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CLOSURES OF STEINBERG FIBERS IN TWISTED WONDERFUL COMPACTIFICATIONS

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Abstract. By a case-free approach we give a precise description of the closure of a Steinberg fiber within a twisted wonderful compactification of a simple linear algebraic group. In the nontwisted case this description was earlier obtained by the first author.

1. Introduction

Let G be a simple linear algebraic group over an algebraically closed field k. Let B be a Borel subgroup of G and let $T \subset B$ denote a maximal torus. Let W denote the associated Weyl group and let I denote the associated set of simple roots. For a subset J of I we let W_J denote the subgroup of W generated by the simple roots in J.

The wonderful compactification X of G (see, e.g., [DP], [Str]) is a smooth projective $(G \times G)$-variety containing G as an open subset. The $(G \times G)$-orbits in X are indexed by the subsets J of I, and we fix certain base points h_J for these orbits. Let σ denote a diagram automorphism of G and let X_σ be the associated twisted wonderful compactification of G, i.e., as a variety X_σ is just X but the $(G \times G)$-action is twisted by σ on the second coordinate. Let $h_{J,\sigma}$ denote the point in X_σ identified with $h_{\sigma(J)}$ in X. Then the collection $h_{\sigma(J)}$, $J \subset I$, are representatives for the $(G \times G)$-orbits in X_σ. A *G-stable piece* in X_σ is then a locally closed and smooth subvariety in X of the form $Z^w_{J,\sigma} = \mathrm{diag}(G)(Bw, 1)h_{J,\sigma}$, where $w \in W^{\sigma(J)}$ is a minimal length coset representative of $W/W_{\sigma(J)}$ and $\mathrm{diag}(G)$ denotes the diagonal in $G \times G$. We then have a decomposition $X_\sigma = \bigsqcup_{J \subset I, w \in W^J} Z^w_J$ (see [L2, 12.3] and [H2, 1.12]).

The G-stable pieces were first introduced by Lusztig to study the G-orbits and parabolic character sheaves. However, his original definition was based on some inductive method. The (equivalent) definition that we used above is due to the first author in [H1]. What we need in this paper is that the dimension of $Z^w_{J,\sigma}$ is equal to $\dim(G) - l(w) - |I - J|$, where $l(w)$ is the length of w and $|I - J|$ is the cardinality of the set $I - J$ (see [L2, 8.20]). More properties about the G-stable pieces can be found in [L2] and [H2]. The G-stable pieces were also used by Evens and Lu in [EL] to study the Poisson structure and symplectic leaves.

Consider G as a $(G \times G)$-variety by left and right translation and define G_σ, similar

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to the definition of X_σ, by twisting the G-structure of G on the second factor by σ. A σ-conjugacy class in G_σ is then a diag(G)-orbit in G_σ. The set of elements in G_σ whose semisimple part lies in a fixed σ-conjugacy class is then called a Steinberg fiber of G_σ. In this paper we study the closure of Steinberg fibers within X_σ.

In [L2] Lusztig gave an explicit description for the closure of the unipotent variety in the group compactification when $G = \mathrm{PGL}_2$ or PGL_3. In [Sp2] Springer studied the closure of an arbitrary Steinberg fiber for any connected, simple algebraic group and obtained some partial results. Based on their results, the first author obtained an explicit description of the closure of Steinberg fibers in the nontwisted case. The result in [H1] was formulated using G-stable pieces and the proof was based on a case-by-case checking. The main purpose of this paper is to generalize the result of [H1] to the twisted case with a more conceptual (and easier) proof. More precisely, we prove

Theorem. *Let F be a Steinberg fiber of G_σ and let $\overline{F}$ be its closure in X_σ. Then*

$$\overline{F} - F = \bigsqcup_{J \subset I} \bigsqcup_{\substack{w \in W^{\sigma(J)} \\ \mathrm{supp}_\sigma(w) = I}} Z^w_{J,\sigma},$$

where $\mathrm{supp}_\sigma(w)$ denotes the minimal σ-stable subset of I such that w is contained in $W_{\mathrm{supp}_\sigma(w)}$.

As a consequence, the boundary of the closure is independent of the choice of the Steinberg fiber. Likewise, it may be shown that the boundary of the closure of F within any equivariant embedding of G is independent of the choice of F (see [T]). As a byproduct, we will also give an explicit description of the "nilpotent cones" on X.

2. Wonderful compactifications and G-stable pieces

2.1.

Let G denote a simple linear algebraic group over an algebraically closed field k. We consider G as a $(G \times G)$-variety by left and right translation. Let B be a Borel subgroup of G, B^- be an opposite Borel subgroup, and let $T = B \cap B^-$. The unipotent radical of B (resp., B^-) will be denoted by U (resp., U^-). Let R denote the set of roots defined by T and let R^+ denote the set of positive roots defined by B. Let $(\alpha_i)_{i \in I}$ be the set of simple roots. For $i \in I$, we denote by ω_i and s_i the fundamental weight and the simple reflection corresponding to α_i.

We denote by W the Weyl group associated to T. For any subset J of I, let W_J be the subgroup of W generated by $\{s_j \mid j \in J\}$ and let W^J be the set of minimal length coset representatives of W/W_J.

For $J \subset I$, let $P_J \supset B$ be the standard parabolic subgroup defined by J and let $P_J^- \supset B^-$ be the parabolic subgroup opposite to P_J. Set $L_J = P_J \cap P_J^-$. Then L_J is a Levi subgroup of P_J and P_J^-. The semisimple quotient of L_J of adjoint type will be denoted by G_J. We denote by π_{P_J} (resp., $\pi_{P_J^-}$) the projection of P_J (resp., P_J^-) onto G_J.

2.2.

Let X denote the wonderful compactification of G ([DP], [Str]). Then X is an irreducible, smooth projective $(G \times G)$-variety with finitely many $(G \times G)$-orbits Z_J

indexed by the subsets J of I. As a $(G \times G)$-variety the orbit Z_J is uniquely isomorphic to the product $(G \times G) \times_{P_J^- \times P_J} G_J$, where $P_J^- \times P_J$ acts on $G \times G$ by $(q,p) \cdot (g_1, g_2) = (g_1 q^{-1}, g_2 p^{-1})$ and on G_J by $(q,p) \cdot z = \pi_{P_J^-}(q) z \pi_{P_J}(p)^{-1}$. Let h_J be the image of $(1,1,1)$ in Z_J under this isomorphism.

We denote by $\mathrm{diag}(G)$ the image of the diagonal embedding of G in $G \times G$. For $J \subset I$ and $w \in W^J$, set $Z_J^w = \mathrm{diag}(G)(Bw, 1)h_J$. Then Z_J^w is a locally closed subvariety of X and (see [L2, 12.3] and [H2, 1.12])

$$X = \bigsqcup_{\substack{J \subset I \\ w \in W^J}} Z_J^w.$$

We call Z_J^w a G-stable piece.

3. Twisted actions

3.1.

An automorphism σ of G which stabilizes the Borel subgroup B and the maximal torus T will induce a permutation of I. When the order of σ as an automorphism of G coincides with the order of the associated permutation of I, we say that σ is a diagram automorphism. From now on σ will denote a diagram automorphism of G. We also denote by σ the corresponding bijection on I and W.

Let G_σ be the $(G \times G)$-variety which as a variety is isomorphic to G and where the $G \times G$ action is twisted by the morphism $G \times G \to G \times G$, $(g,h) \mapsto (g, \sigma(h))$ for $g, h \in G$. Then we define the wonderful compactification X_σ of G_σ to be the $(G \times G)$-variety which as a variety is isomorphic to the wonderful compactification X of G and where the $G \times G$ action is twisted in the same way as above. Notice that we may regard G_σ as a connected component of the semidirect product $G \rtimes \langle \sigma \rangle$. In this case, X_σ is the completion of G_σ considered in [L2, 12].

The $(G \times G)$-orbits in X_σ coincide with the associated orbits in X and we let $Z_{J,\sigma}$ denote the orbit coinciding with $Z_{\sigma(J)}$. Accordingly, we let $h_{J,\sigma}$ denote the point in $Z_{J,\sigma}$ identified with the base point $h_{\sigma(J)}$ of $Z_{\sigma(J)}$. For $J \subset I$ and $w \in W^{\sigma(J)}$, set $Z_{J,\sigma}^w = \mathrm{diag}(G)(Bw, 1)h_{J,\sigma}$. Then

$$X_\sigma = \bigsqcup_{J \subset I} \bigsqcup_{w \in W^{\sigma(J)}} Z_{J,\sigma}^w.$$

We call $(Z_{J,\sigma}^w)_{J \subset I, w \in W^{\sigma(J)}}$ the G-stable pieces of X_σ (see [L2, 12.3] and [H2, 1.12]).

3.2.

The orbits of $\mathrm{diag}(G)$ on G_σ are called σ-conjugacy classes. Let $G /\!\!/_\sigma G$ be the affine variety whose algebra is the subalgebra $k[G]^{G,\sigma}$ of functions in $k[G]$ invariant under σ-conjugacy. The inclusion $k[G]^{G,\sigma} \to k[G]$ induces a morphism $\mathrm{St} : G_\sigma \to G /\!\!/_\sigma G$. If σ is trivial, then St is just the Steinberg morphism of G. Thus, for arbitrary σ, we call St the Steinberg morphism of G_σ and the fibers the Steinberg fibers of G_σ.

An element $g \in G_\sigma$ is σ-conjugate to an element in B [Ste2, Lemma 7.3]. Write $b = tu$ where $t \in T$ and u is an element of the unipotent radical U of B. It is then easily seen that there exists an element $t_1 \in T$, such that $t_1 t\sigma(t_1)^{-1} \in T^\sigma$. Hence, g is σ-conjugate to some element in $T^\sigma U$, i.e., we may assume that $t \in T^\sigma$. Notice that t is contained in the closure of the σ-conjugacy class of tu and, thus, by geometric invariant theory, we find $\mathrm{St}(tu) = \mathrm{St}(t)$. Moreover, considering $t\sigma$ as an element of the semisimple group $G \rtimes \langle\sigma\rangle$ it follows that $t\sigma$ is quasi-semisimple in the sense of [Ste2, Sec. 9], i.e., the automorphism of G obtained by conjugation by $t\sigma$ will fix a Borel subgroup and a maximal torus thereof. As a consequence, the σ-conjugacy class of t in G_σ is closed [Spa, II.1.15(f)]. We conclude that any Steinberg fiber of G_σ is of the form $\bigcup_{g\in G} g(tU)\sigma(g)^{-1}$ for some $t \in T^\sigma$. In particular, any Steinberg fiber is irreducible.

3.3.

Let G_{sc} be the connected, simply connected, linear algebraic group associated to G, and let B_{sc} (resp., T_{sc}) denote the Borel subgroup (resp., maximal torus) of G_{sc} associated to B (resp., T). By [Ste2, 9.16] the automorphism σ of G may be lifted to an automorphism of G_{sc}, which we also denote by σ. We then define the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-variety $G_{\mathrm{sc},\sigma}$ similar to the definition of G_σ. We may also form the quotient $G_{\mathrm{sc}} /\!\!/_\sigma G_{\mathrm{sc}}$ and define Steinberg fibers in $G_{\mathrm{sc},\sigma}$ similar to the considerations in Section 3.2 for G_σ.

The automorphism σ of G_{sc} induces a natural action of σ on the set Λ of T_{sc}-characters, and we let Λ_+^σ denote the set of σ-invariant dominant weights. Moreover, we let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_l$ denote the σ-orbits in I, and set $\omega_{\mathcal{C}_j} = \sum_{i\in\mathcal{C}_j} \omega_i$. Then the set of elements $\omega_{\mathcal{C}_j}$, $j = 1, \dots, l$, is a generating set of the semigroup Λ_+^σ.

3.4.

To any dominant weight $\lambda = \sum_{i\in I} a_i\omega_i$ we associate the subset $I(\lambda) = \{i \in I \mid a_i \neq 0\}$ of I. For $w \in W$, let $\mathrm{supp}(w) \subset I$ be the set of simple roots whose associated simple reflections occur in some (or, equivalently, any) reduced decomposition of w (see [Bou, Prop. IV.1.7]) and let $\mathrm{supp}_\sigma(w) = \bigcup_{k\geqslant 0} \sigma^k(\mathrm{supp}(w))$. We have the following characterization of $\mathrm{supp}(w)$.

Lemma 3.5. *Let $w \in W$ and $i \in I$. Then $w\omega_i \neq \omega_i$ if and only if $i \in \mathrm{supp}(w)$. Hence for a dominant weight λ, $w\lambda \neq \lambda$ if and only if $I(\lambda) \cap \mathrm{supp}(w) \neq \varnothing$.*

Proof. If $i \notin \mathrm{supp}(w)$, then $w\omega_i = \omega_i$. Now we fix a reduced expression $w = s_{i_1} \cdots s_{i_n}$. Assume that $i \in \mathrm{supp}(w)$. We show that $w\omega_i \neq \omega_i$ by induction on n. If $i_n \neq i$, then we are done by induction in n. Hence, we may assume that $i_n = i$. But then $w\alpha_i$ is a negative root. Thus $1 = \langle \omega_i, \alpha_i^\vee \rangle = \langle w\omega_i, (w\alpha_i)^\vee \rangle$ and, in particular, we cannot have $w\omega_i = \omega_i$. □

3.6.

For any dominant weight $\lambda \in \Lambda_+$ let $\mathrm{H}(\lambda)$ denote the dual Weyl module for G_{sc} with lowest weight $-\lambda$. We then define ${}^\sigma\mathrm{H}(\lambda)$ to be the G_{sc}-module which as a vector space is $\mathrm{H}(\lambda)$ and with G_{sc}-action twisted by the automorphism σ of G_{sc}. Notice that up to a nonzero constant there exists a unique G_{sc}-isomorphism ${}^\sigma\mathrm{H}(\lambda) \simeq \mathrm{H}(\sigma(\lambda))$. In particular, when $\lambda \in \Lambda_+^\sigma$ is σ-invariant there exists a G_{sc}-equivariant isomorphism $f_\lambda \colon \mathrm{H}(\lambda) \to {}^\sigma\mathrm{H}(\lambda)$. Fix f_λ such that its restriction to the lowest weight space $k_{-\lambda}$ in $\mathrm{H}(\lambda)$ is the identity map (here we use the identification of ${}^\sigma\mathrm{H}(\lambda)$ with $\mathrm{H}(\lambda)$ as vector spaces).

4. The "nilpotent cone" of X

4.1.

For any dominant weight λ there exists (see [DS, 3.9]) a $(G\times G)$-equivariant morphism

$$\rho_\lambda : X \longrightarrow \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$$

which extends the morphism $G \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$ defined by $g \mapsto g[\mathrm{Id}_\lambda]$, where $[\mathrm{Id}_\lambda]$ denotes the class representing the identity map on $\mathrm{H}(\lambda)$ and g acts by the left action. By the definition of X_σ we obtain a $(G\times G)$-equivariant morphism

$$X_\sigma \longrightarrow \mathbb{P}\big(\mathrm{Hom}_k({}^\sigma\mathrm{H}(\lambda), \mathrm{H}(\lambda))\big).$$

When $\lambda \in \Lambda_+^\sigma$ we may apply f_λ to obtain an induced map

$$\rho_{\lambda,\sigma} : X_\sigma \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$$

which is $(G\times G)$-equivariant.

4.2.

An element in $\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$ is said to be nilpotent if it may be represented by a nilpotent endomorphism of $\mathrm{H}(\lambda)$. For $\lambda \in \Lambda_+^\sigma$ we let

$$\mathcal{N}(\lambda)_\sigma = \{z \in X_\sigma \mid \rho_{\lambda,\sigma}(z) \text{ is nilpotent}\},$$

and call $\mathcal{N}(\lambda)_\sigma$ the nilpotent cone of X_σ associated to the dominant weight λ. In Section 4.4, we will give an explicit description of $\mathcal{N}(\lambda)_\sigma$.

4.3.

Define ht to be the height map on the root lattice, i.e., the linear map on the root lattice which maps all the simple roots to 1.

Now fix $\lambda \in \Lambda_+$. Choose a basis $v_1, \dots, v_m$ for $\mathrm{H}(\lambda)$ consisting of T-eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_m$ satisfying $\mathrm{ht}(\lambda_j + \lambda) \geqslant \mathrm{ht}(\lambda_i + \lambda)$ whenever $j \leqslant i$. Then B is upper triangular with respect to this basis.

Let A_J be a representative of $\rho_\lambda(h_J)$ in $\mathrm{End}(\mathrm{H}(\lambda))$. Then when $\lambda_j + \lambda$ is a linear combination of the simple roots in J we have that $A_J v_j \in k^\times v_j$. If $\lambda_j + \lambda$ is not a linear combination of the simple roots in J, then $A_J v_j = 0$. Assuming that λ is σ-invariant we obtain, by the definitions in Section 4.1, a similar description for a representative $A_{J,\sigma}$ of $\rho_{\lambda,\sigma}(h_{J,\sigma})$: if $\lambda_j + \lambda$ is a linear combination of the simple roots in J, then we have that $A_{J,\sigma} v_j \in k^\times f_\lambda(v_j)$; otherwise, $A_{J,\sigma} v_j = 0$. Notice that we regard $f_\lambda(v_j)$ as an element of $\mathrm{H}(\lambda)$ and, as such, $f_\lambda(v_j)$ is a T-eigenvector of weight $\sigma(\lambda_j)$.

We now obtain

Proposition 4.4. *Let $\lambda \in \Lambda_+^\sigma$, then*

$$\mathcal{N}(\lambda)_\sigma = \bigsqcup_{J \subset I} \; \bigsqcup_{\substack{w \in W^{\sigma(J)} \\ I(\lambda)\cap \mathrm{supp}(w) \neq \varnothing}} Z^w_{J,\sigma}.$$

Proof. Let $w \in W^{\sigma(J)}$. Assume that $w\lambda \neq \lambda$. Note that if x is a nonnegative linear combination of the simple roots in J, then $\mathrm{ht}(w\sigma(x)) \geqslant \mathrm{ht}(x)$. Hence,

$$\mathrm{ht}\big(w\sigma(-\lambda + x) + \lambda\big) = \mathrm{ht}(w\sigma(x)) + \mathrm{ht}(-w\lambda + \lambda) > \mathrm{ht}(x).$$

Therefore, $\rho_{\lambda,\sigma}((w, 1)h_{J,\sigma})$ is represented by a strictly upper triangular matrix with respect to the chosen basis in Section 4.3 above. As a consequence, for any $b \in B$, $\rho_{\lambda,\sigma}((bw, 1)h_{J,\sigma})$ is also represented by a strictly upper triangular matrix. So $(Bw, 1)h_{J,\sigma} \subset \mathcal{N}(\lambda)_\sigma$. Since $\mathcal{N}(\lambda)_\sigma$ is $\mathrm{diag}(G)$-stable it follows that

$$Z^w_{J,\sigma} = \mathrm{diag}(G)(Bw, 1)h_{J,\sigma} \subset \mathcal{N}(\lambda)_\sigma.$$

Now assume that $w\lambda = \lambda$. Let $b \in B$ and $z = (bw, 1)h_{J,\sigma}$. Denote by A a representative of $\rho_{\lambda,\sigma}(z)$ in $\mathrm{End}(\mathrm{H}(\lambda))$. Let V be the subspace of $\mathrm{H}(\lambda)$ spanned by $v_1, \dots, v_{m-1}$. Then $Av_m \in k^\times v_m + V$ and $AV \subset V$. Hence, $A^n v_m \neq 0$ for all $n \in \mathbb{N}$. Thus $z \notin \mathcal{N}(\lambda)_\sigma$. □

Corollary 4.5. *Let $\lambda, \mu \in \Lambda^\sigma_+$, then*

$$\mathcal{N}(\lambda + \mu)_\sigma = \mathcal{N}(\lambda)_\sigma \cup \mathcal{N}(\mu)_\sigma.$$

Proof. This follows from the relation $I(\lambda + \mu) = I(\lambda) \cup I(\mu)$. □

5. A compactification of G_{sc}

5.1.

Consider the morphism $\psi_i : G_{\mathrm{sc}} \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i)) \oplus k\big)$ defined by $g \mapsto [(g \cdot \mathrm{Id}_{\mathrm{H}(\omega_i)}, 1)]$, where $\mathrm{Id}_{\mathrm{H}(\omega_i)}$ denotes the identity map on $\mathrm{H}(\omega_i)$ and g acts on $\mathrm{End}(\mathrm{H}(\omega_i))$ by the left action. Let, furthermore, $\pi : G_{\mathrm{sc}} \to X$ denote the the natural $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-equivariant morphism and let X_{sc} denote the closure of the image of the product map

$$\Big(\pi, \prod_{i \in I} \psi_i\Big) : G_{\mathrm{sc}} \longrightarrow X \times \prod_{i \in I} \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i) \oplus k).$$

Then X_{sc} is a projective $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-equivariant variety containing G_{sc} as an open subset. Unlike X the variety X_{sc} need not be smooth and, in general, it is not even normal (e.g., in type A_3). By abuse of notation we use the notation π and ψ_i, $i \in I$, for the natural extensions of the corresponding maps to X_{sc}.

Lemma 5.2. *The projective morphism $\pi : X_{\mathrm{sc}} \to X$ defines a bijection between $X_{\mathrm{sc}} - G_{\mathrm{sc}}$ and $X - G$. In particular, π is a finite morphism.*

Proof. Notice that the $(G_{\mathrm{sc}} \times G_{\mathrm{sc}})$-invariant homogeneous polynomial function on $\mathrm{End}(\mathrm{H}(\omega_i)) \oplus k$ defined by $(f, a) \mapsto \det(f) - a^{\dim_k(\mathrm{H}(\omega_i))}$ vanishes at $(\mathrm{Id}_{\mathrm{H}(\omega_i)}, 1)$. Thus,

$$X_{\mathrm{sc}} \subset X \times \prod_{i \in I} \Big(\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i) \oplus k) - \mathbb{P}(0 \oplus k)\Big),$$

and we may consider the following commutative diagram

$$\begin{CD}
X_{\mathrm{sc}} @>>> X \times \prod_{i\in I}\big(\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i) \oplus k) - \mathbb{P}(0 \oplus k)\big)\\
@V{\pi}VV @VVV\\
X @>{\mathrm{id}_X \times \prod_{i\in I}\rho_{\omega_i}}>> X \times \prod_{i\in I}\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i))\big)
\end{CD},$$

where all the maps are the natural ones. Assume now that x is an element of the boundary $X_{\mathrm{sc}} - G_{\mathrm{sc}}$. As the dimensions of G_{sc} and X_{sc} coincide, the $(G_{\mathrm{sc}}, 1)$-stabilizer of x has strictly positive dimension. In particular, the images $\psi_i(x) = [(f_i, a_i)]$, $i \in I$, have the same property. Thus, the endomorphism f_i is not invertible and thus $a_i = 0$. This proves that

$$X_{\mathrm{sc}} - G_{\mathrm{sc}} \subset X \times \prod_{i\in I}\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i)),$$

and hence π maps $X_{\mathrm{sc}} - G_{\mathrm{sc}}$ injectively to the boundary $X - G$. As π is dominant and projective, and thus surjective, this proves the first assertion. Finally, π is finite as it is projective and quasifinite. □

5.3.

For a dominant weight $\lambda \in \Lambda_+$ let $\psi_\lambda : G_{\mathrm{sc}} \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda)) \oplus k\big)$ be the morphisms defined by $\psi_\lambda(g) = [(g \cdot \mathrm{Id}_{\mathrm{H}(\lambda)}, 1)]$ for $g \in G_{\mathrm{sc}}$. Then we let X_{sc}^λ denote the closure of the image of the map

$$G_{\mathrm{sc}} \longrightarrow X_{\mathrm{sc}} \times \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda)) \oplus k\big)$$

defined as the product of the inclusion $G_{\mathrm{sc}} \subset X_{\mathrm{sc}}$ and ψ_λ. We claim

Lemma 5.4. *The canonical morphism $\pi^\lambda : X_{\mathrm{sc}}^\lambda \to X_{\mathrm{sc}}$ is an isomorphism.*

Proof. Let X_0 be the complement of the union of the closures $\overline{Bs_iB^-}$, $i \in I$, within X, and let $X_0' = \overline{T} \cap X_0$. Then, by [BK, Prop. 6.2.3(i)], the natural morphism

$$U \times U^- \times X_0' \longrightarrow X_0,$$
$$(g, h, x) \mapsto (g, h) \cdot x,$$

is an isomorphism of varieties. Let $X_{\mathrm{sc},0} = \pi^{-1}(X_0)$ and let $X_{\mathrm{sc},0}'$ denote the (scheme-theoretic) inverse image $\pi^{-1}(X_0')$. As π is $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-equivariant we obtain an induced isomorphism

$$U \times U^- \times X_{\mathrm{sc},0}' \to X_{\mathrm{sc},0}.$$

In particular, $X_{\mathrm{sc},0}'$ is an irreducible closed subvariety of the open subvariety $X_{\mathrm{sc},0}$ containing T_{sc} as an open subset. Thus, $X_{\mathrm{sc},0}'$ is contained in the closure of T_{sc} within X_{sc}. Let π_λ denote the composition of π and π^λ. Defining $X_{\mathrm{sc},0}^\lambda = \pi_\lambda^{-1}(X_0)$ and $(X_{\mathrm{sc},0}^\lambda)' = \pi_\lambda^{-1}(X_0')$ we, similarly, obtain an isomorphism

$$U \times U^- \times (X_{\mathrm{sc},0}^\lambda)' \to X_{\mathrm{sc},0}^\lambda.$$

Notice that $X_{\mathrm{sc},0}^\lambda = \pi^\lambda(X_{\mathrm{sc},0})$. Moreover, the $(G \times G)$-translates of X_0 cover X [BK, Theorem 6.1.8]. Thus, it suffices to show that the morphism $(X_{\mathrm{sc},0}^\lambda)' \to X_{\mathrm{sc},0}'$ induced by π^λ is an isomorphism. This will follow if π^λ induces an isomorphism between the closures of T_{sc} in X_{sc} and X_{sc}^λ. Determining the latter closures of T_{sc} and checking that they are isomorphic is now an easy exercise. □

It follows that we may consider ψ_λ as the extended morphism

$$\psi_\lambda : X_{\mathrm{sc}} \longrightarrow \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda)) \oplus k\big),$$

which we will do in the following. As in the proof of Lemma 5.2 we may prove that

$$\psi_\lambda(X_{\mathrm{sc}}) \subset \Big(\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda)) \oplus k\big) - \mathbb{P}\big(0 \oplus k\big)\Big)$$

and that the induced map $X_{\mathrm{sc}} \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$ is compatible with $\pi : X_{\mathrm{sc}} \to X$ and the map $\rho_\lambda : X \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$.

5.5.

The variety X_{sc} is a compactification of G_{sc} with the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$ action defined in the natural way. Let $X_{\mathrm{sc},\sigma}$ be the $(G_{\mathrm{sc}} \times G_{\mathrm{sc}})$-variety which as a variety is isomorphic to X_{sc} and where the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-action is twisted by the morphism $G_{\mathrm{sc}} \times G_{\mathrm{sc}} \to G_{\mathrm{sc}} \times G_{\mathrm{sc}}$, $(g,h) \mapsto (g, \sigma(g))$ for $g, h \in G_{\mathrm{sc}}$. Thus we may identify $G_{\mathrm{sc},\sigma}$ of Section 3.3 with an open subset of $X_{\mathrm{sc},\sigma}$. Notice that by Lemma 5.2 we may identify the boundaries of $X_{\mathrm{sc},\sigma}$ and X_σ and we may therefore also regard $Z^w_{J,\sigma}$, for $J \neq I$, as subsets of $X_{\mathrm{sc},\sigma}$. Similar to the way that $\rho_{\lambda,\sigma}$ is defined from ρ_λ when λ is σ-invariant, we may now also define a $(G_{\mathrm{sc}} \times G_{\mathrm{sc}})$–equivariant map

$$\psi_{\lambda,\sigma} \colon X_{\mathrm{sc}} \longrightarrow \mathbb{P}(\mathrm{End}(\mathrm{H}(\lambda)) \oplus k),$$

from ψ_λ.

6. Steinberg fibers and trace maps

6.1.

Let Tr_i denote the trace function on $\mathrm{End}(\mathrm{H}(\omega_{\mathcal{C}_i}))$. To each $a_i \in k$ we may associate a global section (Tr_i, a_i) of the line bundle $\mathcal{O}_i(1) := \mathcal{O}_{\mathbb{P}(\mathrm{End}(\mathrm{H}(\omega_{\mathcal{C}_i}))\oplus k)}(1)$ on $\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_{\mathcal{C}_i})) \oplus k\big)$. The pullback of (Tr_i, a_i) to $X_{\mathrm{sc},\sigma}$, by the morphism $\psi_{\omega_{\mathcal{C}_i},\sigma}$, is then a global section f^σ_{i,a_i} of a line bundle on $X_{\mathrm{sc},\sigma}$. In the following we will study the common zero set $Z(a_1, \dots, a_l)$ of the sections f^σ_{i,a_i} for varying $a_i \in k$. By choosing a trivialization of the pullback of $\mathcal{O}_i(1)$ to $G_{\mathrm{sc},\sigma}$ we may think of f^σ_{i,a_i} as a function on $G_{\mathrm{sc},\sigma}$ and, by abuse of notation, we also denote this function by f^σ_{i,a_i}. We assume that the trivialization is chosen such that $f^\sigma_{i,a} - f^\sigma_{i,0} = a$ as functions on G_{sc} (then the trivialization is actually unique). Then f^σ_{i,a_i} is invariant under σ-conjugation by G_{sc} and thus f^σ_{i,a_i} induces a morphism $\overline{f}^\sigma_{i,a_i} : G_{\mathrm{sc}} /\!\!/_\sigma G_{\mathrm{sc}} \to k$. We then claim

Proposition 6.2. *The product morphism*

$$(\overline{f}^\sigma_{1,0}, \overline{f}^\sigma_{2,0}, \dots, \overline{f}^\sigma_{l,0}) \colon G_{\mathrm{sc}} /\!\!/_\sigma G_{\mathrm{sc}} \longrightarrow \mathbb{A}^l$$

is an isomorphism.

Proof. Let f_i^σ denote the restriction of $f_{i,0}^\sigma$ to $T_{\rm sc}$. An easy calculation shows that f_i^σ equals

$$-w_0\omega_{\mathcal{C}_i} + \sum_{\substack{\lambda\in\Lambda^\sigma,\lambda\neq -w_0\omega_{\mathcal{C}_i} \\ \mathrm{H}(\omega_{\mathcal{C}_i})_\lambda\neq 0}} q_{i,\lambda}\lambda.$$

Thus f_i^σ is contained in the semigroup algebra $k[\Lambda^\sigma]$ generated by the σ-invariant weights of Λ. Moreover, f_i^σ is invariant under the action of the group W^σ of σ-invariant elements of W. Hence, f_i^σ is an element of the polynomial ring $k[\Lambda^\sigma]^{W^\sigma}$ in l variables [Sp3, Proof of Cor. 2], generated (as a k-algebra) by the elements

$$\mathrm{sym}(\omega_{\mathcal{C}_i}) := \sum_{w\in W^\sigma} w\omega_{\mathcal{C}_i}, \quad i=1,\dots,l.$$

As in [Ste1, Proof of Lemma 6.3] we conclude that also f_i^σ, $i=1,\dots,l$, generates $k[\Lambda^\sigma]^{W^\sigma}$ as a k-algebra. Now apply [Sp3, Theorem 1]. □

Corollary 6.3. *The intersection of $Z(a_1,\dots,a_l)$ with the boundary $X_{{\rm sc},\sigma} - G_{{\rm sc},\sigma}$ of $X_{{\rm sc},\sigma}$ is independent of $a_1,\dots,a_l$. Moreover, the intersection $Z(a_1,\dots,a_l)\cap G_{{\rm sc},\sigma}$ is a single Steinberg fiber.*

Proof. As in the proof of Lemma 5.2 it follows that x is an element of $X_{{\rm sc},\sigma} - G_{{\rm sc},\sigma}$ exactly when the image $\psi_{\omega_{\mathcal{C}_i},\sigma}(x)$ is of the form $[(f,0)]$ for all $i=1,\dots,l$. Thus, the section f_{i,a_i}^σ coincides with $f_{i,0}^\sigma$ on the boundary of $X_{{\rm sc},\sigma}$. This proves the first statement. The latter statement follows by Proposition 6.2. □

7. Proofs of the main results

Lemma 7.1. *Let $J\subsetneq I$, $w\in W^{\sigma(J)}$, and $b\in B$. If $f_{i,0}^\sigma((bw,1)h_{J,\sigma})=0$, then: either (1) $w\omega_{\mathcal{C}_i}\neq\omega_{\mathcal{C}_i}$; or (2) $\mathcal{C}_i\subset J$ and $w\alpha_j=\alpha_j$ for all $j\in\mathcal{C}_i$.*

Proof. Assume that $w\omega_{\mathcal{C}_i}=\omega_{\mathcal{C}_i}$. Then the diagonal entry of a representative A of $\rho_{\omega_{\mathcal{C}_i},\sigma}((bw,1)h_{J,\sigma})$ associated to the lowest weight space is nonzero. In particular, the relation $f_{i,0}^\sigma((bw,1)h_{J,\sigma})=0$ cannot be satisfied unless there exists a weight $x-\omega_{\mathcal{C}_i}\neq -\omega_{\mathcal{C}_i}$ of $\mathrm{H}(\omega_{\mathcal{C}_i})$ satisfying $x=\sum_{j\in J}a_j\alpha_j$, with $a_j\in\mathbb{N}\cup\{0\}$ and $w\sigma(x)=x$.

Let $K\subseteq J$ denote the set of $j\in J$ such that $a_j\neq 0$. As $x-\omega_{\mathcal{C}_i}$ is a weight of $\mathrm{H}(\omega_{\mathcal{C}_i})$ we know that $\mathcal{C}_i\cap K$ is nonempty. Now $\sum_{j\in K}a_j w\alpha_{\sigma(j)}=\sum_{j\in K}a_j\alpha_j$ and thus $\sum_{j\in K}a_j(\mathrm{ht}(w\alpha_{\sigma(j)})-\mathrm{ht}(\alpha_j))=0$. As $w\in W^{\sigma(J)}$ we conclude that $\mathrm{ht}(w\alpha_{\sigma(j)})\geqslant 1$ and, consequently, $w\alpha_{\sigma(j)}$ is a simple root for all $j\in K$. By the assumption $w\omega_{\mathcal{C}_i}=\omega_{\mathcal{C}_i}$ we know that $w\alpha_{\sigma(j)}=\alpha_{\sigma(j)}$ for each $j\in\mathcal{C}_i\cap K$. In particular, when $j\in\mathcal{C}_i\cap K$, then $a_{\sigma(j)}=a_j$. Hence, $\mathcal{C}_i\cap K$ is invariant under σ and as $\mathcal{C}_i$ is a single σ-orbit we have $\mathcal{C}_i\cap K=\mathcal{C}_i$. This ends the proof. □

Lemma 7.2. *Let $J\subsetneq I$. Then*

$$Z(a_1,\dots,a_l)\cap Z_{J,\sigma} = \bigsqcup_{\substack{w\in W^{\sigma(J)} \\ \mathrm{supp}_\sigma(w)=I}} Z_{J,\sigma}^w.$$

Proof. By Corollary 6.3 it suffices to consider the case when all a_i are zero. By Proposition 4.4,

$$\bigsqcup_{J\subset I}\ \bigsqcup_{\substack{w\in W^{\sigma(J)}\\ \mathrm{supp}_\sigma(w)=I}} Z^w_{J,\sigma}=\bigcap_i \mathcal{N}(\omega_{\mathcal{C}_i})_\sigma \subset Z(0,\dots,0).$$

This proves one inclusion. For $z\in Z(0,\dots,0)\cap Z_{J,\sigma}$, we have that $z=(g,g)(bw,1)h_{J,\sigma}$ for some $g\in G, b\in B$, and $w\in W^{\sigma(J)}$. Then $f^\sigma_{i,0}((bw,1)h_{J,\sigma})=0$ for all $i=1,\dots,l$. It suffices to prove that $\mathrm{supp}_\sigma(w)=I$.

If $w=1$, then, by Lemma 7.1, $\mathcal{C}_i\subset J$ for each σ-orbit $\mathcal{C}_i$. Thus $I=J$, which contradicts our assumption. Now assume that $w\neq 1$ and that $\mathrm{supp}_\sigma(w)\neq I$. As G is assumed to be simple, there exist simple roots α_i and α_j with $n=-\langle\alpha_j,\alpha_i^\vee\rangle\neq 0$ satisfying that $i\in\mathrm{supp}_\sigma(w)$ and $j\notin\mathrm{supp}_\sigma(w)$. Let $\mathcal{C}_i$ and $\mathcal{C}_j$ denote the associated σ-orbits of α_i and α_j. As $\mathrm{supp}_\sigma(w)$ is σ-stable it follows that $\mathcal{C}_i\subset\mathrm{supp}_\sigma(w)$ and $\mathcal{C}_j\subset I-\mathrm{supp}_\sigma(w)$.

Now there exists $m\in\mathbb{N}$, such that $\sigma^m(i)\in\mathrm{supp}(w)$ and thus $w\omega_{\sigma^m(i)}\neq\omega_{\sigma^m(i)}$. Hence redefining, if necessary, α_i and α_j, we may assume that $w\omega_i\neq\omega_i$. Consider then the relation $\alpha_j=2\omega_j-n\omega_i-\lambda$ with λ denoting a dominant weight. Then Lemma 7.1, applied to $\mathcal{C}_j$, implies that $w\alpha_j=\alpha_j$ and $w\omega_j=\omega_j$ and thus $w(n\omega_i+\lambda)=n\omega_i+\lambda$. As both ω_i and λ are dominant we conclude that $w\omega_i=\omega_i$, which is a contradiction. □

Now we will prove the main theorem.

Theorem 7.3. *Let F be a Steinberg fiber of G_σ and $\overline{F}$ its closure in X_σ. Then*

$$\overline{F}-F=\bigsqcup_{J\subset I}\ \bigsqcup_{\substack{w\in W^{\sigma(J)}\\ \mathrm{supp}_\sigma(w)=I}} Z^w_{J,\sigma},$$

which also coincides with the set $Z(a_1,\dots,a_l)\cap(X_{\mathrm{sc},\sigma}-G_{\mathrm{sc},\sigma})$ for all $a_1,\dots,a_l$.

Proof. Let C be an irreducible component of $Z(a_1,\dots,a_l)$. Then by Krull's principal ideal theorem, $\dim(C)\geqslant\dim(G_{\mathrm{sc}})-l$. By [L2, 8.20],

$$\dim(Z^w_{J,\sigma})=\dim(G)-l(w)-|I-J|<\dim(G_{\mathrm{sc}})-l$$

for $J\neq I$ and $w\in W^{\sigma(J)}$ with $\mathrm{supp}_\sigma(w)=I$. Thus, by Lemma 7.2,

$$\dim\big(C\cap(X_{\mathrm{sc},\sigma}-G_{\mathrm{sc},\sigma})\big)<\dim(G_{\mathrm{sc}})-l\leqslant\dim(C).$$

Hence $C\cap G_{\mathrm{sc},\sigma}$ is dense in C. But, by Corollary 6.3, the intersection $Z(a_1,\dots,a_l)\cap G_{\mathrm{sc},\sigma}$ is a single Steinberg fiber $F(a_1,\dots,a_n)$ which, as in Section 3.2, is irreducible. We conclude that C is contained in the closure of $F(a_1,\dots,a_l)$, and thus the closure of $F(a_1,\dots,a_l)$ is $Z(a_1,\dots,a_l)$. In particular, $Z(a_1,\dots,a_l)$ is irreducible.

Let F be a Steinberg fiber of G_σ. Then $F=\pi(F(a_1,\dots,a_l))$ for some $a_1,\dots,a_l\in k$. Hence $\overline{F}$ equals to the closure of $\pi(Z(a_1,\dots,a_l))$. The statement now follows from Lemmas 7.2 and 5.2. □

Remark. 1. We call an element $w \in W$ a σ-twisted Coxeter element if $l(w) = l$ and $\mathrm{supp}_\sigma(w) = I$. (The notation of twisted Coxeter elements was first introduced by Springer in [Sp1]. Our definition is slightly different from his.) It follows easily from Theorem 7.3 that $\overline{Z^w_{I-\{i\},\sigma}}$ are the irreducible components of $\overline{F} - F$, where $i \in I$ and w runs over all σ-twisted Coxeter elements that are contained in $W^{I-\{\sigma(i)\}}$.

2. By the proof of Theorem 7.3 we may also deduce that the closure of a Steinberg fiber F within $X_{\mathrm{sc},\sigma}$ coincides with $Z(a_1, \dots, a_l)$ for certain uniquely determined $a_1, \dots, a_l$ depending on F. This result may be considered as an extension of Corollary 2 in [Sp3] to the compactification $X_{\mathrm{sc},\sigma}$ of $G_{\mathrm{sc},\sigma}$. More precisely, notice that the statement of [Sp3, Cor. 2] is equivalent to saying that a Steinberg fiber F of $G_{\mathrm{sc},\sigma}$ is the common zero set of the functions f^σ_{i,a_i} for uniquely determined $a_1, \dots, a_l$. Here we think of f^σ_{i,a_i} as regular functions on $G_{\mathrm{sc},\sigma}$ as explained in Section 6.1. When generalizing to $X_{\mathrm{sc},\sigma}$ the only difference is that we have to regard $\overline{f}^\sigma_{i,a_i}$ as sections of certain line bundles on $X_{\mathrm{sc},\sigma}$.

Similar to [H1, 4.6], we have the following consequence.

Corollary 7.4. *Assume that G_σ is defined and split over $\mathbb{F}_q$, then for any Steinberg fiber F of G_σ, the number of $\mathbb{F}_q$-rational points of $\overline{F} - F$ is*

$$\left(\sum_{w \in W} q^{l(w)}\right)\left(\sum_{\mathrm{supp}_\sigma(w)=I} q^{l(w_0 w)+L(w_0 w)}\right),$$

where w_0 is the maximal element of W and, for $w \in W$, $l(w)$ is its length and $L(w)$ is the number of simple roots α satisfying $w\alpha < 0$.

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