(H^n)^* \otimes H^n \rightarrow \mathcal{M}_m$, we get that the first Chern form of the Calabi-Yau moduli \mathcal{M}_m are integrable with the Hodge metric.

3.1 Calabi-Yau Moduli and Hodge Bundles

In this section, we briefly review the construction of the moduli space of polarized Calabi-Yau manifolds with level m structure with $m \geq 3$ and its basic properties. For the concept of Kuranishi family of compact complex manifolds, we refer to [10, Pages 8-10], [7, Page 94] or [12, Page 19] for equivalent definitions and more details. If a complex analytic family $\pi : \mathcal{X} \rightarrow S$ of compact complex manifolds is complete at each point of S and versal at the point $0 \in S$, then the family $\pi : \mathcal{X} \rightarrow S$ is called a Kuranishi family of the complex manifold $X = \pi^{-1}(0)$. The base space S is called the Kuranishi space. If the family is complete at each point of a neighbourhood of $0 \in S$ and versal at 0 , then this family is called a local Kuranishi family at $0 \in S$. In particular, by definition, if the family is versal at each point of S , then it is local Kuranishi at each point of the base S .

A basis of the quotient space $(H_n(X, \mathbb{Z})/\text{Tor})/m(H_n(X, \mathbb{Z})/\text{Tor})$ is called a level m structure on the polarized Calabi-Yau manifold (X, L) , where we always assume $m \geq 3$. For deformations of polarized Calabi-Yau manifolds with level m structure, we have the following theorem, which is a reformulation of [9, Theorem 2.2]. One can also look at [7] and [12] for more details about the construction of the moduli space of Calabi-Yau manifolds.

Theorem 3.1. *Let (X, L) be a polarized Calabi-Yau manifold with level m structure with $m \geq 3$, then there exists a quasi-projective complex manifold \mathcal{M}_m with a versal family of Calabi-Yau manifolds,*

$$\mathcal{X}_{\mathcal{M}_m} \longrightarrow \mathcal{M}_m, \quad (3.1)$$

containing X as a fiber, and polarized by an ample line bundle $\mathcal{L}_{\mathcal{M}_m}$ on the versal family $\mathcal{X}_{\mathcal{M}_m}$.

Let us define $\mathcal{T}_L(X)$ to be the universal cover of the base space \mathcal{M}_m of the versal family above,

$$\pi : \mathcal{T}_L(X) \longrightarrow \mathcal{M}_m$$

and the family

$$\mathcal{U} \longrightarrow \mathcal{T}_L(X)$$

to be the pull-back of the family (3.1) by the projection π , which can be considered as a family of polarized and marked Calabi-Yau manifolds. Recall that a marking on a Calabi-Yau manifold is given by an integral basis of $H_n(X, \mathbb{Z})/\text{Tor}$. For simplicity, we will denote $\mathcal{T}_L(X)$ by \mathcal{T} , which has the following property:

Proposition 3.2. *The Teichmüller space \mathcal{T} is a simply connected smooth complex manifold, and the family*

$$\mathcal{U} \longrightarrow \mathcal{T} \quad (3.2)$$

containing X as a fiber, is local Kuranishi at each point of the Teichmüller space \mathcal{T} .

Note that the Teichmüller space \mathcal{T} does not depend on the choice of level m . In fact, let m_1, m_2 be two different positive integers, $\mathcal{U}_1 \rightarrow \mathcal{T}_1$ and $\mathcal{U}_2 \rightarrow \mathcal{T}_2$ are two versal families constructed via level m_1 and level m_2 respectively as above, both of which contain X as a fiber. By using the fact that \mathcal{T}_1 and \mathcal{T}_2 are simply connected and the definition of versal families, we have a biholomorphic map $f : \mathcal{T}_1 \rightarrow \mathcal{T}_2$, such that the versal family $\mathcal{U}_1 \rightarrow \mathcal{T}_1$ is the pull-back of the versal family $\mathcal{U}_2 \rightarrow \mathcal{T}_2$ by the map f . Thus these two families are isomorphic to each other.

In this paper, we call \mathcal{M}_m with $m \geq 3$ the Calabi-Yau moduli for simplicity. Given any point $p \in \mathcal{M}_m$, the corresponding fiber X_p in the versal family $\mathcal{U} \rightarrow \mathcal{T}$ is a polarized Calabi-Yau manifold (X_p, L_p) . Hence, the flat holomorphic bundle H^n over \mathcal{M}_m whose fibers are the primitive cohomology group $PH^n(X_p), p \in \mathcal{M}_m$, endowed with the Gauss-Manin connection, carries a polarized Hodge structure of weight n .

The flat bundle H^n contains a flat real subbundle $H_{\mathbb{R}}^n$, whose fiber corresponds to the subspaces $PH^n(X_p, \mathbb{R}) \subset PH^n(X_p)$; and $H_{\mathbb{R}}^n$, in turn, contains a flat lattice bundle $H_{\mathbb{Z}}^n$, whose fibers are the images of $PH^n(X_p, \mathbb{Z})$ in $PH^n(X_p, \mathbb{R})$. Moreover, there exist C^∞ subbundles $H^{p,q} \subset H^n$ with $p+q=n$, whose fibers over $p \in \mathcal{M}_m$ are $PH^{p,q}(X_p)$. For $0 \leq k \leq n$, $F^k = \bigoplus_{i \geq k} H^{i, n-i}$ are then holomorphic subbundles of H^n . Thus the bundle H^n defines a variation of polarized Hodge structure over \mathcal{M}_m , which is defined over \mathbb{Z} . Thus, by the functorial construction of variation of polarized Hodge structure, the holomorphic bundle $(H^n)^* \otimes H^n \rightarrow \mathcal{M}_m$ defines a variation of polarized Hodge structure over \mathcal{M}_m , which is defined over \mathbb{Z} .

3.2 Period Map and the Hodge Metric on Calabi-Yau Moduli

For any point $p \in \mathcal{T}$, let (X_p, L_p) be the corresponding fiber in the versal family $\mathcal{U} \rightarrow \mathcal{T}$, which is a polarized Calabi–Yau manifold. A canonical identification of the middle dimensional cohomology of X_p to that of the background manifold M , that is, $H^n(M) \simeq H^n(X_p)$ can be used to identify $H^n(X_p)$ for all fibers over \mathcal{T} . Thus we get a canonically trivial bundle $H^n(M) \times \mathcal{T}$. The period map from \mathcal{T} to D is defined by assigning to each point $p \in \mathcal{T}$ the Hodge structure on X_p , that is

$$\Phi_{\mathcal{T}} : \mathcal{T} \rightarrow D, \quad p \mapsto \Phi(p) = \{F^n(X_p) \subset \cdots \subset F^0(X_p)\}$$

For the Calabi-Yau moduli \mathcal{M}_m , we have the following period map:

$$\Phi : \mathcal{M}_m \longrightarrow D/\Gamma, \tag{3.3}$$

where Γ denotes the global monodromy group which acts properly and discontinuously on the period domain D . By going to finite covers of \mathcal{M}_m and D/Γ , we may also assume D/Γ is smooth without loss of generality.

In [3], Griffiths and Schmid studied the Hodge metric on the period domain D which is the natural homogeneous metric on D . We denote it by h . In particular, this Hodge metric is a complete homogeneous metric. By local Torelli theorem for Calabi–Yau manifolds, we know that $\Phi_{\mathcal{T}}, \Phi$ are both locally injective. Thus it follows from [3] that the pull-backs of h by $\Phi_{\mathcal{T}}$ and Φ on \mathcal{T} and \mathcal{M}_m respectively are both well-defined Kähler metrics. By abuse of notation, we still call these pull-back metrics the Hodge metrics. For explicit formula of the Hodge metric over moduli space of polarized Calabi-Yau manifolds, especially for threefolds, the reader can refer to [L99], [L01-1] and [L01-2] for details.

The period map has several good properties, and one may refer to Chapter 10 in [13] for details. Among them, one of the most important is the following Griffiths transversality: the period map Φ is a holomorphic map and its tangent map satisfies that

$$\Phi_*(v) \in \bigoplus_{k=1}^n \text{Hom}(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k) \quad \text{for any } p \in \mathcal{T} \text{ and } v \in T_p^{1,0}\mathcal{T}$$

with $F^{n+1} = 0$, or equivalently, $\Phi_*(v) \in \bigoplus_{k=0}^n \text{Hom}(F_p^k, F_p^{k-1})$. And, by the local Torelli theorem for Calabi-Yau manifolds, the map Φ_* is injective. So we have

Proposition 3.3. *Let \mathcal{M}_m be the moduli space of polarized Calabi-Yau manifolds with level m structure with $m \geq 3$. Then $(H^n)^* \otimes H^n$ defines a variation of polarized Hodge structure over \mathcal{M}_m , which is defined over \mathbb{Z} . Moreover, with the induced Hodge metric over the Calabi-Yau moduli \mathcal{M}_m , the tangent bundle*

$$T\mathcal{M}_m \hookrightarrow (H^n)^* \otimes H^n, \tag{3.4}$$

is a holomorphic subbundle of $(H^n)^ \otimes H^n$ over \mathcal{M}_m with the induced Hodge metric.*

3.3 Chern Forms of Calabi-Yau Moduli with the Hodge Metric

As the Calabi-Yau moduli \mathcal{M}_m is quasi-projective, see Theorem 3.1, we know that there is a compact projective manifold $\overline{\mathcal{M}}_m$ such that $\overline{\mathcal{M}}_m - \mathcal{M}_m$ is a normal crossing divisor. Also, the local monodromy of the variation of polarized Hodge structure around the divisor is at least quasi-unipotent. Thus after passing to a finite ramified cover if necessary, the local monodromy becomes unipotent. Therefore, without loss of generality, we can assume the Hodge bundles have canonical extensions, which are vector bundles over the compactification $\overline{\mathcal{M}}_m$ of the Calabi-Yau moduli \mathcal{M}_m , due to Proposition 2.2. Then, by Theorem 2.3 and Theorem 2.4, we have

Theorem 3.4. *The first Chern form of the Calabi-Yau moduli \mathcal{M}_m with the induced Hodge metric define currents over the compactification $\overline{\mathcal{M}}_m$ with normal crossing boundary divisors. Moreover, let R_H be the curvature form of $T\mathcal{M}_m$ with the induced Hodge metric, then we have*

$$\left(\frac{-1}{2\pi i}\right)^N \int_{T\mathcal{M}_m} (\text{tr} R_H)^N = c_1(\widetilde{T\mathcal{M}_m})^N$$

where $N = \dim_{\mathbb{C}} \mathcal{M}_m$.

Proof. By Proposition 3.3, with the Hodge metric, the tangent bundle $T\mathcal{M}_m$ of the Calabi-Yau moduli \mathcal{M}_m is a holomorphic subbundle of the variation of polarized Hodge structure $(H^n)^* \otimes H^n \rightarrow \mathcal{M}_m$. So $T\mathcal{M}_m$ has the canonical extension, which give us a holomorphic vector bundle $\widetilde{T\mathcal{M}_m} \subset \widetilde{(H^n)^* \otimes H^n}$ over $\overline{\mathcal{M}}_m$. Therefore, by Theorem 2.4, the first Chern form of $T\mathcal{M}_m \rightarrow \mathcal{M}_m$ define currents over the compactification $\overline{\mathcal{M}}_m$ of \mathcal{M}_m , which represent the first Chern class of the vector bundle $\widetilde{T\mathcal{M}_m} \rightarrow \overline{\mathcal{M}}_m$ with the induced Hodge metric. \square

4 Chern Forms of Calabi-Yau Moduli with the Weil-Petersson Metric

In this section, we first review the Weil-Petersson geometry of the Calabi-Yau moduli. Then, by the standard isomorphism $T\mathcal{M}_m \cong (F^n)^* \otimes F^{n-1}/F^n$, we get that the Chern forms of Calabi-Yau moduli \mathcal{M}_m are integrable with the Weil-Petersson metric.

For each fiber X_s , we assign the Calabi-Yau metric $g(s)$ in the polarization Kähler class. Then on the fiber $X_s = X$, the Kodaira-Spencer theory gives rise to an isomorphism $\rho : T_s S \rightarrow H^1(X, T^{1,0} X) \cong \mathbb{H}^{0,1}(T^{1,0} X)$, the space of the harmonic Beltrami differentials. So for $v, w \in T_s S$, one defines the Weil-Petersson metric on S by:

$$g_{WP}(v, w) := \int_X \langle \rho(v), \rho(w) \rangle_{g(s)} d\text{vol}_{g(s)}$$

Let $\dim X = n$. Using the fact that the global holomorphic n -form $\Omega = \Omega(s)$ is flat with respect to $g(s)$, it can be shown that

$$g_{WP}(v, w) = -\frac{\tilde{Q}(i_v\Omega, \overline{i_w\Omega})}{\tilde{Q}(\Omega, \overline{\Omega})}. \quad (4.1)$$

Here, for convenience, we write $\tilde{Q}(\cdot, \cdot) = (\sqrt{-1})^n Q(\cdot, \cdot)$, where Q is the intersection product. Therefore, \tilde{Q} has alternating signs in the successive primitive cohomology groups $PH^{p,q} \subset H^{p,q}$ with $p+q = n$. In particular, g_{WP} is Kähler and is independent of the choice of Ω . In fact, g_{WP} is also independent of the choice of the polarization. The reader can refer to [6] for details of the definition.

Formula (4.1) of the Weil-Petersson metric implies that the natural map $H^1(X, T_X) \rightarrow \text{Hom}(F^n, F^{n-1}/F^n)$ via the interior product $v \mapsto i_v\Omega$ is an isometry from the tangent bundle $T\mathcal{M}_m$ with the Weil-Petersson metric to the Hodge bundle $(F^n)^* \otimes F^{n-1}/F^n$ with the induced Hodge metric. So the Weil-Petersson metric is precisely the metric induced from the first piece of the Hodge metric on the horizontal tangent bundle over the period domain.

More precisely, for the Calabi-Yau moduli \mathcal{M}_m , we have the following period map from the moduli space to the period domain of Hodge structures:

$$\Phi : \mathcal{M}_m \rightarrow D/\Gamma \quad (4.2)$$

And the differential of the period map Φ gives us the infinitesimal period map at $p \in \mathcal{M}_m$:

$$\Phi_* : T_p\mathcal{M}_m \rightarrow \text{Hom}(F^n, F^{n-1}/F^n) \oplus \text{Hom}(F^{n-1}/F^n, F^{n-2}/F^{n-1}) \oplus \dots$$

is an isomorphism in the first piece. By using this isomorphism and Theorem 2.4, we have the following result, which is [5, Theorem 6.3]. Our proof is different and much simpler.

Theorem 4.1. *The Chern forms of the Calabi-Yau moduli \mathcal{M}_m with the Weil-Petersson metric define currents over the compactification $\overline{\mathcal{M}}_m$ of \mathcal{M}_m . Moreover, the first Chern form represent the first Chern class of the quotient bundle $(F^n)^* \otimes F^{n-1}/(F^n)^* \otimes F^n \rightarrow \overline{\mathcal{M}}_m$.*

Proof. Equipped with the Weil-Petersson metric, the tangent bundle $T\mathcal{M}_m$ of the Calabi-Yau moduli \mathcal{M}_m is isomorphic to

$$(F^n)^* \otimes F^{n-1}/F^n \cong (F^n)^* \otimes F^{n-1}/(F^n)^* \otimes F^n,$$

which is a quotient of subbundles of the variation of polarized Hodge structure $(H^n)^* \otimes H^n \rightarrow \mathcal{M}_m$. Here the Hodge bundles $F^{k'}$'s are all equipped with their natural Hodge metrics. So, by Theorem 2.3, the Chern forms of $T\mathcal{M}_m$ define currents over the compactification $\overline{\mathcal{M}}_m$ of \mathcal{M}_m . Moreover, the first Chern form of the tangent bundle $T\mathcal{M}_m$ with the Weil-Petersson metric represent the first Chern class of the canonical extension

$$\widetilde{T\mathcal{M}_m} \cong (F^n)^* \otimes F^{n-1}/(F^n)^* \otimes F^n.$$

□

As a corollary, we have the following result on the Chern numbers,

Corollary 4.2. *Let f be an invariant polynomial on $\text{Hom}(T\mathcal{M}_m, T\mathcal{M}_m)$ and R_{WP} represent the curvature form of the Weil-Petersson metric on the Calabi-Yau moduli \mathcal{M}_m . Then we have*

$$\int_{\mathcal{M}_m} \text{tr}(f(R_{WP})) < \infty. \quad (4.3)$$

Proof. The proof follows directly from Theorem 4.1. □

As pointed out in the introduction, it follows from Theorem 2.3 easily that the first Chern form of all of the Hodge bundles with Hodge metrics also represent the Chern classes of their canonical extensions. Finally note that the Kähler form of the Weil-Petersson metric is equal to the first Chern form of the Hodge bundle F^n with its Hodge metric,

$$\omega_{WP} = c_1(F^n)_H,$$

so we easily deduce that the Weil-Petersson volume is finite and is a rational number, as proved in [6] and [11] by computations.

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