GEOMETRY OF $B \times B$-ORBIT CLOSURES IN EQUIVARIANT EMBEDDINGS

XUHUA HE AND JESPER FUNCH THOMSEN

Abstract. Let X denote an equivariant embedding of a connected reductive group G over an algebraically closed field k. Let B denote a Borel subgroup of G and let Z denote a $B \times B$-orbit closure in X. When the characteristic of k is positive and X is projective we prove that Z is globally F-regular. As a consequence, Z is normal and Cohen-Macaulay for arbitrary X and arbitrary characteristics. Moreover, in characteristic zero it follows that Z has rational singularities. This extends earlier results by the second author and M. Brion.

1. Introduction

Let G denote a connected and reductive linear algebraic group over an algebraically closed field k. Let B denote a Borel subgroup of G. An (equivariant) embedding X of G is a normal $G \times G$-variety which contains an open subset which is $G \times G$-equivariantly isomorphic to G. Here we think of G as a $G \times G$-variety through left and right translation. In this paper we study the geometry of $B \times B$-orbit closures in X. Examples of such varieties include all toric varieties, all (generalized) Schubert varieties and all large Schubert varieties (see [B-P]).

The geometry of $B \times B$-orbit closures within equivariant embeddings has been the subject of several earlier papers. In [B] it was realized that such orbit closures were mostly singular with singular locus of codimension 2. In the special case of the *wonderful compactification* of a semisimple group G of adjoint type, this was later strengthened in [B-P], where it was proved that closures of orbits of the form BgB, for $g \in G$, are normal and Cohen-Macaulay. Closures of this form are called *large Schubert varieties*. Using the concept of global F-regularity the latter result was generalized to arbitrary X and G in [B-T]. For arbitrary $B \times B$-orbit closures it seems that normality and Cohen-Macaulayness is only known for the wonderful compactifications [B2, Rem.1]. In the present paper we show that all $B \times B$-orbit closures for arbitrary X and G will be normal and Cohen-Macaulay. Moreover, when the field k has characteristic 0 we will show that such orbit closures have rational singularities. As in [B-T] the main technical tool will be that of global F-regularity.

2000 *Mathematics Subject Classification.* 14M17, 14L30, 14B05.
The first author is supported by NSF grant DMS-0111298.

Global F-regularity was introduced by K. Smith in [S2]. By definition a projective variety Z over a field of positive characteristic is globally F-regular if every ideal of some homogeneous coordinate ring of Z is tightly closed. Any globally F-regular variety will be normal and Cohen-Macaulay. Moreover, every homogeneous coordinate ring of Z will share the same properties. Another consequence is that the higher cohomology groups of nef line bundles on Z will be zero. Known classes of globally F-regular varieties include projective toric varieties [S2], (generalized) Schubert varieties [L-P-T] and projective large Schubert varieties [B-T]. In this paper we prove that every $B \times B$-orbit closure in a projective embedding X of a reductive group G is globally F-regular. Notice that varieties of this form include the mentioned classes above.

The paper is organized as follows. In Section 2 we introduce notation. In Section 3 we give a short introduction to Frobenius splitting, canonical Frobenius splitting and global F-regularity. In section 4 we present the main technical result (Proposition 4.1) which relates the mentioned concepts from Section 3. In section 5 we describe the $G \times G$-orbit closures in a toroidal embedding. Section 6 describes the decomposition of the closure of a $B \times B$-orbit into the union of some $B \times B$-orbits for toroidal embeddings. This is a generalization of Springer's result in [Sp] on the wonderful compactification. As a by-product of this description we obtain, that any Frobenius splitting of a toroidal embedding X which compatibly Frobenius splits the boundary components and the large Schubert varieties of codimension 1, will automatically compatibly Frobenius split all $B \times B$-orbit closures in X. This is used in Section 7 to conclude that all $B \times B$-orbit closures in a toroidal embedding are simultaneous canonical Frobenius split. In section 8 we prove that any $B \times B$-orbit closure in a projective embedding (over a field of positive characteristic) is globally F-regular. The proof of this proceeds by reducing to the case when X is toroidal and then using the results of the previous sections. Finally in Section 9 we treat the characteristic 0 case by descending the results from Section 8 to positive characteristic.

2. Notation

Throughout this paper G will denote a connected reductive linear algebraic group over an algebraically closed field k. The associated semisimple and connected group of adjoint type will be denoted by G_{ad}. The associated canonical morphism is denoted by $\pi_{\mathrm{ad}} : G \to G_{\mathrm{ad}}$. We will fix a maximal torus T and a Borel subgroup $B \supset T$ of G.

The set of roots determined by T will be denoted by R and we define the subset of positive roots R^+ of R to be the set of roots $\alpha \in R$ such that the α-weight space of the Lie algebra of B is nonzero. The set $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of simple positive roots will be indexed by

$I = \{1, \ldots, l\}$. For each subset $J \subset I$ we let $P_J \supset B$ denote the corresponding parabolic subgroup of G. The associated Levi subgroup containing T will be denoted by L_J while we use the notation U_J to denote the unipotent radical of P_J. The notation U_J^- and L_J^- will be used for the equivalent subgroups in the parabolic subgroup P_J^- opposite to P_J. When J is empty we simple denote P_J^- by B^- and U_J by U. The semisimple group of adjoint type associated with L_J is denoted by G_J.

To each root $\alpha \in R$ there is an associated reflection s_α in the Weyl group $W = {}^{N_G(T)}/_T$. The reflection associated with the simple root α_i is called simple and will be simply written as s_i. We may then write each element w in W as a product of simple reflection and the minimal number of factors in such a product is the length of w and will be denoted by $l(w)$. The unique element of maximal length will be denoted by w_0. For $J \subset I$, we denote by W_J the subgroup of the Weyl group W generated by the simple reflections s_i for, $i \in J$, and by W^J the set of minimal length coset representatives of W/W_J. The element in W_J of longest length is denoted by w_0^J. For an element $w \in W$ we let $\dot{w}$ denote a representative for w in the normalizer of T. Moreover, we define $R(w) = \{\alpha \in R^+ : w\alpha \in R^+\}$, and denote by U_w the subgroup of B generated by the root subgroups U_α for $\alpha \in R(w)$. Then we let B_w denote the subgroup TU_w of B.

By a variety over k we mean a reduced and separated scheme of finite type over k. In particular, a variety need not be irreducible.

3. Generalities on Frobenius splitting

Let X be a scheme of finite type over an algebraically closed field k of positive characteristic $p > 0$. The *absolute Frobenius morphism* $F : X \to X$ on X is the morphism of schemes which on the level of points is the identity map and where the associated map of sheaves

$$F^\sharp : \mathcal{O}_X \to F_*\mathcal{O}_X,$$

is the p-th power map. A *Frobenius splitting* of X is an $\mathcal{O}_X$-linear morphism

$$s : F_*\mathcal{O}_X \to \mathcal{O}_X,$$

such that the composition $s \circ F^\sharp$ is the identity map.

3.1. **Compatibly split subschemes.** Let Y denote a closed subscheme of X with sheaf of ideals $\mathcal{I}_Y$. A Frobenius splitting s of X is said to *compatibly Frobenius split* Y if $s(\mathcal{I}_Y) \subset \mathcal{I}_Y$. In this case there exists an induced Frobenius splitting of Y. When Y is compatibly Frobenius split by s then any irreducible component of Y will also be compatibly Frobenius split by s. Moreover, if Y' is another (by s) compatibly Frobenius split closed subscheme then the scheme theoretic intersection $Y \cap Y'$ will also be compatibly Frobenius split by s.

3.2. **Push-forward.** Let $f : X \to X'$ denote a morphism of schemes of finite type over k. Assume that X admits a Frobenius splitting s which compatibly splits a closed subscheme Y. If the induced map $f^\sharp : \mathcal{O}_{X'} \to f_*\mathcal{O}_X$, is an isomorphism, then s induces by push-forward a Frobenius splitting of X' which compatibly Frobenius splits the scheme theoretic image of Y.

3.3. **Stable Frobenius splitting along divisors.** Let D denote an effective Cartier divisor on X and let s_D denote the canonical section of the associated line bundle $\mathcal{O}_X(D)$. Then X is said to admit a stable Frobenius splitting along D if there exists a positive integer e and an $\mathcal{O}_X$-linear morphism

$$s : F_*^e\mathcal{O}_X(D) \to \mathcal{O}_X,$$

such that $s(s_D) = 1$. Notice that in this case the composition of s with the canonical map $\mathcal{O}_X \to F_*^e\mathcal{O}_X(D)$, defined by s_D, is a Frobenius splitting of X. If D' is another effective divisor then it is known (see e.g. [B-T, Lemma 3.1]) that X is stably Frobenius split along the sum $D + D'$ if and only if X is stably Frobenius split along both D and D'.

When X admits a stable Frobenius splitting s along D and Y is a closed subscheme of X, then we say that s compatibly Frobenius splits Y if $s\big(F_*^e\big(\mathcal{I}_Y \otimes \mathcal{O}_X(D)\big)\big) \subset \mathcal{I}_Y$ and, moreover, none of the components of Y are contained in the support of D.

3.4. **Canonical Frobenius splitting.** Let now G be a connected reductive linear algebraic group. Fix a Borel subgroup B and a maximal torus $T \subset B$ of G. When X is a B-variety there is an induced action of B on the set of $\mathcal{O}_X$-linear maps $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$. More precisely, when $b \in B$ and $f \in \mathcal{O}_X(V)$, for V open in X, then we define $b \cdot f$ to be the function on bV defined by $(b \cdot f)(v) = f(b^{-1}v)$. Then for $s : F_*\mathcal{O}_X \to \mathcal{O}_X$ we define $(b \star s) : F_*\mathcal{O}_X \to \mathcal{O}_X$ by

$$(b \star s)(f) = b \cdot s(b^{-1} \cdot f).$$

We regard $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ as a k-vectorspace by letting $z \in k$ act on $s : F_*\mathcal{O}_X \to \mathcal{O}_X$ as

$$(z.s)(f) = z^{1/p}s(f).$$

We may then define the following important concept : a Frobenius splitting s of X is said to be (B, T)-canonical if :

- $t \star s = s$, $\forall t \in T$.
- Let $\alpha \in \Delta$ and let $x_\alpha : k \to G$ be the associated homomorphism of algebraic groups satisfying $tx_\alpha(z)t^{-1} = x_\alpha(\alpha(t)z)$, $t \in T$. Then
$$x_\alpha(z) \star s = \sum_{i=1}^{p-1} z^i.s_{i,\alpha} \text{ , for all } z \in k,$$
for certain fixed $s_{i,\alpha} \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$.

When X is a B-variety we define the variety $G \times_B X$ to be the quotient of $G \times X$ by the B-action defined by $b.(g, x) = (gb^{-1}, bx)$ for $b \in B, g \in G$ and $x \in X$. With this notation we have the following crucial result connected with canonical Frobenius splittings (see e.g. [B-K, 4.1.E(4)])

Proposition 3.1. *Let X be a variety admitting a (B, T)-canonical Frobenius splitting s. Then the variety $G \times_B X$ admits a (B, T)-canonical Frobenius splitting such that $\overline{B\dot{w}B} \times_B X$ is compatibly Frobenius split for all $w \in W$ and such that $G \times_B Y$ is compatibly Frobenius split for all B-stable subvarieties of X which are compatibly Frobenius split by s.*

3.5. **Strong F-regularity.** A general reference for this subsection is [H-H]. Let K be a field of positive characteristic $p > 0$ and let R denote a commutative K-algebra essentially of finite type, i.e. equal to some localization of a finitely generated K-algebra. We say that R is *strongly F-regular* if for each $s \in R$, not contained in a minimal prime of R, there exists a positive integer e such that the R-linear map $F_s^e : R \to F_*^e R$, $r \mapsto r^{p^e} s$, is split. When R is strongly F-regular then R is normal and Cohen-Macaulay. Moreover, all ideals in R will be tightly closed and thus R will be F-rational, i.e. every parameter ideal is tightly closed.

The ring R is strongly F-regular if and only if all of its localized rings are strongly F-regular. Thus, we define a scheme X of finite type over K to be strongly F-regular if all of its local rings $\mathcal{O}_{X,x}$, for $x \in X$, are strongly F-regular. Then the affine scheme Spec(R) (when R is a finitely generated K-algebra) is strongly F-regular precisely when R is strongly F-regular.

3.6. **Global F-regularity.** Consider an irreducible projective variety X over k. For an ample line bundle $\mathcal{L}$ on X we define the associated section ring to be

$$R = R(X, \mathcal{L}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{L}^n).$$

We then say that X is *globally F-regular* if the ring $R(X, \mathcal{L})$ is strongly F-regular for some (or equivalently, any) ample invertible sheaf $\mathcal{L}$ on X. Global F-regularity was introduced by K. Smith in [S2]. When X is globally F-regular then X is also strongly F-regular. In particular, globally F-regular varieties are normal, Cohen-Macaulay and locally F-rational.

The following important result by Smith [S2, Theorem 3.10] connects global F-regularity, Frobenius splitting and strong F-regularity.

Theorem 3.2. *If X is an irreducible projective variety over k then the following are equivalent:*

(1) *X is globally F-regular.*
(2) *X is stably Frobenius split along an ample effective Cartier divisor D and the (affine) complement $X \setminus D$ is strongly F-regular.*
(3) *X is stably Frobenius split along every effective Cartier divisor.*

The connection between (1) and (3) in this theorem leads to the following result which can be found in [L-P-T].

Corollary 3.3. *Let $f : \tilde{X} \to X$ be a morphism of projective varieties. If the connected map $f^\sharp : \mathcal{O}_X \to f_* \mathcal{O}_{\tilde{X}}$ is an isomorphism and $\tilde{X}$ is globally F-regular then X is also globally F-regular.*

4. Some criteria for globally F-regularity

Throughout this section we assume that k has positive characteristic. The following result connects canonical Frobenius splitting and global F-regularity.

Proposition 4.1. *Let Y be an irreducible projective B-variety. Let $y \in Y$ and $w \in W$. Define $Y' = Y - B \cdot y$ and assume that*

(1) *$B_w \cdot y = B \cdot y$ and $B \cdot y$ is dense in Y.*
(2) *Y admits a (B,T)-canonical Frobenius splitting which compatibly splits the subvariety Y'.*
(3) *Y is strongly F-regular.*

Write $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ as a reduced product of simple reflections and define

$$Z = P_1 \times_B P_2 \times_B \cdots \times_B P_n \times_B Y,$$

where $P_j = B \cup B \dot{s}_{i_j} B$ is a minimal parabolic subgroup. Then Z is globally F-regular.

Proof. Let $\mathcal{L}$ denote an ample line bundle on Z. Since Y is strongly F-regular, Y is normal. Moreover the Picard group of B is trivial. Thus we may consider $\mathcal{L}$ as a B-linearized line bundle. In particular, B acts linearly on the finite dimensional vector space $\mathrm{H}^0(Z, \mathcal{L})$ of global sections of $\mathcal{L}$ and we may thus find a nonzero global section s which is B-invariant up to scalars.

Let $z = [\dot{s}_{i_1}, \dots, \dot{s}_{i_n}, y] \in Z$. Then by assumption (1) the orbit $B \cdot z$ is dense in Z with complement equal to the union of the subsets

$$Z_i = P_1 \times_B \cdots \times_B B \times_B \cdots \times_B P_n \times_B Y, \ i = 1, \dots, n,$$

$$Z'_j = P_1 \times_B P_2 \times_B \cdots \times_B P_n \times_B Y'_j, \ j = 1, \dots, m,$$

where Z_i is defined by substituting B with P_i in the definition of Z and Y'_j, $j = 1, \dots, m$, denotes the components of Y'. As the support $\mathrm{supp}(s)$ of s is B-stable and of codimension 1 in Z it follows that $\mathrm{supp}(s)$ is contained in $Z - B \cdot z$, i.e in the union of Z_i, $i = 1, \dots, n$ and

Z'_j, $j = 1, \dots, m$. In particular, we may choose nonnegative integers n_i and m_j such that the zero divisor of s in Z equals

$$Z(s) = \sum_{i=1}^{n} n_i Z_i + \sum_{j=1}^{m} m_j Z'_j.$$

By assumption (2) and Proposition 3.1 the variety Z admits a Frobenius splitting which compatibly Frobenius splits Z'_j, $j = 1, \dots, m$ and Z_i, $i = 1, \dots, n$. Let Y^0 denote the (B-invariant) nonsingular locus in Y. As Y is normal the complement $Y - Y^0$ is of codimension ≥ 2. Now define

$$Z^0 = P_1 \times_B P_2 \times_B \cdots \times_B P_n \times_B Y^0.$$

Then Z^0 is a smooth variety which admits a Frobenius splitting compatibly splitting the divisors $Z_i \cap Z^0$, $i = 1, \dots, n$ and the subvarieties $Z'_j \cap Z^0$, $j = 1, \dots, m$. As Z^0 is smooth this implies (see e.g. [L-P-T, Lemma 1.1]) that Z^0 admits a stable Frobenius splitting along the effective Cartier divisor :

$$\sum_{i=1}^{n} (Z_i \cap Z^0) + \sum_{j=1}^{m} \delta_j (Z'_j \cap Z^0),$$

where $\delta_j = 0$ if Z'_j is not a divisor and else $\delta_j = 1$. As a consequence, Z^0 admits a stable Frobenius splitting along

$$\sum_{i=1}^{n} n_i (Z_i \cap Z^0) + \sum_{j=1}^{m} m_j (Z'_j \cap Z^0).$$

In other words, Z^0 is stably Frobenius split along the Cartier divisor defined by the restriction of s to Z^0. Thus the morphism

$$\mathcal{O}_{Z^0} \to F^e_* \mathcal{O}_{Z^0}(Z(s) \cap Z^0),$$

defined by the restriction of s to Z^0 splits for some sufficiently large integer e.

As Y is normal so is Z. Moreover, $Z - Z^0$ has codimension ≥ 2 and thus $i_* i^* \mathcal{M}$ for any line bundle $\mathcal{M}$ on Z where i denotes the inclusion map of Z^0 in Z. Applying the functor i_* to the stable splitting above we find that Z admits a stable Frobenius splitting along the effective Cartier divisor defined by s. Moreover, as Y is strongly F-regular also Z and hence $Z - \mathrm{supp}(s)$ is strongly F-regular (see e.g. [L-S, Lemma 4.1]). This proves that Z is globally F-regular and ends the proof. $\square$

For convenience of the reader we include the following result (see [R, Lemma 2.11]) which we will use in the proof of the next proposition.

Lemma 4.2. *Let $f : X \to Y$ denote a projective morphism of irreducible varieties and let X' denote a closed irreducible subvariety of X. Consider the image $Y' = f(X')$ as a closed subvariety of Y. Let $\mathcal{L}$ denote an ample line bundle on Y and assume*

(1) $f_* \mathcal{O}_X = \mathcal{O}_Y$.
(2) $\mathrm{H}^i(X, f^*\mathcal{L}^n) = \mathrm{H}^i(X', f^*\mathcal{L}^n) = 0$ *for* $i > 0$ *and* $n >> 0$.
(3) *The restriction map* $\mathrm{H}^0(X, f^*\mathcal{L}^n) \to \mathrm{H}^0(X', f^*\mathcal{L}^n) = 0$ *is surjective for* $n >> 0$.

Then the induced map $f' : X' \to Y'$ *is a rational morphism, i.e.* $f'_* \mathcal{O}_{X'} = \mathcal{O}_{Y'}$ *and* $\mathrm{R}^i f'_* \mathcal{O}_{X'} = 0$, $i > 0$.

Proposition 4.3. *Let X denote an irreducible G-variety and let Y denote a closed irreducible B-subvariety of X. Assume that X admits a (B,T)-canonical Frobenius splitting which compatibly splits Y. Let $P_1, \dots, P_n$ denote a collection of minimal parabolic subgroups of G. Then the natural map*

$$f : Z = P_1 \times_B \cdots \times_B P_n \times_B Y \to (P_1 \cdots P_n) \cdot Y \subset X,$$

is a rational morphism, i.e. $R^i f_* \mathcal{O}_Z = 0$, $i > 0$, $f_* \mathcal{O}_Z = \mathcal{O}_{f(Z)}$.

Proof. Define $Z_X = P_1 \times_B \cdots \times_B P_n \times_B X$. As X is a G-variety we may identify Z_X with the product $Z(P_1, \dots, P_n) \times X$, where $Z(P_1, \dots, P_n)$ denotes the Bott-Samelson variety $P_1 \times_B \cdots \times_B P_n / B$. We define $g : Z_X \to X$ to be the associated projection map. As $Z(P_1, \dots, P_n)$ is an irreducible projective variety we have $g_* \mathcal{O}_{Z_X} = \mathcal{O}_X$.

Let $Z_{X,i}$, $i = 1, \dots, n$, denote the Cartier divisor

$$Z_{X,i} = P_1 \times_B \cdots \times_B B \times_B \cdots \times_B P_n \times_B X,$$

in Z_X, where P_i in the definition of Z_X is substituted by B. Then, by Proposition 3.1, the variety Z_X admits a Frobenius splitting s which compatibly splits the subvariety Z and the divisors $Z_{X,i}$, $i = 1, \dots, n$. Thus by [L-P-T, Lem.1.1] the Frobenius splitting $s : F_* \mathcal{O}_{Z_X} \to O_{Z_X}$ maps through the morphism

$$F_* \mathcal{O}_{Z_X} \to F_* \Big(\mathcal{O}_{Z_X} \Big(\sum_{i=1}^n Z_{X,i} \Big) \Big)$$

defined by the product of the canonical sections of the Cartier divisors $Z_{X,i}$, $i = 1, \dots, n$. Thus we may regard s as a stable Frobenius splitting of X along $\sum_{i=1}^n Z_{X,i}$ which compatibly splits Z. By [T, Lem.4.3, Lem.4.4] we conclude that Z_X admits a stable Frobenius splitting along any divisor of the form

$$\sum_{i=1}^n n_i Z_{X,i},$$

with n_i being positive integers, which compatibly Frobenius splits Z.

Let $\mathcal{L}$ denote any ample line bundle on X. Choose n_i, $i = 1, \dots, n$, such that the line bundle

$$\mathcal{L}'_m = g^* \mathcal{L}^{p^m} \otimes \mathcal{O}_{Z_X} \big(\sum_{i=1}^n n_i Z_{X,i} \big)$$

is ample on Z_X for all $m > 0$ (that this is possible follows e.g. from [L-T, Lem.6.1]). By [T, Lem.4.8] there exists, for some m, an embedding of abelian groups

$$\mathrm{H}^j\big(Z_X, \mathcal{I}_Z \otimes g^*\mathcal{L}\big) \subseteq \mathrm{H}^j\big(Z_X, \mathcal{I}_Z \otimes \mathcal{L}'_m\big),$$

for all j. So by [B-K, Thm.1.2.8] and the ampleness of $\mathcal{L}'_m$ it follows that $\mathrm{H}^j\big(Z_X, \mathcal{I}_Z \otimes g^*\mathcal{L}\big)$ is zero for $j > 0$. Similarly (with Z substituted with Z_X) we may conclude that $\mathrm{H}^j\big(Z_X, g^*\mathcal{L}\big)$ is zero for $j > 0$. Together these two latter statements imply that $\mathrm{H}^j\big(Z, g^*\mathcal{L}\big)$ is also zero for $j > 0$.

Applying Lemma 4.2 now ends the proof. $\square$

Combining the two propositions above with Corollary 3.3 we find

Theorem 4.4. *Let X denote an irreducible G-variety and let Y be a closed irreducible B-subvariety of X. Assume that X admits a (B, T)-canonical Frobenius splitting which compatibly splits Y. Let $y \in Y$ and $w \in W$ and assume that the triple (Y, y, w) satisfies the assumptions in Proposition 4.1. Then $(\overline{B\dot{w}B})Y$ is globally F-regular.*

5. The $G \times G$-orbit closures in toroidal embeddings

Consider G as a $G \times G$-variety by left and right translation. An *equivariant G-embedding* (or simply a G-embedding) is a normal $G \times G$-variety X containing an open subset which is $G \times G$-equivariantly isomorphic to G.

5.1. **Wonderful compactifications.** When $G = G_{\mathrm{ad}}$ is of adjoint type there exists a distinguished equivariant embedding $\mathbf{X}$ of G which is called the *wonderful compactification* (see e.g. [B-K, 6.1]).

The boundary $\boldsymbol{X} - G$ of $\boldsymbol{X}$ is a union of irreducible divisors $\boldsymbol{X}_i$, $i \in I$, which intersect transversally. For a subset $J \subset I$ we denote the intersection $\cap_{j \in J} \boldsymbol{X}_j$ by $\boldsymbol{X}_J$. Then $\boldsymbol{Y} := \boldsymbol{X}_I$ is the unique closed $G \times G$-orbit in $\boldsymbol{X}$. As a $G \times G$-variety $\boldsymbol{Y}$ is isomorphic to ${}^{G}/_{B} \times {}^{G}/_{B}$.

5.2. **Toroidal embeddings.** An embedding X of a reductive group G is called *toroidal* if the canonical map $\pi_{\mathrm{ad}} : G \to G_{\mathrm{ad}}$ admits an extension $\pi : X \to \mathbf{X}$ into the wonderful compactification $\mathbf{X}$ of the group G_{ad} of adjoint type.

5.3. **The $G \times G$-orbits.** For the rest of this section we assume that X is a toroidal embedding of G. The boundary $X - G$ is of pure codimension 1 (see [H, Prop.3.1]). Let $X_1, \dots, X_n$ denote the boundary divisors. For each $G \times G$-orbit closure Y in X we then associate the set

$$K_Y = \{i \in \{1, \dots, n\} \mid Y \subset X_i\},$$

where by definition $K_Y = \varnothing$ when $Y = X$. Then by [B-K, Prop.6.2.3], $Y = \cap_{i \in K_Y} X_i$. Moreover, we define

$$\mathcal{I} = \{K_Y \subset \{1, \dots, n\} \mid Y \text{ a } G \times G\text{-orbit closure in } X\ \},$$

and write $X_K := \cap_{i\in K} X_i$ for $K \in \mathcal{J}$. Then $(X_K)_{K\in\mathcal{J}}$ are the closures of $G\times G$-orbits in X. When X is the wonderful compactification of G_{ad} then $\mathcal{J} = \mathcal{P}(I)$, where $\mathcal{P}(I)$ denotes the set of subsets of I. Moreover, the $G\times G$-equivariant map $\pi : X \to \boldsymbol{X}$ induces a map $p : \mathcal{J} \to \mathcal{P}(I)$ such that $\pi(X_K) = \boldsymbol{X}_{p(K)}$.

5.4. **The base points.** Let X' denote the closure of T within X and let similarly $\mathbf{X}'$ denote the closure of $T_{\mathrm{ad}} = \pi_{\mathrm{ad}}(T)$ within $\mathbf{X}$. Let X_0 denote the complement of the union of the closures $\overline{B\dot{s}_iB^-}$, $i = 1, \dots, l$, within X. Then X_0 is an open $B\times B^-$-stable subset of X. Moreover, if we let X'_0 denote the intersection of X' and X_0 then the map

$$U \times U^- \times X'_0 \to X_0,$$

$$(u, v, x) \mapsto (u, v)z.$$

is an isomorphism (see [B-K, Prop.6.2.3(i)]). With similar definitions for $\mathbf{X}$ we also obtain an isomorphism

$$U \times U^- \times \mathbf{X}'_0 \to \mathbf{X}_0.$$

The above defined subsets are related in the way that $\pi^{-1}(\mathbf{X}'_0) = X'_0$ and consequently also $\pi^{-1}(\mathbf{X}_0) = X_0$.

The set $\mathbf{X}'_0$ is a toric variety (with respect to T_{ad}). In particular, it contains finitely many $T\times T$-orbits. The $T\times T$-orbits are classified by the set $\mathcal{P}(I)$ of subsets of I. We may choose representatives $\boldsymbol{h}_J$, $J \subset I$, for these orbits such that $\boldsymbol{h}_J$ is invariant under the groups $U^-_{I-J}\times U_{I-J}$, $\mathrm{diag}(L_{I-J})$ and $Z(L_{I-J}) \times Z(L_{I-J})$ (see e.g. [Sp, 1.1]). Such a representative $\boldsymbol{h}_J$ is then uniquely determined.

Each $G\times G$-orbit in $\mathbf{X}$ intersects $\mathbf{X}'_0$ in a unique $T\times T$-orbit (see [B-K, Prop.6.2.3(ii)]). In particular, the elements $\boldsymbol{h}_J$ are also representatives for the $G\times G$-orbits in $\mathbf{X}$. Moreover, $(G\times G)\cdot \boldsymbol{h}_J$ is the open dense $G\times G$-orbit in $\boldsymbol{X}_J$.

Now for the toroidal embedding X and $K \in \mathcal{J}$, we may pick a point h_K in the open $G\times G$-orbit of X_K which maps to $\boldsymbol{h}_{p(K)}$. Then $(h_K)_{K\in\mathcal{J}}$ is a set of representatives of the $G\times G$-orbits in X. Notice that $h_K \in \pi 1(\boldsymbol{X}'_0) \subset X'_0$.

5.5. **The structure of $G\times G$-orbit closures.** The following result should be well known but, as we have not been able to find a reference to it, we include a proof.

Lemma 5.1. *Let H denote a linear algebraic over the field k and let Y denote a homogeneous H-variety. Let $y \in Y$ and let $p_y : H \to Y$ denote the associated orbit map. Let H_y denote the stabilizer group scheme of y. Then $Y \simeq H/H_y$ as homogeneous H-varieties. Moreover, if H_y is a normal subgroup scheme of H then Y may be given a structure of a linear algebraic group such that p_y is a morphism of algebraic groups.*

Proof. By [D-G, Prop.III.3.5.2.] it follows that we may identify H/H_y with a locally closed subscheme Y' of Y. As Y is a homogeneous H-variety we conclude that $Y = Y'$. In particular, H/H_y is a variety.

Consider now the case when H_y is a normal subgroup scheme of H. Then by [D-G, Prop.III.3.5.6] the quotient H/H_y is an affine group scheme. Thus the isomorphism $Y \simeq H/H_y$ induces a desired algebraic group structure on Y. □

Proposition 5.2. *Let X be a toroidal embedding. Let $K \in \mathfrak{I}$, $J = p(K)$ and $h = h_K$. Then*

1. *h is invariant under the groups $U^-_{I-J} \times U_{I-J}$ and* $\mathrm{diag}(L_{I-J})$.
2. *We simply write $(L_{I-J} \times \{1\}) \cdot h$ as $L_{I-J} \cdot h$. The closure $\overline{L_{I-J} \cdot h}$ in X is $L_{I-J} \times L_{I-J}$-equivariantly isomorphic to a toroidal equivariant embedding of a quotient L_{I-J}/H_{I-J} of L_{I-J} by some (not necessarily reduced) subgroup H_{I-J} of the (scheme theoretical) center of L_{I-J}.*
3. *The natural morphism*
$$\phi_K : (G \times G) \times_{P^-_{I-J} \times P_{I-J}} \overline{L_{I-J} \cdot h} \to X_K,$$
is a birational and bijective $G \times G$-equivariant morphism. Moreover, when the characteristic of k is positive then ϕ_K is an isomorphism.
4. *For $v \in W^{I-J}$ and $w \in W$ define $[K, v, w] := (B\dot{v}, B\dot{w}) \cdot h$. Then*
$$(G \times G) \cdot h = \bigsqcup_{v \in W^{I-J}, w \in W} [K, v, w].$$

Proof. The statements holds if $X = \mathbf{X}$ (see [Sp, 1.1]).

Now let $V = \pi^{-1}(\boldsymbol{h}_J) \subset \pi^{-1}(\mathbf{X}'_0) = X'_0$. Let $U_1 \simeq \mathbb{G}_a$ be a 1-dimensional additive subgroup in $G \times G$ normalized by $T \times T$ which acts trivially on $\boldsymbol{h}_J$. Then $U_1 \cdot h \subset V \subset X'_0$. By [B-K, Prop.6.2.3(ii)] the $G \times G$-orbit of h intersects X'_0 in a single $T \times T$-orbit. Hence, $U_1 \cdot h \subset (T \times T) \cdot h$ and thus U_1 leaves $(T \times T) \cdot h$ invariant. But $(T \times T) \cdot h \simeq (k^*)^n$, for some n, and any action of $\mathbb{G}_a$ on $(k^*)^n$ is trivial. In particular, U_1 leaves h invariant. This proves that h is invariant under $U^-_{I-J} \times U_{I-J}$ and the semisimple part of $\mathrm{diag}(L_J)$. Now (1) follows as any element in the toric variety X' is invariant under $\mathrm{diag}(T)$.

We identify X_0 with $U \times U^- \times X'_0$ and simply write $(T \times \{1\}) \cdot h$ as $T \cdot h$. Then $(U \cap L_{I-J}) \times (U^- \cap L_{I-J}) \times (\overline{T \cdot h} \cap X'_0)$ is a closed irreducible subset of X_0 contained in $\overline{L_{I-J} h}$ and of the same dimension as $\overline{L_{I-J} h}$. Hence $\overline{L_{I-J} \cdot h} \cap X_0 \simeq (U \cap L_{I-J}) \times (U^- \cap L_{I-J}) \times (\overline{T \cdot h} \cap X'_0)$. As $X' = \overline{T}$ is a toric variety every T-orbit closure in X' is normal. Hence $\overline{L_{I-J} \cdot h} \cap X_0$ is normal. As a consequence, every intersection of the form $\overline{L_{I-J} \cdot h} \cap x X_0$, for $x \in L_{I-J}$, is also normal.

We claim that the union $\cup_{x \in L_{I-J}} x X_0$ contains $\overline{L_{I-J} \cdot h}$. To see this it suffices to prove that the union $\cup_{x \in L_{I-J}} x \mathbf{X}_0$ contains the wonderful compactification $\overline{G_{I-J}} = \overline{L_{I-J} \cdot \boldsymbol{h}_J}$ (see [Sp, 1.1] for this equality) of G_{I-J}. But $\mathbf{X}_0$ contains (by definition) the corresponding open subset $(\overline{G_{I-J}})_0$ of $\overline{G_{I-J}}$ and, moreover, $\overline{G_{I-J}}$ is covered by the L_{I-J}-translates of the subset $(\overline{G_{I-J}})_0$. This proves the claim and as a consequence $\overline{L_{I-J} \cdot h}$ is normal.

As $\pi(h) = \boldsymbol{h}_J$ it follows that the scheme theoretic L_{I-J}-stabilizer $(L_{I-J})_h$ of h is a closed subgroup scheme of the L_{I-J}-stabilizer of $\boldsymbol{h}_J$. The latter stabilizer coincides with the scheme theoretic center of L_{I-J}. So applying Lemma 5.1 we conclude that $L_{I-J} \cdot h$ is isomorphic to the reductive group $L_{I-J}/(L_{I-J})_h$. As a consequence, $\overline{L_{I-J} \cdot h}$ is an equivariant embedding of $L_{I-J}/(L_{I-J})_h$. Moreover the map π induces a morphism $\pi : \overline{L_{I-J} \cdot h} \to \overline{L_{I-J} \cdot \boldsymbol{h}_J} \simeq \overline{G_{I-J}}$, so $\overline{L_{I-J} \cdot h}$ is even a toroidal embedding of $L_{I-J}/(L_{I-J})_h$. This proves statement (2).

Consider the commutative diagram

$$\begin{tikzcd}
(G \times G) \times_{P^-_{I-J} \times P_{I-J}} \overline{L_{I-J} \cdot h} \arrow[r, "\phi_K"] \arrow[d] & X_K \arrow[d] \\
(G \times G) \times_{P^-_{I-J} \times P_{I-J}} \overline{L_{I-J} \cdot \boldsymbol{h}_J} \arrow[r, "\boldsymbol{\phi}_J"] & \mathbf{X}_J
\end{tikzcd},$$

where all the maps are the natural ones. As $\boldsymbol{\phi}_J$ is an isomorphism it follows that ϕ_K is injective. As ϕ_K is a projective morphism this implies that ϕ_K is finite. Moreover, as $\overline{L_{I-J} \cdot h}$ is closed in X_K and invariant under $P^-_{I-J} \times P_{I-J}$ the image of ϕ_K is closed. Therefore ϕ_K is surjective and hence bijective. Moreover, due to the identification $\overline{L_{I-J}h} \cap X_0 \simeq (U \cap L_{I-J}) \times (U^- \cap L_{I-J}) \times (\overline{T \cdot h} \cap X'_0)$ it follows that ϕ_K is birational. This proves the first part of statement (3). When the characteristic is positive then X_K is Frobenius split (see e.g. [B-K, Thm.6.2.7]) and thus weakly normal (see e.g. [B-K, Thm.1.2.5]). It follows that ϕ_K is an isomorphism which ends the proof of statement (3).

By statement (1) and the Bruhat decomposition it easily follows that the union of $[K, v, w]$, for $v \in W^{I-J}, w \in W$, equals $(G \times G) \cdot h$. Moreover, when $X = \boldsymbol{X}$ then by [Sp, Lemma 1.3(i)] this union is disjoint (notice that our notation is slightly different from the notation used in [Sp] : the subset $[J, v, w]$ in [Sp] corresponds to $[I - J, x, w]$ in the present paper). As $\pi([K, v, w])$ equals the associated $B \times B$-orbit $[J, v, w]$ in $\boldsymbol{X}$ this proves statement (4) in general. $\square$

Remark 5.3. *Statement (3) in Proposition 5.2 above is also correct in characteristic* 0*. This follows from Theorem 9.1 which proves that X_K is normal and thus, by Zariski's main theorem, that ϕ_K is an isomorphism. A result similar to (3) for some special (i.e. regular) embeddings has earlier been obtained in* [B, Sect.2.1].

6. $B \times B$-orbit closures

In this section we will study inclusions between $B \times B$-orbit closures in a toroidal embedding X of G. We will state a precise description of when a $B \times B$-orbit $[K, v, w]$ is contained in the closure of another $B \times B$-orbit $[K', v', w']$. This generalizes the corresponding results of T. Springer for $X = \boldsymbol{X}$ given in [Sp, Sect.2]. As a consequence we will be able to prove that any $B \times B$-orbit closure Z in X of codimension ≥ 2 is a component of an intersection of $B \times B$-orbit closures distinct from Z. By standard Frobenius splitting techniques this will enable us to prove that each $B \times B$-orbit closure admits a canonical Frobenius splitting.

6.1. **Inclusions between $B \times B$-orbit closures.** Let $K \in \mathcal{I}$ and $J = p(K)$. Let $B_J = B \cap L_{I-J}$ and $B' = (w_0^{I-J} w_0) B (w_0^{I-J} w_0)^{-1}$. Define $\pi_J : B \to B_J$ by $\pi_J(bu) = b$, for $b \in B_J$ and $u \in U_{I-J}$, and $\pi'_J : B' \to B_J$ by $\pi'_J(bu) = b$, for $b \in B_J$ and $u \in U^-_{I-J}$. By Proposition 5.2(1) the base point h_K is invariant under $\mathrm{diag}(B_J)$. In particular, we may define a $B' \times B$ action on $G \times G \times \overline{B_J \cdot h_K}$ by

$$(b_1, b_2)(g_1, g_2, z) = (g_1 b_1^{-1}, g_2 b_2^{-1}, (\pi'_J(b_1), \pi_J(b_2))z),$$

for $b_1 \in B', b_2 \in B$, $g_1, g_2 \in G$ and $z \in \overline{B_J \cdot h_K}$. The associated quotient is denoted by $(G \times G) \times_{B' \times B} \overline{B_J \cdot h_K}$. The map $G \times G \times \overline{B_J \cdot h_K} \to X$, $(g_1, g_2, z) \mapsto (g_1, g_2)z$, induces a projective surjective morphism

$$p_K : (G \times G) \times_{B' \times B} \overline{B_J \cdot h_K} \to X_K.$$

which can be used to prove

Lemma 6.1. *Let $v, v', w, w' \in W$. Assume that $v w_0^{I-J} \leq v' w_0^{I-J}$ and $w' \leq w$ in the Bruhat order on W. Then*

$$(B\dot{v}', B\dot{w}') \cdot (B_J \cdot h_K) \subset \overline{(B\dot{v}, B\dot{w}) \cdot (B_J \cdot h_K)}.$$

Proof. By restricting the map p_K above we obtain a projective and surjective map

$$(\overline{B\dot{v}B'} \times \overline{B\dot{w}B}) \times_{B' \times B} \overline{B_J \cdot h_K} \to \overline{(B\dot{v}, B\dot{w}) \cdot (B_J \cdot h_K)}.$$

For the above statement to be true it thus suffices to have $Bv'B' \subset \overline{BvB'}$ and $Bw'B \subset \overline{BwB}$, which is clearly satisfied under the stated conditions. $\square$

Notice that when $v \in W^{I-J}$ then the set $(B\dot{v}, B\dot{w}) \cdot (B_J \cdot h_K)$, in Lemma 6.1, coincides with the orbit $[K, v, w]$.

Proposition 6.2. *Let $K, K' \in \mathcal{I}$, $v \in W^{I-p(K)}$, $v' \in W^{I-p(K')}$ and $w, w' \in W$. Then $[K', v', w']$ is contained in $\overline{[K, v, w]}$ if and only if $K \subset K'$ and there exists $u \in W_{I-p(K')}$ and $u' \in W_{I-p(K)} \cap W^{I-p(K')}$ such that $v u' u^{-1} \leq v'$, $w' u \leq w u'$.*

Proof. Notice $\overline{[K,v,w]} \subset \pi_1(\pi(\overline{[K,v,w]})) \cap X_K$. Thus if $[K',v',w'] \subset \overline{[K,v,w]}$, then $K \subset K'$ and $[p(K'),v',w'] \subset \overline{[p(K),v,w]}$. By [Sp, 2.4], there exists $u \in W_{I-p(K')}$ and $u' \in W_{I-p(K)} \cap W^{I-p(K')}$ such that $vu'u_1 \leq v'$, $w'u \leq wu'$.

On the other hand, assume that $v' \in W^{I-p(K')}$, $w' \in W$, $u \in W_{I-p(K')}$ and $u' \in W_{I-p(K)} \cap W^{I-p(K')}$ such that $vu'u_1 \leq v'$, $w'u \leq wu'$. Assume, moreover, that $K \subset K'$. By the one to one correspondence between the set of $G \times G$-orbits in X and the set of T-orbits in $\overline{X'_0}$ [B-K, Prop.6.2.3(ii)], it follows that $h_{K'} \in \overline{T \cdot h_K}$. Thus $(\dot{x}, \dot{x})h_{K'} \in \overline{T \cdot h_K}$ for all $x \in W_{I-p(K)}$ by Proposition 5.2(i). Therefore, with $J' = p(K')$, we find by use of Lemma 6.1,

$$[K',v',w'] = (B\dot{v}'\dot{u}, B\dot{w}'\dot{u}) \cdot h_{K'} \subset \overline{(B\dot{v}\dot{u}', B\dot{w}\dot{u}') \cdot (B_{J'} \cdot h_{K'})}.$$

As $u' \in W^{I-J'}$ we have $u'B_{J'} \subset Bu'$. Thus the right hand side of the above inclusion is contained in

$$\overline{(B\dot{v}B, B\dot{w}) \cdot ((u',u')h_{K'})} \subset \overline{(B\dot{v}, B\dot{w}) \cdot (B_J \cdot h_K)} = \overline{[K,v,w]},$$

which ends the proof. □

We may reformulate the above proposition to a slightly simpler version.

Proposition 6.3. *Let $K, K' \in \mathfrak{I}$, $v \in W^{I-p(K)}$, $v' \in W^{I-p(K')}$ and $w, w' \in W$. Then $[K',v',w'] \subset \overline{[K,v,w]}$ if and only if $K' \supset K$ and there exists $u \in W_{I-p(K)}$ such that $vu \leq v'$, $w' \leq wu$.*

Proof. If $[K',v',w'] \subset \overline{[K,v,w]}$, then in $\boldsymbol{X}$ we have

$$[I,v',w'] \subset \overline{[p(K'),v',w']} \subset \overline{[p(K),v,w]}.$$

By Proposition 6.2 there exists $u \in W_{I-p(K)}$ such that $vu \leq v'$, $w' \leq wu$. On the other hand, assume that $K' \supset K$ and there exists $u \in W_{I-p(K)}$ such that $vu \leq v'$, $w' \leq wu$. Write u as $u = u_1u_2$ for $u_1 \in W_{I-p(K)} \cap W^{I-p(K')}$ and $u_2 \in W_{I-p(K')}$. By [He, Cor.3.4], there exists $u'_2 \leq u_2$ such that $w'(u'_2)_1 \leq wu_1$. Moreover, $vu_1u'_2 \leq vu_1u_2 \leq v'$. Hence by Proposition 6.2, $[K',v',w'] \subset \overline{[K,v,w]}$ and the proposition is proved. □

For later reference we state the following easy consequences of the above propositions.

Corollary 6.4. *Let $K, K' \in \mathfrak{I}$, $v \in W^{I-p(K)}$, $v' \in W^{I-p(K')}$ and $w, w' \in W$.*

(1) *If $\overline{[K',v',w']} \subset \overline{[K,v,w]}$ then $v \leq v'$.*
(2) *$\overline{[K,v,w']} \subset \overline{[K',v,w]}$ if and only if $w' \leq w$ and $K' \subset K$.*

6.2. **Intersection of $B \times B$-orbit closures.** In this section we will prove.

Proposition 6.5. *Let $Z \neq X$ denote a $B \times B$-orbit closure in X. If Z has codimension 1 in X the Z is either a boundary divisor X_i, $1 \leq i \leq n$, of X or else Z coincides with the closure of a codimension 1 Bruhat cell $B\dot{s}_i\dot{w}_0B$, $1 \leq i \leq l$, within X. If the codimension of Z is ≥ 2 then there exist $B \times B$-orbit closures $Z_1 \neq Z$ and $Z_2 \neq Z$ in X such that Z is a component of the intersection $Z_1 \cap Z_2$.*

The proof of Proposition 6.5 will depend on the following 4 lemmas.

Lemma 6.6. *Let $w \in W$ be an element of length $l(w) < l(w_0) - 1$. Then there exist elements w' and w'' distinct from w such that $\overline{[\varnothing, 1, w]}$ is an irreducible component of $\overline{[\varnothing, 1, w']} \cap \overline{[\varnothing, 1, w'']}$.*

Proof. Choose simple reflections s_i and s_j such that $l(ws_i) = l(s_jw) = l(w) + 1$. If ws_i and s_jw are distinct then the statement follows by setting $w' = ws_i$ and $w'' = s_jw$. If $ws_i = s_jw$, then we choose a simple reflection s_k such that $l(ws_is_k) = l(ws_i) + 1 = l(w) + 2$. Then $k \neq i$. As $ws_is_k = s_jws_k$, we conclude that $l(ws_k) = l(w) + 1$. The statement follows by setting $w' = ws_i$ and $w'' = ws_k$. $\square$

Lemma 6.7. *For $K \in \mathfrak{I}$ and $w \in W$, $\overline{[K, 1, w]}$ is an irreducible component of $\overline{[\varnothing, 1, w]} \cap \overline{[K, 1, w_0]}$.*

Proof. By Proposition 6.3, $\overline{[K, 1, w]} \subset \overline{[\varnothing, 1, w]} \cap \overline{[K, 1, w_0]}$. As X is a finite union of $B \times B$-orbits each irreducible component of the intersection $\overline{[\varnothing, 1, w]} \cap \overline{[K, 1, w]}$ will be the closure of a $B \times B$-orbit in X. Assume that $K' \in \mathfrak{I}$, $v \in W^{I-p(K')}$ and $w' \in W$ satisfy

$$\overline{[K, 1, w]} \subset \overline{[K', v, w']} \subset \overline{[\varnothing, 1, w]} \cap \overline{[K, 1, w_0]}.$$

Then by Corollary 6.4(1) we have $v = 1$. Moreover, Proposition 6.3 implies that $K' = K$. Then Corollary 6.4(2) shows that $w' = w$, which ends the proof. $\square$

Lemma 6.8. *Let $v, v' \in W^{I-p(K)}$ with $v = s_iv'$ for some $i \in I$ and $l(v) = l(v') + 1$. Then $\overline{[K, v, w_0]}$ is an irreducible component of $\overline{[K, v', w_0]} \cap \overline{[\varnothing, 1, w_0v\imath]}$.*

Proof. By Proposition 6.3 we easily conclude $\overline{[K, v, w_0]} \subset \overline{[\varnothing, 1, w_0v\imath]}$ and $\overline{[K, v, w_0]} \subset \overline{[K, v', w_0]}$. Assume that $w \in W^{I-p(K)}$ and $w' \in W$ satisfy

$$\overline{[K, v, w_0]} \subset \overline{[K, w, w']} \subset \overline{[K, v', w_0]} \cap \overline{[\varnothing, 1, w_0v\imath]}.$$

Then by Corollary 6.4 (i), $v' \leq w \leq v$. So $w = v'$ or $w = v$. Moreover, by Proposition 6.3 there exists $u \in W_{I-p(K)}$ such that $wu \leq v$ and $w_0 \leq w'u$. As $v \in W^{I-p(K)}$ we conclude that $u = 1$ and $w' = w_0$. Then, by Proposition 6.3, there exists $u' \in W$ such that $u' \leq w$ and

$w_0 \leq w_0 v\imath u'$. Thus $u' = v$ and w must then be equal to v. The lemma is proved. $\square$

Lemma 6.9. *We keep the assumptions on v and v' from the previous Lemma 6.8. Then for $w \in W$, $\overline{[K, v, w]}$ is an irreducible component of $\overline{[K, v, w_0]} \cap \overline{[K, v', w]}$.*

Proof. By Proposition 6.3 we have $\overline{[K, v, w]} \subset \overline{[K, v, w_0]} \cap \overline{[K, v', w]}$. Assume that $u \in W^{I-p(K)}$ and $w' \in W$ satisfy $\overline{[K, v, w]} \subset \overline{[K, u, w']} \subset \overline{[K, v, w_0]} \cap \overline{[K, v', w]}$. Then, by Corollary 6.4(i), $u = v$ and hence by Corollary 6.4(ii) we have $w \leq w'$. Moreover, by Proposition 6.3 there exists $u' \in W_{I-p(K)}$ such that $v'u' \leq v$ and $w' \leq wu'$. We conclude that $u = 1$ and as a consequence that $w' = w$. $\square$

We can now prove Proposition 6.5.

Proof. Let $K \in \mathfrak{I}$, $v \in W^{I-p(K)}$ and $w \in W$ such that $Z = \overline{[K, v, w]}$. Notice that by Proposition 6.3 the closure $\overline{[\varnothing, 1, w_0]}$ contains all $B \times B$-orbit closures and hence it will be equal to X.

We first consider the situation when $w \neq w_0 :$ if there exists a simple reflection s_i such that $l(s_i v) = l(v) - 1$ then by Lemma 6.9 we may use $Z_1 = \overline{[K, v, w_0]}$ and $Z_2 = \overline{[K, s_i v, w]}$ (notice that this makes sense as $s_i v \in W^{I-p(K)}$). So we may assume that $v = 1$. If now $K \neq \varnothing$ then by Lemma 6.7 we may use $Z_1 = \overline{[\varnothing, 1, w]}$ and $Z_2 = \overline{[K, 1, w_0]}$. So we may assume that $Z = \overline{[\varnothing, 1, w]}$. If $l(w) < l(w_0) - 1$ then we may apply Lemma 6.6 to define Z_1 and Z_2. This leaves us with the cases $\overline{[\varnothing, 1, s_i w_0]}$, $i = 1, \dots, l$, which are equal to the closures of the Bruhat cells $B\dot{s}_i \dot{w}_0 B \subseteq G$ within X.

Next assume that $w = w_0 :$ if there exists a simple reflection s_i such that $l(s_i v) = l(v) - 1$ then by Lemma 6.8 we may use $Z_1 = \overline{[K, s_i v, w_0]}$ and $Z_2 = \overline{[\varnothing, 1, w_0 v\imath]}$. So we may assume that $v = 1$. As $Z \neq X$ we have that $Z = \overline{[K, 1, w_0]}$ with K a nonempty set. In particular, by Proposition 6.3, Z coincides with the $G \times G$-orbit closure X_K. Now let $K' \subset K$ be a minimal subset such that X_K is an irreducible component of $\cap_{i \in K'} X_i$. If $|K'| = 1$ then Z coincides with a boundary divisor, so we may assume that $|K'| > 1$. Let $j \in K'$ and let $Y_1, \dots, Y_s$ denote the irreducible components of the intersection $\cap_{i \in K' - \{j\}} X_i$. Then, by minimality of K', each Y_i, for $i = 1, \dots, s$, is a $G \times G$-orbit closure distinct from Z. Moreover, there exists an i such that Z is an irreducible component of the intersection $Y_i \cap X_j$. Now use $Z_1 = Y_i$ and $Z_2 = X_j$. $\square$

7. Frobenius splitting of $B \times B$-orbit closures

Let X denote an equivariant embedding of the reductive group G over a field of positive characteristic $p > 0$. As above the boundary divisors of X will be denoted by $X_1, \dots, X_n$. Moreover, we will use the notation D_i, $i = 1, \dots, l$, to denote the closures of the Bruhat cells $B\dot{s}_i\dot{w}_0B$, $i = 1, \dots, l$, within X.

Proposition 7.1. *The equivariant embedding X admits a $(B \times B, T \times T)$-canonical Frobenius splitting which compatibly splits the closure of all $B \times B$-orbits.*

Proof. First of all X admits a $(B \times B, T \times T)$-canonical Frobenius splitting s which compatibly splits all boundary component X_j, $j = 1, \dots, n$, and the subvarieties D_i, $i = 1, \dots, l$ (see [B-K, Thm.6.2.7]).

Consider, for a moment, the case when X is toroidal. We claim that s compatibly Frobenius splits all $B \times B$-orbit closures. If this is not the case, then there exists a $B \times B$-orbit closure Z of maximal dimension which is not compatibly Frobenius split by s. By Proposition 6.5 the codimension of Z must be ≥ 2. In particular, we can find orbit closures $Z_1 \neq Z$ and $Z_2 \neq Z$ such that Z is a component of the intersection $Z_1 \cap Z_2$. By the maximality assumption on Z the orbit closures Z_1 and Z_2 will be compatibly Frobenius split by s. But then every component of $Z_1 \cap Z_2$, and thus Z, will also be compatibly Frobenius split by s, which is a contradiction. This ends the proof when X is toroidal.

For an arbitrary embedding X we may find a toroidal embedding X' of G and a birational projective morphism $f : X' \to X$ extending the identity map on G (see e.g. [B-K, Prop.6.2.5]). Now X' admits a $(B \times B, T \times T)$-canonical Frobenius splitting s' which compatibly Frobenius splits all $B \times B$-orbit closures. By Zariski's main theorem the map $f^\sharp : \mathcal{O}_{X'} \to f_*\mathcal{O}_X$ induced by f is an isomorphism. In particular, s' induces by push forward a $(B \times B, T \times T)$-canonical Frobenius splitting s of X. Moreover, the image in X of every $B \times B$-orbit closure in X' will be compatibly Frobenius split by s. But any $B \times B$-orbit closure in X is the image of a similar orbit closure in X'. This ends the proof. $\square$

7.1. **Cohomology vanishing.** As a direct consequence of Proposition 7.1 we conclude the following vanishing result (see e.g. [B-K, Thm.1.2.8]).

Proposition 7.2. *Let X be a projective equivariant embedding of G. Let Z denote a $B \times B$-orbit closures in X and let $\mathcal{L}$ denote an ample line bundle on Z. Then*

$$\mathrm{H}^i\big(Z, \mathcal{L}\big) = 0, \ \ i > 0.$$

Moreover, if $Z' \subset Z$ is another $B \times B$-orbit closure then the restriction map

$$\mathrm{H}^0\big(Z, \mathcal{L}\big) \to \mathrm{H}^0\big(Z', \mathcal{L}_{Z'}\big),$$

is surjective.

Later (Corollary 8.5) we will see that the vanishing part of Propositione 7.2 remains true when the line bundle $\mathcal{L}$ is only assumed to be nef, i.e. when $\mathcal{L} \otimes \mathcal{M}$ is an ample line bundle for every ample line bundle $\mathcal{M}$.

8. Global F-regularity of $B \times B$-orbit closures

We are now ready to state and prove the main result of the paper.

Theorem 8.1. *Let X denote a projective equivariant embedding of a reductive group G over a field of positive characteristic $p > 0$. Let Z denote a $B \times B$-orbit closure in X. Then Z is globally F-regular.*

We will divide the proof of Theorem 8.1 into 2 parts. The first part concerns the case when X is toroidal.

Lemma 8.2. *Let X be a projective toroidal embedding. Then any $B \times B$-orbit closure $\overline{[K, v, w]}$ in X is globally F-regular.*

Proof. Keep the notation of Section 6.1. As a consequence of Proposition 7.1, X admits a $(B' \times B, T \times T)$-canonical Frobenius splitting s which compatibly Frobenius splits every $B' \times B$-orbit closure.

Let $Y = \overline{(B' \times B)h_K}$ and $Y' = Y - (B' \times B)h_K$. Then s induces a $(B' \times B, T \times T)$-canonical Frobenius splitting s_Y of Y which compatibly Frobenius splits Y'. Notice that by Proposition 5.2(1), $Y = \overline{B_J \cdot h_K}$. Thus by Proposition 5.2(2), Y is the closure of the Borel subgroup B_J of L_{I-J} within some equivariant embedding of L_{I-J}. Hence, Y is a large Schubert variety for some equivariant embedding of L_{I-J} and, as such, Y is globally F-regular [B-T, Thm.4.3]. Define $v' = w_0^{I-p(K)} w_0 v$. Then $(B' \times B)_{(v', w)}$ contains the group $B_J \times \{1\}$ (notice that the set of positive roots on the first coordinate is defined with respect to B') and thus by Proposition 5.2(1)

$$(B' \times B)h_K = (B_J \times \{1\})h_K = (B' \times B)_{(v',w)}h_K.$$

The above statements proves that the triple $(Y, h_K, (v', w))$ satisfies the requirements of Proposition 4.1. Now Theorem 4.4 shows that the closed subvariety

$$(\overline{B'\dot{v}'B'}, \overline{B\dot{w}B})\overline{B_J \cdot h_K} = (\dot{w}_0^{I-J}\dot{w}_0, 1)\overline{[K, v, w]},$$

is globally F-regular. Thus also $\overline{[K, v, w]}$ must be globally F-regular. □

8.1. The general case.

Let X denote an arbitrary equivariant projective embedding of G. To handle the proof of Theorem 8.1 for X we start by the following construction : Consider the natural $G \times G$-equivariant embedding

$$f : G \to X \times \mathbf{X}.$$

and let Y denote the normalization of the closure of the image of f. Then Y is a projective equivariant toroidal embedding of G. We let ϕ :

$Y \to X$ denote the associated $G \times G$-equivariant projective morphism to X. Then

Lemma 8.3. *Let Z' denote the closure of a $B \times B$-orbit within Y and let Z denote its image $\phi(Z')$ within X. Then the induced morphism $\phi' : Z' \to Z$ is a rational morphism.*

Proof. We will prove this using Lemma 4.2. Notice first of all that ϕ is birational and X is normal, so by Zariski's main theorem we have $\phi_* \mathcal{O}_Y = \mathcal{O}_X$. Let now $\mathcal{L}$ denote a very ample line bundle on X. Then by Lemma 8.2 and [S2, Cor.4.3],

$$\mathrm{H}^i\big(Y, \phi^* \mathcal{L}\big) = \mathrm{H}^i\big(Z', \phi^* \mathcal{L}\big) = 0, \; i > 0,$$

as $\phi^* \mathcal{L}$ is globally generated and thus nef.

Let $\tilde{\boldsymbol{D}}_i$, $i = 1, \dots, l$, denote the closures $\overline{B^- \dot{s}_i \dot{w}_0 B^-}$ in $\boldsymbol{X}$. Then the divisor $\tilde{\boldsymbol{D}} = \sum_{i=1}^{l} \tilde{\boldsymbol{D}}_i$ is ample [B-K, Prop.6.1.11]. Let $\mathcal{M} = \mathcal{O}_{\boldsymbol{X}}(\tilde{\boldsymbol{D}})$ denote the associated line bundle and let $\mathcal{M}' = \phi^* \mathcal{M}$ be its pull back to Y. Let $\boldsymbol{s}$ denote the canonical section of $\mathcal{M}$ and let s' denote its pull back to Y. Let V denote an irreducible component of the support of s'. If V is contained in the boundary of Y then the support of s' will contain a closed $G \times G$-orbit. In particular, also the support $\cup_i \tilde{\mathbf{D}}_i$ of s will contain a closed $G \times G$-orbit. As the latter is not the case we conclude that each component of the support of s' will intersect G. Moreover, the support of s' is $B^- \times B^-$-stable. As a consequence, we conclude that the divisor of zeroes of s' equals

$$\sum_{i=1}^{l} n_i \tilde{D}'_i,$$

for some positive integers n_i and with $\tilde{D}'_i$, $i = 1, \dots, l$, denoting the closure $\overline{B^- \dot{s}_i \dot{w}_o B^-}$ in Y.

Let Y_j, $j = 1, \dots, n$, denote the boundary components in Y and let D'_i, $i = 1, \dots, l$, denote the closures $\overline{B \dot{s}_i \dot{w}_0 B}$ in Y. Let Y^0 denote the smooth locus of Y. Then Y^0 admits a Frobenius splitting which compatibly Frobenius splits the Cartier divisors $Y^0 \cap Y_j$, $j = 1, \dots, n$, and $D'_i \cap Y^0$ and $\tilde{D}'_i \cap Y^0$, $i = 1, \dots, l$ [B-K, Thm.6.2.7]. As in the proof of Proposition 4.3 we conclude that Y^0 admits a stable Frobenius splitting along the effective divisor

$$\operatorname{div}(s') \cap Y^0 = \sum_{i=1}^{l} n_i (\tilde{D}'_i \cap Y^0),$$

which compatibly Frobenius splits $D'_i \cap Y^0$, $i = 1, \dots, l$, and $Y_j \cap Y^0$, $j = 1, \dots, n$. Let ψ_0 denote such a stable Frobenius splitting; i.e. let e be an integer such that ψ_0 is a splitting of the morphism

$$\mathcal{O}_{Y^0} \to F^e_* \mathcal{M}'_{|Y^0},$$

defined by the restriction of s' to Y^0. Let now $i : Y^0 \to Y$ denote the inclusion morphism. Applying the functor i_* to the above split morphism and using that Y is normal, we find that the morphism

$$\mathcal{O}_Y \to F^e_* \mathcal{M}',$$

defined by s' has an induced splitting ψ. Then ψ defines a stable Frobenius splitting along $\mathrm{div}(s')$ which compatibly Frobenius splits D_i', $i = 1, \dots, l$, and Y_j, $j = 1, \dots, n$ (as the compatibility can be checked on the open dense subsets Y^0).

We now claim that Z' is not contained in any $\tilde{D}_i'$. To see this assume that Z' is contained in $\tilde{D}_i'$ for some i. As Z' is $B \times B$-invariant and as $\tilde{D}_i'$ is $B^- \times B^-$-invariant it follows that $(B^-B, B^-B)Z'$ is contained in $\tilde{D}_i'$. But then also $(G, G)Z'$ must be contained in $\tilde{D}_i'$. We conclude that $\tilde{\mathbf{D}}_i'$ contains a closed $G \times G$-orbit which is a contradiction. Hence, Z' is not contained in the support of s'. As in the proof of Proposition 7.1 we may then use Proposition 6.5 to show that Z' is compatibly Frobenius split by the stable Frobenius splitting ψ. By [T, Lem.4.8] it follows that we have an embedding

$$\mathrm{H}^1\big(Y, \mathcal{I}_{Z'} \otimes \phi^* \mathcal{L}\big) \subset \mathrm{H}^1\big(Y, \mathcal{I}_{Z'} \otimes \phi^* \mathcal{L}^{p^e} \otimes \mathcal{M}'\big)$$

of abelian groups, where $\mathcal{I}_{Z'}$ denotes the sheaf of ideals associated to Z'. But $\mathcal{L}^{p^e} \boxtimes \mathcal{M}$ is ample on $X \times \boldsymbol{X}$ and, as the map $Y \to X \times \boldsymbol{X}$ is finite, we conclude that $\phi^* \mathcal{L}^{p^e} \otimes \mathcal{M}'$ is ample on Y. Applying [B-K, Thm.1.2.8] it follows that $\mathrm{H}^1\big(Y, \mathcal{I}_{Z'} \otimes \phi^* \mathcal{L}\big)$ is zero.

As all the requirement in Lemma 4.2 are now satisfied this ends the proof. $\square$

We may now prove Theorem 8.1

Proof. By Corollary 3.3 and Lemma 8.3 we may assume that X is toroidal. Now apply Lemma 8.2. $\square$

8.2. **Applications.** As the main application of Theorem 8.1 we find.

Corollary 8.4. *Let X denote an equivariant embedding of a reductive group G over a field of positive characteristic. Then every $B \times B$-orbit closure in X is strongly F-regular. In particular, every $B \times B$-orbit closure is normal, Cohen-Macaulay and locally F-rational.*

Proof. As in the proof of [B-T, Cor.4.2] we may reduce to the case when X is projective. Then by Theorem 8.1 every $B \times B$-orbit closure is globally F-regular and thus strongly F-regular. This ends the proof. $\square$

We also obtain the following strengthening of Proposition 7.2.

Corollary 8.5. *Let X denote a projective equivariant embedding of a reductive group G over a field of positive characteristic. Let Z denote* *the closure of a $B \times B$-orbit and let $\mathcal{L}$ be a nef line bundle on Z. Then the cohomology $\mathrm{H}^i\big(Z, \mathcal{L}\big)$ vanishes for $i > 0$.*

Proof. Just apply [S2, Cor.4.3]. $\square$

9. The characteristic 0 case

Let X denote a scheme of finite type over a field K of characteristic 0. Then there exists a finitely generated $\mathbb{Z}$-algebra A and a flat scheme X_A of finite type over A, such that the base change of X_A to K may be naturally identified with X. Moreover, when $m \subset A$ is a maximal ideal we may form the base change $X_{k(m)}$ of X_A to the finite field $k(m) = {}^{A}\!/_{m}$. We then say that the scheme X is of *strongly F-regular type* (resp. *F-rational type*) if $X_{k(m)}$ is strongly F-regular (resp. F-rational) for all maximal ideals m in a dense open subset of Spec(A).

Any scheme X of strongly F-regular type will also be of F-rational type. Thus, by [S, Thm.4.3], schemes of strongly F-regular type will have rational singularities, in particular, they will be normal and Cohen-Macaulay.

In the proof of the next result we will use the following observation (see e.g. [H-H3, Thm.5.5(e)]: let $\overline{k}(m)$ denote an algebraic closure of the field $k(m)$. If the base change $X_{\overline{k}(m)}$ is strongly F-regular then also $X_{k(m)}$ is strongly F-regular.

We can now prove the characteristic 0 version of Corollary 8.4.

Theorem 9.1. *Let X denote an equivariant embedding of a reductive group G over an algebraically closed field k of characteristic 0. Then every $B \times B$-orbit closure in X is of strongly F-regular type. In particular, every $B \times B$-orbit closure in X has rational singularities.*

Proof. We may assume that there exists a split $\mathbb{Z}$-form $G_{\mathbb{Z}}$ of G over which B is defined by a closed subscheme $B_{\mathbb{Z}}$. Let Z denote a $B \times B$-orbit closure in X. The complete data consisting of the $G \times G$-action on X, the open embedding $G \subset X$, the $B \times B$-stability of Z, the closed embedding $Z \subset X$ and the irreducibility of X and Z may all be descended to some finitely generated $\mathbb{Z}$-algebra A (see e.g. [H-H2, Sect.2] for this kind of technique). This means that there exists schemes $G_A := G_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(A)$, $B_A := B_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(A)$, X_A and Z_A flat and of finite type over $\mathrm{Spec}(A)$ satisfying, that for every maximal ideal $m \subseteq A$ the associated base changes $G_{\overline{k}(m)}$, $B_{\overline{k}(m)}$, $X_{\overline{k}(m)}$ and $Z_{\overline{k}(m)}$, to an algebraic closure $\overline{k}(m)$ of the field $k(m) = {}^{A}\!/_{m}$, share the same structure; i.e. $G_{\overline{k}(m)}$ is a reductive linear algebraic group, $X_{\overline{k}(m)}$ is an irreducible $G_{\overline{k}(m)} \times G_{\overline{k}(m)}$-variety containing $G_{\overline{k}(m)}$ as an open subset and $Z_{\overline{k}(m)}$ is an irreducible $B_{\overline{k}(m)} \times B_{\overline{k}(m)}$-stable subvariety of $X_{\overline{k}(m)}$. As X is normal we may even assume that $X_{\overline{k}(m)}$ is normal (see [H-H2, Thm.2.3.17]). In particular, $X_{\overline{k}(m)}$ is then an equivariant embedding of the reductive group $G_{\overline{k}(m)}$. Moreover, by the finiteness of the number of $B_{\overline{k}(m)} \times B_{\overline{k}(m)}$-orbits in $X_{\overline{k}(m)}$ we conclude that $Z_{\overline{k}(m)}$ is the closure of such an orbit.

Applying Corollary 8.4 and the observation above, we conclude that Z is of strongly F-regular type and thus also of F-rational type. Finally, as mentioned above, the latter statement implies that Z has rational singularities. □

We may now generalize Corollary 8.5 to arbitrary characteristics.

Corollary 9.2. *Let X denote a projective equivariant embedding of a reductive group G over a field of arbitrary characteristic. Let Z denote the closure of a $B \times B$-orbit and let $\mathcal{L}$ be a nef line bundle on Z. Then the cohomology $\mathrm{H}^i\big(Z, \mathcal{L}\big)$ vanishes for $i > 0$.*

Proof. Apply Corollary 8.5 and [S2, Cor.5.5]. □

For a discussion of other kinds of vanishing results for varieties of globally F-regular type we refer to [S2, Sect.5].

References

[B-K] M. Brion and S. Kumar, *Frobenius Splittings Methods in Geometry and Representation Theory*, Progress in Mathematics (2004), Birkhäuser, Boston.

[B] M. Brion, *The behaviour at infinity of the Bruhat decomposition*, Comment. Math. Helv. **73** (1998), 137–174.

[B2] M. Brion, *Multiplicity-free subvarieties of flag varieties*, in *Commutative Algebra*, Contemporary Math. **331** (2003) 13–23.

[B-P] M. Brion and P. Polo, *Large Schubert Varieties*, Represent. Theory **4** (2000), 97–126.

[B-T] M. Brion and J. F. Thomsen, *F-regularity of large Schubert varieties*, math.AG/0408180.

[D-G] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, 1970.

[H] R .Hartshorne, *Ample Subvarieties of Algebraic Varieties*, Lecture notes in math. **156** (1970), Springer-Verlag.

[H-H] M. Hochster and C. Huneke, *Tight closure and strong F-regularity*, Mémoires de la Société Mathématique de France **38** (1989), 119–133.

[H-H2] M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62.

[H-H3] M. Hochster and C. Huneke, *Tight closure in equal characteristic zero*, preprint.

[He] X. He, *The G-stable pieces of the wonderful compactification*, math.RT/0412302.

[L-S] G. Lyubeznik and K. E. Smith, *Weak and strong F-regularity are equivalent in graded rings*, Amer. Math. Soc. **121** (1999), 1279–1290.

[L-T] N. Lauritzen and J. F. Thomsen, *Line bundles on Bott-Samelson varieties*, J. Alg. Geom. **13** (2004), 461–473.

[L-P-T] N. Lauritzen, U. R. Pedersen and J. F. Thomsen, *Global F-regularity of Schubert varieties with applications to D-modules*, math.AG/0402052.

[R] A, Ramanathan, *Equations defining Schubert varieties and Frobenius splitting of diagonals*, Inst. Hautes Études Sci. Publ. Math. **65** (1987), 61–90.

[S] K. E. Smith, *F-rational rings have rational singularities*, Amer. J. Math. **119** (1997), 159–180.

[S2] K. E. Smith, *Globally F-regular varieties: Applications to vanishing theorems for quotients of Fano varieties*, Michigan Math. J. **48** (2000), 553–572.

[Sp] T. A. Springer, *Intersection cohomology of $B \times B$-orbits closures in group compactifications*, J. Alg. **258** (2002), 71–111.

[T] J.F. Thomsen, *Frobenius splitting of equivariant closures of regular conjugacy classes*, math.AG/0502114.

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

E-mail address: `hugo@math.ias.edu`

Institut for matematiske fag, Aarhus Universitet, 8000 Århus C, Denmark

E-mail address: `funch@imf.au.dk`