ON THE AFFINENESS OF DELIGNE-LUSZTIG VARIETIES

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Abstract. We prove that the Deligne-Lusztig variety associated to minimal length elements in any δ-conjugacy class of the Weyl group is affine, which was conjectured by Orlik and Rapoport in [10].

1.1 Notations. Let $\mathbf{k}$ be an algebraic closure of the finite prime field $\mathbb{F}_p$ and G be a connected reductive algebraic group over $\mathbf{k}$ with an endomorphism $F : G \to G$ such that some power F^d of F is the Frobenius endomorphism relative to a rational structure over a finite subfield $\mathbf{k}_0$ of $\mathbf{k}$. Let q be the positive number with $q^d = |\mathbf{k}_0|$.

We fix a F-stable Borel subgroup B and a F-stable maximal torus $T \subset B$. Let Φ be the set of roots and $(\alpha_i)_{i \in I}$ be the set of simple roots corresponding to (B, T). For $i \in I$, let $\omega_i^\vee$ be the corresponding fundamental coweight. Let $W = N(T)/T$ be the Weyl group and $(s_i)_{i \in I}$ be the set of simple reflections. For $w \in W$, let $l(w)$ be the length of w. Since (B, T) is F-stable, F induces a bijection on I and an automorphism on W. We denote the induced maps on W and I by δ. Now δ also induces isomorphisms on the set of characters $X = \mathrm{Hom}(T, G_m)$ and the set of cocharacters $X^\vee = \mathrm{Hom}(G_m, T)$ which we also denote by δ. Then it is easy to see that $F^* \mu = q \cdot \delta^{-1}(\mu)$ for $\mu \in X^\vee$.

For $J \subset I$, let Φ_J be the set of roots generated by $\{\alpha_j\}_{j \in J}$ and W_J be the subgroup of W generated by $\{s_j\}_{j \in J}$. Let W^J be the set of minimal length coset representatives for W/W_J. The unique maximal element in W will be denoted by w_0 and the unique maximal element in W_J will be denoted by w_0^J.

Let $\alpha_0 = \sum_{i \in I} n_i \alpha_i$ be the highest root and $n_0 = \sum_{i \in I} n_i$.

1.2. Let $\mathcal{B}$ be the set of Borel subgroups of G. For $w \in W$, let $O(w) = \{({}^g B, {}^{g \dot{w}} B); g \in G\}$ be the G-orbit on $\mathcal{B} \times \mathcal{B}$ that corresponding to w. Set

$$X(w) = \{B' \in \mathcal{B}; (B', F(B')) \in O(w)\}.$$

This is the Deligne-Lusztig variety associated to w (see [2, 1.4]). It is known that $X(w)$ is a variety of pure dimension $l(w)$ (see *loc.cit.*) and

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is quasi-affine (see [7]). It is also known that when $q \geqslant h$ (where h is the Coxeter number), then $X(w)$ is affine (see [2, Theorem 9.7]).

The main result we will prove in this note is the following

Theorem 1.3. *Let $w \in W$ be a minimal length element in the δ-conjugacy class $\{xw\delta(x)^{-1}; x \in W\}$. Then $X(w)$ is affine.*

Remark. *The case where w is a Coxeter element was proved by Lusztig in* [9, Corollary 2.8] *in a geometric way and the cases for split classical groups were proved by Orlik and Rapoport in* [10, section 5] *by finding a minimal length element in each δ-conjugacy that satisfies the criterion* [2, Theorem 9.7]. *Our approach is motivated by the approach of Orlik and Rapoport. However, a main difference is the way of choosing the minimal length elements. We will discuss it in more detail in 1.14.*

Before discussing the proof of the theorem above, we first recall some results on the minimal length elements.

1.4. We follow the notations in [6, section 3.2].

Let $w, w' \in W$ and $j \in I$, we write $w \xrightarrow{s_j}_\delta w'$ if $w' = s_j w \delta(s_j)$ and $l(w') \leqslant l(w)$. If $w = w_0, w_1, \cdots, w_n = w'$ is a sequence of elements in W such that for all k, we have $w_{k-1} \xrightarrow{s_j}_\delta w_k$ for some $j \in I$, then we write $w \to_\delta w'$.

We call $w, w' \in W$ *elementarily strongly δ-conjugate* if $l(w) = l(w')$ and there exists $x \in W$ such that $w' = xw\delta(x)^{-1}$ and $l(xw) = l(x) + l(w)$ or $l(w\delta(x)^{-1}) = l(x) + l(w)$. We call w, w' *strongly δ-conjugate* if there is a sequence $w = w_0, w_1, \cdots, w_n = w'$ such that w_{i-1} is elementarily strongly δ-conjugate to w_i. We will write $w \sim_\delta w'$ if w and w' are strongly δ-conjugate.

If $w \sim_\delta w'$ and $w \to_\delta w'$, then we say that w and w' are in the same δ-cyclic shift class and write $w \approx_\delta w'$. For $w \in W$, set

$$\mathrm{Cyc}_\delta(w) = \{w' \in W; w \approx_\delta w'\}.$$

The following result was proved in [5, Theorem 1.1] for the usual conjugacy classes and in [4, Theorem 2.6] for the twisted conjugacy classes.

Theorem 1.5. *Let $\mathcal{O}$ be a δ-conjugacy class in W and $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then*

(1) For each $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \to_\delta w'$.

(2) Let $w, w' \in \mathcal{O}_{\min}$, then $w \sim_\delta w'$.

1.6. In general, $\mathcal{O}_{\min}$ might be a union of several δ-cyclic shift classes. However, for some special δ-conjugacy classes, we have a better result. Let us first introduce some notations.

For $w \in W$, set $\mathrm{supp}_\delta(w) = \cup_{n \geqslant 0} \delta^n \mathrm{supp}(w)$. Then $\mathrm{supp}_\delta(w)$ is the minimal δ-stable subset of I such that $w \in W_{\mathrm{supp}_\delta(w)}$.

A δ-conjugacy class $\mathcal{O}$ of W is called *cuspidal* if $\mathcal{O} \cap W_J = \varnothing$ for all proper δ-stable subset J of I.

The following result was proved in [6, Theorem 3.2.7] for the usual conjugacy classes, in [4, section 6] for twisted conjugacy classes of exceptional groups and in [8, Theorem 7.5] for twisted conjugacy classes of classical groups.

Theorem 1.7. *Let $\mathcal{O}$ be a δ-conjugacy class and $w \in \mathcal{O}_{\min}$. Then*
(1) If $\text{supp}_\delta(w) = I$, then $\mathcal{O}$ is cuspidal.
(2) If $\mathcal{O}$ is cuspidal, then $\mathcal{O}_{\min} = \text{Cyc}_\delta(w)$.

1.8. By [10, Lemma 2.6], if $w \approx_\delta w'$, then $X(w)$ and $X(w')$ are universally homeomorphic. In particular, if $X(w)$ is affine, then $X(w')$ is also affine for any $w' \approx_\delta w$. (However, it is unknown if the same result holds when $w' \sim_\delta w$.)

By [2, Theorem 9.7], to prove our main theorem, it suffices to prove the following result.

Proposition 1.9. *Let $C = \{\mu \in X^\vee \otimes \mathbb{R}; \alpha_i(\mu) > 0 \text{ for } i \in I\}$ be the fundamental chamber corresponding to B. Then for any δ-conjugacy class $\mathcal{O}$ of W and $w' \in \mathcal{O}_{\min}$, there exists $w \approx_\delta w'$ and $\mu \in X^\vee \otimes \mathbb{R}$ such that $\alpha(\mu) > 0$ for $\alpha > 0$ with $w\alpha < 0$ and $F^*\mu - w \cdot \mu \in C$.*

1.10 Reduction to cuspidal classes. Assume that the Proposition 1.9 holds in the case where $\mathcal{O}$ is cuspidal. We will prove now that it holds in general.

Let $\mathcal{O}$ be an δ-conjugacy class of W and $w' \in \mathcal{O}_{\min}$. Let $J = \text{supp}_\delta(w')$ and $\mathcal{O}'$ be the δ-conjugacy class of W_J that contains w'. Then $w' \in \mathcal{O}'_{\min}$ and $\mathcal{O}'_{\min} \subset \mathcal{O}_{\min}$. By Theorem 1.7 (1), $\mathcal{O}'$ is a cuspidal δ-conjugacy class of W_J. Notice that if $x \in W_J$ and $\alpha \in \Phi$ with $\alpha > 0$ and $x\alpha < 0$, then $\alpha \in \Phi_J$. By our hypothesis, there exist $w \in \mathcal{O}'_{\min}$ and $\mu = \sum_{i \in J} m_i \omega_i^\vee$ for some $m_i \in \mathbb{R}$ such that $\alpha(\mu) > 0$ for $\alpha > 0$ with $w\alpha < 0$ and $\alpha_i\big(F^*\mu - w \cdot \mu\big) > 0$ for all $i \in J$. Set $\lambda = \mu + m \sum_{i \notin J} \omega_i^\vee$ for $m \gg 0$. Then $\alpha(\lambda) = \alpha(\mu)$ for $\alpha > 0$ with $w\alpha < 0$.

If $i \in J$, then $\alpha_i\big(F^*\lambda - w \cdot \lambda\big) = \alpha_i\big(F^*\mu - w \cdot \mu\big) > 0$.

If $i \notin J$, then $w^{-1}(\alpha_i) = \alpha_i + \sum_{j \in J} a_j \alpha_j$ for some $a_j \in \mathbb{N}$ with $\sum_{j \in J} a_j \leqslant n_0$. Hence

$$\alpha_i\big(F^*\lambda - w \cdot \lambda\big) = qm - w^{-1}(\alpha_i)(\lambda) > (q-1)m - n_0 \max_{i \in J} |m_i| > 0.$$

Therefore, to prove Proposition 1.9, it suffices to prove the following statement.

Lemma 1.11. *For any cuspdial δ-conjugacy class $\mathcal{O}$ of W, there exists $w \in \mathcal{O}_{\min}$ and $\mu \in X^\vee \otimes \mathbb{R}$ such that $\alpha(\mu) > 0$ for $\alpha > 0$ with $w\alpha < 0$ and $F^*\mu - w \cdot \mu \in C$.*

1.12 Reduction to irreducible types. Assume that Lemma 1.11 holds in the case where (Φ, I) is irreducible. We will prove now that it holds in general.

Step 1. Assume that $\Phi = \Phi_1 \sqcup \Phi_2 \sqcup \cdots \sqcup \Phi_r$, where each Φ_i is irreducible and generated by $I_i = I \cap \Phi_i$ such that $\delta(I_i) = I_{i+1}$ for $i < r$ and $\delta(I_r) = I_1$. Then $W = W_1 \times W_2 \times \cdots \times W_r$ and we may regard W_1 as a subgroup of W in the natural way. In this case, $\mathcal{O} \cap W_1$ is a cuspidal δ^r-conjugacy class of W_1. Notice that if $x \in W_1$ and $\alpha \in \Phi$ with $\alpha > 0$ and $x\alpha < 0$, then $\alpha \in \Phi_1$. By assumption, there exists $w \in \mathcal{O}_{\min} \cap W_1 = (\mathcal{O} \cap W_1)_{\min}$ and $\mu = \sum_{i \in I_1} m_i \omega_i^\vee$ for some $m_i \in \mathbb{R}$ such that $\alpha(\mu) > 0$ for $\alpha > 0$ with $w\alpha < 0$ and $\alpha_i\big((F^*)^r \mu - w \cdot \mu\big) > 0$ for all $i \in I_1$. Therefore, there exists $\epsilon_i > 0$ with $(1 - \epsilon_i) \operatorname{sgn}(m_i) > 0$ for all $i \in I_1$ such that $\alpha_i\big(\sum_{j \in I_1} (F^*)^r \epsilon_j^{r-1} m_j \omega_j^\vee - w \cdot \mu\big) > 0$ for all $i \in I_1$. Now set $\lambda = \sum_{i \in I_1} \sum_{k=0}^{r-1} \epsilon_i^k (F^*)^k m_i \omega_i^\vee$. Then $\alpha(\lambda) = \alpha(\mu) > 0$ for all $\alpha > 0$ with $w\alpha < 0$. Moreover, for $i \in I_1$ and $0 \leqslant n < r$,

$$\alpha_{\delta^{-n} i}(F^* \lambda - w \cdot \lambda) = \begin{cases} \alpha_{\delta^{-n} i}(F^* \lambda - \lambda) = q^n m_i \epsilon_i^{n-1}(1 - \epsilon_i), & \text{if } n \neq 0; \\ \alpha_i\big(\sum_{j \in I_1} (F^*)^r \epsilon_j^{r-1} m_j \omega_j^\vee - w \cdot \mu\big), & \text{if } n = 0. \end{cases}$$

Therefore, $\alpha_{\delta^{-n} i}(F^* \lambda - w \cdot \lambda) > 0$ for all $i \in I$ and $0 \leqslant n < r$.

Step 2. Assume that $\Phi = \Phi_1 \sqcup \Phi_2$, where Φ_k is generated by $I_k = I \cap \Phi_k$ and $\delta(I_k) = I_k$ for $k = 1, 2$. Then $W = W_1 \times W_2$ and $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$, where $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) is a cuspidal δ-conjugacy class of W_1 (resp. W_2). By assumption, there exists $w_k \in (\mathcal{O}_k)_{\min}$ and $\mu_k = \sum_{i \in I_i} m_i \omega_i^\vee$ for some $m_i \in \mathbb{R}$ such that $\alpha(\mu_k) > 0$ for $\alpha > 0$ with $w_k \alpha < 0$ and $\alpha_i(F^* \mu - w_k \cdot \mu) > 0$ for all $i \in I_k$. Now set $w = (w_1, w_2) \in \mathcal{O}_{\min}$ and $\lambda = \mu_1 + \mu_2$. Then $\alpha(\lambda) = \alpha(\mu_k) > 0$ for $\alpha \in \Phi_k$ with $\alpha > 0$ and $w\alpha < 0$. Also $\alpha_i\big(F^* \lambda - w \cdot \lambda\big) = \alpha_i\big(F^* \mu_k - w_k \cdot \mu_k\big) > 0$ for $i \in I_k$.

Step 3. Now we consider the general case. Here $\Phi = \sqcup \Phi^i$ and $\Phi^i = \sqcup \Phi^i_j$, where each Φ^i_j is irreducible and generated by $I^i_j = I \cap \Phi^i_j$ and for each i, δ permutes $\{I^i_j\}$ cyclically. So we may apply Step 1 to each Φ^i and then apply Step 2 to Φ. It is easy to see that Lemma 1.11 holds in general.

1.13 Reduction to the condition (J, w_1)**.** It is easy to see that the map $w \mapsto w^{-1}$ sends an δ-conjugacy class $\mathcal{O}$ to a δ^{-1}-conjugacy class $\mathcal{O}^*$. If $\mathcal{O}$ is cuspidal, then so is $\mathcal{O}^*$. If $w \in \mathcal{O}_{\min}$, then $w^{-1} \in \mathcal{O}^*_{\min}$. We will prove the following variant of Lemma 1.11.

(a). Let $\mathcal{O}$ be a cuspidal δ^{-1}-conjugacy class. Then there exists $w \in \mathcal{O}_{\min}$ and $\mu \in X^\vee \otimes \mathbb{R}$ such that $\alpha(\mu) > 0$ for $\alpha > 0$ with $w^{-1}\alpha < 0$ and $q\alpha_i(\mu) - (w\alpha_{\delta^{-1}(i)})(\mu) > 0$ for all $i \in I$.

In fact, we will show that for most of the cases,

(b). there exists $w \in \mathcal{O}_{\min}$ and $\mu \in C$ such that

$$q\alpha_i(\mu) - (w\alpha_{\delta^{-1}(i)})(\mu) > 0, \qquad \text{for all } i \in I.$$

The idea is as follows.

For $J \subset I$ and $x \in W^{\delta^{-1}(J)}$, set

$$I(J, x, \delta^{-1}) = \max\{K \subset J; \mathrm{Ad}(x)\delta^{-1}(K) = K\}.$$

(In fact, if $K_1, K_2 \subset J$ with $\mathrm{Ad}(x)\delta^{-1}(K_i) = K_i$ for $i = 1, 2$, then $\mathrm{Ad}(x)\delta^{-1}(K_1 \cup K_2) = K_1 \cup K_2$. Thus $\{K \subset J; \mathrm{Ad}(x)\delta^{-1}(K) = K\}$ contains a unique maximal element.)

By an observation in [8, section 7], for each cuspidal δ^{-1}-conjugacy class $\mathcal{O}$ in a Weyl group of classical type, there exists a maximal subset $J \subsetneq I$, $w_1 \in W^{\delta^{-1}(J)}$ and a cuspidal $\mathrm{Ad}(w_1)\delta^{-1}$-conjugacy class $\mathcal{O}'$ in $W_{I(J,w_1,\delta^{-1})}$ such that for any $v \in \mathcal{O}'_{\min}$, $vw_1 \in \mathcal{O}_{\min}$. We will see later in 1.15 that the observation is also valid for exceptional groups.

The following condition plays an essential role in our proof.

Condition (J, w_1): there exists $m_i \in \mathbb{R}_{>0}$ for $i \notin I(J, w_1, \delta^{-1})$ such that

$$(*) \qquad q\alpha_i(\sum_{j \notin I(J,w_1,\delta^{-1})} m_j \omega_j^\vee) - (w_1\alpha_{\delta^{-1}(i)})(\sum_{j \notin I(J,w_1,\delta^{-1})} m_j \omega_j^\vee) > 0$$

for $i \notin I(J, w_1, \delta^{-1})$.

Claim. *Keep notations as above. Suppose that the condition (J, w_1) is true and that 1.13 (b) holds for $(\Phi_{I(J,w_1,\delta^{-1})}, \mathrm{Ad}(w_1)\delta^{-1}, \mathcal{O}')$. Then 1.13(b) holds for $(\Phi_I, \delta^{-1}, \mathcal{O})$.*

We simply write K for $I(J, w_1, \delta^{-1})$. By our assumption, there exists $v \in \mathcal{O}'_{\min}$ and $m_i \in \mathbb{R}_{>0}$ for $i \in I$ such that for $i \in K$,

$$\begin{aligned} &q\alpha_i(\sum_{j \in K} m_j \omega_j^\vee) - (v\alpha_{w_1\delta^{-1}(i)})(\sum_{j \in K} m_j \omega_j^\vee) \\ &= q\alpha_i(\sum_{j \in K} m_j \omega_j^\vee) - (vw_1\alpha_{\delta^{-1}(i)})(\sum_{j \in K} m_j \omega_j^\vee) > 0. \end{aligned}$$

and for $i \notin K$,

$$q\alpha_i(\sum_{j \notin K} m_j \omega_j^\vee) - (w_1\alpha_{\delta^{-1}(i)})(\sum_{j \notin K} m_j \omega_j^\vee) > 0.$$

Set $w = vw_1$ and $\lambda = \sum_{j \in K} m_j \omega_j^\vee + m \sum_{j \notin K} m_j \omega_j^\vee$ for $m \gg 0$. Notice that $w_1 \Phi_{\delta^{-1}(K)} = \Phi_K$. Thus for any $i \notin K$, $w_1\alpha_{\delta^{-1}(i)} \notin \Phi_K$ and $w\alpha_{\delta^{-1}(i)} = vw_1\alpha_{\delta^{-1}(i)} = w_1\alpha_{\delta^{-1}(i)} + \sum_{j \in K} a_j \alpha_j$ for $a_j \in \mathbb{Z}$ with $\sum_{j \in K} |a_j| \leqslant n_0$. Now for $i \in K$,

$$q\alpha_i(\lambda) - (w\alpha_{\delta^{-1}(i)})(\lambda) = q\alpha_i(\sum_{j \in K} m_j \omega_j^\vee) - (vw\alpha_{\delta^{-1}(i)})(\sum_{j \in K} m_j \omega_j^\vee) > 0$$

and for $i \notin K$,

$$
\begin{aligned}
& q\alpha_i(\lambda) - (w\alpha_{\delta^{-1}(i)})(\lambda) \\
& = m\big(q\alpha_i(\sum_{j \notin K} m_j \omega_j^\vee) - (w_1\alpha_{\delta^{-1}(i)})(\sum_{j \notin K} m_j \omega_j^\vee)\big) - \sum_{j \in K} a_j m_j \\
& \geqslant m\big(q\alpha_i(\sum_{j \notin K} m_j \omega_j^\vee) - (w_1\alpha_{\delta^{-1}(i)})(\sum_{j \notin K} m_j \omega_j^\vee)\big) - n_0 \max_{j \in K} m_j > 0.
\end{aligned}
$$

Therefore the 1.13(b) holds for $(\Phi_I, \delta^{-1}, \mathcal{O})$.

1.14. We will show below that except the cases labelled with ♠ (case 12 for type E_8, case 3 for type F_4, case 1 for type G_2, case 2 and case 4 for type ${}^2F_4$), the condition (J, w_1) are satisfied. Hence by induction on $|I|$, we can show that 1.13(b) holds for these cases. For the cases labelled with ♠, we will prove 1.13(a) instead.

In [10, Lemma 5.4 & Lemma 5.7], Orlik and Rapoport checked the condition (J, w_1^{-1}) for type A_n and B_n. In fact, in the case where $\delta = id$, the condition (J, w_1^{-1}) plays the same role in the proof of Lemma 1.11 as the condition (J, w_1) does in the proof of 1.13(a). However, there are some big difference between the condition (J, w_1) and the condition (J, w_1^{-1}). In fact, there are more cases in the exceptional groups in which the condition (J, w_1^{-1}) is not satisfied and for some of these cases, Lemma 1.11 is not easy to check directly. This is the reason why we prove 1.13(a) and the condition (J, w_1) instead of Lemma 1.11 and the condition (J, w_1^{-1}).

1.15. We use the same labelling of Dynkin diagram as in [1]. We will use the same list of representatives of minimal length elements for all the cuspidal δ^{-1}-conjugacy classes for the classical groups as in [8, 7.12-7.22]. For the exceptional groups, we will also list a representative of minimal length elements for each cuspidal δ^{-1}-conjugacy class. The representatives are presented as vw_1 for $w_1 \in W^{\delta^{-1}(J)}$ and v is a minimal length element in the $\mathrm{Ad}(w_1)\delta^{-1}$-conjugacy class of $W_{I(J,w,\delta^{-1})}$ that contains v. (These representatives are obtained by direct calculation based on the tables in [6, Appendix B] and [4, section 6]).

Set

$$
s_{[a,b]} = \begin{cases} s_a s_{a-1} \cdots s_b, & \text{if } a \geqslant b; \\ 0, & \text{otherwise }. \end{cases}
$$

Type A_n

Set $J = I - \{1\}$. Here $w_1 = s_{[n,1]}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequality $(*)$ is just $qm_i - m_{i-1} > 0$ for $i \neq 1$. So we may take $m_i = 1$ for all i.

Type ${}^2A_n$

Set $J = I - \{n\}$. Here $w_1 = s_{[n+1-a,1]}$ for some $a \leqslant \frac{n}{2} + 1$ and $I(J, w_1, \delta^{-1}) = \{a, a+1, \cdots, n-a\}$. The inequalities $(*)$ are just $qm_i - m_{n+1-i} > 0$ for $i < a-1$, $qm_{a-1} - m_{n+1-a} - m_{n+2-a} > 0$ and $qm_i - m_{n-i} > 0$ for $n - a < i < n$. So we may take

$$m_i = \begin{cases} 2, & \text{if } i = a-1 \text{ or } n+1-a; \\ 1 & \text{otherwise.} \end{cases}$$

Type B_n and C_n

Set $J = I - \{1\}$.

Case 1. $w_1 = s_{[n-1,a]}^{-1} s_{[n,1]}$ for some $1 \leqslant a < n$ and $I(J, w_1, \delta^{-1}) = \{a+1, a+2, \cdots, n\}$. The inequalities $(*)$ are just $qm_i - m_{i-1} > 0$ for $1 < i < a$ and $qm_a - m_{a-1} - m_a > 0$. So we may take $m_i = 1$ for $i < a$ and $m_a = 2$.

Case 2. $w_1 = s_{[n,1]}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_i - m_{i-1} > 0$ for $1 < i < n$ and $qm_n - m_n - \epsilon m_{n-1} > 0$, where

$$\epsilon = \begin{cases} 1, & \text{type } B_n; \\ 2, & \text{type } C_n. \end{cases}$$

So we may take $m_n = 3$ and $m_i = 1$ for $i < n$.

Type D_n and ${}^2D_n$

Set $J = I - \{\delta(1)\}$.

Case 1. $w_1 = s_{[n-2,a]}^{-1} s_{[n,1]}$ for some $a \leqslant n-2$ and $I(J, w_1, \delta^{-1}) = \{a+1, a+2, \cdots, n\}$. The inequalities $(*)$ are just $qm_i - m_{i-1} > 0$ for $1 < i < a$ and $qm_a - m_{a-1} - m_a > 0$. So we may take $m_a = 1$ for $i < a$ and $m_a = 2$.

Case 2. $w_1 = s_{[n,1]}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_i - m_{i-1} > 0$ for $1 < i \leqslant n-2$, $qm_{\delta(n-1)} - m_{n-2} - m_n > 0$ and $qm_{\delta(n)} - m_{n-2} - m_{n-1} > 0$. So we may take $m_{n-1} = m_n = 2$ and $m_i = 1$ for $i \leqslant n-2$.

Case 3. $w_1 = s_{[n-1,1]}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_i - m_{i-1} > 0$ for $1 < i \leqslant n-2$, $qm_{\delta(n-1)} - m_{n-2} > 0$ and $qm_{\delta(n)} - m_{n-2} - m_{n-1} - m_n > 0$. So we may take $m_{\delta(n)} = 3$ and $m_i = 1$ for $i \neq \delta(n)$.

Type ${}^3D_4$

Here $\delta^{-1}(s_1) = s_3$, $\delta^{-1}(s_3) = s_4$ and $\delta^{-1}(s_4) = s_1$. Set $J = I - \{4\}$.

Case 1. $w_1 = s_2 s_1$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_1 - m_2 - m_3 > 0$, $qm_2 - m_1 > 0$ and $qm_3 - m_2 - m_4 > 0$. So we may take $(m_1, m_2, m_3, m_4) = (3, 2, 2, 1)$.

Case 2. $w_1 = s_{[3,1]}$ and $I(J, w_1, \delta^{-1}) = \{1, 2\}$. The inequality $(*)$ is just $qm_3 - m_3 - m_4 > 0$. So we may take $m_3 = 2$ and $m_4 = 1$.

Case 3. $w_1 = s_1 s_2 s_{[4,1]}$ and $I(J, w_1, \delta^{-1}) = \{2, 3\}$. The inequality $(*)$ is just $qm_1 - m_4 > 0$. So we may take $m_1 = m_4 = 1$.

Type E_6

Set $J = I - \{6\}$.

Case 1. $w_1 = s_{[6,1]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_1 - m_3 > 0$, $qm_2 - m_1 - m_3 - m_4 > 0$, $qm_3 - m_2 - m_4 > 0$, $qm_4 - m_5 > 0$ and $qm_5 - m_6 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6) = (2, 4, 3, 1, 1, 1)$.

Case 2. $w_1 = s_3 s_4 s_{[6,1]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$.. The inequalities $(*)$ are just $qm_1 - m_4 > 0$, $qm_2 - m_1 > 0$, $qm_3 - m_2 > 0$, $qm_4 - m_3 - m_4 - m_5 > 0$ and $qm_5 - m_6 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6) = (5, 3, 2, 9, 1, 1)$.

Case 3. $w_1 = s_2 s_4 s_5 s_3 s_4 s_{[6,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{3, 4\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $v = s_3$ or $s_3 s_4 s_3$. The inequalities $(*)$ are just $qm_1 - m_5 > 0$, $qm_2 - m_1 > 0$ and $qm_5 - m_1 - m_5 - m_6 > 0$. So we may take $(m_1, m_2, m_5, m_6) = (3, 2, 5, 1)$.

Case 4. $w_1 = w_0 w_0^J$, $I(J, w_1, \delta^{-1}) = \{2, 3, 4, 5\}$, $\mathrm{Ad}(w_1)$ is of order 3 on $I(J, w_1, \delta^{-1})$ and $\mathcal{O}'$ is cuspidal with $l(v) = 8$. The inequality $(*)$ is just $qm_1 - m_6 > 0$. So we may take $m_1 = m_6 = 1$.

Type ${}^2E_6$

Set $J = I - \{1\}$.

Case 1. $w_1 = s_2 s_{[6,4]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_2 - m_4 > 0$, $qm_3 - m_6 > 0$, $qm_4 - m_5 > 0$, $qm_5 - m_2 - m_3 - m_4 > 0$ and $qm_6 - m_1 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6) = (1, 2, 1, 3, 5, 1)$.

Case 2. $w_1 = s_4 s_{[6,2]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_2 - m_3 > 0$, $qm_3 - m_6 > 0$, $qm_4 - m_4 - m_5 > 0$, $qm_5 - m_2 > 0$ and $qm_6 - m_1 - m_3 - m_4 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6) = (1, 3, 5, 3, 2, 9)$.

Case 3. $w_1 = s_5 s_4 s_{[6,2]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{4\}$ and $v = s_4$. The inequalities $(*)$ are just $qm_2 - m_3 > 0$, $qm_3 - m_5 - m_6 > 0$, $qm_5 - m_2 > 0$ and $qm_6 - m_1 - m_3 - m_5 > 0$. So we may take $(m_1, m_2, m_3, m_5, m_6) = (1, 3, 5, 2, 7)$.

Case 4. $w_1 = s_{[6,4]} s_{[6,2]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{2, 3, 4, 5\}$, $\mathrm{Ad}(w_1)\delta^{-1}$ is of order 3 on $I(J, w_1, \delta^{-1})$, sending s_2 to s_3, s_3 to s_5 and s_5 to s_2 and $\mathcal{O}'$ is cuspidal with $l(v) = 4$ or 6. The inequality $(*)$ is just $qm_6 - m_1 - m_6 > 0$. So we may take $m_1 = 1$ and $m_6 = 2$.

Case 5. $w_1 = s_{[5,3]} s_{[6,4]} s_{[6,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{3, 4, 6\}$, $\mathrm{Ad}(w_1)\delta^{-1}$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $v = s_3 s_4 s_3 s_6$. The inequalities $(*)$ is just $qm_2 - m_1 > 0$. So we may take $(m_1, m_2, m_5) = (1, 1, 1)$.

Case 6. $w_1 = s_3 s_1 w_0 w_0^J$, $I(J, w_1, \delta^{-1}) = \{5, 6\}$, $\mathrm{Ad}(w_1)\delta^{-1}$ acts trivially on $I(J, w_1, \delta^{-1})$ and $v = s_5 s_6$. The inequalities $(*)$ are just $qm_2 - m_1 > 0$, $qm_3 - m_2 > 0$ and $qm_4 - m_3 - m_4 > 0$. So we may take $(m_1, m_2, m_3, m_4) = (1, 1, 1, 2)$.

Case 7. $w_1 = s_1 w_0 w_0^J$, $I(J, w_1, \delta^{-1}) = \{4, 5, 6\}$, $\mathrm{Ad}(w_1)\delta^{-1}$ acts trivially on $I(J, w_1, \delta^{-1})$ and $v = s_{[6,4]}$. The inequalities $(*)$ are just $qm_2 - m_1 - m_3 > 0$ and $qm_3 - m_2 > 0$. So we may take $m_1 = 1$ and $m_2 = m_3 = 2$.

Case 8. $w_1 = w_0 w_0^{\delta^{-1}(J)}$ and $v = w_0^{\delta^{-1}(J)}$. The inequalities $(*)$ are always satisfied.

Type E_7

Set $J = I - \{7\}$.

Case 1. $w_1 = s_{[7,1]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_1 - m_3 > 0$, $qm_2 - m_1 - m_3 - m_4 > 0$, $qm_3 - m_2 - m_4 > 0$, $qm_4 - m_5 > 0$, $qm_5 - m_6 > 0$ and $qm_6 - m_7 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = (2, 4, 3, 1, 1, 1, 1)$.

Case 2. $w_1 = s_3 s_4 s_{[7,1]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_1 - m_4 > 0$, $qm_2 - m_1 > 0$, $qm_3 - m_2 > 0$, $qm_4 - m_3 - -m_4 - m_5 > 0$, $qm_5 - m_6 > 0$ and $qm_6 - m_7 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = (5, 3, 2, 9, 1, 1, 1)$.

Case 3. $w_1 = s_4 s_3 s_5 s_4 s_{[7,1]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_1 - m_5 > 0$, $qm_2 - m_1 > 0$, $qm_3 - m_2 - m_4 > 0$, $2m_4 - m_3 > 0$, $qm_5 - m_4 - -m_5 - m_6 > 0$ and $qm_6 - m_7 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = (5, 3, 3, 2, 4, 1, 1)$.

Case 4. $w_1 = s_2 s_4 s_3 s_5 s_4 s_{[7,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{3, 4\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $v = s_3$ or $s_3 s_4 s_3$. The inequalities $(*)$ are just $qm_1 - m_5 > 0$, $qm_2 - m_1 > 0$, $qm_5 - m_2 - m_5 - m_6 > 0$ and $qm_6 - m_7 > 0$. So we may take $(m_1, m_2, m_5, m_6, m_7) = (3, 2, 5, 1, 1)$.

Case 5. $w_1 = s_3 s_4 s_2 s_{[5,3]} s_{[6,4]} s_{[7,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{4\}$ and $v = s_4$. The inequalities $(*)$ are just $qm_1 - m_6 > 0$, $qm_2 - m_1 - m_3 > 0$, $qm_3 - m_5 > 0$, $qm_5 - m_2 > 0$ and $qm_6 - m_3 - m_5 - m_6 - m_7 > 0$. So we may take $(m_1, m_2, m_3, m_5, m_6, m_7) = (7, 5, 2, 3, 7, 1)$.

Case 6. $w_1 = s_1 s_3 s_4 s_2 s_{[5,3]} s_{[6,4]} s_{[7,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{2, 3, 4, 5\}$, $\mathrm{Ad}(w_1)$ sends α_2 to α_3, α_3 to α_5, α_4 to α_4, α_5 to α_2 and $\mathcal{O}'$ is cuspidal with $l(v) = 4$, 6 or 8. The inequalities $(*)$ are just $qm_1 - m_6 > 0$ and $qm_6 - m_1 - m_6 - m_7 > 0$. So we may take $(m_1, m_6, m_7) = (3, 5, 1)$.

Case 7. $w_1 = s_2 s_4 s_3 s_5 s_4 s_2 s_{[6,3]} s_{[7,4]} s_{[1,7]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{3, 4, 5, 6\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $v = w_0^{I(J, w_1, \delta^{-1})}$. The inequalities $(*)$ are just $qm_1 - m_7 > 0$ and $qm_2 - m_1 - m_2 > 0$. So we may take $m_1 = m_7 = 1$ and $m_2 = 2$.

Case 8. $w_1 = s_{[6,4]} s_{[5,2]}^{-1} s_1 s_3 s_4 s_2 s_{[6,3]} s_{[7,4]} s_{[7,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{2, 3, 4, 5\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta^{-1})$ exchanging α_3 and α_5 and $v = s_3 s_5 s_4 s_3 s_5 s_4 s_2$. The inequalities $(*)$ are just $qm_1 - m_6 - m_7 > 0$ and $qm_6 - m_1 > 0$. So we may take $(m_1, m_6, m_7) = (2, 2, 1)$.

Case 9. $w_1 = w_0 w_0^J$ and $v = w_0^J$. The inequalities $(*)$ are always satisfied.

Type E_8

Set $J = I - \{8\}$.

Case 1. $w_1 = s_{[8,1]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_1 - m_3 > 0$, $qm_2 - m_1 - m_3 - m_4 > 0$, $qm_3 - m_2 - m_4 > 0$, $qm_4 - m_5 > 0$, $qm_5 - m_6 > 0$, $qm_6 - m_7 > 0$ and $qm_7 - m_8 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8) = (2, 3, 4, 1, 1, 1, 1, 1)$.

Case 2. $w_1 = s_3 s_4 s_{[8,1]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_1 - m_4 > 0$, $qm_2 - m_1 > 0$, $qm_3 - m_2 > 0$, $qm_4 - m_3 - m_4 - m_5 > 0$, $qm_5 - m_6 > 0$, $qm_6 - m_7 > 0$ and $qm_7 - m_8 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8) = (5, 3, 2, 9, 1, 1, 1, 1)$.

Case 3. $w_1 = s_4 s_5 s_3 s_4 s_{[8,1]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_1 - m_5 > 0$, $qm_2 - m_1 > 0$, $qm_3 - m_2 - m_4 > 0$, $qm_4 - m_3 > 0$, $qm_5 - m_4 - m_5 - m_6 > 0$, $qm_6 - m_7 > 0$, $qm_7 - m_8 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8) = (5, 3, 3, 2, 4, 1, 1, 1)$.

Case 4. $w_1 = s_2 s_4 s_3 s_5 s_4 s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{3, 4\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $v = s_3$ or $s_3 s_4 s_3$. The inequalities $(*)$ are just $qm_1 - m_5 > 0$, $qm_2 - m_1 > 0$, $qm_5 - m_2 - m_5 - m_6 > 0$, $qm_6 - m_7 > 0$ and $qm_7 - m_8 > 0$. So we may take $(m_1, m_2, m_5, m_6, m_7, m_8) = (3, 2, 5, 1, 1, 1)$.

Case 5. $w_1 = s_4 s_2 s_{[5,3]} s_{[6,4]} s_{[8,1]}^{-1}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_1 - m_6 > 0$, $qm_2 - m_1 > 0$, $qm_3 - m_5 > 0$, $qm_4 - m_3 - m_4 > 0$, $qm_5 - m_2 > 0$, $qm_6 - m_4 - m_5 - m_6 - m_7 > 0$, $qm_7 - m_8 > 0$. So we may take $(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8) = (9, 5, 2, 3, 3, 17, 1, 1)$.

Case 6. $w_1 = s_3 s_4 s_2 s_{[5,3]} s_{[6,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = s_4$ and $v = s_4$. The inequalities $(*)$ are just $qm_1 - m_6 > 0$, $qm_2 - m_1 - m_3 > 0$, $qm_3 - m_5 > 0$, $qm_5 - m_2 > 0$, $qm_6 - m_3 - m_5 - m_6 - m_7 > 0$ and $qm_7 - m_8 > 0$. So we may take $(m_1, m_2, m_3, m_5, m_6, m_7, m_8) = (7, 5, 2, 3, 13, 1, 1)$.

Case 7. $w_1 = s_1 s_3 s_4 s_2 s_{[5,3]} s_{[6,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{2, 3, 4, 5\}$, $\mathrm{Ad}(w_1)$ sends α_2 to α_3, α_3 to α_5, α_4 to α_4 and α_5 to α_2 and $\mathcal{O}'$ is cuspidal with $l(v) = 2$, 4, 6 or 8. The inequalities $(*)$ are just $qm_1 - m_6 > 0$, $qm_6 - m_1 - m_6 - m_7 > 0$ and $qm_7 - m_8 > 0$. So we may take $(m_1, m_6, m_7, m_8) = (3, 5, 1, 1)$.

Case 8. $w_1 = s_4 s_3 s_5 s_4 s_2 s_{[6,3]} s_{[7,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{3, 6\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta)$ and $v = s_3$. The inequalities $(*)$ are just $qm_1 - m_7 > 0$, $qm_2 - m_1 - m_4 > 0$, $qm_4 - m_5 > 0$, $qm_5 - m_2 - m_4 > 0$ and $qm_7 - m_4 - m_5 - m_7 - m_8 > 0$. So we may take $(m_1, m_2, m_4, m_5, m_7, m_8) = (8, 6, 3, 5, 15, 1)$.

Case 9. $w_1 = s_2 s_4 s_3 s_5 s_4 s_2 s_{[6,3]} s_{[7,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{3, 4, 5, 6\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $v = s_3 s_4$ or $s_4 s_5 s_4 s_3$ or $w_0^{I(J, w_1, \delta)}$. The inequalities $(*)$ are just $qm_1 - m_7 > 0$, $qm_2 - m_1 - m_2 > 0$ and $qm_7 - m_2 - m_7 - m_8 > 0$. So we may take $(m_1, m_2, m_7, m_8) = (4, 5, 7, 1)$.

Case 10. $w_1 = s_5 s_4 s_{[7,2]}^{-1} s_1 s_3 s_4 s_2 s_{[5,3]} s_{[6,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{2, 4\}$, $\mathrm{Ad}(w_1)$ acts trivially on $I(J, w_1, \delta^{-1})$ and $v = s_2 s_4$. The inequalities $(*)$ are just $qm_1 - m_7 > 0$, $qm_3 - m_5 - m_6 > 0$, $qm_5 - m_3 > 0$, $qm_6 - m_1 > 0$, $qm_7 - m_3 - 2m_5 - m_6 - m_7 - m_8 > 0$. So we may take $(m_1, m_3, m_5, m_6, m_7, m_8) = (17, 7, 4, 9, 33, 1)$.

Case 11. $w_1 = s_{[6,1]} s_4 s_3 s_5 s_4 s_2 s_{[6,3]} s_{[7,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{2, 3, 4, 5\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta^{-1})$ exchanging α_3 and α_5 and $v = s_2 s_4 s_5$ or $s_4 s_5 s_3 s_4 s_2 s_5 s_3$. The inequalities $(*)$ are just $qm_1 - m_6 - m_7 > 0$, $qm_6 - m_1 > 0$ and $qm_7 - 2m_6 - m_7 - m_8 > 0$. So we may take $(m_1, m_6, m_7, m_8) = (9, 5, 12, 1)$.

Case 12 ♠. $w_1 = s_{[7,1]} s_4 s_3 s_5 s_4 s_2 s_{[6,3]} s_{[7,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = I - \{7, 8\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $\mathcal{O}'$ is cuspidal with $l(v) = 12$, 14, 16, 18 or 36. In this case, the inequalities $(*)$ is never satisfied if $q = 2$. However, notice that $w_1^{-1} v^{-1} \alpha_8 = w_1^{-1} \alpha_8 > 0$ for all $v \in W_{I(J, w_1, \delta^{-1})}$. Thus if we choose v to be a representative listed above in type ${}^2E_6$ and $m_1, m_2, \cdots, m_6$ be the corresponding positive numbers there and take $m_7 \gg -m_8 \gg \max_{i=1,2,\cdots,6}\{m_i\}$, then one can see that 1.13(a) holds for $w = vw_1$ and $\mu = \sum_{i=1}^{8} m_i \omega_i^\vee$.

Case 13. $w_1 = s_{[6,4]} s_{[6,2]}^{-1} s_{[7,4]} s_{[6,2]}^{-1} s_1 s_3 s_4 s_2 s_{[5,3]} s_{[8,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{2, 3, 4, 5, 7\}$, $\mathrm{Ad}(w_1)$ sending α_2 to α_3, α_3 to α_5, α_4 to α_4, α_5 to α_2, α_7 to α_7 and $\mathcal{O}'$ is cuspidal with $l(v) = 9$. The inequalities $(*)$ are just $qm_1 - m_6 - m_8 > 0$ and $qm_6 - m_1 - m_6 > 0$. So we may take $(m_1, m_6, m_8) = (3, 4, 1)$.

Case 14. $w_1 = s_3 s_4 s_2 s_{[7,5]}^{-1} s_{[6,4]}^{-1} s_{[5,3]}^{-1} s_{[3,1]}^{-1} s_{[4,1]} s_{[5,3]} s_{[6,4]} s_2 s_{[7,3]} s_{[8,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{4, 5, 6, 7\}$, $\mathrm{Ad}(w_1)$ acts trivially on $I(J, w_1, \delta^{-1})$ and $v = s_{[7,4]}$. The inequalities $(*)$ are just $qm_1 - m_3 - m_8 > 0$, $qm_2 - m_1 - m_3 > 0$ and $qm_3 - m_2 > 0$. So we may take $(m_1, m_2, m_3, m_8) = (3, 3, 2, 1)$.

Case 15. $w_1 = s_1 s_3 s_4 s_2 s_{[7,5]}^{-1} s_{[6,4]}^{-1} s_3 s_4 s_{[4,1]}^{-1} s_{[5,1]} s_4 s_3 s_{[6,4]} s_2 s_{[7,3]} s_{[8,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{2, 3, 4, 5, 6, 7\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $v = s_3 s_4 s_{[5,2]}^{-1} s_{[6,4]} s_{[7,2]}^{-1}$ or $s_{[4,2]}^{-1} s_{[5,2]}^{-1} s_4 s_{[5,2]}^{-1} s_{[6,4]} s_{[7,2]}^{-1}$. The inequality $(*)$ is just $qm_1 - m_1 - m_8 > 0$. So we may take $m_1 = 2$ and $m_8 = 1$.

Case 16. $w_1 = s_{[7,1]} s_4 s_3 s_5 s_4 s_2 s_{[6,3]} s_{[7,4]} s_{[7,1]}^{-1} s_{[8,1]} s_4 s_3 s_5 s_4 s_2 s_{[6,3]} s_{[7,4]} s_{[8,1]}^{-1}$, $I(J, w_1, \delta^{-1}) = \{1, 2, 3, 4, 5, 6\}$, $\mathrm{Ad}(w_1)$ acts trivially on $I(J, w_1, \delta^{-1})$ and $\mathcal{O}'$ is cuspidal with $l(v) = 24$. The inequality $(*)$ is just $qm_7 - m_7 - m_8 > 0$. So we may take $m_7 = 2$ and $m_8 = 1$.

Case 17. $w_1 = w_0 w_0^J$ and $v = w_0^J$. The inequalities $(*)$ are always satisfied.

Type F_4

Set $J = I - \{4\}$.

Case 1. $w_1 = s_{[4,1]}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_2 - m_1 > 0$, $qm_3 - m_2 - m_3 - m_4 > 0$ and $qm_4 - m_3 > 0$. So we may take $(m_1, m_2, m_3, m_4) = (1, 1, 5, 3)$.

Case 2. $w_1 = s_3 s_2 s_{[4,1]}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are just $qm_2 - m_1 - m_2 - 2m_3 > 0$, $qm_3 - m_4 > 0$ and $qm_4 - m_2 - m_3 > 0$. So we may take $(m_1, m_2, m_3, m_4) = (1, 12, 5, 9)$.

Case 3 ♠. $w_1 = s_2 s_3 s_2 s_{[4,1]}$, $I(J, w_1, \delta^{-1}) = \{3, 4\}$, $\mathrm{Ad}(w_1)$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $v = s_3$ or $s_3 s_4 s_3$. In this case, the inequalities $(*)$ is never satisfied if $q = 2$. However, notice that $w_1^{-1} v^{-1} \alpha_1 = w_1^{-1} \alpha_1 > 0$ for all $v \in W_{I(J, w_1, \delta^{-1})}$. Thus if we choose v to be a representative listed above in type ${}^2A_n$ and m_3, m_4 be the corresponding positive numbers there and take $m_2 \gg -m_1 \gg \max\{m_3, m_4\}$, then one can see that 1.13(a) holds for $w = v w_1$ and $\mu = \sum_{i=1}^4 m_i \omega_i^\vee$.

Case 4. $w_1 = s_{[3,1]} s_3 s_2 s_{[4,1]}$, $I(J, w_1, \delta^{-1}) = \{2\}$ and $v = s_2$. The inequalities $(*)$ are just $qm_3 - m_3 - m_4 > 0$ and $qm_4 - m_1 - m_3 > 0$. So we may take $(m_1, m_3, m_4) = (1, 4, 3)$.

Case 5. $w_1 = s_{[4,1]} s_3 s_2 s_{[4,1]}$, $I(J, w_1, \delta^{-1}) = \{2, 3\}$, $\mathrm{Ad}(w_1)$ acts trivially on $I(J, w_1, \delta^{-1})$ and $v = s_2 s_3$ or $s_2 s_3 s_2 s_3$. The inequality $(*)$ is just $qm_4 - m_1 - m_4 > 0$. So we may take $m_1 = 1$ and $m_4 = 2$.

Case 6. $w_1 = s_1 w_0 w_0^{\delta(J)}$, $I(J, w_1, \delta^{-1}) = \{3, 4\}$, $\mathrm{Ad}(w_1)$ acts trivially on $I(J, w_1, \delta^{-1})$ and $v = s_3 s_4$. The inequality $(*)$ is just $qm_2 - m_1 - m_2 > 0$. So we may take $m_1 = 1$ and $m_2 = 2$.

Case 7. $w_1 = w_0 w_0^J$ and $v = w_0^J$. The inequalities $(*)$ are always satisfied.

Type G_2

Set $J = \{1\}$.

Case 1 ♠. $w_1 = s_1 s_2$ and $I(J, w_1, \delta^{-1}) = \emptyset$. In this case, the inequalities $(*)$ is never satisfied if $q = 2$. However, take $m_1 \gg -m_2 > 0$, then 1.13(a) holds for w_1 and $\mu = m_1 \omega_1^\vee + m_2 \omega_2^\vee$.

Case 2. $w_1 = s_1 s_2 s_1 s_2$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequality $(*)$ is just $qm_1 - m_1 - m_2 > 0$. So we may take $m_1 = 2$ and $m_2 = 1$.

Case 3. $w_1 = w_0$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequalities $(*)$ are always satisfied.

Type ${}^2B_2$

Set $J = \{1\}$.

Case 1. $w_1 = s_1$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequality $(*)$ is just $qm_1 - m_1 - m_2 > 0$. So we may take $m_1 = 3$ and $m_2 = 1$.

Case 2. $w_1 = s_1 s_2 s_1$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequality $(*)$ is just $qm_1 - m_2 > 0$. So we may take $m_1 = m_2 = 1$.

Type ${}^2G_2$

Set $J = \{2\}$.

Case 1. $w_1 = s_2$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequality $(*)$ is just $qm_2 - m_1 - m_2 > 0$. So we may take $m_1 = 1$ and $m_2 = 2$.

Case 2. $w_1 = s_2s_1s_2$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequality $(*)$ is just $qm_2 - 2m_1 - m_2 > 0$. So we may take $m_1 = 1$ and $m_2 = 3$.

Case 3. $w_1 = s_2s_1s_2s_1s_2$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequality $(*)$ is just $qm_2 - m_1 > 0$. So we may take $m_1 = m_2 = 1$.

Type $^2F_4$

Set $J = I - \{4\}$.

Case 1. $w_1 = s_2s_1$ and $I(J, w_1, \delta^{-1}) = \emptyset$. The inequality $(*)$ are just $qm_1 - m_4 > 0$, $qm_2 - m_2 - m_3 > 0$ and $qm_3 - m_1 > 0$. So we may take $(m_1, m_2, m_3, m_4) = (1, 3, 1, 1)$.

Case 2 ♠. $w_1 = s_2s_{[3,1]}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. In this case, the inequalities $(*)$ is never satisfied if $q = \sqrt{2}$. However, take $-m_4 \gg m_2 = m_3 \gg m_1 > 0$, then 1.13(a) holds for w_1 and $\mu = \sum_{i=1}^4 m_i\omega_i^\vee$.

Case 3. $w_1 = s_1s_2s_{[3,1]}$, $I(J, w_1, \delta^{-1}) = \{2, 3\}$, $\text{Ad}(w_1)\delta^{-1}$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $v = s_2$ or $s_2s_3s_2$. The inequality $(*)$ is just $qm_1 - m_1 - m_4 > 0$. So we may take $m_1 = 3$ and $m_4 = 1$.

Case 4 ♠. $w_1 = s_{[3,1]}s_2s_3s_2s_{[4,1]}$ and $I(J, w_1, \delta^{-1}) = \emptyset$. In this case, the inequalities $(*)$ is never satisfied if $q = \sqrt{2}$. However, 1.13(a) holds for $w = w_1$ and $\mu = 3\omega_1^\vee + \omega_2^\vee + 3\omega_3^\vee - 3\omega_4^\vee$.

Case 5. $w_1 = s_2s_{[3,1]}s_2s_3s_2s_{[4,1]}$, $I(J, w_1, \delta) = \{1, 3\}$, $\text{Ad}(w_1)\delta^{-1}$ is of order 2 on $I(J, w_1, \delta^{-1})$ and $v = s_2$. The inequality $(*)$ is just $qm_2 - m_2 - m_4 > 0$. So we may take $m_2 = 3$ and $m_4 = 1$.

Case 6. $w_1 = w_0w_0^{\delta^{-1}(J)}$, $I(J, w_1, \delta^{-1}) = \{2, 3\}$, $\text{Ad}(w_1)\delta^{-1}$ is of order 2 on $W_{I(J,w_1,\delta^{-1})}$ and $v = s_2s_3s_2$. The inequality $(*)$ is just $qm_1 - m_4 > 0$. So we may take $m_1 = m_4 = 1$.

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