Three semi-discrete integrable systems related to orthogonal polynomials and their generalized determinant solutions

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Abstract. In this paper, we present a generalized Toeplitz determinant solution for the generalized Schur flow and propose a mixed form of the two known relativistic Toda chains together with its generalized Toeplitz determinant solution. Besides, we also give a Hankel type determinant solution for a nonisospectral Toda lattice. All these results are obtained by technical determinant operations. As a bonus, we finally obtain some new combinatorial numbers based on the moment relations with respect to these semi-discrete integrable systems and give the corresponding combinatorial interpretations by means of the lattice paths.

AMS classification scheme numbers: 37K10, 05A15, 11C20

PACS numbers: 02.30.Ik, 02.10.Ox

Keywords: Generalized determinant solutions, Combinatorial interpretations, Semi-discrete integrable systems, Orthogonal polynomials

Submitted to: *Nonlinearity*

1. Introduction

In the literature, there have been many connections found between certain classes of orthogonal polynomials and integrable systems. We can list a variety of integrable systems related to orthogonal polynomials. In the fully discrete case, the classical examples are the discrete-time Toda chain [24] and the discrete-time Lotka-Volterra chain [42]. These two discrete-time integrable systems can be derived from the compatibility of some orthogonal polynomials and symmetric orthogonal polynomials respectively. As for the relations with semi-discrete integrable systems, the orthogonal polynomials appear as wave functions of the Lax pair of the Toda lattice due to a one-parameter deformation of the measure [15, 32, 37]. Besides, the Camassa-Holm equation [13, 19], as a typical representative, has a closed form solution expressed in terms of the orthogonal polynomials [8], which bridges the relation between orthogonal polynomials and continuous integrable systems. For more examples, please consult [34, 35, 41, 48].

In this paper, we specially focus on three semi-discrete integrable systems, which have direct links with orthogonal polynomials.

The first one is the Schur flow. In [30], Mukaihira and Nakamura discussed monic orthogonal polynomials on the unit circle, which are named as the Szegő orthogonal polynomials [46]. The Szegő orthogonal polynomials admit recurrence relations, where the coefficients are the so-called Schur parameters α_k (also referred to as Verblunsky coefficients [39, 40]). It was shown that, if some time-dependent measure is involved, the Schur parameters satisfy the complex semi-discrete modified KdV equation, namely, the Schur flow:

$$\partial_t\, \alpha_k = \beta_k(\alpha_{k-1} - \alpha_{k+1}), \qquad \beta_k := 1 - |\alpha_k|^2\,, \quad k = 1, 2, \ldots, \quad \alpha_0 = 1.$$

For this Schur flow, they gave its solution in terms of the Toeplitz type determinant. It is noted that the real Schur flow appeared in [1, 2] under the name discrete modified KdV equation, as a spatial discretization of the modified Korteweg-de Vries equation $f_t = 6f^2 f_x - f_{xxx}$. In [18], Faybusovich and Gekhtman dealt with finite real Schur flows and suggested two more distinct Lax equations based on the Hessenberg matrix representation of the multiplication operator (see also [6, 10]).

The second one is related to the relativistic Toda chain, which has intimate connections with the Laurent biorthogonal polynomials [29, 38, 45, 47]. In [29, 47], by considering the two different evolution relations of the Laurent polynomials with respect to time t, Zhedanov el al. obtained the following two distinct semi-discrete integrable flows:

$$\partial_t\, d_n = \frac{b_{n+1}}{d_{n+1}} - \frac{b_n}{d_{n-1}}, \qquad \partial_t\, b_n = b_n(\frac{1}{d_n} - \frac{1}{d_{n-1}}), \tag{1}$$

and

$$\partial_t d_n = -d_n(b_{n+1} - b_n), \qquad \partial_t b_n = -b_n(b_{n+1} - b_{n-1} + d_{n-1} - d_n). \tag{2}$$

These two semi-discrete integrable flows are both equivalent to the restricted relativistic Toda chain. The parameters b_n and d_n appeared as the coefficients of the three-term recurrence relation satisfied by the Laurent biorthogonal polynomials. In what follows we respectively name the equations (1) and (2) as the first form of the relativistic Toda chain and the second form of the relativistic Toda chain without confusion. It is noted that these two forms of the relativistic Toda chain (1) and (2) admit the solutions taking the same explicit expressions in terms of ratios of Toeplitz determinants, where the moments of the Toeplitz determinants satisfy different time evolution relations [47].

The last semi-discrete integrable system discussed in this paper is the nonisospectral Toda lattice which is in the framework of orthogonal polynomials as well. In paper [9], Berezansky and Shmoish considered one-parametric family deformations of the measures of the orthogonal polynomials with respect to time t, and presented nonisospectral deformations of the finite and semi-infinite Jacobi matrices governed by a generalized Lax equation. Such an equation is equivalent to a wide class of generalized Toda lattices. Among this wide class of generalized Toda lattices, there is a special nonisospectral Toda lattice presented as follows:

$$\partial_t a_n = \frac{1}{2} a_n [b_{n+1}(3+2n) - b_n(2n-1)], \tag{3}$$

$$\partial_t b_n = 2a_{n-1}^2(1-n) + 2a_n^2(1+n) + b_n^2. \tag{4}$$

To the best of our knowledge, the determinant solution (with time evolution of the moments given by (50)) of such nonisospectral Toda lattice has not been considered so far.

Aiming at the above three semi-discrete integrable systems, we design this paper to seek their generalized determinant solutions respectively. In paper [25], Kajiwara et al. showed us the generalized Hankel type determinant solutions of the Toda and discrete Toda equations based on the obtained determinant formulae for the τ functions of the Painlevé equations. The key point in their work is introducing the convolution term for the evolution of moments $\{c_n\}_{n=0}^{\infty}$ in the Hankel determinants, i.e.

$$\partial_t c_{n-1} = c_n - \psi \sum_{i=0}^{n-2} c_i c_{c-2-i}, \qquad c_0 = \phi,$$

with two arbitrary functions $\phi(t)$ and $\psi(t)$. Moreover, in paper [37], from the point of view of the standard orthogonal polynomials and a (nonmatrix) Stieltjes function, Zhedanov et al. got that the equations of motion for the Toda chain are equivalent to a Riccati equation for the Stieltjes function as well as to a recurrence relation satisfied by the moments. Motivated by their ideas, on one hand, we turn to search the recurrence relations satisfied by the moments of the Toeplitz type determinants. On the other hand, we adjust the recurrence relation admitted by the moments of the Hankel type determinant to the nonisospectral case. Then the generalized determinant solutions with respect to the above three semi-discrete integrable systems are deduced. Our results are obtained with the help of the determinant identity techniques, especially of the *Jacobi identity* and the *Schwein identity*.

The first purpose of this paper is to derive a generalized Toeplitz determinant formula of the generalized Schur flow, where the moments of the Toeplitz determinant satisfy recurrence relations.

Our second task aims to obtain a generalized form of the relativistic Toda chain based on the forms (1) and (2) as well as its corresponding generalized Toeplitz determinant solution. The moments of the Toeplitz determinant solution admit similar recurrence relations to those got in the generalized Schur flow. In the sequel, we name this generalized form of the relativistic Toda chain as the mixed form of the relativistic Toda chain. It is shown that this mixed form of the relativistic Toda chain can be reduced to the original forms (1) and (2) respectively after choosing some appropriate parameters.

The next objective of this paper is to derive a Hankel determinant solution for the nonisopectral Toda lattice (3) and (4). For the sake of convenience, we actually choose to consider the deformations of (3) and (4) in subsection 3.3, that is

$$\partial_t u_n = u_n\left[(2n+1)v_{n+1} - (2n-3)v_n\right],$$
$$\partial_t v_n = v_n^2 + 2nu_n - 2(n-2)u_{n-1}.$$

It turns out that, the evolution of the moments with respect to time t involves the index variable n while it doesn't happen in isospectral case.

Finally, recall that some classical combinatorial numbers satisfy recurrence relations. For instance:

Catalan numbers: $c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k} \qquad (n \geq 0), \quad c_0 = 1.$

Motzkin numbers: $M_{n+1} = M_n + \sum_{k=0}^{n-1} M_k M_{n-1-k} \qquad (n \geq 0), \quad M_0 = 1.$

Large Schröder numbers: $S_{n+1} = S_n + \sum_{k=0}^{n} S_k S_{n-k} \qquad (n \geq 0), \quad S_0 = 1.$

It is well known that these combinatorial numbers all can be interpreted via the view point of lattice paths [4, 14, 16, 17, 21, 43].

Inspired by this idea, as a bonus, in section 4, we get some new combinatorial numbers based on the moments relations with respect to the above three semi-discrete integrable systems and then address their corresponding combinatorial interpretations by means of the lattice paths.

This paper is organized as follows: Since our results are obtained by use of determinant identity techniques, we firstly introduce two important determinant identity formulae in section 2. In section 3, we consider the generalized determinant solutions with regard to the three semi-discrete integrable systems. More precisely, in section 3.1, we are committed to giving a generalized solution of the generalized Schur flow in terms of Toeplitz type determinant. Then, in section 3.2, we propose a mixed form of the relativistic Toda chain from the two known relativistic Toda forms together with the corresponding generalized Toeplitz determinant solution. In section 3.3, we construct a Hankel determinant solution for the nonisospectral Toda lattice. In section 4, some new combinatorial numbers and their combinatorial interpretations are obtained. Section 5 is devoted to conclusion and discussion. It is noted that we place the proofs of all the lemmas in the Appendices A-C.

2. Determinant formulae

The determinant identity techniques are standard in the theory of integrable systems. As we mentioned in section 1, these techniques make a great contribution to the acquisition of our main results in this paper, so it is necessary to list them independently.

As a given determinant D, the celebrated *Jacobi identity* [5, 12] reads

$$D \cdot D\begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix} = D\begin{bmatrix} i_1 \\ j_1 \end{bmatrix} \cdot D\begin{bmatrix} i_2 \\ j_2 \end{bmatrix} - D\begin{bmatrix} i_1 \\ j_2 \end{bmatrix} \cdot D\begin{bmatrix} i_2 \\ j_1 \end{bmatrix}, \tag{5}$$

where $D\begin{bmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{bmatrix}$ with $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ denotes the determinant of the matrix obtained from D by removing the rows with indices $i_1, i_2, \ldots, i_k$ and the columns with indices $j_1, j_2, \ldots, j_k$.

The *Schwein identity* [5, 12] of size n can be expressed as follows.

$$\begin{aligned}
&\begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \end{vmatrix} \begin{vmatrix} \hat{b}_1 & \hat{b}_2 & \cdots & \hat{b}_{n-1} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-1} \end{vmatrix} \\
-&\begin{vmatrix} \hat{b}_1 & \hat{b}_2 & \cdots & \hat{b}_n \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \end{vmatrix} \begin{vmatrix} b_1 & b_2 & \cdots & b_{n-1} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-1} \end{vmatrix} \\
=&\begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ \hat{b}_1 & \hat{b}_2 & \cdots & \hat{b}_n \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n} \end{vmatrix} \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix},
\end{aligned} \tag{6}$$

where the elements of each matrices can be any numbers. In fact, it is easy to check that this *Schwein identity* follows immediately from applying the *Jacobi identity* (5) to

$$D = \begin{vmatrix} b_1 & b_2 & \cdots & b_n & 0 \\ \hat{b}_1 & \hat{b}_2 & \cdots & \hat{b}_n, & 1 \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} & 0 \end{vmatrix}$$

with $i_1 = 1$, $i_2 = n + 1$, $j_1 = n$ and $j_2 = n + 1$ respectively.

3. Generalized determinant solutions of the three semi-discrete integrable systems

3.1. Generalized Toeplitz determinant solution of the generalized Schur flow

In [30], Mukaihira and Nakamura considered the orthogonal polynomials $\{\Phi_n(z)\}_{n=0}^{\infty}$ on the unit circle $\{z|z = e^{i\theta}, 0 \leq \theta \leq 2\pi\}$ introduced by Szegő [46], where the inner product is defined by

$$< \Phi_k|d\sigma|\Phi_l >= \int_0^{2\pi} \Phi_l \overline{\Phi}_k d\sigma(\theta) = h_k\delta_{k,l}, \quad k, l = 0, 1, \ldots$$

with a finite positive Borel measure $\sigma(\theta)$. Note that, throughout this paper, $\overline{X}$ means the complex conjugation of X.

The polynomials $\Phi_k(z)$ and the supplementary polynomials $\Phi_k^*(z) := z^k\overline{\Phi}_k(1/z)$ satisfy the following relations [3]:

$$\begin{pmatrix} \Phi_{k+1} \\ \Phi_{k+1}^* \end{pmatrix} = \begin{pmatrix} z & \alpha_{k+1} \\ z\bar{\alpha}_{k+1} & 1 \end{pmatrix} \begin{pmatrix} \Phi_k \\ \Phi_k^* \end{pmatrix}, \qquad k = 0, 1, \ldots$$

with $\Phi_0 = 1$ and $\Phi_0^* = 1$. Here α_k denote $\alpha_k := \Phi_k(0)$, $k = 1, 2, \ldots$, and are referred as the Schur parameters (or Verblunsky coefficients [39, 40]) of the orthogonal polynomials $\Phi_k(z)$ on the unit circle.

Consider the following one-parameter deformation of the measure

$$d\sigma(\theta; t) = \exp\{(z + \frac{1}{z})t\}d\sigma(\theta; 0), \quad z = e^{i\theta}, \qquad t \in R,$$

where $\sigma(\theta; 0) = \sigma(\theta)$. The coefficients $\alpha_k(t)$ satisfy the complex semi-discrete modified KdV equation, namely, the Schur flow:

$$\partial_t \alpha_k = \beta_k(\alpha_{k-1} - \alpha_{k+1}), \quad \beta_k := 1 - |\alpha_k|^2, \quad k = 1, 2, \ldots, \quad \alpha_0 = 1. \quad (7)$$

The Schur flow (7) has the Toeplitz determinant representation [30]

$$\alpha_k(t) := \frac{\hat{\Delta}_k(t)}{\Delta_k(t)},$$

where

$$\Delta_k(t) := \begin{vmatrix} s_0 & s_1 & \cdots & s_{k-1} \\ s_{-1} & s_0 & \cdots & s_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1-k} & s_{2-k} & \cdots & s_0 \end{vmatrix}, \quad \hat{\Delta}_k(t) := \begin{vmatrix} s_1 & s_2 & \cdots & s_k \\ s_0 & s_1 & \cdots & s_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{2-k} & s_{3-k} & \cdots & s_1 \end{vmatrix}, \quad (8)$$

and the moments $s_j(t) = \int_0^{2\pi} z^j d\sigma(\theta; t)$ admit the following relations:

$$\partial_t s_j = s_{j-1} + s_{j+1}, \quad s_j = \bar{s}_{-j}, \quad s_0 : real, \quad j = 0, \pm 1, \ldots . \quad (9)$$

Based on (9), we consider a generalization of the evolution relations of the moments with a convolution term and obtain a generalized Toeplitz type determinant solution for a generalized Schur flow with the help of some determinant techniques. We present our result as the following theorem:

Theorem 3.1.1. *Consider the generalized Schur flow*

$$\partial_t \alpha_k = \mu\beta_k(\alpha_{k-1} - \alpha_{k+1}) + i\psi\zeta\alpha_k, \quad \beta_k := 1 - |\alpha_k|^2, \, k = 1, 2, \ldots, \tag{10}$$

with the initial conditions $\alpha_0 = 1$ *and* $\alpha_1 = \frac{\phi(t)}{\zeta(t)}$, *where* $\mu(t)$, $\psi(t)$, $\varphi(t)$ *and* $\zeta(t)$ *are real functions of* t, *and* $\phi(t)$ *is a complex function of* t, *and there hold* $\partial_t\zeta(t) = \mu(t)[\overline{\phi}(t) + \phi(t)] + \varphi(t)\zeta(t)$. *It admits the solution having the form of*

$$\alpha_k(t) := \frac{\hat{\Delta}_k(t)}{\Delta_k(t)}. \tag{11}$$

Here $\Delta_k(t)$ *and* $\hat{\Delta}_k(t)$ *are the Toeplitz determinants given as (8) with the convention* $\Delta_0 = \hat{\Delta}_0 = 1$. *The moments* s_k *of* $\Delta_k(t)$ *and* $\hat{\Delta}_k(t)$ *satisfy the following recurrence relations:*

$$\begin{cases} \partial_t s_k = \mu(s_{k-1} + s_{k+1}) + i\psi \displaystyle\sum_{j=0}^{k-1} s_j s_{k-j} + \varphi s_k, \quad k = 1, 2, 3, \ldots, \\ s_{-k} = \overline{s}_k, \end{cases} \tag{12}$$

with $s_0 = \zeta(t)$ *and* $s_1 = \phi(t)$.

Remark 3.1.2. *For* $\mu \equiv 1$, *equation (10) is transformed to the Schur flow in "usual" form,*

$$\partial_t \sigma_k = \eta_k(\sigma_{k-1} - \sigma_{k+1}), \qquad \eta_k := 1 - |\sigma_k|^2,$$

by applying the following transformation to α_k *:*

$$\sigma_k = \alpha_k e^{-i\int \zeta(t)\psi(t)dt}.$$

Since σ_k *involves an arbitrary function* ψ, *(11) and (12) with* $\mu \equiv 1$ *give a generalized determinant solution of the original Schur flow (7).*

In order to prove this theorem, firstly we need to get the following lemma, which concerns the derivatives of $\hat{\Delta}_k(t)$ and $\Delta_k(t)$ respectively. The detailed proof is presented in the Appendix A.

Lemma 3.1.3. *There hold*

$$\partial_t\hat{\Delta}_k(t) = \mu(|A_0, A_2, A_3, \cdots, A_k| + |A_1, A_2, \cdots, A_{k-1}, A_{k+1}|) + (i\psi s_0 + k\varphi)\hat{\Delta}_k,$$
$$\partial_t\Delta_k(t) = \mu(|A_{-1}, A_1, A_2, \cdots, A_{k-1}| + |A_0, A_1, \cdots, A_{k-2}, A_k|) + k\varphi\Delta_k,$$

where the abbreviation A_j *denotes* $A_j = (s_j, s_{j-1}, \cdots, s_0, \cdots, s_{j+1-k})^T$.

Proof of Theorem 3.1.1. Applying the *Jacobi identity* (5) to $D = \Delta_{k+1}(t)$ and noticing that

$$D\begin{bmatrix} 1 & k+1 \\ 1 & 2 \end{bmatrix} = \hat{\Delta}_{k-1},$$

$$D\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Delta_k, \quad D\begin{bmatrix} k+1 \\ 2 \end{bmatrix} = |A_0, A_2, A_3, \cdots, A_k|,$$

$$D\begin{bmatrix} 1 \\ 2 \end{bmatrix} = |A_{-1}, A_1, A_2, \cdots, A_{k-1}|, \quad D\begin{bmatrix} k+1 \\ 1 \end{bmatrix} = \hat{\Delta}_k,$$

we obtain

$$|A_0, A_2, A_3, \cdots, A_k|\,\Delta_k - |A_{-1}, A_1, A_2, \cdots, A_{k-1}|\,\hat{\Delta}_k = \Delta_{k+1}\hat{\Delta}_{k-1}. \tag{13}$$

Similarly, applying the *Jacobi identity* (5) to $D = \hat{\Delta}_{k+1}(t)$ with $i_1 = 1$, $i_2 = k+1$, $j_1 = k$, $j_2 = k+1$, we get

$$|A_0, A_1, \cdots, A_{k-2}, A_k|\,\hat{\Delta}_k - |A_1, A_2, \cdots, A_{k-1}, A_{k+1}|\,\Delta_k = \hat{\Delta}_{k+1}\Delta_{k-1}. \tag{14}$$

Then, from (11) and making use of (13), (14) as well as Lemma 3.1.3, we have

$$\begin{aligned}
&\Delta_k^2 \partial_t\, \alpha_k = \partial_t \hat{\Delta}_k \Delta_k - \hat{\Delta}_k \partial_t \Delta_k \\
&= \mu(|A_0, A_2, A_3, \cdots, A_k|\,\Delta_k - |A_{-1}, A_1, A_2, \cdots, A_{k-1}|\,\hat{\Delta}_k) \\
&\quad - \mu(|A_0, A_1, \cdots, A_{k-2}, A_k|\,\hat{\Delta}_k - |A_1, A_2, \cdots, A_{k-1}, A_{k+1}|\,\Delta_k) + i\psi s_0 \hat{\Delta}_k \Delta_k \\
&= \mu(\Delta_{k+1}\hat{\Delta}_{k-1} - \hat{\Delta}_{k+1}\Delta_{k-1}) + i\psi s_0 \hat{\Delta}_k \Delta_k.
\end{aligned}$$

Besides, substituting the expression (11) of α_k into the right-hand side of (10), we easily obtain

$$\begin{aligned}
&\Delta_k^2\left[\mu\beta_k(\alpha_{k-1} - \alpha_{k+1}) + i\psi s_0 \alpha_k\right] \\
&= \frac{\Delta_k^2 - \left|\hat{\Delta}_k\right|^2}{\Delta_{k+1}\Delta_{k-1}} \mu(\Delta_{k+1}\hat{\Delta}_{k-1} - \Delta_{k-1}\hat{\Delta}_{k+1}) + i\psi s_0 \hat{\Delta}_k \Delta_k.
\end{aligned}$$

Notice that if there holds

$$\Delta_{k+1}\Delta_{k-1} = \Delta_k^2 - \left|\hat{\Delta}_k\right|^2,$$

then

$$\partial_t\, \alpha_k = \mu\beta_k(\alpha_{k-1} - \alpha_{k+1}) + i\psi s_0 \alpha_k, \qquad k = 1, 2, \ldots$$

is confirmed.

Actually, if we apply the *Jacobi identity* (5) to $D = \Delta_{k+1}(t)$ with $i_1 = 1$, $i_2 = k+1$, $j_1 = 1$ and $j_2 = k+1$ and notice the condition $s_{-k} = \bar{s}_k$, we can obtain:

$$\Delta_{k+1}\Delta_{k-1} = \Delta_k^2 - |A_{-1}, A_0, A_1, \cdots, A_{k-2}|\,\hat{\Delta}_k = \Delta_k^2 - \bar{\hat{\Delta}}_k \hat{\Delta}_k = \Delta_k^2 - \left|\hat{\Delta}_k\right|^2.$$

Therefore, the result is proved. □

3.2. Generalized determinant solution for the mixed form of the relativistic Toda chain

The relativistic Toda chain [29, 38, 47] has intimate relations with the Laurent biorthogonal polynomials [23, 36, 47]. Let L be a functional on the Laurent polynomials span and $c_j = L(z^j)$, $j = 0, \pm 1, \pm 2, \cdots$. Define the monic Laurent biorthogonal polynomials (LBP) $P_n(z)$ by the determinant

$$P_n(z) = (\Delta_n)^{-1} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1-n} & c_{2-n} & \cdots & c_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \tag{15}$$

where Δ_n denotes the Toeplitz determinant of size n:

$$\Delta_n := \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1-n} & c_{2-n} & \cdots & c_0 \end{vmatrix}. \tag{16}$$

The polynomials $P_n(z)$ satisfy the biorthogonal relation

$$L\{P_n(z)Q_m(1/z)\} = h_n \delta_{nm}, \tag{17}$$

where the partner polynomials $Q_n(z)$ have the formula

$$Q_n(z) = (\Delta_n)^{-1} \begin{vmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_{-1} \\ 1 & z & \cdots & z^n \end{vmatrix}. \tag{18}$$

And $P_n(z)$ satisfy the three-term recurrece relation [23, 47]

$$P_{n+1}(z) + (d_n - z)P_n(z) = z b_n P_{n-1}(z), \qquad n \geq 1,$$

where b_n, d_n can be expressed as

$$d_n = \frac{T_{n+1}\Delta_n}{T_n \Delta_{n+1}}, \qquad b_n = \frac{T_{n+1}\Delta_{n-1}}{T_n \Delta_n},$$

with

$$T_n := \begin{vmatrix} c_1 & c_2 & \cdots & c_n \\ c_0 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{2-n} & c_{3-n} & \cdots & c_1 \end{vmatrix}. \tag{19}$$

If the moments c_n depend on an additional (so-called time) parameter t, then the LBP may be connected with the restricted relativistic Toda chain [47]. More precisely, one ansatz

$$\partial_t c_n = c_{n-1}, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{20}$$

lead to the first form of the relativistic Toda chain

$$\partial_t\, d_n = \frac{b_{n+1}}{d_{n+1}} - \frac{b_n}{d_{n-1}}, \qquad \partial_t\, b_n = b_n\left(\frac{1}{d_n} - \frac{1}{d_{n-1}}\right), \tag{21}$$

and the other condition

$$\partial_t c_n = c_{n+1}, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{22}$$

yield the second form of the relativistic Toda chain

$$\partial_t d_n = -d_n(b_{n+1} - b_n), \qquad \partial_t b_n = -b_n(b_{n+1} - b_{n-1} + d_{n-1} - d_n). \tag{23}$$

Thus, these two forms of relativistic Toda chain (21) and (23) admit the solutions taking the same explicit expressions with different moments evolution with respect to time t.

Here, we mention that there are important relations between the Schur parameters α_n and the relativistic Toda parameters b_n and d_n. As a matter of fact [47], when all the Toeplitz determinants are positive $\Delta_n > 0$ and moreover the moments satisfy the condition $c_{-n} = \bar{c}_n$, then the monic biorthogonal Laurent polynomials $P_n(z)$ are just reduced to the Szegő polynomials, i.e. the orthogonal polynomials on the unit circle. In this case, the biorthogonal partners $Q_n(z)$ (18) coincide with complex conjugated polynomials $Q_n(z) = \overline{P}_n(z)$ and the orthogonality relation is reduced to

$$\int_0^{2\pi} P_n(z)\overline{P}_m(1/z) d\sigma(\theta) = h_n \delta_{nm}, \quad z = e^{i\theta} \tag{24}$$

by letting $L(z^n) = \int_0^{2\pi} e^{i\theta n} d\sigma(\theta)$. Correspondingly, the Szegő polynomials $P_n(z)$ admit the relation

$$P_{n+1}(z) = zP_n(z) + \alpha_{n+1} z^n \overline{P}_n(\frac{1}{z}),$$

where the coefficients $\alpha_{n+1} = P_{n+1}(0)$ are just the Schur parameters as we have discussed in subsection 3.1.

Analogous to the work we have done in subsection 3.1, it is natural to expect a general determinant solution of the relativistic Toda chain.

Actually, we shall show in this subsection that there exists a generalized form of the relativistic Toda chain, i.e. the mixed form of the relativistic Toda chain. It covers the two known forms (21) and (23) as special cases and has a generalized Toeplitz type determinant solution. In view of the connections between the Schur parameters α_n and the relativistic Toda parameters b_n and d_n, it is not surprising to find that there is a great similarity in the moment relations between the mixed form of the relativistic Toda chain and the generalized Schur flow. We present our result as the following theorem.

Theorem 3.2.1. *Consider the mixed form of the relativistic Toda chain*

$$\partial_t\, d_n = \mu(\frac{b_{n+1}}{d_{n+1}} - \frac{b_n}{d_{n-1}}) - \nu d_n(b_{n+1} - b_n), \tag{25}$$

$$\partial_t\, b_n = \mu b_n(\frac{1}{d_n} - \frac{1}{d_{n-1}}) - \nu b_n(b_{n+1} - b_{n-1} + d_{n-1} - d_n) \tag{26}$$

with the initial conditions $b_0 = 0$, $d_0 = \frac{\zeta(t)}{\varrho(t)}$ *and* $\frac{b_1}{d_1} = 1 - \frac{\zeta(t)\phi(t)}{\varrho(t)^2}$, *where*

$$\partial_t \varrho(t) = \mu(t)\phi(t) + \nu(t)\zeta(t) + \varphi(t)\varrho(t).$$

It has the solution as follows:

$$d_n = \frac{T_{n+1}\Delta_n}{T_n\Delta_{n+1}}, \qquad b_n = \frac{T_{n+1}\Delta_{n-1}}{T_n\Delta_n}. \tag{27}$$

Here $\Delta_n(t)$ and $T_n(t)$ are given as (16) and (19) with the convention $\Delta_0 = T_0 = 1$. The moments $c_n(t)$ satisfy the relations

$$\begin{cases} \partial_t c_n = \mu c_{n-1} + \nu c_{n+1} - \psi \sum_{j=0}^{-n-1} c_{-j} c_{n+j} + \varphi c_n, & \text{if} \quad n < 0, \\ \partial_t c_n = \mu c_{n-1} + \nu c_{n+1} + \psi \sum_{j=0}^{n-1} c_j c_{n-j} + \varphi c_n, & \text{if} \quad n > 0 \end{cases} \tag{28}$$

with $c_{-1} = \phi(t)$, $c_0 = \rho(t)$ and $c_1 = \zeta(t)$. All the above functions appearing in this theorem may be complex functions of t.

Remark 3.2.2. *When we take $\mu \equiv 1$ and $\nu \equiv 0$, since $\partial_t c_n$ involves two arbitrary functions ψ and φ, formulae (27) and (28) give a generalized solution of the first form of the relativistic Toda chain (21). The same is true for the second form of the relativistic Toda chain (23) if we choose $\mu \equiv 0$ and $\nu \equiv 1$.*

In order to prove this theorem, we need to give several auxiliary results. For the sake of convenience, we shall use the following abbreviations:

$$F_n^m = \begin{vmatrix} c_m & c_{m+2} & \cdots & c_{m+n} \\ c_{m-1} & c_{m+1} & \cdots & c_{m+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+1-n} & c_{m+3-n} & \cdots & c_{m+1} \end{vmatrix},$$

$$G_n^m = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+n-2} & c_{m+n} \\ c_{m-1} & c_m & \cdots & c_{m+n-3} & c_{m+n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m+1-n} & c_{m+2-n} & \cdots & c_{m-1} & c_{m+1} \end{vmatrix},$$

$$H_n^m = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+n-1} \\ c_{m-1} & c_m & \cdots & c_{m+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+1-n} & c_{m+2-n} & \cdots & c_m \end{vmatrix},$$

with the convention that $F_0^m = G_0^m = 0$ and $H_0^m = 1$. It is noted that there holds $H_n^0 = \Delta_n$, $H_n^1 = T_n$ in this case. However, in the latter of this subsection we still use the notations Δ_n and T_n instead of H_n^0 and H_n^1.

The following two lemmas play a key role in the proof of Theorem 3.2.1. The detailed proofs can be found in Appendix A and B, respectively.

Lemma 3.2.3. *For integer $n \geq 1$, the derivatives of T_n and Δ_n with respect to t admit*

$$\partial_t T_n = \mu F_n^0 + \nu G_n^1 + (\psi c_0 + n\varphi) T_n, \tag{29}$$

$$\partial_t \Delta_n = \mu F_n^{-1} + \nu G_n^0 + n\varphi \Delta_n. \tag{30}$$

Lemma 3.2.4. *For integer $n \geq 0$, there hold*

$$F_{n+1}^0 T_n - T_{n+1} F_n^0 = \Delta_{n+1} H_n^2, \tag{31}$$

$$F_{n+1}^{-1}\Delta_n - \Delta_{n+1}F_n^{-1} = H_{n+1}^{-1}T_n, \tag{32}$$
$$T_{n+2}\Delta_n = G_{n+1}^0 T_{n+1} - \Delta_{n+1}G_{n+1}^1, \tag{33}$$
$$\Delta_{n+2}T_n = \Delta_{n+1}F_{n+1}^0 - F_{n+1}^{-1}T_{n+1}, \tag{34}$$
$$T_{n+2}T_n + \Delta_{n+1}H_{n+1}^2 = (T_{n+1})^2, \tag{35}$$
$$\Delta_{n+2}\Delta_n + T_{n+1}H_{n+1}^{-1} = (\Delta_{n+1})^2, \tag{36}$$
$$G_{n+1}^1 T_n = G_n^1 T_{n+1} + H_{n+1}^2\Delta_n, \tag{37}$$
$$G_{n+1}^0\Delta_n = G_n^0\Delta_{n+1} + H_n^{-1}T_{n+1}. \tag{38}$$

From Lemma 3.2.3 and Lemma 3.2.4, we obtain the following corollary for the derivative relations of T_n and Δ_n.

Corollary 3.2.5. *For integer $n \geq 1$, there hold*

$$\partial_t T_{n+1}T_n - T_{n+1}\partial_t T_n = \mu\Delta_{n+1}H_n^2 + \nu\Delta_n H_{n+1}^2 + \varphi T_{n+1}T_n,$$
$$\Delta_n\partial_t\Delta_{n+1} - \Delta_{n+1}\partial_t\Delta_n = \mu T_n H_{n+1}^{-1} + \nu T_{n+1}H_n^{-1} + \varphi\Delta_n\Delta_{n+1},$$
$$\partial_t T_{n+1}\Delta_{n+1} - T_{n+1}\partial_t\Delta_{n+1} = \mu\Delta_{n+2}T_n - \nu T_{n+2}\Delta_n + \psi c_0 T_{n+1}\Delta_{n+1}.$$

Now, we are ready to give the proof of Theorem 3.2.1.

Proof of Theorem 3.2.1. Substitute the expressions of d_n and b_n in (27) into both sides of the equations (25) and (26) respectively.

Firstly, using Corollary 3.2.5, we get $\partial_t d_n$ can be expressed as

$$\partial_t d_n = \mu\frac{\Delta_n H_n^2}{T_n^2} - \mu\frac{T_{n+1}H_{n+1}^{-1}}{(\Delta_{n+1})^2} + \nu\frac{H_{n+1}^2\Delta_{n+1}(\Delta_n)^2 - T_{n+1}^2T_nH_n^{-1}}{(T_n\Delta_{n+1})^2}. \tag{39}$$

Secondly, with the help of the identities (35) and (36), the right hand side of (25) reads

$$\begin{aligned}
&\mu(\frac{b_{n+1}}{d_{n+1}} - \frac{b_n}{d_{n-1}}) - \nu d_n(b_{n+1} - b_n)\\
&= \mu[\frac{\Delta_n\Delta_{n+2}}{(\Delta_{n+1})^2} - \frac{T_{n+1}T_{n-1}}{T_n^2}] - \nu\frac{(\Delta_n)^2T_{n+2}T_n - T_{n+1}^2\Delta_{n-1}\Delta_{n+1}}{(T_n\Delta_{n+1})^2}\\
&= \mu[\frac{(\Delta_{n+1})^2 - T_{n+1}H_{n+1}^{-1}}{(\Delta_{n+1})^2} - \frac{T_n^2 - \Delta_nH_n^2}{T_n^2}]\\
&\quad - \nu\frac{(\Delta_n)^2(T_{n+1}^2 - H_{n+1}^2\Delta_{n+1}) - T_{n+1}^2[(\Delta_n)^2 - H_n^{-1}T_n]}{(T_n\Delta_{n+1})^2}\\
&= \mu\frac{\Delta_nH_n^2}{T_n^2} - \mu\frac{T_{n+1}H_{n+1}^{-1}}{(\Delta_{n+1})^2} - \nu\frac{T_{n+1}^2H_n^{-1}T_n - (\Delta_n)^2H_{n+1}^2\Delta_{n+1}}{(T_n\Delta_{n+1})^2}.
\end{aligned} \tag{40}$$

Hence, (25) is verified after comparing (39) with (40).

Similarly, from Corollary 3.2.5, $\partial_t b_n$ admits:

$$\partial_t b_n = \mu\frac{\Delta_{n+1}\Delta_{n-1}}{(\Delta_n)^2}\frac{\Delta_nH_n^2}{T_n^2} - \mu\frac{T_{n+1}T_{n-1}}{T_n^2}\frac{H_n^{-1}T_n}{(\Delta_n)^2} + \nu\frac{H_{n+1}^2\Delta_{n-1}}{T_n^2} - \nu\frac{T_{n+1}H_{n-1}^{-1}}{(\Delta_n)^2}.$$

By employing the identities (35) and (36), there holds

$$\partial_t b_n = \mu\frac{\Delta_{n+1}\Delta_{n-1}}{(\Delta_n)^2} - \mu\frac{T_{n+1}T_{n-1}}{T_n^2} + \nu\frac{\Delta_{n-1}(T_{n+1}^2 - T_{n+2}T_n)}{T_n^2\Delta_{n+1}}$$

$$-\nu \frac{T_{n+1}[(\Delta_{n-1})^2 - \Delta_n \Delta_{n-2}]}{T_{n-1}(\Delta_n)^2}. \tag{41}$$

And the right hand side of (41) is just

$$\mu b_n(\frac{1}{d_n} - \frac{1}{d_{n-1}}) - \nu b_n(b_{n+1} - b_{n-1} + d_{n-1} - d_n).$$

Thus, (26) holds and we complete the proof Theorem 3.2.1. □

3.3. Generalized determinant solution of the nonisospectral Toda lattice

In this subsection, we turn to the generalized Hankel type determinant solution to a nonisospectral Toda lattice. First of all, let's briefly review some results about the nonisospectral Toda in [9].

Consider a sequence of orthonormal polynomials $P_k(\lambda)$ with the orthonormality

$$\int_R P_j(\lambda)P_k(\lambda)d\rho(\lambda) = \delta_{jk}, \qquad j, k = 0, 1, \ldots$$

and introduce a one-parametric family of measures $d\rho(\lambda, t)$, which depends on $\Phi(\lambda, t)$ and $\Psi(\lambda, t)$, the polynomials in λ with continuously differentiable t-dependent coefficients:

$$\Phi(\lambda, t) = \sum_{i=0}^{l} \varphi_i(t)\lambda^i,$$
$$\Psi(\lambda, t) = \sum_{i=0}^{m} \psi_i(t)\lambda^i, \qquad \lambda \in R.$$

Then, the semi-infinite Jacobi matrix $L(t)$ satisfies the generalized Lax equation of the form:

$$\partial_t L(t) = \Phi(L(t), t) + [L(t), A(t)], \tag{42}$$

where $A(t) = A(L(t), t)$ is a skew-symmetric matrix whose form is determined by this evolution. See [9] for the detailed description of the t-dependent measure corresponding to the polynomials Φ, Ψ and the skew-symmetric matrix A. Note that we also use Ψ in place of Θ, which coincides with Section 7 of [9].

In the case of

$$\Phi(\lambda, t) = \varphi_0 + \varphi_1\lambda + \varphi_2\lambda^2, \qquad \Psi(\lambda, t) = \psi_0 + \psi_1\lambda,$$

the equation (42) is equivalent to the following a wide class of generalized Toda lattices (see [9, p.140, (7.7) and (7.8)]):

$$\begin{cases} \partial_t a_n = \frac{1}{2}a_n[2\varphi_1 + 2\varphi_2(b_n + b_{n+1}) + \psi_1(b_{n+1} - b_n) + 2n\varphi_2(b_{n+1} - b_n)], \\ \partial_t b_n = \varphi_0 + \varphi_1 b_n + a_{n-1}^2[\varphi_2 - \psi_1 - 2\varphi_2(n-1)] + a_n^2(\varphi_2 + \psi_1 + 2\varphi_2 n) + \varphi_2 b_n^2, \\ n = 0, 1, \ldots, \quad a_{-1} = 0, \end{cases}$$

where φ_i and ψ_j are real constants. In particular, if taking $\varphi_0 = \varphi_1 = \varphi_2 = \psi_0 = 0$ and $\psi_1 = 1$, i.e. $\Phi(\lambda, t) = 0$ (isospectral deformation) and $\Psi(\lambda, t) = 1$, it reduces to the classical semi-infinite Toda lattice:

$$\partial_t a_n(t) = \frac{1}{2}a_n(b_{n+1} - b_n), \quad a_{-1} = 0, \tag{43}$$

$$\partial_t b_n(t) = a_n^2 - a_{n-1}^2, \quad n = 0, 1, \dots \quad . \tag{44}$$

When taking $\varphi_0 = \varphi_1 = \psi_0 = 0$ and $\varphi_2 = \psi_1 = 1$, the above wide class of generalized Toda lattices is reduced to the following nonisospectral Toda lattice:

$$\partial_t a_n = \frac{1}{2}a_n[b_{n+1}(3 + 2n) - b_n(2n - 1)], \tag{45}$$

$$\partial_t b_n = 2a_{n-1}^2(1 - n) + 2a_n^2(1 + n) + b_n^2, \tag{46}$$

$$n = 0, 1, \dots \ , \quad a_{-1} = 0.$$

It is well known that the isospectral Toda lattice (43) and (44) have the solution of the Hankel type determinant [15, 25, 37, 48]. Then, does such determinant structure exist for the solution of the nonisospectral Toda lattice?

Indeed, we present in this subsection that there exists a generalized Hankel type determinant solution for the following nonisospectral Toda lattice:

$$\partial_t u_n = u_n \left[(2n + 1)v_{n+1} - (2n - 3)v_n\right],$$

$$\partial_t v_n = v_n^2 + 2nu_n - 2(n - 2)u_{n-1},$$

which can be transformed to (45) and (46) by the following transformation

$$u_n = a_{n-1}^2, \qquad v_n = b_{n-1}.$$

In order to show this result, to begin with, let us introduce some determinant notations. We define

$$\tau_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix}, \qquad \sigma_n = \begin{vmatrix} a_0 & \dots & a_{n-2} & a_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & \dots & a_{2n-4} & a_{2n-3} \\ a_n & \dots & a_{2n-2} & a_{2n-1} \end{vmatrix},$$

$$h_n = \begin{vmatrix} a_0 & \dots & a_{n-2} & a_n \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & \dots & a_{2n-4} & a_{2n-2} \\ a_n & \dots & a_{2n-2} & a_{2n} \end{vmatrix}, \quad g_n = \begin{vmatrix} a_0 & \dots & a_{n-2} & a_{n+1} \\ a_1 & \dots & a_{n-1} & a_{n+2} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & \dots & a_{2n-3} & a_{2n} \end{vmatrix},$$

$$f_n = \begin{vmatrix} a_0 & \dots & a_{n-3} & a_{n-1} & a_n \\ a_1 & \dots & a_{n-2} & a_n & a_{n+1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-1} & \dots & a_{2n-4} & a_{2n-2} & a_{2n-1} \end{vmatrix},$$

with the convention that $\tau_{-1} = \sigma_0 = h_0 = f_0 = f_1 = g_0 = 0$, $\tau_0 = 1$, $\sigma_1 = a_1$, $h_1 = g_1 = a_2$ and $f_2 = a_1a_3 - a_2^2$.

Then, we express the main result of this subsection in the following theorem.

Theorem 3.3.1. *The nonisospectral Toda lattice*

$$\partial_t u_n = u_n \left[(2n + 1)v_{n+1} - (2n - 3)v_n\right], \tag{47}$$

$$\partial_t v_n = v_n^2 + 2nu_n - 2(n - 2)u_{n-1}, \tag{48}$$

with the initial conditions $u_0 = 0$ *and* $v_1 = \frac{\partial_t \phi(t)}{\phi(t)}$ *has the solution expressed as*

$$u_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}, \quad v_n = \frac{\sigma_n}{\tau_n} - \frac{\sigma_{n-1}}{\tau_{n-1}}. \tag{49}$$

Here, we use the conventions $\tau_0 = 1$, $\sigma_1 = a_1$, $\tau_{-1} = \sigma_0 = 0$, *and the moments* a_n *of the Hankel determinants are defined recursively as*

$$\partial_t a_{n-1} = n a_n - \psi \sum_{i+j=n-2} a_i a_j, \qquad a_0 = \phi, \tag{50}$$

where $\psi(t)$ *and* $\phi(t)$ *are any complex functions of* t.

Before the proof of this theorem, firstly we present the following lemma about the closed bilinear forms of the nonisospectral Toda lattice (47) and (48). The detailed proof is placed in the Appendix C.

Lemma 3.3.2. *For any integer* $n \geq 0$, *there hold*

$$\partial_t \tau_n = (2n-1)\sigma_n, \tag{51}$$

$$\partial_t \sigma_{n+1} = 2n h_{n+1} + 2g_{n+1} - \phi\psi\tau_{n+1}, \tag{52}$$

$$h_n \tau_n - (\sigma_n)^2 = \tau_{n+1}\tau_{n-1}, \tag{53}$$

$$f_{n+1}\tau_n + g_n \tau_{n+1} - \sigma_{n+1}\sigma_n = 0, \tag{54}$$

$$f_n + g_n = h_n. \tag{55}$$

Proof of Theorem 3.3.1. Firstly, (47) can be easily checked by substituting the expression of u_n in (49) and the derivative relations (51) into both sides of the equation (47).

Secondly, from relations (51), (52) and the identities (53) and (55), we obtain

$$\begin{aligned} &\partial_t v_n \\ &= \frac{(2n-1)(h_n\tau_n - \sigma_n^2) - (h_n - g_n)\tau_n + g_n\tau_n}{\tau_n^2} \\ &\quad - \frac{(2n-3)(h_{n-1}\tau_{n-1} - \sigma_{n-1}^2) - (h_{n-1} - g_{n-1})\tau_{n-1} + g_{n-1}\tau_{n-1}}{\tau_{n-1}^2} \\ &= \frac{(2n-1)\tau_{n-1}\tau_{n+1} - f_n\tau_n + g_n\tau_n}{\tau_n^2} \\ &\quad - \frac{(2n-3)\tau_{n-2}\tau_n - f_{n-1}\tau_{n-1} + g_{n-1}\tau_{n-1}}{\tau_{n-1}^2}. \end{aligned} \tag{56}$$

On the other hand, by employing the identities (53), (54) and (55), the right hand side of (48) admits

$$\begin{aligned} &v_n^2 + 2nu_n - 2(n-2)u_{n-1} \\ &= \frac{\sigma_n^2}{\tau_n^2} + 2n\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} + \frac{\sigma_{n-1}^2}{\tau_{n-1}^2} - 2(n-2)\frac{\tau_{n-2}\tau_n}{\tau_{n-1}^2} - 2\frac{\sigma_n\sigma_{n-1}}{\tau_n\tau_{n-1}} \\ &= \frac{h_n\tau_n - \tau_{n-1}\tau_{n+1}}{\tau_n^2} + 2n\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} + \frac{h_{n-1}\tau_{n-1} - \tau_{n-2}\tau_n}{\tau_{n-1}^2} \end{aligned}$$

$$
\begin{aligned}
&\quad - 2(n-2)\frac{\tau_{n-2}\tau_n}{\tau_{n-1}^2} - 2\frac{f_n\tau_{n-1} + g_{n-1}\tau_n}{\tau_n\tau_{n-1}} \\
&= \frac{h_n\tau_n}{\tau_n^2} + (2n-1)\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} + \frac{h_{n-1}\tau_{n-1}}{\tau_{n-1}^2} - (2n-3)\frac{\tau_{n-2}\tau_n}{\tau_{n-1}^2} - 2\frac{f_n}{\tau_n} - 2\frac{g_{n-1}}{\tau_{n-1}} \\
&= \frac{(f_n+g_n)\tau_n}{\tau_n^2} + (2n-1)\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} + \frac{(f_{n-1}+g_{n-1})\tau_{n-1}}{\tau_{n-1}^2} \\
&\quad - (2n-3)\frac{\tau_{n-2}\tau_n}{\tau_{n-1}^2} - 2\frac{f_n}{\tau_n} - 2\frac{g_{n-1}}{\tau_{n-1}} \\
&= \frac{(g_n - f_n)\tau_n + (2n-1)\tau_{n-1}\tau_{n+1}}{\tau_n^2} \\
&\quad + \frac{(f_{n-1} - g_{n-1})\tau_{n-1} - (2n-3)\tau_{n-2}\tau_n}{\tau_{n-1}^2}.
\end{aligned} \tag{57}
$$

Then we immediately get (48) by comparing (56) with (57).

Thus the proof is completed. □

4. Combinatorial numbers based on the moments relations and their combinatorial interpretations

In this section, we are devoted to getting some new combinatorial numbers and addressing their corresponding combinatorial interpretations. Indeed, some reports on the solutions to integrable systems and the combinatorial numbers have exist, such as [14, 32, 33].

To begin with, we restate some conventional definitions [7, 16, 43].

Consider the lattice paths in the Cartesian plane, starting and ending on the x-axis, never going below the x-axis. The permitted step types of the lattice paths are the three steps: $U = (1, 1)$, the up diagonal step; $H = (r, 0)$, the horizontal step and $D = (1, -1)$, the down diagonal step. Furthermore, denote the weight of each type of steps as $w(U), w(H), w(D)$ respectively, then we have:

Definition 4.0.3. *The weight of a lattice path P is the product of the weights of its steps.*

Definition 4.0.4. *The weight of a set of paths S is the sum of the weights of the paths in S.*

Now, we are ready to get some new combinatorial numbers based on the moment relations proposed in the previous section. At the same time, we give their combinatorial interpretations by means of the lattice paths. It's noted that the moment recurrence relations of the generalized Schur flow are actually a special case of those in the mixed form of the relativistic Toda chain. In the following subsection, we only discuss the new combinatorial numbers derived from the mixed form of the relativistic Toda chain.

4.1. Combinatorial numbers derived from the moment relations of the mixed form of the relativistic Toda chain

Consider the case of $n > 0$ in (28), that is:

$$\partial_t c_n = \mu c_{n-1} + \nu c_{n+1} + \psi \sum_{j=0}^{n-1} c_j c_{n-j} + \varphi c_n, \tag{58}$$

where μ, ν are real numbers and ψ and φ are real functions of t.

We can choose some special functions so as to get different combinatorial numbers.

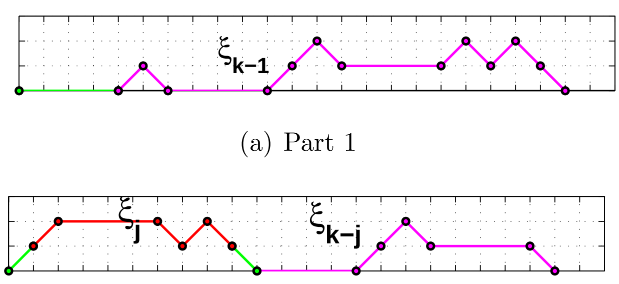
(I) If $\nu \neq 0$, taking $\nu = 1$ without loss of generality, we choose the special functions

$$\begin{cases} \psi(t) := -\gamma e^{-\eta t}, \\ \varphi(t) := \eta - \gamma, \\ c_n := \xi_n e^{\eta t}, \\ \mu := -\mu, \end{cases}$$

where γ, η are arbitrary real constants and $\{\xi_n\}_{(n \in N)}$ is a real number sequence. In what follows, all these notations have the same meanings. Substituting all above functions into the moment relation (58) and letting $\xi_0 = 1$, $\xi_1 = \gamma$, we can immediately find that the number sequence ξ_n satisfies the recurrence relation as follows:

$$\xi_{n+1} = \mu \xi_{n-1} + \gamma \sum_{j=0}^{n} \xi_j \xi_{n-j}. \tag{59}$$

Here, ξ_n can be interpreted via the view of the lattice paths. Assume the horizontal step is taken as $H = (4, 0)$ and we impose the weight $w(H) = \mu$ on each H-step, $w(D) = 1$ to each D-step and $w(U) = \gamma$ to each U-step. Then ξ_n denotes the weight of the set of the above lattice paths running from $(0, 0)$ to $(2n, 0)$.



(a) Part 1

(b) Part 2

Figure 1. Lattice paths of ξ_n in (59) for $\nu \neq 0$

The proof is a fairly elementary argument in enumerative combinatorics. All valid steps from $(0, 0)$ to $(2n + 2, 0)$ can be divided into two types, one of which begins with H (Figure 1(a)) and the other one starts with U (Figure 1(b)). Obviously, the weight of the set of the lattice paths in the first type is $\mu \xi_{n-1}$. One can also show that the total weight for the second is $\sum_{j=0}^{n} \gamma \xi_j \xi_{n-j}$. Thus, taking into account both of the two parts, we get the weight of the set of the lattice paths running from $(0,0)$ to $(2n+2,0)$ is just that indicated in (59).

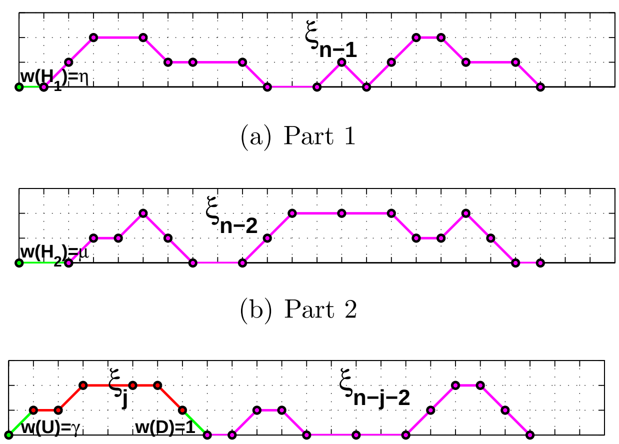
(II) In the case of $\nu \neq 0$, (taking $\nu = 1$), another choice is making

$$\begin{cases} \psi(t) := -\gamma e^{-\eta t}, \\ \varphi(t) := 0, \\ c_n := \xi_{n-1} e^{\eta t}, \\ \mu := -\mu. \end{cases}$$

Substitute these functions into (58) and take $\xi_{-1} = 0$, $\xi_0 = 1$, then there holds:

$$\xi_n = \eta \xi_{n-1} + \mu \xi_{n-2} + \gamma \sum_{j=0}^{n-2} \xi_j \xi_{n-j-2}. \tag{60}$$

Assume the horizontal steps include two kinds: $H_1 = (1,0)$, and $H_2 = (2,0)$. Then, ξ_n can be regarded as the weight of the set of all the paths running from $(0,0)$ to $(n,0)$ associated with $w(H_1) = \eta$, $w(H_2) = \mu$, $w(D) = 1$ and $w(U) = \gamma$. In this case, the proof can be achieved by noting that all paths can be divided into three parts according to the first step. The first part is the set of the paths that begin with H_1. The paths of the second part start with H_2. The third ones begin with U. An illustration is shown by Figure 2.



(a) Part 1

(b) Part 2

(c) Part 3

Figure 2. Another choice for $\nu \neq 0$: lattice paths of ξ_n in (60)

(III) For $\nu = 0$, we choose the following special functions:

$$\begin{cases} \psi(t) := \gamma e^{-\eta t}, \\ \varphi(t) := \eta - 1, \\ c_n := \xi_{n-1} e^{\eta t}. \end{cases}$$

Substitute these functions into

$$\partial_t c_n = \mu c_{n-1} + \psi \sum_{j=0}^{n-1} c_j c_{n-j} + \varphi c_n,$$

and let $\xi_{-1} = 0$, $\xi_0 = 1$, then we can easily obtain that the number ξ_n satisfies the following recurrence relation:

$$\xi_n = \mu\xi_{n-1} + \gamma \sum_{j=0}^{n-1} \xi_j \xi_{n-j-1}. \tag{61}$$

The numbers ξ_n from (61) are nothing but the weighted large Schröder numbers [21, 43].

4.2. Combinatorial numbers derived from the moment relations of the nonisospectral Toda lattice

In this subsection, we shall discuss some combinatorics numbers derived from the recurrence relation of the moments of the nonisospectral Toda lattice.

Consider

$$\partial_t a_{n-1} = n a_n - \psi \sum_{i+j=n-2} a_i a_j, \qquad a_0 = \phi,$$

with

$$\begin{cases} \phi = 1, \\ \psi = \gamma e^{-\eta t}, \\ a_n = \dfrac{\xi_n}{n!} e^{\eta t}. \end{cases}$$

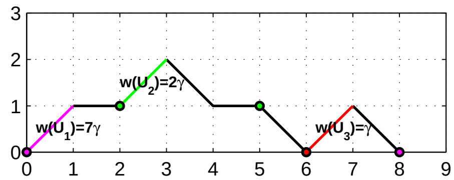
Then ξ_n satisfy the recurrence relation

$$\xi_n = \eta\xi_{n-1} + \gamma(n-1) \sum_{i=0}^{n-2} \begin{pmatrix} n-2 \\ i \end{pmatrix} \xi_i \xi_{n-2-i}, \tag{62}$$

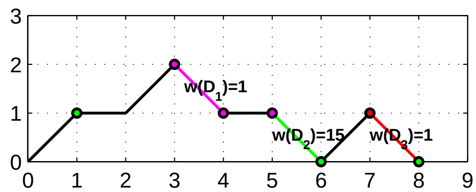
with $\xi_0 = 1$.

In this case, ξ_n may count some non-constant weighted paths. Let P be a path from $(0, 0)$ to $(n, 0)$ with steps in $S = \{(1, 0), (1, 1), (1, -1)\}$. Let $\overrightarrow{st}$ be a step of P from $s = (i_1, j_1)$ to $t = (i_2, j_2)$. If $\overrightarrow{st}$ is a horizontal step $(1, 0)$, we assign its weight as $w(\overrightarrow{st}) = \eta$. However, for each up diagonal step or down diagonal step, the weight will depend on the position. For the up diagonal step $\overrightarrow{st}$ of P from (i_1, j_1) to (i_1+1, j_1+1), we define the *forward path length* of $\overrightarrow{st}$, denoted by $f_P(\overrightarrow{st})$, as the maximal length of partial paths of P from (i_1, j_1) to (i_3, j_1) with $i_1 \le i_3$ such that no step of which is below $y = j_1$, i.e. $f_P(\overrightarrow{st}) = \max(i_3) - i_1$. Then, we assign the weight of the up diagonal step $\overrightarrow{st}$ as $w(\overrightarrow{st}) = \gamma(f_P(\overrightarrow{st}) - 1)$.

As for the down diagonal step $\overrightarrow{st}$ from $(\tilde{i}_1, \tilde{j}_1)$ to $(\tilde{i}_1+1, \tilde{j}_1-1)$, we define the *forward path length* of $\overrightarrow{st}$ as the maximal length of partial paths of P from $(\tilde{i}_1+1, \tilde{j}_1-1)$ to $(\tilde{i}_2, \tilde{j}_1-1)$ with $\tilde{i}_1+1 \le \tilde{i}_2$ such that no step of which is below $y = \tilde{j}_1 - 1$. Here, we still denote it as $f_P(\overrightarrow{st})$ and there holds $f_P(\overrightarrow{st}) = \max(\tilde{i}_2) - (\tilde{i}_1+1)$. The *backward path length* of $\overrightarrow{st}$, denoted by $b_P(\overrightarrow{st})$, is the maximal length of partial paths of P from $(\tilde{i}_0, \tilde{j}_1)$ to $(\tilde{i}_1, \tilde{j}_1)$ with $\tilde{i}_0 \le \tilde{i}_1$ such that no step of which is below $y = \tilde{j}_1$. There holds $b_P(\overrightarrow{st}) = \tilde{i}_1 - \min(\tilde{i}_0)$. Then, we define the weight of the down diagonal step $\overrightarrow{st}$ as the binomial number $w(\overrightarrow{st}) = \begin{pmatrix} b_P(\overrightarrow{st}) + f_P(\overrightarrow{st}) \\ b_P(\overrightarrow{st}) \end{pmatrix}$ with the convention $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1$.



(a) The weights of the up diagonal steps for $n = 8$



(b) The weights of the down diagonal steps for $n = 8$

Figure 3. Two examples of the non-constant weighted lattice paths

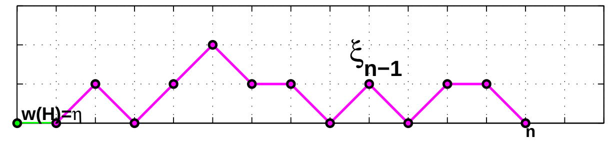
For instance, Figure 3(a) shows the *forward path lengths* and the corresponding weights of the three up diagonal steps:

$s = (i_1, j_1)$	$t = (i_1 + 1, j_1 + 1)$	$(\max(i_3), j_1)$	$f_P(\overrightarrow{st})$	$w(\overrightarrow{st})$
(0,0)	(1,1)	(8,0)	8	7γ
(2,1)	(3,2)	(5,1)	3	2γ
(6,0)	(7,1)	(8,0)	2	γ

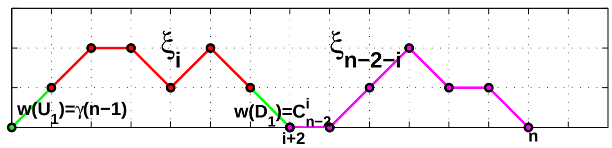
And Figure 3(b) describes the *forward path lengths*, the *backward path lengths* and the corresponding weights of the three down diagonal steps:

$s = (\tilde{i}_1, \tilde{j}_1)$	$t = (\tilde{i}_1 + 1, \tilde{j}_1 - 1)$	$(\min(\tilde{i}_0), \tilde{j}_1)$	$(\max(\tilde{i}_2), \tilde{j}_1 - 1)$	$b_P(\overrightarrow{st})$	$f_P(\overrightarrow{st})$	$w(\overrightarrow{st})$
(3,2)	(4,1)	(3,2)	(5,1)	0	1	1
(5,1)	(6,0)	(1,1)	(8,0)	4	2	15
(7,1)	(8,0)	(7,1)	(8,0)	0	0	1

Under the above assumptions, ξ_n can be interpreted as the weight of the set of the lattice paths running from $(0, 0)$ to $(n, 0)$. The proof can be obtained following a similar way as before. All valid steps from $(0, 0)$ to $(n, 0)$ consist of two types. One type begins with a horizontal step (Figure 4(a)) and the other one starts with an up diagonal step (Figure 4(b)). One can show that the weight of the set of the lattice paths in the first type is $\eta\xi_{n-1}$ by noting η as the weight of the first horizontal step. And the weight for the second one is $\gamma(n-1)\sum_{i=0}^{n-2}\begin{pmatrix} n-2 \\ i \end{pmatrix}\xi_i\xi_{n-2-i}$, since the weight of the first up diagonal step is $\gamma(n-1)$ and the first down diagonal from $(i+1, 1)$ to $(i+2, 0)$ admits the weight $\begin{pmatrix} n-2 \\ i \end{pmatrix}$. Thus, the total weight of the set of the lattice paths running from $(0, 0)$ to $(n, 0)$ is the summation to these two cases, which shows (62).



(a) Part 1



(b) Part 2

Figure 4. Lattice paths of ξ_n in (62)

4.3. A class of more generalized combinatorial numbers

Actually, the combinatorial numbers in the previous subsections can be extended to more general cases. For example, we may consider a generalization for the relatively unusual form (62).

Let P be a path from $(0,0)$ to $(n,0)$ with steps in $S = \{(1,0),(1,1),(1,-1)\}$ and $\overrightarrow{st}$ be a step of P from $s = (i_1, j_1)$ to $t = (i_2, j_2)$. The *forward path length* and the *backward path length* of the up diagonal step and the down diagonal step $\overrightarrow{st}$ have the same definitions as we defined in the previous subsection. We denote them as $f_P(\overrightarrow{st})$ and $b_P(\overrightarrow{st})$ as before respectively. Besides, we define the *forward path length* of the horizontal step $\overrightarrow{st}$ in the same manner as that in a up diagonal step case, denoted by $f_P(\overrightarrow{st})$.

Now we define the weight of each step. Given two single-variable functions p, q, and a two-variables function r, then the weight of $\overrightarrow{st}$ of P is

$$\omega_P(\overrightarrow{st}) = \begin{cases} p(f_P(\overrightarrow{st})) & \text{if } \overrightarrow{st} \text{ is a horizonal step,} \\ q(f_P(\overrightarrow{st})) & \text{if } \overrightarrow{st} \text{ is a up diagonal step,} \\ r(b_P(\overrightarrow{st}), f_P(\overrightarrow{st})) & \text{if } \overrightarrow{st} \text{ is a down diagonal step.} \end{cases}$$

Let ξ_n be the weight of the set of the lattice paths running from $(0,0)$ to $(n,0)$ with steps in S. We still divide all the paths into two parts according to the initial step (a horizontal step or an up diagonal step). Taking the above non-constant weight into account and following the similar argument in the previous subsection, we will obtain the following recurrence relation for ξ_n:

$$\xi_n = p(n)\xi_{n-1} + q(n)\sum_{i=0}^{n-2} r(i, n-2-i)\xi_i\xi_{n-2-i}.$$

In particular, let $p(x)=\eta$, $q(x)=\gamma(x-1)$ and $r(x,y)=\begin{pmatrix} x+y \\ x \end{pmatrix}$, then we have

$$\xi_n=\eta\xi_{n-1}+\gamma(n-1)\sum_{i=0}^{n-2}\begin{pmatrix} n-2 \\ i \end{pmatrix}\xi_i\xi_{n-2-i},$$

which is exactly (62).

Furthermore, let $\xi_0=\xi_1=1$, for the following cases we have $\xi_n=n!$.

(i). $p(x)=1$, $q(x)=x-1$ and $r(x,y)=\begin{pmatrix} x+y \\ x \end{pmatrix}$;

(ii). $p(x)=x-1$, $q(x)=1$ and $r(x,y)=\begin{pmatrix} x+y \\ x \end{pmatrix}$;

(iii). $p(x)=q(x)=1$, $r(x,y)=\begin{pmatrix} x+y+2 \\ x \end{pmatrix}$.

Let $\xi_0=\xi_1=1$, for the case $p(x)=q(x)=1$ and $r(x,y)=\begin{pmatrix} x+y+1 \\ x \end{pmatrix}$, the sequence $\{\xi_n\}$ is 1, 1, 2, 5, 16, 61, 272. . . , which appears as the sequence in "The On-Line Encyclopedia of Integer Sequences" (see A000111).

5. Conclusion and Discussion

In this paper, we have presented generalized Toeplitz determinant solutions for the generalized Schur flow and the mixed form of the relativistic Toda chain. Furthermore, we have also given a general Hankel determinant solution for the nonisospectral Toda lattice. Finally, some new combinatorial numbers have been obtained from the recurrence relations of the moments proposed in the present paper. These new combinatorial numbers can be interpreted by means of the lattice paths.

Mukaihira and Nakamura have considered an integrable discrete-time Schur flow [30]. It admits a solution in terms of the ratios of the Toeplitz determinants and can be applied to compute a Perron-Carathéodory continued fraction in a polynomial time. Moreover, Kajiwara et al. presented a generalized Hankel type determinant solution of the discrete-time Toda lattice [25]. It is known that many discrete-time integrable systems with the determinant solutions can be used to design numerical algorithms [11, 22, 31, 44]. Therefore, it is possible and worthwhile to seek the integrable discrete-time chains with the generalized determinant solutions corresponding to the three semi-discrete integrable systems discussed in the present paper.

Besides, one can express some determinants in terms of non-intersecting paths by employing the well-known combinatorial method of Gessel-Viennot-Lindström [20, 28, 43]. Therefore, one can set up the corresponding results for some Hankel type determinants and Toeplitz type determinants with the elements in Section 4. Actually, any Toeplitz type determinant can be rewritten as a Hankel determinant. Furthermore, it is also an interesting topic to connect the integrable equations and the non-intersecting paths directly when the solutions may be expressed in terms of Hankel or Toeplitz determinants. We refer the readers to [14, 26, 27, 33, 49]. As for our present case, work is still in progress although some partial results have been done.

Acknowledgement

We thank the anonymous referees for their valuable suggestions. X.K. Chang and X.B. Hu are grateful to Prof. Yeong-Nan Yeh and Prof. Jyh-Hao Lee, Institute of Mathematics, Academia Sinica, for inviting them for a stay during which part of this work was done. Besides, X.K. Chang would like to thank Dr. Hongchuan Lei for his helpful discussions. This work was partially supported by the National Natural Science Foundation of China (Grant no. 11331008, 11371251); the Knowledge Innovation Program of LSEC, the Institute of Computational Mathematics, AMSS, CAS; and the Shandong Provincial Natural Science Foundation, China(Grant no. ZR2013AQ024).

Appendix A. Proof of Lemma 3.1.3

Proof. These two formulae can be proved by applying the rule of the determinant derivative directly and employing relations (12) in a similar manner. Here we only give the detailed proof to the case of $\partial_t \hat{\Delta}_k(t)$.

By virtue of the recurrence relations (12), we have

$$\partial_t A_j = \mu(A_{j-1} + A_{j+1}) + \varphi A_j + i\psi B_j,$$

where

$$B_j = \begin{pmatrix} \sum_{p=0}^{j-1} s_p s_{j-p} \\ \sum_{p=0}^{j-2} s_p s_{j-1-p} \\ \vdots \\ 0 \\ \vdots \\ -\sum_{p=0}^{k-j-2} s_{-p} s_{j+1+p-k} \end{pmatrix},$$

with the convention $\sum_{i=p}^{q}\{\} = 0$ for $p > q$. It is noted that we have used

$$\partial_t s_k = \partial_t \bar{s}_{-k} = \mu(s_{k-1} + s_{k+1}) - i\psi \sum_{j=0}^{-k-1} s_{-j} s_{k+j} + \varphi s_k, \qquad k = -1, -2, -3, \ldots,$$

which is obtained from (12).

Then, by use of the rule of determinant derivative, we have

$$\partial_t \hat{\Delta}_k(t) = \sum_{j=1}^{k} |A_1, A_2, \cdots, A_{j-1}, \partial_t A_j, A_{j+1}, \cdots, A_k|$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} |A_1, A_2, \cdots, A_{j-1}, \mu(A_{j-1}+A_{j+1}) + \varphi A_j + i\psi B_j, A_{j+1}, \cdots, A_k| \\
&= \mu(|A_0, A_2, A_3, \cdots, A_k| + |A_1, A_2, \cdots, A_{k-1}, A_{k+1}|) + k\varphi\hat{\Delta}_k \\
&\quad + i\psi \sum_{j=1}^{k} |A_1, A_2, \cdots, A_{j-1}, B_j, A_{j+1}, \cdots, A_k| .
\end{aligned}
$$

Thus, in order to prove the result, it is sufficient to show that

$$
\sum_{j=1}^{k} |A_1, A_2, \cdots, A_{j-1}, B_j, A_{j+1}, \cdots, A_k| = s_0\hat{\Delta}_k.
$$

Rewrite

$$
\begin{aligned}
&|A_1, A_2, \cdots, A_{j-1}, B_j, A_{j+1}, \cdots, A_k| \\
&= \begin{vmatrix}
s_1 & s_2 & \cdots & s_{j-1} & \sum_{p=0}^{j-1} s_p s_{j-p} & s_{j+1} & \cdots & s_k \\
s_0 & s_1 & \cdots & s_{j-2} & \sum_{p=0}^{j-2} s_p s_{j-1-p} & s_j & \cdots & s_{k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
s_{1-j} & s_{2-j} & \cdots & s_{-1} & 0 & s_1 & \cdots & s_{k-j} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
s_{2-k} & s_{3-k} & \cdots & s_{j-k} & -\sum_{p=0}^{k-j-2} s_{-p} s_{j+1+p-k} & s_{j+2-k} & \cdots & s_1
\end{vmatrix}.
\end{aligned}
$$

Adding the $p-$th column multiplied by $-s_{j-p}$ to the $j-$th column ($j \geq 2$) for $p = 1, 2, \ldots, j-1$, we have

$$
\begin{aligned}
&|A_1, A_2, \cdots, A_{j-1}, B_j, A_{j+1}, \cdots, A_k| \\
&= \begin{vmatrix}
s_1 & \cdots & s_{j-1} & s_0 s_j & s_{j+1} & \cdots & s_k \\
s_0 & \cdots & s_{j-2} & 0 & s_j & \cdots & s_{k-1} \\
s_{-1} & \cdots & s_{j-3} & -s_{-1} s_{j-1} & s_{j-1} & \cdots & s_{k-2} \\
s_{-2} & \cdots & s_{j-4} & -\sum_{p=-2}^{-1} s_p s_{j-3-p} & s_{j-2} & \cdots & s_{k-3} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
s_{1-j} & \cdots & s_{-1} & -\sum_{p=1-j}^{-1} s_p s_{-p} & s_1 & \cdots & s_{k-j} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
s_{2-k} & \cdots & s_{j-k} & -\sum_{p=2-k}^{-1} s_p s_{j+1-k-p} & s_{j-k+2} & \cdots & s_1
\end{vmatrix} \\
&= (-1)^{j+1} s_0 s_j \hat{\Delta}_k \begin{bmatrix} 1 \\ j \end{bmatrix} - \sum_{q=2-k}^{-1} (-1)^{2-q+j} (\sum_{p=q}^{-1} s_p s_{j-1+q-p}) \hat{\Delta}_k \begin{bmatrix} 2-q \\ j \end{bmatrix},
\end{aligned}
$$

where the last step is a consequence of expanding the determinant with respect to the $j-$th column. It is easy to check that the above result also holds for $j = 1$. And then,

$$
\begin{aligned}
&\sum_{j=1}^{k} |A_1, A_2, \cdots, A_{j-1}, B_j, A_{j+1}, \cdots, A_k| \\
&= \sum_{j=1}^{k} (-1)^{j+1} s_0 s_j \hat{\Delta}_k \begin{bmatrix} 1 \\ j \end{bmatrix} - \sum_{j=1}^{k} \sum_{q=2-k}^{-1} (-1)^{2-q+j} (\sum_{p=q}^{-1} s_p s_{j-1+q-p}) \hat{\Delta}_k \begin{bmatrix} 2-q \\ j \end{bmatrix}
\end{aligned}
$$

$$= s_0 \sum_{j=1}^{k} (-1)^{j+1} s_j \hat{\Delta}_k \left[\begin{array}{c} 1 \\ j \end{array} \right] - \sum_{q=2-k}^{-1} (-1)^{2-q} \sum_{p=q}^{-1} s_p \sum_{j=1}^{k} (-1)^j s_{j-1+q-p} \hat{\Delta}_k \left[\begin{array}{c} 2-q \\ j \end{array} \right]$$
$$= s_0 \sum_{j=1}^{k} (-1)^{j+1} s_j \hat{\Delta}_k \left[\begin{array}{c} 1 \\ j \end{array} \right]$$
$$= s_0 \hat{\Delta}_k.$$

Here we have used the classic result, that is, for any $n \times n$ determinant $D = det(d_{ij})$, there holds $\sum_{i=1}^{n} d_{ik} D_{ij} = \delta_{k,j} D$, where D_{ij} is the (i,j)-cofactor of the determinant D.

Thus, we confirm the first formula and the proof is completed. □

Lemma 3.2.3 can be also proved by following the similar process to the above proof. We omit its detailed proof.

Appendix B. Proof of Lemma 3.2.4

Proof. The first two identities (31) and (32) are nothing but applying the *Schwein identity* (6) to Δ_{n+1} with $b_j = c_{1-j}$, $\hat{b}_j = c_{2-j}$, $a_{i,j} = c_{i+2-j}$, and to H_{n+1}^{-1} with $b_j = c_{-j}$, $\hat{b}_j = c_{1-j}$, $a_{i,j} = c_{i+1-j}$ respectively, where $i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, n+1$.

For the third one (33), applying the *Jacobi identity* (5) to $D = T_{n+2}$ and noting that

$$T_{n+2} \left[\begin{array}{cc} 1 & n+2 \\ n+1 & n+2 \end{array} \right] = \Delta_n,$$
$$T_{n+2} \left[\begin{array}{c} 1 \\ n+2 \end{array} \right] = \Delta_{n+1}, \quad T_{n+2} \left[\begin{array}{c} 1 \\ n+1 \end{array} \right] = G_{n+1}^0,$$
$$T_{n+2} \left[\begin{array}{c} n+2 \\ n+2 \end{array} \right] = T_{n+1}, \quad T_{n+2} \left[\begin{array}{c} n+2 \\ n+1 \end{array} \right] = G_{n+1}^1,$$

we obtain

$$T_{n+2} \Delta_n = G_{n+1}^0 T_{n+1} - \Delta_{n+1} G_{n+1}^1.$$

Similarly, formulae (34), (35) and (36) are the consequences of applying the *Jacobi identity* (5) to $D = \Delta_{n+2}$ (with $i_1 = 1$, $i_2 = n+2$, $j_1 = 1$, $j_2 = 2$), $D = T_{n+2}$ (with $i_1 = 1$, $i_2 = n+2$, $j_1 = 1$, $j_2 = n+2$) and $D = \Delta_{n+2}$ (with $i_1 = 1$, $i_2 = n+2$, $j_1 = 1$, $j_2 = n+2$) respectively.

As for the last two identities, (37) follows immediately after applying the *Jacobi identity* (5) to

$$D = \left| \begin{array}{cccccc} c_1 & c_2 & \cdots & c_n & c_{n+1} & c_{n+2} \\ c_0 & c_1 & \cdots & c_{n-1} & c_n & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{1-n} & c_{2-n} & \cdots & c_0 & c_1 & c_2 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{array} \right| = -G_{n+1}^1,$$

with $i_1 = 1$, $i_2 = n+2$, $j_1 = 1$, $j_2 = n+2$. Similarly, the *Jacobi identity* (5) yields the last identity (38) by putting

$$D = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} & c_n & c_{n+1} \\ c_{-1} & c_0 & \cdots & c_{n-2} & c_{n-1} & c_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{-n} & c_{1-n} & \cdots & c_{-1} & c_0 & c_1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{vmatrix} = -G^0_{n+1},$$

with $i_1 = 1$, $i_2 = n+2$, $j_1 = 1$, $j_2 = n+2$.

Thus, we complete the proof. □

Appendix C. Proof of Lemma 3.3.2

Proof. First we introduce the following notations for convenience.

$$\tilde{A}_k = \begin{pmatrix} a_k \\ a_{k+1} \\ \vdots \\ a_{k+n-1} \end{pmatrix}, \qquad \tilde{A}^*_k = \begin{pmatrix} ka_k \\ (k+1)a_{k+1} \\ \vdots \\ (k+n-1)a_{k+n-1} \end{pmatrix},$$

$$\tilde{B}_k = \begin{pmatrix} \sum_{i+j=k-1} a_i a_j \\ \sum_{i+j=k} a_i a_j \\ \vdots \\ \sum_{i+j=k+n-2} a_i a_j \end{pmatrix}, \quad \tilde{D}_k = \begin{pmatrix} \sum_{i+j=k-1} a_i a_j \\ \vdots \\ \sum_{i+j=k+n-2} a_i a_j \\ \sum_{i+j=k+n} a_i a_j \end{pmatrix},$$

$$\tilde{C}_k = \begin{pmatrix} a_k \\ \vdots \\ a_{k+n-1} \\ a_{k+n+1} \end{pmatrix}, \qquad \tilde{C}^*_k = \begin{pmatrix} ka_k \\ \vdots \\ (k+n-1)a_{k+n-1} \\ (k+n+1)a_{k+n+1} \end{pmatrix}.$$

Then from the recurrence relation (50), we have

$$\partial_t \tilde{A}_k = \tilde{A}^*_{k+1} - \psi \tilde{B}_k, \quad \partial_t \tilde{C}_k = \tilde{C}^*_{k+1} - \psi \tilde{D}_k.$$

Next we will prove (51)–(55) respectively.

(I) Consider the derivative of τ_n,

$$\begin{aligned} \partial_t \tau_n &= \sum_{k=0}^{n-1} \left| \tilde{A}_0, \ldots, \partial_t \tilde{A}_k, \ldots, \tilde{A}_{n-1} \right| \\ &= \sum_{k=0}^{n-1} \left| \tilde{A}_0, \ldots, \tilde{A}_{k-1}, \tilde{A}^*_{k+1}, \tilde{A}_{k+1}, \ldots, \tilde{A}_{n-1} \right| \\ &\quad - \psi \sum_{k=0}^{n-1} \left| \tilde{A}_0, \ldots, \tilde{A}_{k-1}, \tilde{B}_k, \tilde{A}_{k+1}, \ldots, \tilde{A}_{n-1} \right|. \end{aligned}$$

Noticing that $\tilde{A}^*_{k+1} = (k+n)\tilde{A}_{k+1} - \widehat{A}_k$, where $\widehat{A}_k = ((n-1)a_{k+1}, (n-2)a_{k+2}, \ldots, a_{k+n-1}, 0)^T$, and expanding the determinant $\left|\tilde{A}_0, \ldots, \widehat{A}_k, \ldots, \tilde{A}_{n-1}\right|$ with the $(k+1)$-th column, we have

$$\begin{aligned}
&\sum_{k=0}^{n-1} \left|\tilde{A}_0, \ldots, \tilde{A}_{k-1}, \tilde{A}^*_{k+1}, \tilde{A}_{k+1}, \ldots, \tilde{A}_{n-1}\right| \\
&= (2n-1)\left|\tilde{A}_0, \ldots, \tilde{A}_{n-2}, \tilde{A}_n\right| - \sum_{k=0}^{n-1}\left|\tilde{A}_0, \ldots, \widehat{A}_k, \ldots, \tilde{A}_{n-1}\right| \\
&= (2n-1)\sigma_n - \sum_{k=1}^{n}\sum_{i=1}^{n-1}(-1)^{i+k}(n-i)a_{k+i-1}\tau_n \begin{bmatrix} i \\ k \end{bmatrix} \\
&= (2n-1)\sigma_n - \sum_{i=1}^{n-1}(n-i)\sum_{k=1}^{n}(-1)^{i+k}a_{k+i-1}\tau_n \begin{bmatrix} i \\ k \end{bmatrix} \\
&= (2n-1)\sigma_n.
\end{aligned}$$

As for the second term in the expression of $\partial_t \tau_n$, adding the j-th column multiplied by $-a_{k-j}$ to the $(k+1)$-th column for $j = 1, 2, \cdots, k$, and expanding the determinant with the $(k+1)$-th column, we get

$$\begin{aligned}
&\sum_{k=0}^{n-1} \left|\tilde{A}_0, \cdots, \tilde{A}_{k-1}, \tilde{B}_k, \tilde{A}_{k+1}, \cdots, \tilde{A}_{n-1}\right| \\
&= \sum_{k=0}^{n-1} \begin{vmatrix} a_0 & \ldots & a_{k-1} & 0 & a_{k+1} & \ldots & a_{n-1} \\ a_1 & \ldots & a_k & a_0 a_k & a_{k+2} & \ldots & a_n \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & \ldots & a_{k+n-2} & \sum_{i=0}^{n-2} a_i a_{k+n-2-i} & a_{k+n} & \ldots & a_{2n-2} \end{vmatrix} \\
&= \sum_{k=0}^{n-1}\sum_{i=0}^{n-2}(-1)^{k+i+1}\left(a_0 a_{k+i} + \cdots + a_i a_k\right)\tau_n \begin{bmatrix} i+2 \\ k+1 \end{bmatrix} \\
&= \sum_{i=0}^{n-2}\left(a_0\sum_{k=0}^{n-1}(-1)^{k+i+1}a_{k+i}\tau_n \begin{bmatrix} i+2 \\ k+1 \end{bmatrix} + \ldots + a_i\sum_{k=0}^{n-1}(-1)^{k+i+1}a_k\tau_n \begin{bmatrix} i+2 \\ k+1 \end{bmatrix}\right) \\
&= 0.
\end{aligned}$$

Consequently, we obtain $\partial_t \tau_n = (2n-1)\sigma_n$.

(II) Consider the derivative of σ_{n+1},

$$\begin{aligned}
\partial_t \sigma_{n+1} &= \sum_{k=0}^{n} \left|\tilde{C}_0, \cdots, \partial_t \tilde{C}_k, \cdots, \tilde{C}_n\right| \\
&= \sum_{k=0}^{n} \left|\tilde{C}_0, \cdots, \tilde{C}_{k-1}, \tilde{C}^*_{k+1}, \tilde{C}_{k+1}, \cdots, \tilde{C}_n\right| \\
&\quad - \psi\sum_{k=0}^{n} \left|\tilde{C}_0, \cdots, \tilde{C}_{k-1}, \tilde{D}_k, \tilde{C}_{k+1}, \cdots, \tilde{C}_n\right|.
\end{aligned}$$

Since $\tilde{C}^*_{k+1} = (k+n)\tilde{C}_{k+1} - \widehat{C}_k$, where $\widehat{C}_k = ((n-1)a_{k+1}, \cdots, a_{k+n-1}, 0, -2a_{k+n+2})^T$, we have

$$
\begin{aligned}
&\sum_{k=0}^{n}\left|\tilde{C}_0,\cdots,\tilde{C}_{k-1},\tilde{C}^*_{k+1},\tilde{C}_{k+1},\cdots,\tilde{C}_n\right| \\
&=2n\left|\tilde{C}_0,\cdots,\tilde{C}_{n-1},\tilde{C}_{n+1}\right|-\sum_{k=0}^{n}\left|\tilde{C}_0,\cdots,\widehat{C}_k,\cdots,\tilde{C}_n\right| \\
&=2nh_{n+1}-\sum_{k=0}^{n}\sum_{i=2}^{n}(n+1-i)a_{k+i-1}(-1)^{i+k}\sigma_{n+1}\begin{bmatrix} i-1 \\ k+1 \end{bmatrix} \\
&\quad+\sum_{k=0}^{n}2a_{k+n+2}(-1)^{k+n}\sigma_{n+1}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \\
&=2nh_{n+1}-\sum_{i=2}^{n}(n+1-i)\sum_{k=0}^{n}(-1)^{k+i}a_{k+i-1}\sigma_{n+1}\begin{bmatrix} i-1 \\ k+1 \end{bmatrix} \\
&\quad+2\sum_{k=0}^{n}(-1)^{k+n}a_{k+n+2}\sigma_{n+1}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \\
&=2nh_{n+1}+2g_{n+1},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{n}\left|\tilde{C}_0,\cdots,\tilde{C}_{k-1},\tilde{D}_k,\tilde{C}_{k+1},\cdots,\tilde{C}_n\right| \\
&=\sum_{k=0}^{n}\begin{vmatrix} a_0 & \cdots & a_{k-1} & 0 & a_{k+1} & \cdots & a_n \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{k+n-2} & \sum_{i=0}^{n-2}a_ia_{k+n-2-i} & a_{k+n} & \cdots & a_{2n-1} \\ a_{n+1} & \cdots & a_{k+n} & \sum_{i=0}^{n}a_ia_{k+n-i} & a_{k+n+2} & \cdots & a_{2n+1} \end{vmatrix} \\
&=\sum_{k=0}^{n}\sum_{i=2}^{n}\left(a_0a_{k-2+i}+\cdots+a_{i-2}a_k\right)(-1)^{k+1+i}\sigma_{n+1}\begin{bmatrix} i \\ k+1 \end{bmatrix} \\
&\quad+\sum_{k=0}^{n}(-1)^{n+k}\left(a_0a_{k+n}+\cdots+a_na_k\right)\sigma_{n+1}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \\
&=a_0\sum_{k=0}^{n}(-1)^{n+k}a_{k+n}\sigma_{n+1}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}=\phi\tau_{n+1},
\end{aligned}
$$

which lead to $\partial_t\sigma_{n+1}=2nh_{n+1}+2g_{n+1}-\phi\psi\tau_{n+1}$.

(III) Applying the *Jacobi identity* (5) to τ_{n+1} and observing that

$$
\begin{aligned}
&\tau_{n+1}\begin{bmatrix} n & n+1 \\ n & n+1 \end{bmatrix}=\tau_{n-1}, \\
&\tau_{n+1}\begin{bmatrix} n \\ n \end{bmatrix}=h_n,\quad \tau_{n+1}\begin{bmatrix} n+1 \\ n+1 \end{bmatrix}=\tau_n,
\end{aligned}
$$

$$\tau_{n+1}\begin{bmatrix} n \\ n+1 \end{bmatrix} = \tau_{n+1}\begin{bmatrix} n+1 \\ n \end{bmatrix} = \sigma_n,$$

we obtain the formula (53) immediately.

(IV) Define

$$D = \begin{vmatrix} a_0 & \cdots & a_n & a_{n+1} \\ \vdots & \ddots & \vdots & \vdots \\ a_n & \cdots & a_{2n} & a_{2n+1} \\ 0 & \cdots & 0 & 1 \end{vmatrix}.$$

It is obvious that

$$D = \tau_{n+1}, \quad D\begin{bmatrix} n+1 & n+2 \\ n & n+1 \end{bmatrix} = g_n,$$

$$D\begin{bmatrix} n+1 \\ n \end{bmatrix} = \sigma_n, \quad D\begin{bmatrix} n+2 \\ n+1 \end{bmatrix} = \sigma_{n+1},$$

$$D\begin{bmatrix} n+1 \\ n+1 \end{bmatrix} = \tau_n, \quad D\begin{bmatrix} n+2 \\ n \end{bmatrix} = f_{n+1}.$$

Then equation (54) can be obtained by applying the *Jacobi identity* (5) to D with $i_1 = n+1$, $i_2 = n+2$, $j_1 = n$ and $j_2 = n+1$.

(V) Let $f(x,t) = \sum_{k=0}^{+\infty} a_k(t) x^k/k!$, then it is easy to see that $a_k(t) = \frac{\partial^k}{\partial x^k} f(0,t)$. Define a $n \times n$ determinant as $F_n = det\left(f^{(i+j-2)}\right)_{0 \le i,j \le n-1}$, and we have

$$\frac{\partial^2 F_n}{\partial x^2} = \begin{vmatrix} f & \cdots & f^{(n-2)} & f^{(n)} \\ \vdots & \ddots & \vdots & \vdots \\ f^{(n-2)} & \cdots & f^{(2n-4)} & f^{(2n-2)} \\ f^{(n)} & \cdots & f^{(2n-2)} & f^{(2n)} \end{vmatrix}$$

$$= \begin{vmatrix} f & \cdots & f^{(n-3)} & f^{(n-1)} & f^{(n)} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ f^{(n-1)} & \cdots & f^{(2n-4)} & f^{(2n-2)} & f^{(2n-1)} \end{vmatrix} + \begin{vmatrix} f & \cdots & f^{(n-2)} & f^{(n+1)} \\ \vdots & \ddots & \vdots & \vdots \\ f^{(n-1)} & \cdots & f^{(2n-3)} & f^{(2n)} \end{vmatrix}.$$

Set $x = 0$ in the above expression, then we get $h_n = f_n + g_n$ immediately.

Thus, the proof is completed. □

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