GREEN POLYNOMIALS OF WEYL GROUPS, ELLIPTIC PAIRINGS, AND THE EXTENDED DIRAC INDEX

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Abstract. We provide a direct connection between Springer theory, via Green polynomials, the irreducible representations of the pin cover $\widetilde{W}$, a certain double cover of the Weyl group W, and an extended Dirac operator for graded Hecke algebras. Our approach leads to a new and uniform construction of the irreducible genuine $\widetilde{W}$-characters. In the process, we give a construction of the action by an outer automorphism of the Dynkin diagram on the cohomology groups of Springer theory, and we also introduce a q-elliptic pairing for W with respect to the reflection representation V. These constructions are of independent interest. The q-elliptic pairing is a generalization of the elliptic pairing of W introduced by Reeder, and it is also related to S. Kato's notion of (graded) Kostka systems for the semidirect product $A_W = \mathbb{C}[W] \ltimes S(V)$.

1. Introduction

1.1. Graded affine Hecke algebras were defined by Lusztig [19] in his study of representations of reductive p-adic groups and Iwahori-Hecke algebras. A Dirac operator $\mathcal{D}$ for graded affine Hecke algebras was defined in [1], and, by analogy with the setting of Dirac theory for $(\mathfrak{g}, K)$-modules of real reductive groups, the notion of Dirac cohomology was introduced. The Dirac cohomology and the Dirac index in the Hecke algebra setting were further studied in [7, 8]. The Dirac cohomology spaces are representations for a certain double cover ("pin cover") $\widetilde{W}$ of the Weyl group W. The irreducible representations of $\widetilde{W}$ had been classified case by case in the work of Schur, Morris, Read and others, see for example [23, 25, 34]. Recently, it was remarked in [6] (again case by case) that there is a close relation between the representation theory of $\widetilde{W}$ and the geometry of the nilpotent cone in semisimple Lie algebras $\mathfrak{g}$.

In this paper, we provide a direct link between:

(a) the Springer W-action on cohomology groups and an extension of it to a $W \rtimes \langle\delta\rangle$-action, for the automorphism δ given by the action of the long Weyl group element;
(b) the irreducible representations of $\widetilde{W}$, and
(c) an extended Dirac index for tempered modules of graded Hecke algebras.

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In particular, our approach leads to a new, uniform construction of the irreducible genuine $\widetilde{W}$-characters, see Theorem 1.3 below. The starting point is a reinterpretation of the Lusztig-Shoji algorithm in terms of certain q-elliptic pairings for W with respect to the reflection representation V (see section 2.1); here q is an indeterminate. This is a generalization of the elliptic pairing of W introduced in [26], see also [24, 8, 7]. The q-elliptic pairing is also related to Kato's notion of (graded) Kostka systems for the semidirect product $A_W = \mathbb{C}[W] \ltimes S(V)$ and the graded Euler-Poincaré pairing.

1.2. Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and Weyl group W. Let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{g}$ and $\mathcal{N}^{\mathsf{sol}}$ be the subset consisting of those nilpotent elements whose connected centralizers $Z_G(e)^0$ are solvable. For every element $e \in \mathcal{N}$, let $A(e) = Z_G(e)/Z_G(e)^0$ be the component group of its centralizer and $\widehat{A(e)}_0$ be the set of $A(e)$-representations of Springer type. For $\phi \in \widehat{A(e)}_0$, we consider certain q-graded representations of W, denoted $X_q(e,\phi)$, see section 2.2, defined using the Springer action [31] on cohomology groups $H^*(\mathcal{B}_e)^\phi = \mathrm{Hom}_{A(e)}[\phi, H^*(\mathcal{B}_e)]$. By analyzing the Lusztig-Shoji algorithm [18, 29], we prove first:

Theorem 1.1. *Let $e, e' \in \mathcal{N}$.*

1. *If $G \cdot e \neq G \cdot e'$, then $X_q(e,\phi)$ and $X_q(e',\phi')$ are orthogonal with respect to the q-elliptic pairing on W, for all $\phi \in \widehat{A(e)}_0$, $\phi' \in \widehat{A(e')}_0$.*
2. *The map $X_q(e,\phi) \to \phi$ is an isometry with respect to the q-elliptic pairing of W, and a certain (q, M)-elliptic pairing of $A(e)$, where M is the q-graded $A(e)$-representation defined in (2.8.1). When $e \in \mathcal{N}^{\mathsf{sol}}$, M is the natural representation of $A(e)$ on the space of complex characters of the central torus in $Z_G(e)^0$.*

The case $q = 1$ was known before from [26], where it was obtained by different methods.

The main application to $\widetilde{W}$-representations is the specialization $q = -1$. This case is related to an action on $H^*(\mathcal{B}_e)$ of the extended group $W_\# = W \rtimes \langle \delta \rangle$, where δ is the automorphism of G corresponding to $-w_0$. Here w_0 is the longest Weyl group element. We define this action in section 4, by extending the Springer action, and relate it to the results in [2, 14, 29] to obtain the following theorem.

Theorem 1.2. *For every $e \in \mathcal{N}$ and $i \geqslant 0$,*

$$\mathrm{tr}(\delta w, H^{2i}(\mathcal{B}_e)) = (-1)^i \mathsf{sgn}(w_0)\, \mathrm{tr}(w_0 w, H^{2i}(\mathcal{B}_e)).$$

The main technical difficulty that one needs to bypass is that δ does not immediately act on $\mathcal{B}_e$ since $\delta(e) \neq e$ in general. Theorem 1.2 turns out to be also intrinsically related to the δ-extended trace and the index of a Dirac operator on tempered modules of the extended graded Hecke algebra $\mathbb{H}_\# = \mathbb{H} \rtimes \langle \delta \rangle$, see section 6, in particular Theorem 6.10. The extended Dirac operator and its index are natural complements to the case studied in [1, 7, 8], and when w_0 is not central, they provide more information. For example, when $G = PGL(n)$, the only tempered $\mathbb{H}_n^A$-module with nonzero Dirac index is the Steinberg module St. On the other hand, the tempered modules with nonzero extended Dirac index are those of the form $\mathrm{Ind}_{\mathbb{H}_J^A}^{\mathbb{H}_n^A}(\mathsf{St})$, where $\mathbb{H}_J^A = \prod_{j=1}^k \mathbb{H}_{m_j}^A$, with $\sum_{j=1}^k m_j = n$ and all m_j distinct.

1.3. We explain next the main results concerning $\widetilde{W}$-representations that we obtain via this approach.

Let $R(\widetilde{W})$ be the (complexification of the) Grothendieck group of finite dimensional $\widetilde{W}$-representations, and $R(\widetilde{W})_{\mathsf{gen}}$ the subspace spanned by the genuine irreducible $\widetilde{W}$-representations, i.e., those which do not factor through to W. Let V be the reflection representation of W. Let $C(V)$ be the Clifford algebra of V with respect to a W-invariant inner product $(\ ,\ )$, and let $\mathcal{S}$ be the unique simple spin $C(V)$-module (when $\dim V$ is even), respectively the sum of the two simple spin $C(V)$-modules (when $\dim V$ is odd). For every $e \in \mathcal{N}$ and every $\phi \in \widehat{A(e)}_0$, set

$$\widetilde{\Sigma}(e,\phi) = X_{-1}(e,\phi) \otimes \mathcal{S}, \tag{1.3.1}$$

By definition, this is a (virtual) character in $R(\widetilde{W})_{\mathsf{gen}}$, self dual under tensoring with sgn. Theorem 1.2 implies that

$$\mathrm{tr}(w, X_{-1}(e,\phi)) = (-1)^{\dim \mathcal{B}_e} \mathsf{sgn}(w_0)\, \mathrm{tr}(ww_0\delta, H^*(\mathcal{B}_e)^\phi). \tag{1.3.2}$$

We also remark that $\widetilde{\Sigma}(e,\phi)$ depends only on the image of ϕ in

$$\overline{R}_{-1}(A(e)) = R_{-1}(A(e))/\mathrm{rad}\langle\ ,\ \rangle^{-1}_{A(e)}.$$

To see this, denote

$$W_{\text{(-1)-ell}} = \{w \in W : \det_V(1+w) \neq 0\}. \tag{1.3.3}$$

As we explain in section 5, the radical of the (-1)-elliptic form $\langle\ ,\ \rangle^{-1}_W$ on $R(W)$ can be identified with the space of characters supported on the complement $W \setminus W_{\text{(-1)-ell}}$. On the other hand, $\mathcal{S}$ is supported precisely on the preimage of $W_{\text{(-1)-ell}}$ in $\widetilde{W}$, by (6.5.5), thus tensoring with $\mathcal{S}$ kills the radical of $\langle\ ,\ \rangle^{-1}_W$ on $R(W)$. Because of this, it makes sense to use the notation $\widetilde{\Sigma}(e,[\phi])$, where $[\phi]$ is the image of ϕ in $\overline{R}_{-1}(A(e))$. Moreover, we may extend the definition of $\widetilde{\Sigma}(e,[\phi])$ linearly with respect to $\overline{R}_{-1}(A(e))$ and thus talk about $\widetilde{\Sigma}(e,[\chi])$ for an arbitrary element $[\chi] \in \overline{R}_{-1}(A(e))$ in the span of $\widehat{A(e)}_0$. The main results of section 7 may be summarized as follows. Set $a_V = 1$, if $\dim V$ is even, and $a_V = 2$, if $\dim V$ is odd.

Theorem 1.3.

(1) *Let $e \in \mathcal{N}$ and $\phi \in \widehat{A(e)}_0$ be given. The character $\widetilde{\Sigma}(e,[\phi]) \neq 0$ if and only if $e \in \mathcal{N}^{\mathsf{sol}}$. In this case, $\widetilde{\Sigma}(e,[\phi])$ is the character of a genuine $\widetilde{W}$-representation.*

(2) *Every genuine irreducible $\widetilde{W}$-character $\widetilde{\sigma}$ occurs in a $\widetilde{\Sigma}(e,[\phi])$, $e \in \mathcal{N}^{\mathsf{sol}}$. Moreover, the G-orbit of e is uniquely determined by $\widetilde{\sigma}$.*

(3) *If $e \in \mathcal{N}^{\mathsf{sol}}$, then for all $[\chi], [\chi']$,*

$$\langle \widetilde{\Sigma}(e,[\chi]), \widetilde{\Sigma}(e,[\chi'])\rangle_{\widetilde{W}} = a_V \langle [\chi],[\chi']\rangle^{-1}_{A(e)}.$$

We calculate the structure of the spaces $\overline{R}_{-1}(A(e))$, $e \in \mathcal{N}^{\mathsf{sol}}$, and we refine part (3) of Theorem 1.3 in Appendix A. We show that for every $e \in \mathcal{N}^{\mathsf{sol}}$, there exists an orthogonal basis $\{[\chi_1],\dots,[\chi_k]\}$ of $\overline{R}_{-1}(A(e))$ such that

$$\widetilde{\tau}(e,[\chi_j]) = \frac{1}{a_e}\widetilde{\Sigma}(e,[\chi_j]), \qquad \text{here } a_e \text{ is a certain power of 2,} \tag{1.3.4}$$

is either an irreducible (sgn self dual) $\widetilde{W}$-character or the sum of two (sgn dual to each other) irreducible $\widetilde{W}$-characters, with two interesting exceptions: one family of nilpotent orbits in type D_n and one orbit in E_7 (see Appendix A, particularly Remark A.7).

The proof of Theorem 1.3 is independent of the results of [6], and in particular, together with Corollary 7.4, it recovers uniformly [6, Theorem 1.0.1]. It is also independent of the previous known classifications, e.g., [23, 25, 34]. Our proof relies on Theorem 1.1 with $q = -1$, and on Theorem 1.2 and its relation with the extended Dirac index in section 6, together with "Vogan's conjecture" [1, Theorem 4.2].

1.4. We mention two applications of our results.

By comparing Theorem 1.3 and Appendix A with [6], the characters $\widetilde{\tau}(e, [\phi])$ can be easily identified in terms of the previous known classifications. From this point of view, (1.3.1) can immediately be interpreted to give a character formula of $X_{-1}(e, \phi)$ on $w \in W_{\text{(-1)-ell}}$, or alternatively, using (1.3.2) and Theorem 1.2, as a character formula of $H^*(\mathcal{B}_e)^\phi$ on δ-twisted elliptic conjugacy classes (see Lemma 5.5 for the definition). In this way, one obtains an extension of [8, Theorem 1.1]. See section 7.4 for details.

Theorem 1.3 can also be used to give a solution in terms of Kostka-type numbers to the problem of decomposing tensor products $\sigma \otimes \mathcal{S}$, $\sigma \in \widehat{W}$. See (7.5.3) and Corollary 7.9.

1.5. We conclude the introduction by giving a brief summary of the structure of the paper. In section 2, we recall certain elements of the Lusztig-Shoji algorithm, and prove Theorem 1.1. In section 3, we study the nilpotent elements with solvable connected centralizer, i.e., $\mathcal{N}^{\mathsf{sol}}$. In particular, following a suggestion of Lusztig, we prove that a nilpotent element u is in $\mathcal{N}^{\mathsf{sol}}$ if and only if it is δ-quasidistinguished, in the sense of Definition 3.2. (This definition is the natural generalization of the notion of quasidistinguished from [26].)

In section 4, we define the action of $W_\#$ on the cohomology groups $H^*(\mathcal{B}_e)$, extending the Springer action, and prove Theorem 1.2. In section 5, we relate the (-1)-elliptic pairing with a δ-twisted elliptic pairing, and consider the corresponding spaces of virtual elliptic characters.

In section 6, we introduce the extended Dirac operator for the extended graded Hecke algebras $\mathbb{H}_\#$, and define its index. Using Lusztig's geometric realization of irreducible $\mathbb{H}$-modules ([20, 21]), we relate the index of tempered modules with the character formula in Theorem 1.2. In section 7, we prove the results about $\widetilde{W}$-representations, in particular, Theorem 1.3. In Appendix A, we compute explicitly the spaces $\overline{R}_{-1}(A(e))$ and the associated spin representations $\widetilde{\tau}(e, [\chi])$. In Appendix B, we present a relation between q-elliptic pairings of W and the Kostka systems of [16].

2. The Lusztig-Shoji algorithm and the q-elliptic pairing

2.1. If Γ is a finite group, let $R(\Gamma)$ denote the Grothendieck group of finite dimensional $\mathbb{C}[\Gamma]$-modules. Let $\langle\ ,\ \rangle_\Gamma$ be the character pairing of Γ. If q is an indeterminate, set $R_q(\Gamma) = R(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ and extend $\langle\ ,\ \rangle$ $\mathbb{Z}[q]$-linearly to $R_q(\Gamma)$.

Let U be a finite dimensional $\mathbb{C}$-representation of Γ and $\wedge^i U$ the i-th exterior power of U viewed as a Γ-representation. Denote

$$\wedge^{-q} U = \sum_{i \geqslant 0} (-q)^i \wedge^i U \in R_q(\Gamma). \tag{2.1.1}$$

Define the q-elliptic product in $R_q(\Gamma)$ to be:

$$\langle \chi, \chi' \rangle_\Gamma^q := \langle \chi \otimes \wedge^{-q} U, \chi' \rangle_\Gamma \in \mathbb{Z}[q]. \tag{2.1.2}$$

The case of interest for us will be when Γ is a Weyl group W acting on the reflection representation $U = V$.

2.2. In the rest of this section, let $\mathbb{F}$ be a finite field and let G be a connected semisimple algebraic group split over $\mathbb{F}$. Let $F : G \to G$ be the corresponding Frobenius map and $G^F = G(\mathbb{F})$ be the corresponding finite group of Lie type.

We assume furthermore that the characteristic of F is sufficiently large. Specialize q from the previous section to the order of finite field $\mathbb{F}$. Let $e \in \mathcal{N}^F$ be given, and denote $\mathcal{O}_e$ the nilpotent orbit of e. We set $A(e) = Z_G(e)/Z_G(e)^0$. Then F acts trivially on $A(e)$ and there is a one-to-one correspondence between G^F-orbits in $\mathcal{O}_e^F$ and conjugacy classes in $A(e)$.

For every $e \in \mathcal{N}$ denote $\mathcal{B}_e$ the variety of Borel subalgebras of $\mathfrak{g}$ containing e and let d_e be its dimension. Springer [31] defined an action of W on the cohomology groups (with rational coefficients) $H^j(\mathcal{B}_e)$. This action commutes with the natural $A(e)$-action. Moreover, it is known that $H^j(\mathcal{B}_e) = 0$ unless j is even ([2, 10, 29]). Set

$$\widehat{A(e)}_0 = \{\phi \in \widehat{A(e)} : H^{2d_e}(\mathcal{B}_e)^\phi \neq 0\}, \tag{2.2.1}$$

the set of $A(e)$-representations of Springer type. For every pair (e, ϕ), $\phi \in \widehat{A(e)}_0$, let $\sigma(e, \phi) \in \widehat{W}$ denote the irreducible Springer representation afforded by $H^{2d_e}(\mathcal{B}_e)^\phi$. For latter use, encode the Springer correspondence as the bijective map:

$$\Psi : G \backslash \{(e, \phi) : e \in \mathcal{N}, \phi \in \widehat{A(e)}_0\} \to \widehat{W}, \quad \Psi((e, \phi)) = \sigma(e, \phi). \tag{2.2.2}$$

Define

$$\begin{gathered} X_q(e) = \sum_{i \geqslant 0} q^{d_e - i} H^{2i}(\mathcal{B}_e) \otimes \mathsf{sgn} \in R_q(W), \\ X_q(e, \phi) = \mathrm{Hom}_{A(e)}[\phi, X_q(e)] \in R_q(W). \end{gathered} \tag{2.2.3}$$

Thus $\sigma(e, \phi)$ occurs in degree 0 in $X_q(e, \phi)$.

Define $R_q(W)^e$ to be the subspace of $R_q(W)$ spanned by $\{X_q(e, \phi) : \phi \in \widehat{A(e)}_0\}$.

2.3. The Lusztig-Shoji algorithm [18, 29] gives a solution to a matrix equation

$$K(q)\Lambda(q)K(q)^t = \Omega(q), \tag{2.3.1}$$

where the matrices $K(q), \Lambda(q), \Omega(q)$ are square matrices of size $\# G \backslash \{(e, \phi) : e \in \mathcal{N}, \phi \in \widehat{A(e)}_0\}$ with entries in $\mathbb{Z}[q]$ to be defined next. We notice from the start that since our normalization of $X_q(e, \phi)$ is different than the usual one, in the sense that $\sigma(e, \phi)$ has degree 0 in q rather than top degree, we will need to adjust in the definition of $\Lambda(q)$ below.

Fix a set $\{e\}$ of representatives of G-orbits in $\mathcal{N}$ and for every such e, the set $\{\phi\}$ of representations in $\widehat{A(e)}_0$.

Let $K(q)$ denote the upper uni-triangular matrix whose (e, ϕ), (e', ϕ') entry is given by the graded multiplicity of $\sigma(e, \phi)$ in $X_q(e', \phi')$.

Let $\Omega(q)$ be the symmetric matrix of fake degrees. More precisely, the (e,ϕ), (e',ϕ') entry in $\Omega(q)$ is the graded multiplicity of $\sigma(e,\phi)\otimes\sigma(e',\phi')$ in $X_q(1)$. Recall that $X_q(1)$ can be identified with the graded representation of W on the space of coinvariants of W in $S(V)$. Moreover with this interpretation of $X_q(1)$, a well-known identity of Chevalley is:

$$X_q(1)\otimes\wedge^{-q}V = p(q)\mathsf{triv}, \qquad \text{where } p(q)=\prod_i(1-q^{m_i}); \tag{2.3.2}$$

here m_i are the fundamental degrees of W.

The matrix $\Lambda(q)$ is block-diagonal, with one block of size $|\widehat{A(e)}_0|$ for each e. To define it precisely, we need more notation. For every conjugacy class c of $A(e)$, denote by $\mathcal{O}_e^F(c)$ the corresponding G^F-orbits. We fix an element e_c for each conjugacy class c of $A(e)$. For $c=\{1\}$, we may choose $e_c=e$. We choose an element $g_c\in G$ such that $g_c\cdot e=e_c$ and set $x_c=g_c^{-1}F(g_c)\in Z_G(e)$. Then the image $\bar{x}_c$ of x_c in $A(e)$ is contained in c.

For every $\phi\in\widehat{A(e)}$, define the G^F-class function $f_\phi^e:\mathcal{N}^F\to\mathbb{Q}$, by

$$f_\phi^e(x)=\begin{cases}\operatorname{tr}\phi(c), & \text{if } x\in\mathcal{O}_e^F(c),\\ 0, & \text{if } x\notin\mathcal{O}_e^F.\end{cases} \tag{2.3.3}$$

Define a bilinear form on $\mathbb{Q}$-valued functions on $\mathcal{N}^F$, by

$$(f,f')=\sum_{x\in\mathcal{N}^F}f(x)f'(x). \tag{2.3.4}$$

Define the matrix $\widetilde{\Lambda}$ whose $(e,\phi),(e',\phi')$ entry is

$$(f_\phi^e,f_{\phi'}^{e'})=\delta_{e,e'}\sum_c|\mathcal{O}_e^F(c)|\operatorname{tr}\phi(c)\operatorname{tr}\phi'(c). \tag{2.3.5}$$

For every (e,ϕ), define the function $g_\phi^e:\mathcal{N}^F\to\mathbb{Q}$, constant on G^F-orbits, by

$$g_\phi^e(x)=\frac{1}{|G^F||A(e)|}\begin{cases}\operatorname{tr}\phi(c)|c||Z_G(e_c)^F|, & \text{if } x\in\mathcal{O}_e^F(c),\\ 0, & \text{if } x\notin\mathcal{O}_e^F.\end{cases} \tag{2.3.6}$$

It is immediate that

$$(f_\phi^e,g_{\phi'}^{e'})=\delta_{e,e'}\langle\phi,\phi'\rangle_{A(e)}=\delta_{e,e'}\delta_{\phi,\phi'}; \tag{2.3.7}$$

in other words, $\{g_\phi^e\}$ is the basis dual to $\{f_\phi^e\}$. This means that the inverse matrix $\widetilde{\Lambda}^{-1}$ has entries

$$(g_\phi^e,g_{\phi'}^{e'})=\frac{\delta_{e,e'}}{|A(e)|^2|G^F|}\sum_c|Z_G(e_c)^F|\operatorname{tr}\phi(c)\operatorname{tr}\phi'(c)|c|^2. \tag{2.3.8}$$

We can regard $\widetilde{\Lambda}$ as a matrix in q, and denote it by $\widetilde{\Lambda}(q)$. The relation between $\widetilde{\Lambda}(q)$ and $\Lambda(q)$ is given by Lusztig [18, (24.2.7)] and Shoji [29, section 4]. Since we need to account here for our normalization of the matrix $K(q)$ in section 2.1, the relation is

$$\Lambda(q)=q^{\dim Z_G(e)}\widetilde{\Lambda}(q^{-1}). \tag{2.3.9}$$

2.4. Let $M(q)$ be the symmetric matrix whose entries are $\langle X_q(e,\phi), X_q(e',\phi')\rangle_W^q$. We relate first $M(q)$ and $\Lambda(q)$.

Theorem 2.1.

$$\Lambda(q)M(q) = p(q)\mathsf{id}.$$

Proof. We compute $\Lambda(q)$. We first calculate $\Omega(q)^{-1}$. Using (2.3.2), we have $X_q(1)\otimes\sigma\otimes\wedge^{-q}V = p(q)\sigma$, for every irreducible W-representation σ, and then:

$$\begin{aligned} X_q(1)\otimes\sigma\otimes\wedge^{-q}V &= \sum_{\sigma'}\langle\sigma', X_q(1)\otimes\sigma\otimes\wedge^{-q}V\rangle_W\sigma' \\ &= \sum_{\sigma'}\sum_{\sigma_1,\sigma_2}\langle\sigma_1, X_q(1)\otimes\sigma\rangle_W\langle\sigma_2,\wedge^{-q}V\rangle_W\langle\sigma',\sigma_1\otimes\sigma_2\rangle_W\sigma' \\ &= \sum_{\sigma'}\sum_{\sigma_1,\sigma_2}\langle\sigma_1\otimes\sigma^*, X_q(1)\rangle_W\langle\sigma_2,\wedge^{-q}V\rangle_W\langle\sigma',\sigma_1\otimes\sigma_2\rangle_W\sigma'. \end{aligned}$$

Thus

$$\sum_{\sigma_1,\sigma_2}\langle\sigma_1\otimes\sigma^*, X_q(1)\rangle_W\langle\sigma_2,\wedge^{-q}V\rangle_W\langle\sigma',\sigma_1\otimes\sigma_2\rangle_W = \delta_{\sigma,\sigma'}p(q),$$

and therefore

$$\Omega(q)^{-1}_{\sigma_1,\sigma_2} = p(q)\sum_{\sigma_3}\langle\sigma_3,\wedge^{-q}V\rangle_W\langle\sigma_2,\sigma_1\otimes\sigma_3\rangle_W. \tag{2.4.1}$$

(In the calculations above, σ, σ_i, $i=1,3$, vary over $\widehat{W}$.)

Next, we compute $p(q)K(q)^t\Omega(q)^{-1}K(q)$. The (e,ϕ), (e',ϕ') entry of this matrix equals:

$$\begin{aligned} &p(q)\sum_{\sigma_1,\sigma_2}K(q)^t_{(e,\phi),\Psi^{-1}(\sigma_1)}\Omega(q)^{-1}_{\sigma_1,\sigma_2}K(q)_{\Psi^{-1}(\sigma_2),(e',\phi')} \\ &= p(q)K(q)_{\Psi^{-1}(\sigma_1),(e,\phi)}\Omega(q)^{-1}_{\sigma_1,\sigma_2}K(q)_{\Psi^{-1}(\sigma_2),(e',\phi')} \\ &= \sum_{\sigma_1,\sigma_2,\sigma_3}\langle\sigma_1, X_q(e,\phi)\rangle_W\langle\sigma_3,\wedge^{-q}V\rangle_W\langle\sigma_2,\sigma_1\otimes\sigma_3\rangle_W\langle\sigma_2, X_q(e',\phi')\rangle_W \\ &= \sum_{\sigma_2}\left(\sum_{\sigma_1,\sigma_3}\langle\sigma_1, X_q(e,\phi)\rangle_W\langle\sigma_3,\wedge^{-q}V\rangle_W\langle\sigma_2,\sigma_1\otimes\sigma_3\rangle_W\right)\langle\sigma_2, X_q(e',\phi')\rangle_W \\ &= \sum_{\sigma_2}\langle\sigma_2, X_q(e,\phi)\otimes\wedge^{-q}V\rangle_W\langle\sigma_2, X_q(e',\phi')\rangle_W \\ &= \langle X_q(e,\phi)\otimes\wedge^{-q}V, X_q(e',\phi')\rangle_W = \langle X_q(e,\phi), X_q(e',\phi')\rangle_W^q. \end{aligned}$$

In other words, $p(q)K(q)^t\Omega(q)^{-1}K(q) = M(q)$, and the conclusion follows from (2.3.1). □

Corollary 2.2. *If $e\neq e'$, i.e., they are representatives of distinct G-orbits in $\mathcal{N}$, then*

$$\langle X_q(e,\phi), X_q(e',\phi')\rangle_W^q = 0,$$

for all ϕ,ϕ'. Therefore, the subspaces $R_q(W)^e$ and $R_q(W)^{e'}$ are orthogonal with respect to $\langle\ ,\ \rangle_W^q$.

Proof. This is immediate from Theorem 2.1, since $\Lambda(q)$ is block-diagonal. □

Example 2.3. Suppose $G = SL(3)$, so $W = S_3$ acting on a two dimensional reflection space V. Since all component group representations of Springer type are trivial, we drop them from notation. With our conventions, we have: $X_q(3) = (1^3)$, $X_q(21) = (21) + q(1^3)$, and $X_q(1^3) = (3) + q(21) + q^2(21) + q^3(1^3)$. Then we find

$$M(q) = \mathrm{diag}(1, 1-q, (1-q^2)(1-q^3)).$$

Example 2.4. Suppose $G = Sp(4)$. We have five Green polynomials, with our convention: $X_q(4) = (0 \times 11)$, $X_q((22), \mathsf{triv}) = (1 \times 1) + q(0 \times 11)$, $X_q((22), \mathsf{sgn}) = (11 \times 0)$, $X_q((211)) = (0 \times 2) + q(1 \times 1) + q^2(0 \times 11)$, and

$$X_q(1^4) = (2 \times 0) + q(1 \times 1) + q^2(11 \times 0 + 0 \times 2) + q^3(1 \times 1) + q^4(0 \times 11).$$

Then we find

$$M(q) = \mathrm{diag}\left(1, \begin{pmatrix} 1 & -q \\ -q & 1 \end{pmatrix}, 1-q^2, (1-q^2)(1-q^4)\right).$$

We wish to relate $R_q(W)^e$ with $R_q(A(e))^0$, i.e., the subspace of $R_q(A(e))$ spanned by $\widehat{A(e)}_0$.

2.5. To compute $M(q)$ further, in light of Theorem 2.1, we need to consider $\Lambda(q)^{-1} = q^{\dim Z_G(e)}(\widetilde{\Lambda}^{-1})(q^{-1})$. Notice that

$$|G^F|(q^{-1}) = \prod_i (1 - q^{m_i}) = p(q), \tag{2.5.1}$$

and using (2.3.8), we see that the $(e, \phi), (e', \phi')$ entry in $\Lambda(q)^{-1}$ equals

$$\delta_{e,e'} \frac{1}{|A(e)|^2 p(q)} \sum_c \mathrm{tr}\, \phi(c) \,\mathrm{tr}\, \phi'(c) |c|^2 q^{\dim Z_G(e)} |Z_G(e_c)^F|(q^{-1}). \tag{2.5.2}$$

The map $g \mapsto g_c g g_c^{-1}$ gives an isomorphism from $Z_G(e)$ to $Z_G(e_c)$. Hence

$$\begin{aligned} Z_G(e_c)^F &= \{g \in Z_G(e_c); F(g) = g\} \cong \{g \in Z_G(e); F(g_c g g_c^{-1}) = g_c g g_c^{-1}\} \\ &= \{g \in Z_G(e); F_c(g) = g\} = Z_G(e)^{F_c}. \end{aligned}$$

Here $F_c = \mathrm{Ad}(x_c) \circ F$ is a Frobenius morphism on $Z_G(e)$. It is easy to see that the image of $Z_G(e)^{F_c}$ under the map $Z_G(e) \to A(e)$ is $Z_{A(e)}(\bar{x}_c)$. Hence we have the following short exact sequence

$$1 \longrightarrow (Z_G^0(e))^{F_c} \longrightarrow Z_G(e)^{F_c} \longrightarrow Z_{A(e)}(\bar{x}_c) \longrightarrow 1\ . \tag{2.5.3}$$

Let R_e be the unipotent radical of $Z_G(e)$ and $H_e = Z_G^0(e)/R_e$. Then H_e is a connected reductive group and

$$H_e^{F_c} = (Z_G^0(e))^{F_c} / R_e^{F_c}.$$

So

$$\begin{aligned} |Z_G(e_c)^F| &= |Z_G(e_c)^{F_c}| = |Z_{A(e)}(\bar{x}_c)| \cdot |(Z_G^0(e))^{F_c}| \\ &= \frac{|A(e)|}{|c|} |(Z_G^0(e))^{F_c}| = q^{\dim(R_e)} \frac{|A(e)|}{|c|} |H_e^{F_c}|. \end{aligned}$$

Corollary 2.5. *The $(e, \phi), (e', \phi')$ entry in the matrix $M(q)$ is given by*

$$\delta_{e,e'} \frac{1}{|A(e)|} \sum_c \mathrm{tr}\, \phi(c) \,\mathrm{tr}\, \phi'(c) |c| \zeta_{e,c}(q),$$

with $\zeta_{e,c}(q) = q^{\dim H_e} |H_e^{F_c}|(q^{-1})$.

2.6. Now we follow the approach in [5, section 2.9].

We fix a conjugacy class c of $A(e)$. Let T be an F_c-stable maximal torus of H_e contained in a F_c-stable Borel subgroup. Then we can define the F_c-action on the character group X of T and on $V_e = X_{\mathbb{R}}$. We have that $F_c = qF_{c,0}$ on V where $F_{c,0}$ is an automorphism of finite order.

The group H_e is a product $H_e = H'Z^0$, where H' is a semisimple group and Z^0 is the central torus. Then $T = SZ^0$, where $S = T \cap H'$ is a maximal torus of H' and $S \cap Z^0$ is finite. We have the decomposition $V_e = V_1 \oplus V_2$, where $V_1 = (Z^0)^{\perp}_{\mathbb{R}}$ is the subspace spanned by the roots and $V_2 = S^{\perp}_{\mathbb{R}}$ is a complementary subspace. Both V_1 and V_2 are stable under the action of F_c and $F_{c,0}$. Moreover, let V_Z be the vector space spanned by the characters of Z^0. Then $V_2 \cong V_Z$ as F_c-vector space and

$$|(Z^0)^{F_c}| = \det_{V_2}(q - F_{c,0}) = \det_{V_Z}(q - F_{c,0}).$$

Let W_e be the Weyl group of H_e. The W_e-invariants of the algebra of polynomial functions on V_1 is a polynomial ring and there exists homogeneous elements $I_1, \cdots, I_l$ of degree $d_1, \cdots, d_l$ such that

$$\mathbb{C}[V_1]^{W_e} = \mathbb{C}[I_1, \cdots, I_l]$$

and $F_{c,0}(I_i) = \epsilon_{c,i} I_i$, where $\epsilon_{c,i}$ is a root of unity. Moreover,

$$|H_e^{F_c}| = |Z^{F_c}| q^N \Pi_i (q^{d_i} - \epsilon_{c,i}) = q^n \det_{V_Z}(q - F_{c,0}) \Pi_i (q^{d_i} - \epsilon_{c,i}),$$

where N is the number of positive roots.

We may reformulate the order of $H_e^{F_c}$ in the following way.

For any $d \in \mathbb{N}$, let $M[d]_c$ be the complex vector space spanned by I_i with $d_i = d$. Then $F_{c,0}$ acts on $M[d]_c$ and

$$|H_e^{F_c}| = q^N \det_{V_Z}(q - F_{c,0}) \Pi_d \det_{M(d)_c}(q^d - F_{c,0}).$$

Notice that N and V_2 are independent of the choice of c. Although T and I_i depends on the choice of c, the multiset $\{d_1, \cdots, d_l\}$ is the set of degree of fundamental invariants for H_e and thus for any given d, the dimension of the vector spaces $M(d)_c$ is independent of the choice of c.

2.7. By (2.5.3), each coset of $A(e)$ contains a F-stable element. Moreover, if $g, g' \in Z_G(e)^F$ with $gZ_G(e)^0 = g'Z_G(e)^0$, then the actions of $\mathrm{Ad}(g) \circ F$ and $\mathrm{Ad}(g') \circ F$ on V_Z are the same as Z^0 is the central torus of H_e. In other words, the map

$$Z_G(e)^F \to \mathrm{End}(V_Z), \quad g \mapsto \mathrm{Ad}(g) \circ F \mid_{V_Z}$$

factors through a map $A(e) \to \mathrm{End}(V_Z)$. For any $x \in A(e)$, we denote the corresponding endomorphism on V_Z by F_x. Then $F_x = qF_{x,0}$, where $F_{x,0}$ has finite order.

For any $g, g' \in Z_G(e)^F$, $(\mathrm{Ad}(g) \circ F) \circ (\mathrm{Ad}(g') \circ F) = \mathrm{Ad}(gg') \circ F^2$. Notice that $F = F_1$ acts on V_Z as $q\,\mathsf{id}$. Thus the map $x \mapsto F_{x,0}$ gives a group homomorphism from $A(e) \to GL(V_Z)$.

Proposition 2.6. *Let $d \in \mathbb{N}$. Then there exists a representation $M(d)$ of $A(e)$ such that for any $x \in A(e)$, we have*

$$det_{M(d)}(\lambda - x) = det_{M(d)_{x_c}}(\lambda - F_{x_c,0})$$

as a polynomial on λ. Here x_c is the conjugacy class of x.

Proof. We follow the notations in [17].

We first consider the case where G is a classical group.

If $G = GL_n$, then $A(e) = 1$ and the statement is obvious.

If $G = Sp_n$ and $e = \oplus_i J_i^{r_i}$ be a nilpotent element in G, where J_i denotes a nilpotent Jordan block of length i. By [17, Theorem 3],

$$H_e = \Pi_{i \text{ odd}} Sp_{r_i} \times \Pi_{i \text{ even}} O_{r_i}$$

and $A(e) = (\mathbb{Z}_2)^k$, where $k = \sharp\{i; i \text{ even}, r_i > 0\}$.

For any $d \in \mathbb{N}$, let $M(d)_{\text{odd}} = \oplus_{i \text{ odd}} M(d)^{Sp_{r_i}}$ and $M(d)_i = M(d)^{O_{r_i}}$ for i even. Here for any $H = Sp_{r_i}$ or O_{r_i}, $M(d)^H$ is a complex vector space of dimension equal to $\dim M(d)_1$ for the group H, i.e., the number of the degree of the fundamental invariants for H which equals d.

Let

$$M(d) = M(d)_{\text{odd}} \oplus \oplus_{i \text{ even}} M(d)_i.$$

We define the action of $A(e)$ on $M(d)$ as follows.

The action of $A(e)$ on $M(d)_{\text{odd}}$ is trivial. For any i even with $r_i > 0$, the i-th copy of $\mathbb{Z}_2$ in $A(e)$ acts on acts trivially on $M(d)_{i'}$ unless for $i' = i$, $d = \frac{r_i}{2}$. In the latter case, if $4 \mid r_i$, then $M(d)_i$ is 2-dimensional and the i-th copy of $\mathbb{Z}_2$ in $A(e)$ acts on $M(d)_i$ as permutation representation; if $4 \nmid r_i$, then $M(d)_i$ is 1-dimensional and the i-th copy of $\mathbb{Z}_2$ in $A(e)$ acts on $M(d)_i$ as sign representation.

By [17, Theorem 2.12], this is the desired representation.

The case where $G = O_n$ can be proved in the same way.

Now we assume that G is of exceptional type and the semisimple part of H_e is nontrivial.

If $A(e) = S_2 = \{1, \epsilon\}$, then $F_\epsilon^2 = \text{Ad}(g) \circ F^2$ for some $g \in (Z_G(e)^0)^F$. By Lang's theorem, there exists $h \in Z_G(e)^0$ such that $g = xF^2(x)^{-1}$. So $F_\epsilon^2 = \text{Ad}(x) \circ F^2 \circ \text{Ad}(x)^{-1}$. Since F^2 acts on $M(d)_1$ as $q^2\, \mathsf{id}$, F_ϵ^2 acts on $M(d)_\epsilon$ as $q^2\, \mathsf{id}$. In particular, $F_{\epsilon,0}$ is an automorphism on $M(d)_\epsilon$ with $F_{\epsilon,0}^2 = \mathsf{id}$. The statement holds in this case.

If $A(e) \neq \{1\}$ or S_2, we only have the following cases (see [17, Table 5.1 & 5.2]).

Class $D_4(a_1)$ in $E_8(q)$. Here $A(e) = S_3$, $H_e = D_4$, $M(2)$, $M(6)$ are one-dimensional trivial representations of $A(e)$ and $M(4)$ is the irreducible 2-dimensional representation of $A(e)$.

Class $D_4(a_1)A_1$ in $E_8(q)$. Here $A(e) = S_3$, $H_e = A_1^3$ and $M(2)$ is the permutation representation of $A(e)$.

Class $E_7(a_5)$ in $E_8(q)$. Here $A(e) = S_3$, $H_e = A_1$ and $M(2)$ is one-dimensional trivial representation of $A(e)$.

Class $D_4(a_1)$ in $E_7(q)$. Here $A(e) = S_3$, $H_e = A_1^3$ and $M(2)$ is the permutation representation of $A(e)$. □

2.8. We set

$$M = \wedge^{-q} V_Z \otimes (\otimes_d (\wedge^{-q^d} M(d))). \tag{2.8.1}$$

Then the action of $A(e)$ on V_Z and $M(d)$ extends in a unique way to an action on M and for any $x \in A(e)$,

$$\begin{aligned}
\mathrm{tr}_M(x) &= \mathrm{tr}_{\wedge^{-q} V_Z}(x) \times \Pi_d\, tr_{\wedge^{-q^d} M(d)}(x) \\
&= \det_{V_Z}(1-qx) \times \Pi_d \det_{M(d)}(1-q^d x) \\
&= \det_{V_Z}(1-qx) \times \Pi_d \det_{M(d)_{x_c}}(1-q^d F_{x_c}) \\
&= q^{\dim H_e - N} \det_{V_Z}(q^{-1}-x) \times \Pi_d \det_{M(d)_{x_c}}(q^{-d} - F_{x_c,0}) \\
&= q^{\dim H_e} |H_e^{F_{x_c}}|(q^{-1})
\end{aligned}$$

We define the (q,M)-pairing in $R_q(A(e))$ to be

$$\langle \phi, \phi' \rangle^{q,M}_{A(e)} := \langle \phi \otimes M, \phi' \rangle_{A(e)} \in \mathbb{Z}[q]. \tag{2.8.2}$$

Thus we have proved:

Theorem 2.7. *The map $X_q(e,\phi) \to \phi$ induces a $\mathbb{Z}[q]$-isomorphic isometry with respect to the q-elliptic pairing in $R_q(W)^e$ and the (q,M)-pairing in $R_q(A(e))^0$. More precisely:*

$$\langle X_q(e,\phi), X_q(e,\phi') \rangle^q_W = \langle \phi, \phi' \rangle^{q,M}_{A(e)}.$$

3. Nilpotent elements with solvable connected centralizer

3.1. In this section, we discuss certain nilpotent conjugacy classes that will play an essential role in our study of irreducible representations of $\widetilde{W}$.

Let $e \in \mathcal{N}$. A standard (Lie) triple of e is a a triple $\{e,h,f\} \subset \mathfrak{g}$, such that $[h,e] = 2e$, $[h,f] = -2e$, and $[e,f] = h$. Every such triple corresponds to a Lie algebra homomorphism $\varphi : sl(2) \to \mathfrak{g}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f$. By the Dynkin-Kostant classification, see for example [9, pages 35–36], the map $\{e,h,f\} \to e$ gives a one-to-one correspondence between the set of G-conjugacy classes of Lie triples and G-orbits of nilpotent elements in $\mathfrak{g}$. We refer to the element h as a neutral element for e and when we wish to emphasize the dependence of h on e, we denote it by h_e.

Definition 3.1. An element $e \in \mathcal{N}$ is called distinguished ([5]) is the centralizer $Z_{\mathfrak{g}}(e)$ does not contain any nonzero semisimple element.

An element $e \in \mathcal{N}$ is called quasidistinguished if there exists a semisimple element $t \in Z_G(e)$ such that $Z_G(t)$ is semisimple and e is a distinguished element in the Lie algebra of $Z_G(t)$.

Set $u = \exp(e)$. Then e is quasidistinguished if and only if u is quasidistinguished in the sense of [26], i.e., there exists a semisimple element $t \in G$ such that $u \in Z_G(t)$ and $Z_G(tu)$ does not contain any nontrivial torus. Indeed, one direction is obvious. For the converse, let $g = ut$. By Jordan decomposition, any element that commutes with g also commutes with t and u. Thus $Z_G(g) = Z_{Z_G(t)}(u)$. Hence $Z_G(g)$ doesn't contain a nontrivial torus implies that $Z_G(t)$ is semisimple and u is distinguished in $Z_G(t)$. Hence e is distinguished in the Lie algebra of $Z_G(t)$.

Recall that $\mathcal{N}^{\mathsf{sol}}$ is the set of $e \in \mathcal{N}$ such that $Z_G(e)^0$ is a solvable group. It is clear that every distinguished nilpotent element e is also quasidistinguished and belongs to $\mathcal{N}^{\mathsf{sol}}$. It is proved in [27, Lemma 7.1(1)] that if e is quasidistinguished, then necessarily $e \in \mathcal{N}^{\mathsf{sol}}$. However, $e \in \mathcal{N}^{\mathsf{sol}}$ does not imply that e is quasidistinguished.

3.2. Let δ be the automorphism of G given by the action of $-w_0$ on the root datum, where $w_0 \in W$ is the longest element. More precisely, δ is the order two automorphism of the Dynkin diagram when G is of type A_n, D_{2n+1}, or E_6, and δ is trivial for other simple groups.

We set $G_\# = G \rtimes \langle\delta\rangle$. Following [33, section 9], we call an element $g \in G_\#$ quasi-semisimple if there exists a Borel subgroup B of G and a maximal torus $T \subset B$ such that $gBg^{-1} = B$ and $gTg^{-1} = T$. In this case $Z_G(g)$ is a reductive group. Moreover, by [33, Theorem 7.5], if $g \in G_\#$ is semisimple, then g is quasi-semisimple.

Definition 3.2. An element $e \in \mathcal{N}$ is called δ-quasidistinguished if there exists a semisimple element $t\delta \in Z_{G_\#}(e)$ such that $Z_G(t\delta)$ is semisimple and e is a distinguished element in the Lie algebra of $Z_G(t\delta)$.

Suppose that $t\delta$ is semisimple in $G_\#$. The condition that $Z_G(t\delta)$ be semisimple implies that $t\delta$ is an isolated (torsion) element of $G\delta$ in the terminology of [22, section 2] or [28, section 3.8]. For basic results about the isolated elements, see [22, section 2], particularly [22, Lemma 2.6]. The classification of isolated semisimple elements is known, and we recall it next, following [28, sections 3.8, 4.1-4.5]. Let $\mathfrak{t} \subset \mathfrak{b}$ be δ-stable Cartan and Borel subalgebras, respectively. If Φ is the root system of $\mathfrak{g}$ corresponding to $\mathfrak{t}$, with positive roots given by $\mathfrak{b}$. Call two roots $\alpha, \beta \in \Phi$ δ-equivalent if $\alpha|_{\mathfrak{t}^\delta}, \beta|_{\mathfrak{t}^\delta}$ are proportional via a positive constant. If a is a δ-equivalence class in Φ, then a is a δ-orbit in Φ, except in type A_{2n}, when a could be of the form $\{\alpha, \delta(\alpha), \alpha + \delta(\alpha)\}$. Let

$$\mathfrak{g}_a = \sum_{\alpha \in a} \mathfrak{g}_\alpha, \quad \gamma_a = \sum_{\alpha \in a} \alpha|_{\mathfrak{t}^\delta},\ \beta_a = \frac{1}{f_a}\gamma_a,$$

where $f_a = |a|$, unless a is the exception in type A_{2n}, when $f_a = 4$.

With this notation, the root-space decomposition of $\mathfrak{g}^{t\delta}$, $t = \exp(x)$, $x \in \mathfrak{t}^\delta$ is ([28, Proposition 3.8]):

$$\mathfrak{g}^{t\delta} = \mathfrak{t}^\delta \oplus \sum_a \mathfrak{g}_a^{t\delta}, \tag{3.2.1}$$

where the sum of over the δ-equivalence classes $a \in \Phi/\delta$ such that $\langle\gamma_a, x\rangle \in \{-1, 0, 1\}$. Each $\mathfrak{g}_a^{t\delta}$ is one-dimensional, affording either a root β_a or $2\beta_a$, the latter case may only occur in the exceptional a in A_{2n}.

Proposition 3.3. *A nilpotent element $e \in \mathcal{N}$ is δ-quasidistinguished if and only if $e \in \mathcal{N}^{\mathsf{sol}}$, i.e., the centralizer $Z_G(e)^0$ is solvable.*

Proof. One can prove uniformly that "e is δ-quasidistinguished" implies "$e \in \mathcal{N}^{\mathsf{sol}}$" analogously with the untwisted case [27, Lemma 7.1(1)], as follows. Suppose e is δ-quasidistinguished, and let $t\delta \in G\delta \subset G_\#$ be a semisimple element as in Definition 3.2. Let H_e be the (connected) reductive part of $Z_G(e)$ and let $\mathfrak{h}_e$ be the Lie algebra. Since $t\delta \in G_\#$ acts on H_e by conjugation, one can consider $\mathrm{Ad}(t\delta)|_{\mathfrak{h}_e} : \mathfrak{h}_e \to \mathfrak{h}_e$ and let $\mathfrak{h}_e^{t\delta}$ be the fixed points. The algebra $\mathfrak{h}_e^{t\delta}$ is a reductive Lie algebra, since $t\delta$ is semisimple. However, $\mathfrak{h}_e^{t\delta}$ does not contain nonzero semisimple element. This means that $\mathfrak{h}_e^{t\delta} = 0$. By [33, Corollary 10.12], $\mathfrak{h}_e$ has zero derived subalgebra, which implies H_e is a torus, equivalently $Z_G(e)^0$ is solvable.

The proof of the converse direction is case by case.

In type A, we consider $GL(n)$ rather than $SL(n)$ for simplicity. The nilpotent orbits in $\mathcal{N}^{\mathsf{sol}}$ are in one two one correspondence with partitions of n into distinct parts via the Jordan canonical form. Let λ be such a partition and break λ into λ_0 containing all the even parts and λ_1 containing all the odd parts. Let $2m$ be the sum of parts in λ_0. Let $J_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, where I_m is the identity $m \times m$ matrix. Set $t_m = \mathrm{diag}(J_{2m}, I_{n-2m})$. Then $t_m\delta$ is a semisimple element.

We consider the automorphism $\delta : GL(n) \to GL(n)$ given by $\delta(x) = (x^T)^{-1}$. Thus

$$Z_{GL(n)}(t_m\delta) = Sp(2m) \times O(n-2m).$$

Let e_{λ_0} be a distinguished nilpotent element in $sp(2m)$ parameterized by the even partition λ_0 and e_{λ_1} a distinguished nilpotent element in $o(n-2m)$ parameterized by the odd partition λ_1. Then $e_\lambda = \begin{pmatrix} e_{\lambda_0} & 0 \\ 0 & e_{\lambda_1} \end{pmatrix}$ is a representative of the class in $\mathcal{N}^{\mathsf{sol}}$ labeled by λ and it is δ-quasidistinguished by construction.

In $Sp(2n)$ (resp. $SO(2n+1)$ or $SO(4n)$), the automorphism δ is trivial, and one can see from the classification of nilpotent classes that the classes in $\mathcal{N}^{\mathsf{sol}}$ are parameterized by partitions of $2n$ (resp. $2n+1$ or $4n$) where every part is even (resp. odd) and each part appears with multiplicity at most 2. It is easy to check that every such nilpotent class is quasidistinguished. For example, suppose $\lambda = (a_1, a_1, a_2, a_2, \dots, a_k, a_k, a_{k+1}, \dots, a_\ell)$ is a partition of $2n$, where $a_1 < a_2 < \cdots < a_k < a_{k+1} < \cdots < a_\ell$ are even numbers. Let $t_m \in Sp(2n)$ be a semisimple element whose centralizer is $Sp(2m) \times Sp(2n-2m)$, where $2m = \sum_{i=1}^{\ell} a_i$. We choose a distinguished nilpotent element e_1 in $sp(2m)$ corresponding to the partition $(a_1, a_2, \dots, a_\ell)$ and a distinguished nilpotent element e_2 in $sp(2n-2m)$ corresponding to the partition $(a_1, a_2, \dots, a_k)$. Then $e_\lambda = e_1 \times e_2$ is a representative of the nilpotent class in $sp(2n)$ labeled by λ and it is quasi-distinguished by construction.

If $G = SO(4n+2)$, δ corresponds to the automorphism of order 2 of the Dynkin diagram. Suppose the roots of the corresponding root system of type D_{2n+1} are labeled $\{\alpha_1, \alpha_2, \dots, \alpha_{2n+1}\}$ and that δ acts by interchanging α_{2n} and α_{2n+1} and fixes the other roots. The orbits in $\mathcal{N}^{\mathsf{sol}}$ are parameterized by partitions of $4n+2$ into odd parts, each part of multiplicity at most 2. By [17, Lemma 2.9(iv)] and [12, Table 4.3.1], there exist $n+1$ classes of involutions in $G\delta$ with representatives $t_{i-1}\delta$ having centralizers of type $B_{i-1} \times B_{2n+1-i}$, where $1 \leqslant i \leqslant n+1$. Here $t_0 = 1$, and t_i, $1 \leqslant i \leqslant n$, is the order two element in the standard torus in $SO(4n+2)$ corresponding to the root α_i, see for example [12, (4.4.4)] for the precise definition. In particular, t_{i-1} commutes with δ, so $t_{i-1}\delta$ is semisimple, $1 \leqslant i \leqslant n+1$. The construction of δ-quasidistinguished orbits proceeds then exactly as in the untwisted $Sp(2n)$ example above.

When G is exceptional of type G_2, F_4, E_7, or E_8, the automorphism δ is trivial, and one verifies the claim from the classification of nilpotent orbits and their centralizers. This is an easy direct calculation using the explicit classification of nilpotent classes. We give the results below, but skip the details, since the calculation is very similar to the twisted E_6 example which we'll explain in detail.

If G is of type G_2 or F_4, the only nilpotent orbits in $\mathcal{N}^{\mathsf{sol}}$ are already distinguished, so there is nothing to check.

For groups of type E, we use the following labeling of the Dynkin diagrams (this is not the Bourbaki notation):

$$\begin{array}{ccccccccccccc} & & & & \alpha_4 & & & & & & & & \\ & & & & | & & & & & & & & \\ \alpha_1 & — & \alpha_2 & — & \alpha_3 & — & \alpha_5 & — & \alpha_6 & — & \alpha_7 & — & \alpha_8. \end{array} \tag{3.2.2}$$

For types E_7 and E_8, denote $t_i = \exp(\frac{1}{(\gamma,\omega_i^\vee)}\omega_i^\vee) \in T$, where $\omega_i^\vee$ is the fundamental coweight corresponding to the i-th simple root, and γ is the highest positive root.

If G is of type E_7, the non-distinguished nilpotent orbits in $\mathcal{N}^{\mathsf{sol}}$ are denoted in the Bala-Carter classification [5] by $E_6(a_1)$ and $A_4 + A_1$. They come from the regular nilpotent orbits in $Z_G(t_4) = A_7$ and $Z_G(t_3) = A_3 \times A_3 \times A_1$, respectively.

If G is of type E_8, the non-distinguished nilpotent orbits in $\mathcal{N}^{\mathsf{sol}}$ are $D_5 + A_2$, $D_7(a_1)$, $D_7(a_2)$, and $E_6(a_1) + A_1$. They come from $E_7(a_4)$ in $Z_G(t_8) = E_7 \times A_1$, $E_7(a_3)$ in $Z_G(t_8) = E_7 \times A_1$, the regular nilpotent orbit in $Z_G(t_6) = D_5 \times A_3$, and the regular nilpotent orbit in $Z_G(t_2) = A_7 \times A_1$, respectively.

It remains to analyze the case $G = E_6$ and δ coming from the automorphism of order 2 of the Dynkin diagram. There are seven nilpotent orbits in $\mathcal{N}^{\mathsf{sol}}$ labeled: E_6, $E_6(a_1)$, $E_6(a_3)$, D_5, $D_5(a_1)$, $A_4 + A_1$, and $D_4(a_1)$.

Suppose that $t\delta$ is semisimple. We use (3.2.1) to realize δ-quasidistinguished nilpotent orbits. The explicit cases in E_6 are in [28, section 4.5]. For each $x \in \mathfrak{t}^\delta$ such that $t = \exp(x)$ that appears (there are five cases), we compute the simple roots of the Lie algebra $\mathfrak{g}^{t\delta}$ as in (3.2.1). Then for each distinguished nilpotent element in $\mathfrak{g}^{t\delta}$ we match its Dynkin-Kostant diagram (in $\mathfrak{g}^{t\delta}$) with a diagram in $\mathfrak{g}$. This is done as follows: the Dynkin-Kostant diagram of $e \in \mathfrak{g}^{t\delta}$ gives the values of the simple roots β for $\mathfrak{g}^{t\delta}$ on the neutral element $h_e \in \mathfrak{t}^\delta$, and thus we can determine h_e. Next, one makes h_e dominant with respect to the simple roots in $\mathfrak{g}$ and computes the Dynkin-Kostant diagram in $\mathfrak{g}$. The explicit results are below. We denote by $\omega_i^\vee \in \mathfrak{t}$ the fundamental coweight corresponding to α_i.

(0) $x_0 = 0$, $\mathfrak{g}^{t_0\delta} = F_4$, with simple roots:

$$\alpha_4 \;—\; \alpha_3 \Longrightarrow \tfrac{1}{2}(\alpha_2 + \alpha_5) \;—\; \tfrac{1}{2}(\alpha_1 + \alpha_6).$$

The fixed point group F_4 has four distinguished nilpotent orbits: F_4, $F_4(a_1)$, $F_4(a_2)$, and $F_4(a_3)$ which correspond in E_6 to: E_6, D_5, $E_6(a_3)$, and $D_4(a_1)$, respectively.

(1) $x_1 = \frac{1}{4}(\omega_1^\vee + \omega_6^\vee)$, $\mathfrak{g}^{t_1\delta} = B_3 \times A_1$, with simple roots:

$$\alpha_4 \;—\; \alpha_3 \Longrightarrow \tfrac{1}{2}(\alpha_2 + \alpha_5) \qquad\quad \beta,$$

where $\beta = \frac{1}{2}[(\alpha_1+\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+\alpha_6)+(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6)]$. There is only one distinguished nilpotent orbit in $\mathfrak{g}^{t_1\delta}$, the regular one, which corresponds to $D_5(a_1)$ in E_6.

(2) $x_2 = \frac{1}{6}(\omega_2^\vee + \omega_5^\vee)$, $\mathfrak{g}^{t_2\delta} = A_2 \times A_2$, with simple roots:

$$\alpha_4 \;—\; \alpha_3 \qquad\quad \tfrac{1}{2}(\alpha_1 + \alpha_6) \;—\; \beta',$$

where $\beta' = \frac{1}{2}[(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) + (\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6)]$. There is only one distinguished nilpotent orbit in $\mathfrak{g}^{t_2\delta}$, the regular one, which corresponds to $D_4(a_1)$ in E_6.

(3) $x_3 = \frac{1}{4}\omega_3^\vee$, $\mathfrak{g}^{t_3\delta} = A_3 \times A_1$, with simple roots:

$$\beta'' \longrightarrow \tfrac{1}{2}(\alpha_2+\alpha_5) \longrightarrow \tfrac{1}{2}(\alpha_1+\alpha_6) \longrightarrow \alpha_4,$$

where $\beta'' = \frac{1}{2}[(\alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5)+(\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6)]$. There is only one distinguished nilpotent orbit in $\mathfrak{g}^{t_3\delta}$, the regular one, which corresponds to A_4+A_1 in E_6.

(4) $x_4 = \frac{1}{2}\omega_4^\vee$, $\mathfrak{g}^{t_4\delta} = C_4$, with simple roots:

$$\alpha_3 \Longrightarrow \tfrac{1}{2}(\alpha_2+\alpha_5) \longrightarrow \tfrac{1}{2}(\alpha_1+\alpha_6) \longrightarrow \beta''',$$

where $\beta''' = \frac{1}{2}[(\alpha_2+\alpha_3+\alpha_4)+(\alpha_3+\alpha_4+\alpha_5)]$. There are two distinguished nilpotent orbits in C_4, the regular orbit (8) and the subregular orbit (62), which correspond in E_6 to $E_6(a_1)$ and $E_6(a_3)$, respectively.

This finishes the proof for E_6 and therefore, the proof of the proposition. □

Now we analyze the action of δ on G-orbits of Lie triples in $\mathfrak{g}$.

Lemma 3.4. *Let $\mathcal{O}$ be a nilpotent G-orbit in $\mathfrak{g}$. There exists a δ-stable Levi subalgebra $\mathfrak{m}$ and a Lie triple $\{e,h,f\} \subset \mathfrak{m}$, $e \in \mathcal{O}$, such that e is δ-quasidistinguished in $\mathfrak{m}$.*

Proof. We prove the statement case by case. When $\delta = 1$, the claim is immediate from the Bala-Carter classification, in fact, $\mathfrak{m}$ can be chosen so that e is distinguished in $\mathfrak{m}$.

Let $G = GL(n)$. The nilpotent orbit $\mathcal{O}$ is given via Jordan form by a partition λ of n. Let $I = \{1,2,\dots,n-1\}$ be the indexing set for the simple roots. The automorphism δ acts on I as $\delta(i) = n-i$. We construct a subset $J \subset I$, such that $\delta(J) = J$. Start by setting $J = I$. If λ has only distinct parts, by Proposition 3.3, $\mathcal{O}$ is δ-quasidistinguished in $\mathfrak{g}$, so $\mathfrak{m} = \mathfrak{g}$. Suppose λ has equal parts. Write $\lambda = \lambda' \sqcup \{r_1,r_1\} \sqcup \dots \sqcup \{r_\ell,r_\ell\}$, where λ' is the largest subset of λ having only distinct parts. Set $k = \sum_{j=1}^{\ell} r_j$. Remove from I the indices $r_1, n-r_1, r_1+r_2, n-(r_1+r_2), \dots, k, n-k$, the resulting subset is J. By construction, $\delta(J) = J$. Then $\mathfrak{m}$ is the Levi subalgebra corresponding to J, thus $\mathfrak{m} = \sum_{j=1}^{\ell}(gl(r_j) \oplus gl(r_j)) \oplus gl(n-2k)$. Let $\{e,h,f\} \subset \mathfrak{m}$ be a Lie triple in $\mathfrak{m}$ representing the regular nilpotent orbits on each factor $gl(r_j)$ and the nilpotent orbit parameterized by λ' on $gl(n-2k)$. The latter is δ-quasidistinguished by Proposition 3.3.

Let $\mathfrak{g} = so(2n)$. Let $I = \{1,2,\dots,n\}$ be the indexing set of simple roots, and suppose that the branch point of the Dynkin diagram is at $n-2$, so that $\delta(i) = i$ for all $1 \leqslant i \leqslant n-2$ and $\delta(n-1) = n$. Suppose $\mathcal{O}$ is parameterized by a partition λ of $2n$. Then each even part in $\mathcal{O}$ appears with even multiplicity. Partition λ as $\lambda = \lambda' \sqcup \{r_1,r_1\} \sqcup \dots \sqcup \{r_\ell,r_\ell\}$, where λ' is the largest subset of λ such that λ' has only odd parts and each part occurs with multiplicity at most 2. As before, set $k = \sum_{j=1}^{\ell} r_j$. The subset $J \subset I$, $\delta(J) = J$, is

$$J = \{1,\dots,r_1-1\} \cup \{r_1+1,\dots,r_1+r_2-1\} \cup \{k-r_\ell+1,\dots,k-1\} \cup \{k+1,\dots,n\},$$

and the corresponding Levi subalgebra $\mathfrak{m} = \oplus_{j=1}^{\ell} gl(r_j) \oplus so(2n-2k)$. Let $\{e,h,f\}$ be a Lie triple in $\mathfrak{m}$ such that e represents the regular nilpotent orbits on the $gl(r_j)$ factors and the δ-quasidistinguished orbit parameterized by λ' on the $so(2n-2k)$ factor.

When $\mathfrak{g}$ is of type E_6, we list the orbits with the corresponding $\mathfrak{m}$ and $e \in \mathfrak{m}$ in Table 1. The indexing set $I = \{1, \dots, 6\}$ corresponds to the Dynkin diagram (3.2.2). We do not include in the table the δ-quasidistinguished orbits in E_6: E_6, $E_6(a_1)$, $E_6(a_3)$, D_5, $D_5(a_1)$, $A_4 + A_1$, and $D_4(a_1)$.

TABLE 1. E_6: nilpotent orbits

$\mathcal{O}$	$\mathfrak{m}$	δ-quasidistinsguished $e \in \mathfrak{m}$
A_5	$\{1,2,3,5,6\}$	(6)
D_4	$\{2,3,4,5\}$	$(7,1)$
A_4	$\{1,2,3,5,6\}$	$(5,1)$
$A_3 + A_1$	$\{1,2,3,5,6\}$	$(4,2)$
$2A_2 + A_1$	$\{1,2,4,5,6\}$	regular
A_3	$\{2,3,5\}$	(4)
$A_2 + 2A_1$	$\{1,3,4,6\}$	regular
$2A_2$	$\{1,2,5,6\}$	regular
$A_2 + A_1$	$\{1,2,3,5,6\}$	$(3,2,1)$
A_2	$\{2,3,4,5\}$	$(3,3,1,1)$
$3A_1$	$\{2,4,5\}$	regular
$2A_1$	$\{2,5\}$	regular
A_1	$\{3\}$	regular
0	$\emptyset$	0

□

Proposition 3.5. *Let $\mathcal{O}$ be a nilpotent G-orbit in $\mathfrak{g}$. There exists a Lie triple $\{e, h, f\}$, with $e \in \mathcal{O}$, $h \in \mathfrak{t}^\delta$, and an element $g \in G$ such that $\delta(\phi) = \mathrm{Ad}(g)\phi$, and $\delta(g)g \in Z_G(\phi)^0$.*

Proof. Assume $\delta \neq 1$ (otherwise $g = 1$). First, suppose e is δ-quasidistinguished in $\mathfrak{g}$. By definition, there exists $t\delta$, $t \in T^\delta$, an isolated semisimple element of $G\delta$ such that $e \in Z_{\mathfrak{g}}(t\delta)$. We may choose the Lie triple so that $\phi \subset Z_{\mathfrak{g}}(t\delta)$. Thus $\mathrm{Ad}(t\delta)\phi = \phi$, or equivalently $\delta(\phi) = \mathrm{Ad}(t^{-1})\phi$. So we may choose $g = t^{-1} \in T^\delta$, and then $\delta(g)g = t^{-2}$. We claim that $t^{-2} \in Z_G(\phi)^0$. Indeed, by the proof of Proposition 3.3, $t^2 = 1$ unless $\mathfrak{g}$ is of type A_n or when $\mathfrak{g}$ is of type E_6 and $\mathcal{O}$ is $D_5(a_1)$ or $A_4 + A_1$. But in all of these cases, $Z_G(\phi)$ is connected, so there is nothing to prove.

If e is not δ-quasidistinguished in $\mathfrak{g}$, by Lemma 3.4, there exists a δ-stable Levi $\mathfrak{m}$ such that $\phi \subset \mathfrak{m}$ and e is δ-quasidistinguished in $\mathfrak{m}$. From the proof of Lemma 3.4, we see that whenever $\mathfrak{m}$ has two factors of the same type which are flipped by δ, the two factors are of type A_{r-1} (for some r) and the nilpotent elements on these factors are equal to a regular nilpotent element e_{r-1}. This means that δ fixes the pair (e_{r-1}, e_{r-1}), and therefore by the discussion in the case $\mathfrak{m} = \mathfrak{g}$, there exists $g \in \mathrm{Ad}(\mathfrak{m})$ such that $\delta(g)g \in Z_M(\phi)^0 = M \cap Z_G(\phi)^0$. □

4. An action of $W_\#$ on $H^*(\mathcal{B}_e)$

4.1. Let δ be the automorphism on G and on W defined in section 3.2 and $W_\# = W \rtimes \langle\delta\rangle$. In this section, we construct a natural action of $W_\#$ on $H^*(\mathcal{B}_e)$, which extends the action of W we discussed in section 2.2, such that the following theorem holds.

Theorem 4.1. *For every $e \in \mathcal{N}$ and $0 \leqslant i \leqslant d_e$,*

$$\operatorname{tr}(\delta w, H^{2i}(\mathcal{B}_e)) = (-1)^i \mathsf{sgn}(w_0) \operatorname{tr}(w_0 w, H^{2i}(\mathcal{B}_e)).$$

In particular, we have

Corollary 4.2. *For any $w \in W$,*

$$X_q(e,\phi)(\delta w) = (-1)^{d_e} \mathsf{sgn}(w_0) X_{-q}(e,\phi)(w_0 w).$$

4.2. We assume that $\delta \neq \mathsf{id}$.

We fix a δ-stable Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Let $\mathfrak{g}_{\mathsf{reg}}$ be the set of regular semisimple elements in $\mathfrak{g}$ and $\mathfrak{b}_{\mathsf{reg}} = \mathfrak{g}_{\mathsf{reg}} \cap \mathfrak{b}$. Define the action of B on $G \times \mathfrak{b}$ by $b \cdot (g, b') = (gb^{-1}, \operatorname{Ad}(b)b')$. Let $G \times^B \mathfrak{b}$ be the quotient scheme and $G \times^B \mathfrak{b}_{\mathsf{reg}}$ be the image of $G \times \mathfrak{b}_{\mathsf{reg}}$ in $G \times^B \mathfrak{b}$. The Springer resolution (of $\mathfrak{g}$) is the map

$$q : G \times^B \mathfrak{b} \to \mathfrak{g}, \quad (g, b) \mapsto \operatorname{Ad}(g)b. \tag{4.2.1}$$

Its restriction q_{reg} to $G \times^B \mathfrak{b}_{\mathsf{reg}}$ gives an unramified Galois covering of $\mathfrak{g}_{\mathsf{reg}}$ whose Galois group is W.

We fix a prime number l invertible in $\mathbb{F}$. Let $\overline{\mathbb{Q}_l}_{G\times^B\mathfrak{b}}$ be the trivial sheaf on $G\times^B\mathfrak{b}$ and $\overline{\mathbb{Q}_l}_{G\times^B\mathfrak{b}_{\mathsf{reg}}}$ be the trivial sheaf on $G \times^B \mathfrak{b}_{\mathsf{reg}}$. Set $\Psi = Rq_!\overline{\mathbb{Q}_l}_{G\times^B\mathfrak{b}}[\dim(G)]$ and $\Psi_{\mathsf{reg}} = R(q_{\mathsf{reg}})_!\overline{\mathbb{Q}_l}_{G\times^B\mathfrak{b}_{\mathsf{reg}}}[\dim(G)]$. Since q is a small map, Ψ is the intersection cohomology complex $IC(\mathfrak{g}, \Psi_{\mathsf{reg}})$. Thus $\operatorname{End}(\Psi) = \operatorname{End}(\Psi_{\mathsf{reg}}) \cong \overline{\mathbb{Q}_l}[W]$ and we have a natural action of W on Ψ. Notice that the map q is in fact $G_\#$-equivariant. Hence the automorphism δ on G induces an action $\delta^* : \Psi \to \Psi$. We have that $(\delta^*)^2 = 1$ and $(\delta)^* w = \delta(w)\delta^* : \Psi \to \Psi$.

4.3. Let $e \in \mathcal{N}$. If $g \in G$ such that $\delta(e) = \operatorname{Ad}(g)(e)$. Then $\delta(g)g \in Z_G(e)$. We choose $g \in G$ such that $\delta(e) = \operatorname{Ad}(g)(e)$ and the image of $\delta(g)g$ in $A(e)$ lies in the kernel of ϕ for all $\phi \in \widehat{A(e)}_0$. By Proposition 3.5, such g always exists. Thus $\operatorname{Ad}(g)^* \circ \delta^* : \Psi_e \to \Psi_e$ satisfies

$$(\operatorname{Ad}(g)^* \circ \delta^*)^2 = (\operatorname{Ad}(\delta(g)g))^* = \mathsf{id}. \tag{4.3.1}$$

For any $w \in W$, we have the following commuting diagram

$$\begin{array}{ccccc} \Psi_e & \xrightarrow{\delta^*} & \Psi_{\operatorname{Ad}(g)e} & \xrightarrow{\operatorname{Ad}(g)^*} & \Psi_e \\ \downarrow{\scriptstyle \delta(w)} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\ \Psi_e & \xrightarrow{\delta^*} & \Psi_{\operatorname{Ad}(g)e} & \xrightarrow{\operatorname{Ad}(g)^*} & \Psi_e. \end{array} \tag{4.3.2}$$

From (4.3.1) and (4.3.2), we see that the map

$$w \mapsto w, \quad \delta \mapsto \operatorname{Ad}(g)^* \circ \delta^* \tag{4.3.3}$$

gives an action of $W_\#$ on Ψ_e and hence on the cohomology of Ψ_e. By construction, this action commutes with the action of $A(e)$ on Ψ_e. In particular, for any $i \geqslant 0$ and $\phi \in \widehat{A(e)}_0$, we may regard $H^{2i}(\mathcal{B}_e)^\phi$ as a $W_\#$-module and thus $X_q(e,\phi)$ as a virtual character of $W_\#$.

Notice that the $W_\#$-module structure depends on the choice of g. If we pick a different element $g' \in G$ such that $\delta(e) = \operatorname{Ad}(g')(e)$ and $\delta(g')g' \in Z_G(e)^0$, then $g^{-1}g' \in Z_G(e)$ and thus the actions of δ on $H^{2i}(\mathcal{B}_e)^\phi$ (defined using g and g') differ by $\phi(g^{-1}g')$.

4.4. Now we discuss the choice of g which makes the action of δ on $H^*(\mathcal{B}_e)$ nice. We construct such an element $g = g_1$ without using Proposition 3.5. A similar action of $A(e) \rtimes \langle\delta\rangle$ on $H^*(\mathcal{B}_e)$ is studied by Bezrukavnikov and Mirković in [4].

As before we assume p and q are large. We assume furthermore that $p \equiv 1 \mod 3$ if G is of type E_8. Let $F : G \to G$ be the split Frobenius map. For $e \in \mathcal{N}^F$, $F(\mathcal{B}_e) = \mathcal{B}_e$. We say that e *split* (with respect to F) if all the irreducible components of $\mathcal{B}_e$ are F-stable. By [29, Proposition 3.3] and [2, Section 3], each nilpotent orbit of $\mathfrak{g}$ contains exactly one split G^F-orbit.

Let $F' = F\delta = \delta F : G \to G$ be a twisted Frobenius map. The action of F' on W is conjugation by w_0. The following result is proved by Hotta and Springer for unitary groups in [14, Theorem 3.1], as a consequence of a specialization theorem, by Shoji for the other classical groups in [29, Theorem 4.18], and by Beynon and Spaltenstein for exceptional groups, in particular for E_6 in [2, Theorem 4.1], as a consequence of the Lusztig-Shoji algorithm.

Theorem 4.3. *Let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{g}$. There exists a bijection σ from the set of $G^{F'}$-orbits in $\mathcal{O}^{F'}$ to the set of G^F-orbits in $\mathcal{O}^F$ such that*

$$\mathrm{tr}((F')^* \circ w, H^{2i}(\mathcal{B}_e)) = (-1)^i \mathsf{sgn}(w_0)\, \mathrm{tr}(F^* \circ w_0 w, H^{2i}(\mathcal{B}_{\sigma(e)})).$$

We assume furthermore that $\sigma(e)$ is split with respect to F. Let $g \in G$ such that $\delta(e) = \mathrm{Ad}(g)(e)$. Then $e \in \mathcal{O}^{F_0}$, where $F_0 = \mathrm{Ad}(g^{-1}) \circ F$ is again a split Frobenius morphism. There exists $h \in G$ such that $\mathrm{Ad}(h)(e)$ is split with respect to F_0, i.e. $F_0(\mathrm{Ad}(h)(e)) = \mathrm{Ad}(h)(e)$ and all the irreducible components of $\mathcal{B}_{\mathrm{Ad}(h)(e)} = \mathrm{Ad}(h)\mathcal{B}_e$ are F_0-stable. In other words, $h^{-1}F_0(h) \in Z_G(e)$ and all the irreducible components of $\mathcal{B}_e$ are $\mathrm{Ad}(h^{-1}F_0(h)) \circ F_0 = \mathrm{Ad}(h^{-1}) \circ F_0 \circ \mathrm{Ad}(h)$-stable. Set

$$g_1 = g(h^{-1}F_0(h))^{-1}, \tag{4.4.1}$$

and $F_1 = \mathrm{Ad}(g_1)^{-1} \circ F$. Then $\delta(e) = \mathrm{Ad}(g_1)(e)$ and F_1 is a split Frobenius morphism and e is split with respect to F_1.

By [2, section 5(C)], F^* acts by q^i on $H^{2i}(\mathcal{B}_{\sigma(e)})$ and F_1^* acts by q^i on $H^{2i}(\mathcal{B}_e)$. Thus

$$\begin{aligned}\mathrm{tr}((F')^* \circ w, H^{2i}(\mathcal{B}_e)) &= \mathrm{tr}(F_1^* \circ (\mathrm{Ad}(g_1)^* \circ \delta^*) \circ w, H^{2i}(\mathcal{B}_e)) \\ &= q^i\, \mathrm{tr}((\mathrm{Ad}(g_1)^* \circ \delta^*) \circ w, H^{2i}(\mathcal{B}_e))\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}(F^* \circ w_0 w, H^{2i}(\mathcal{B}_{\sigma(e)})) &= q^i\, \mathrm{tr}(w_0 w, H^{2i}(\mathcal{B}_{\sigma(e)})) \\ &= q^i\, \mathrm{tr}(w_0 w, H^{2i}(\mathcal{B}_e)).\end{aligned}$$

By Theorem 4.3,

$$\mathrm{tr}((\mathrm{Ad}(g_1)^* \circ \delta^*) \circ w, H^{2i}(\mathcal{B}_e)) = (-1)^i \mathsf{sgn}(w_0)\, \mathrm{tr}(w_0 w, H^{2i}(\mathcal{B}_e)). \tag{4.4.2}$$

In particular,

$$\mathrm{tr}((\mathrm{Ad}(g_1)^* \circ \delta^*) \circ w_0, H^{2i}(\mathcal{B}_e)) = (-1)^i \mathsf{sgn}(w_0) \dim(H^{2i}(\mathcal{B}_e)).$$

Since $(\mathrm{Ad}(g_1)^* \circ \delta^*)$ commutes with w_0 by (4.3.2), $(\mathrm{Ad}(g_1)^* \circ \delta^* \circ w_0)^2 = \mathrm{Ad}(\delta(g_1)g_1)^*$ acts on $H^{2i}(\mathcal{B}_e)$ via the image of $\delta(g_1)g_1$ in $A(e)$, and so $\mathrm{Ad}(g_1)^* \circ \delta^* \circ w_0$ acts on $H^{2i}(\mathcal{B}_e)$ as an element of finite order. Hence it acts by the scalar $(-1)^i \mathsf{sgn}(w_0)$.

So $(\mathrm{Ad}(g_1)^* \circ \delta^* \circ w_0)^2 = \mathrm{Ad}(\delta(g_1)g_1)^*$ acts on $H^{2d_e}(\mathcal{B}_e)$ as the identity. Therefore the image of $\delta(g_1)g_1$ in $A(e)$ lies in the kernel of ϕ for all $\phi \in \widehat{A(e)}_0$.

4.5. We proved Theorem 4.1 over a finite field. In order to pass from (large) characteristic p to characteristic 0, first note that the representation of W on $H^{2i}(\mathcal{B}_e)$ is independent of the characteristic [32, section 3]. Now we choose g as in the proof of Proposition 3.5. Then the action of $\mathrm{Ad}(g)^* \circ \delta^*$ is again independent of the characteristic. As explained in section 4.3, there exists $z \in A(e)$ such that for any i and ϕ, $\mathrm{Ad}(g)^* \circ \delta^* \circ w_0$ acts on $H^{2i}(\mathcal{B}_e)^\phi$ as $(-1)^i \mathsf{sgn}(w_0)\phi(z)$. In fact, in characteristic p, z is the image of $g_1^{-1}g$ in $A(e)$. The component group $A(e)$ is independent of the choice of characteristic. In characteristic 0, set $g_1 = gz_1^{-1} \in Z_G(e)$, for a representative $z_1 \in Z_G(e)$ of z. Then $\mathrm{Ad}(g_1)^* \circ \delta^* \circ w_0$ acts on $H^{2i}(\mathcal{B}_e)$ as $(-1)^i\mathsf{sgn}(w_0)$, and the same argument as at the end of section 4.4 shows that the image of $\delta(g_1)g_1$ in $A(e)$ lies in the kernel of ϕ for all $\phi \in \widehat{A(e)}_0$.

Remark 4.4. In the rest of the paper, unless otherwise stated, we regard $H^{2i}(\mathcal{B}_e)^\phi$ as a $W_\#$-module via (4.3.3) with respect to the element g_1. As we discussed in section 4.3, g_1 is uniquely determined by (4.4.2) up to right multiplication by $Z_G(e)^0$.

4.6. Suppose that $\delta = 1$. We set $g_1 = 1$.

When $\mathfrak{g}$ is a classical Lie algebra $sp(2n)$, $so(2n+1)$, or $so(4n)$, Theorem 4.3 is proved in [29, Theorem 4.18] and therefore (4.4.2) holds.

When $\mathfrak{g}$ is exceptional of type G_2, F_4, E_6, E_7, E_8, we do not know an explicit reference for Theorem 4.3. However, the argument in [29, Theorem 4.18] (see also [2]) can be applied in these cases as well, as soon as we construct the correct matching σ so that the analogues of [29, Proposition 1.12 and Lemma 4.20(ii)] hold. This is done as follows.

Let $e \in \mathcal{N}^F$. For every $\phi \in \widehat{A(e)}_0$, let $\mathrm{hdeg}(\sigma(e,\phi))$ denote the lowest harmonic degree of $\sigma(e,\phi)$, and set $d_\phi = (-1)^{\mathrm{hdeg}(\sigma(e,\phi))}$.

Lemma 4.5 (compare with [29, Proposition 1.12])**.** *There exists a conjugacy class c_0 of $A(e)$ such that $\phi(c_0) = d_\phi d_1$, for all $\phi \in \widehat{A(e)}$.*

Proof. When $\mathfrak{g}$ is a simple exceptional Lie algebra not of type E_6, the component group $A(e)$ can be 1, $\mathbb{Z}/2\mathbb{Z}$, or S_n with $n = 3, 4, 5$. The claim is obvious in the case 1 or $\mathbb{Z}/2\mathbb{Z}$. In the S_n cases, by inspection of the tables in [5, pages 429-432], we see that all $\sigma(e,\phi)$ have the same parity of the lowest harmonic degrees. Thus, we may choose $c_0 = 1$ in these cases. □

Define the map $\sigma : G^F\backslash G\cdot e \to G^F\backslash G\cdot e$, by $\mathcal{O}_e^F(c) \mapsto \mathcal{O}_e^F(cc_0)$. This makes sense because in the cases when $A(e)$ is not abelian, we chose $c_0 = 1$. By inspection of tables in [17, Chapter 22], we see that the analogue of [29, Lemma 4.20(ii)] holds:

$$|Z_{G^F}(e_c)|(-q) = |Z_{G^F}(e_{cc_0})|(q). \tag{4.6.1}$$

An alternative argument, again case by case, is as follows. The graded W-representations $H^*(e)^\phi$ are explicitly computed in all the exceptional cases: for G_2, F_4 in [30], and for type E in [3]. One can check that if an irreducible W-representation μ occurs in $H^{2i}(\mathcal{B}_e)^\phi$ then $(-1)^{\mathrm{hdeg}(\mu)} = \mathsf{sgn}(w_0)(-1)^i$. See also [3, page 19, Remark (a)] The lowest harmonic degrees can be read from [5]. Since w_0 is central, it acts on every irreducible μ by $(-1)^{\mathrm{hdeg}(\mu)}$, and (4.4.2) follows.

5. The twisted elliptic form

5.1. Let $\epsilon \in \{+1, -1\}$. Set

$$\overline{R}_\epsilon(W) = R(W)_{\mathbb{C}}/\mathrm{rad}\langle\ ,\ \rangle_W^\epsilon.$$

For any $e \in \mathcal{N}$, let $R_\epsilon(W)^e$ be the image of $R(W)^e$ in $R_\epsilon(W)$. By Corollary 2.2, $R_\epsilon(W) = \oplus_e R_\epsilon(W)^e$, where u runs over nilpotent conjugacy classes of G.

Proposition 5.1. *Let $e \in \mathcal{N}$ be given.*

(1) $\overline{R}_1(W)^e \neq 0$ *if and only if u is quasidistinguished.*
(2) $\overline{R}_{-1}(W)^e \neq 0$ *if and only if u is δ-quasidistinguished, or equivalently, $e \in \mathcal{N}^{\mathsf{sol}}$.*

Proof. If $Z_G(e)^0$ is not solvable, then $Z_G(e)^{F_c}$ contains as a subgroup the $\mathbb{F}_q$-points of a rank one semisimple group, and in particular, $|Z_G(e)^{F_c}|$ (as a polynomial in q) is divisible by (q^2-1). Therefore, $|Z_G(e)^{F_c}|(\epsilon) = 0$ for $\epsilon \in \{+1, -1\}$. By Corollary 2.5, $\overline{R}_1(W)^e = \overline{R}_{-1}(W)^e = 0$.

Thus we may reduce to the case where $e \in \mathcal{N}^{\mathsf{sol}}$. The statements then follow by an analysis similar to the proof of Proposition 2.6. □

5.2. In the rest of this section, we focus on the (-1)-elliptic form $\langle\ ,\ \rangle_W^{-1}$. It is by definition

$$\langle \chi, \chi' \rangle_W^{-1} = \langle \chi \otimes \wedge V, Y \rangle_W, \qquad \text{where } \wedge V = \oplus_{i \geqslant 0} \wedge^i V. \tag{5.2.1}$$

We will relate it to the δ-twisted elliptic form, here δ is the automorphism of (W, V) given by $-w_0$:

$$\delta(\xi) = -w_0(\xi),\ \xi \in V, \quad \delta(w) = w_0 w w_0,\ w \in W. \tag{5.2.2}$$

The δ-twisted elliptic form is defined as follows.

If (σ, X) is a W-representation, let $(\sigma^\delta, X^\delta)$ be the δ-twisted representation, i.e.,

$$\sigma^\delta(w)x = \sigma(\delta(w))x = \sigma(w_0 w w_0)x,\ w \in W,\ x \in X^\delta = X.$$

We choose the intertwininer $\phi : (\sigma^\delta, X^\delta) \to (\sigma, X)$ to be $\phi(x) = \sigma(w_0)x$. Clearly $\phi^2 = 1$. Define the δ-twisted character of σ:

$$\mathrm{tr}_\sigma^\delta(w) = \mathrm{tr}(\sigma(w) \circ \phi^{-1}),$$

and using the explicit form of ϕ, we find

$$\mathrm{tr}_\sigma^\delta(w) = \mathrm{tr}_\sigma(w w_0),\ w \in W. \tag{5.2.3}$$

Definition 5.2. Let $R^\delta(W)$ denote the $\mathbb{Z}$-span of δ-twisted characters $\mathrm{tr}_\sigma^\delta$, for all irreducible W-representations σ. Define the δ-elliptic pairing in $R^\delta(W)$:

$$\langle \chi, \chi' \rangle_W^{\delta-\mathsf{ell}} = \frac{1}{|W|} \sum_{w \in W} \chi(w)\chi'(w) \mathrm{det}_V(1 - w\delta). \tag{5.2.4}$$

Lemma 5.3. *The map $\mathrm{tr}_\sigma^\delta \to \mathrm{tr}_\sigma$ induces a linear isometry between the spaces $(R^\delta(W), \langle\ ,\ \rangle_W^{\delta-\mathsf{ell}})$ and $(R(W), \langle\ ,\ \rangle_W^{-1})$:*

$$\langle \mathrm{tr}_\sigma^\delta, \mathrm{tr}_{\sigma'}^\delta \rangle_W^{\delta-\mathsf{ell}} = \langle \mathrm{tr}_\sigma, \mathrm{tr}_{\sigma'} \rangle_W^{-1}.$$

Proof. This is a straightforward calculation:

$$\begin{aligned}
\langle \mathrm{tr}_\sigma^\delta, \mathrm{tr}_{\sigma'}^\delta \rangle_W^{\delta-\mathsf{ell}} &= \frac{1}{|W|} \sum_{w \in W} \mathrm{tr}_\sigma^\delta(w)\, \mathrm{tr}_{\sigma'}^\delta(w) \mathrm{det}_V(1 - w\delta) \\
&= \frac{1}{|W|} \sum_{w \in W} \mathrm{tr}_\sigma(w w_0)\, \mathrm{tr}_{\sigma'}(w w_0) \mathrm{det}_V(1 + w w_0) \\
&= \frac{1}{|W|} \sum_{w \in W} \mathrm{tr}_\sigma(w)\, \mathrm{tr}_{\sigma'}(w) \mathrm{det}_V(1 + w) = \langle \mathrm{tr}_\sigma, \mathrm{tr}_{\sigma'} \rangle_W^{-1}.
\end{aligned}$$

□

5.3. Set
$$\overline{R}^{\delta}(W) = R(W)_{\mathbb{C}}/\mathrm{rad}\langle\ ,\ \rangle_W^{\delta-\mathsf{ell}}.$$
Under the isometry in Lemma 5.3, $\overline{R}_{-1}(W)$ may be identified with $\overline{R}^{\delta}(W)$.

We give an more explicit description of $\overline{R}^{\delta}(W)$ using the δ-elliptic conjugacy classes.

Definition 5.4. The δ-twisted conjugacy class of $w \in W$ is
$$\mathcal{C}_w = \{w'w\delta(w')^{-1} : w' \in W\}.$$
A twisted class $\mathcal{C}$ is called δ-elliptic if $\mathcal{C} \cap W_J = \emptyset$, for all δ-stable proper parabolic subgroups W_J of W. An element $w \in W$ is called δ-elliptic if it belongs to a δ-elliptic conjugacy class.

The following lemma is well-known, see for example [13].

Lemma 5.5. *The following are equivalent for an element $w \in W$:*

(1) *w is δ-elliptic;*
(2) $\det_V(1 - w\delta) \neq 0$*;*
(3) $V^{w\delta} = 0$.

5.4. For every δ-stable parabolic subgroup W_J of W, let $\delta\text{-}\mathrm{Ind}_{W_J}^W : R^{\delta}(W_J) \to R^{\delta}(W)$ be the induction functor: $\delta\text{-}\mathrm{Ind}_{W_J}^W(\sigma) = \mathbb{C}[W]\delta \otimes_{\mathbb{C}[W_J]\delta} \sigma$. Denote
$$R^{\delta}_{\mathrm{ind}}(W) = \sum \delta\text{-}\mathrm{Ind}_{W_J}^W(R^{\delta}(W_J)),$$
where the sum is over all δ-stable proper parabolic subgroups W_J of W.

The following proposition is a straightforward modification of [26, Proposition (2.2.2)], and we omit the proof.

Proposition 5.6. *We keep the notations as above. Then*

(1) rad$\langle\ ,\ \rangle_W^{\delta-\mathsf{ell}} = R^{\delta}_{ind}(W)_{\mathbb{C}}$.

(2) The dimension of $\overline{R}^{\delta}(W)$ equals the number of δ-elliptic conjugacy classes in W.

Remark 5.7. In light of Proposition 5.6, we record the number of δ-elliptic conjugacy classes in the irreducible Weyl groups (where $\delta = -w_0$), see for example [26, section 3.1], [13, section 7] and [11, section 6].

(1) ${}^2A_{n-1}$: the number of partitions of n into odd parts (this is the same as the number of partitions of n into distinct parts);
(2) B_n: the number of partitions of n;
(3) D_{2n}: the number of partitions of $2n$ into an even number of parts;
(4) ${}^2D_{2n+1}$: the number of partitions of $2n+1$ into an odd number of parts;
(5) G_2: 3; F_4: 9; ${}^2E_6$: 9; E_7: 12; E_8: 30.

6. Extended Dirac operator

6.1. We retain the notation from the previous sections. Fix a W-invariant bilinear form $(\ ,\ )$ on V. Then W is a Weyl group in the Euclidean vector space $V, (\ ,\ )$ with semisimple root system $\Phi \subset V^*$, positive roots Φ^+, and simple roots Π. Let $\mathbf{r}$ denote an indeterminate to be specialized later. Recall the automorphism δ, $\delta^2 = 1$, of the root system given by $-w_0$.

Definition 6.1 (Lusztig,[19])**.** The graded affine Hecke algebra $\mathbb{H}$ with equal parameters attached to (Φ, V, W) is the unique associative $\mathbb{C}[\mathbf{r}]$-algebra with identity generated by $\{\xi \in V_{\mathbb{C}}^*\}$ and $\{w \in W\}$ such that:

(1) $\mathbb{H} \cong \mathbb{C}[\mathbf{r}] \otimes_{\mathbb{C}} \mathbb{C}[W] \otimes_{\mathbb{C}} S(V_{\mathbb{C}}^*)$, as $(\mathbb{C}[\mathbf{r}] \otimes_{\mathbb{C}} \mathbb{C}[W], S(V_{\mathbb{C}}^*))$-bimodules;
(2) $\xi \cdot s_\alpha - s_\alpha \cdot s_\alpha(\xi) = 2\mathbf{r}\xi(\alpha^\vee)$, $\alpha \in \Pi$, $\xi \in V_{\mathbb{C}}^*$.

The center of $\mathbb{H}$ is $\mathbb{C}[\mathbf{r}] \otimes_{\mathbb{C}} S(V_{\mathbb{C}}^*)^W$, hence the central characters are parameterized by W-orbits in $\mathbb{C} \oplus V_{\mathbb{C}}$.

Let $*$ denote the conjugate linear anti-automorphism of $\mathbb{H}$ defined on generators via

$$\mathbf{r}^* = \mathbf{r},\ w^* = w^{-1},\ \xi^* = w_0 \cdot \delta(\xi) \cdot w_0,\ w \in W,\ \xi \in V. \tag{6.1.1}$$

For every $\xi \in V$, define

$$\widetilde{\xi} = \frac{1}{2}(\xi - \xi^*). \tag{6.1.2}$$

It is clear that $\widetilde{\xi}^* = -\widetilde{\xi}$. Moreover, it is known that $w \cdot \widetilde{\xi} \cdot w^{-1} = \widetilde{w(\xi)}$.

We consider the extended algebra $\mathbb{H}_\# = \langle \mathbb{H}, \delta \rangle$, and extend the $*$-operation to $\mathbb{H}_\#$ by setting $\delta^* = \delta$.

6.2. We recall the classification of simple $\mathbb{H}$-modules from [20, 21], in particular [21, section 1]. We denote by $H_*^{G'}()$ and $H^*_{G'}()$ the G'-equivariant homology and cohomology as in the references.

If $x \in \mathfrak{g}$, denote $Z_{G\times\mathbb{C}^\times}(x) = \{(g, \lambda) \in G \times \mathbb{C}^\times : \mathrm{Ad}(g)x = \lambda^2 g\}$.

Let $e \in \mathfrak{g}$ be a nilpotent element, and (s, r_0) a semisimple element in the Lie algebra of $Z_{G\times\mathbb{C}^\times}(e)$. In particular, $[s, e] = 2r_0 e$. Choose $A \subset Z_{G\times\mathbb{C}^\times}(x)$ a torus containing $\exp(s, r_0)$. Let $\mathbb{C}_{s,r_0}$ denote the one dimensional $H_A^*(\{pt\})$-module obtained by evaluation at (s, r_0). (One may identify $H_A^*(\{pt\})$ with the space of polynomials $\mathbb{C}[\mathfrak{a}]$, where $\mathfrak{a}$ is the Lie algebra of A.) From ([20, 10.12], [21, 1.13])

$$E_{e,s,r_0} = H_*(\mathcal{B}_e^s) = \mathbb{C}_{s,r_0} \otimes_{H_A^*(\{pt\})} H_*^A(\mathcal{B}_e) = \mathbb{C}_{s,r_0} \otimes_{H^*_{Z_{G\times\mathbb{C}^\times}(e)^0}(\{pt\})} H_*^{Z_{G\times\mathbb{C}^\times}(e)^0}(\mathcal{B}_e), \tag{6.2.1}$$

where $\mathcal{B}_e^s$ is the variety of Borel subalgebras of $\mathfrak{g}$ containing e and s. The space E_{e,s,r_0} carries an $\mathbb{H}$-action such that $\mathbf{r}$ acts by r_0.

Let $Z(e, s) = Z_{G\times\mathbb{C}^\times}(e) \cap (Z_G(s) \times \mathbb{C}^\times)$. Denote by $A(e)$ and $A(e, s)$ the group of components of $Z_{G\times\mathbb{C}^\times}(e)$ and $Z(e, s)$, respectively. The natural map $A(e, s) \to A(e)$ is an injection, so E_{e,s,r_0} carries an action of $A(e, s)$ obtained by restriction from the natural action of $A(e)$ on $H_*^{Z_{G\times\mathbb{C}^\times}(e)^0}(\mathcal{B}_e)$. Set

$$E_{e,s,r_0,\psi} = \mathrm{Hom}_{A(e,s)}[\psi, E_{e,s,r_0}], \tag{6.2.2}$$

and $\widehat{A(e, s)}_0 = \{\psi : E_{e,s,r_0,\psi} \neq 0\}$. By [20], $\psi \in \widehat{A(s, e)}$ if and only if ψ occurs in the action of $A(s, e)$ on $H_*(\mathcal{B}_e^s)$.

Theorem 6.2 ([21, Theorem 1.15])**.** *Let $r_0 \neq 0$.*

(1) *Let e, s be as above and $\psi \in \widehat{A(e, s)}_0$. The $\mathbb{H}$-module $E_{e,s,r_0,\psi}$ has a unique maximal submodule. Let $\overline{E}_{e,s,r_0,\psi}$ be the irreducible quotient.*
(2) *The map $(e, s, \psi) \to \overline{E}_{e,s,r_0,\psi}$ gives a one-to-one correspondence between the of G-conjugacy classes of triples (e, s, ψ) where $e \in \mathfrak{g}$ is nilpotent, $s \in \mathfrak{g}$ is semisimple with $[s, e] = 2r_0 e$, $\psi \in \widehat{A(s, e)}_0$, and the set of simple $\mathbb{H}$-modules on which $\mathbf{r}$ acts by r_0.*

6.3. We recall next the classification and construction of irreducible tempered $\mathbb{H}$-modules following [21]. Suppose that $r_0 \in \mathbb{R}_{>0}$.

Definition 6.3. Denote $V^{*,+} = \{\xi \in V^* : \xi(\alpha^\vee) > 0, \text{ for all } \alpha \in \Pi\}$ the set of dominant elements of V^*. Notice that $\delta(V^{*,+}) = V^{*,+}$. An irreducible $\mathbb{H}$-module X is called tempered if every $S(V_{\mathbb{C}}^*)$ weight $\nu \in V_{\mathbb{C}}$ of X satisfies

$$\xi(\Re\nu) \leqslant 0, \quad \text{for all } \xi \in V^{*,+}.$$

If all the inequalities are strict, then X is called a discrete series module.

Theorem 6.4 ([21, Theorem 1.21])**.** *Let e, s, ψ be as in Theorem 6.2 and $r_0 \in \mathbb{R}_{>0}$.*

(1) *The simple module $\overline{E}_{e,s,r_0,\psi}$ is tempered if and only if there exists a Lie triple $\{e, h, f\}$ such that $[s, h] = 0$, $[s, f] = -2r_0 f$, and $\mathrm{ad}(s - r_0 h) : \mathfrak{g} \to \mathfrak{g}$ has no real eigenvalues.*
Moreover, in this case $\overline{E}_{e,s,r_0,\psi} = E_{e,s,r_0,\psi}$.
(2) *The module $\overline{E}_{e,s,r_0,\psi}(= E_{e,s,r_0,\psi})$ is a discrete series module if and only if e is distinguished and $s = r_0 h$.*

6.4. In light of Theorem 6.4, fix a Lie triple $\phi = \{e, h, f\}$, and consider the module $E_{e,h,1,\psi}$, with $\psi \in \widehat{A(e,h)}_0$. This is a simple tempered $\mathbb{H}$-module (with real central character) on which $\mathbf{r}$ acts by $r_0 = 1$. So we assume from now on that $r_0 = 1$ and drop it from the notations.

Assume that $\delta \neq \mathsf{id}$. We extend the $\mathbb{H}$-module structure on $\overline{E}_{e,h,\psi}$ to a $\mathbb{H}_\#$-module structure. The construction is similar to section 4. Assume, as we may, that $h \in \mathfrak{t}^\delta$. By [9, Lemma 3.7.3 and Remark 3.7.5 (ii)], $A(e) = A(h, e)$. Then the natural map

$$\{g \in G; \delta(\phi) = \mathrm{Ad}(g)(\phi)\} \to \{g \in G; \delta(e) = \mathrm{Ad}(g)(e)\}$$

induces a bijection on the connected components. Let $g \in G$ with $\delta(\phi) = \mathrm{Ad}(g)(\phi)$ and that the image of g in $\{g \in G; \delta(e) = \mathrm{Ad}(g)(e)\}$ is in the same connected component as the element g_1 in section 4.4. Then $\mathrm{Ad}(g)^* \circ \delta^* : H_*(\mathcal{B}_e^h) \to H_*(\mathcal{B}_e^h)$. Since $\delta(h) = h$, $\mathrm{Ad}(g)^* \circ \delta^* : E_{e,h} \to E_{e,h}$. Similar to 4.3.1, we have

$$(\mathrm{Ad}(g)^* \circ \delta^*)^2 = \mathsf{id}. \tag{6.4.1}$$

We have the following commuting diagram

$$\begin{array}{ccccc} \mathbb{H} \times E_{e,h} & \xrightarrow{(\delta,\delta^*)} & \mathbb{H} \times E_{\delta(e),h} & \xrightarrow{(\mathsf{id},\mathrm{Ad}(g)^*)} & \mathbb{H} \times E_{e,h} \\ \downarrow & & \downarrow & & \downarrow \\ E_{e,h} & \xrightarrow{\delta^*} & E_{\delta(e),h} & \xrightarrow{\mathrm{Ad}(g)^*} & E_{e,h}. \end{array} \tag{6.4.2}$$

Define the action of $\delta \in \mathbb{H}_\#$ on $E_{e,h}$ by $\mathrm{Ad}(g)^* \circ \delta^*$. By (6.4.1) and (6.4.2), this gives an action of $\mathbb{H}_\#$ on $E_{e,h}$. The map $\mathrm{Ad}(g)^* \circ \delta^* : H_*(\mathcal{B}_e^h) \to H_*(\mathcal{B}_e^h)$ induces an action on $\widehat{A(e,h)}_0$, which we denote by δ.

By [9, Lemma 3.7.3 and Remark 3.7.5 (ii)], $A(e) = A(h, e)$. By [20, 10.13], there is an isomorphism of W-modules:

$$E_{e,h,\psi} \cong X_1(e, \psi), \tag{6.4.3}$$

where $X_1(e,\psi) = H^*(\mathcal{B}_e)^\psi \otimes \mathsf{sgn}$ is the Springer W-representation from section 2. Since the W-structure of the module does not change under the δ-twist, (6.4.3) implies that $\delta(\psi) = \psi$.

Hence for any $\psi \in \widehat{A(e)}_0$, $E_{e,h,\psi}$ is an $\mathbb{H}_\#$-module such that (6.4.3) is an isomorphism of $W_\#$-modules, where the δ-action on $X_1(u,\psi)$ is as in section 4.

6.5. Define the Clifford algebra $C(V)$ of $(V, (\ ,\ ))$ to be the real associative algebra with identity generated by $\{\xi \in V\}$ subject to the relations

$$\xi \cdot \xi' + \xi' \cdot \xi = -2(\xi, \xi'), \quad \xi, \xi' \in V. \tag{6.5.1}$$

The algebra $C(V)$ is naturally $\mathbb{Z}_{\geqslant 0}$-filtered, where the n-th space $C_n(V)$ in the filtration is the span of all elements of $C(V)$ which are products of at most n elements of V. The associated graded algebra is $\wedge V$.

The Clifford algebra $C(V)$ is also $\mathbb{Z}/2\mathbb{Z}$-graded $C(V) = C(V)_{\mathsf{even}} + C(V)_{\mathsf{odd}}$ by the parity of the degree of homogeneous elements in the filtration just defined. Let $\epsilon : C(V) \to C(V)$ be the involution which is $+1$ on $C(V)_{\mathsf{even}}$ and -1 on $C(V)_{\mathsf{odd}}$.

Let ${}^t : C(V) \to C(V)$ be the anti-automorphism defined by $\xi^t = -\xi$ for all $\xi \in V$.

Define the pin group

$$\mathsf{Pin}(V) = \{g \in C(V)^\times \mid g^t = g^{-1} \text{ and } \epsilon(g) \cdot \xi \cdot g^{-1} \in V, \text{ for all } \xi \in V\}. \tag{6.5.2}$$

This is a central double extension of $O(V)$ with the projection map $p : C(V) \to O(V)$, $p(g)(\xi) = \epsilon(g) \cdot \xi \cdot g^{-1}$, $g \in \mathsf{Pin}(V)$, $\xi \in V$. Since $W \subset O(V)$, one considers

$$\widetilde{W} = p^{-1}(W) \subset \mathsf{Pin}(V), \tag{6.5.3}$$

a central double extension of W.

When $\dim V$ is even, $C(V)$ is a central simple algebra, and therefore it has a unique complex simple module S of dimension $2^{\dim V/2}$. When $\dim V$ is odd, $Z(C(V))$ is two dimensional and $C(V) = C(V)_{\mathsf{even}} \otimes Z(C(V))$. In this case, the unique simple module of $C(V)_{\mathsf{even}}$ can be extended in two inequivalent ways to $C(V)$, S^+ and S^-. In what follows, we will refer to any one of S, S^+, S^- as a spin module of $C(V)$. For convenience, we also set

$$\mathcal{S} = \begin{cases} S, & \dim V \text{ even}, \\ S^+ + S^-, & \dim V \text{ odd}. \end{cases} \tag{6.5.4}$$

One can restrict every spin module to $\mathsf{Pin}(V)$ and furthermore to $\widetilde{W}$. Since we assumed $V^W = 0$ (semisimple), $\widetilde{W}$ generates $C(V)$, and therefore every spin module is an irreducible $\widetilde{W}$-representation.

We have

$$\mathcal{S} \otimes \mathcal{S} = a_V \wedge V \tag{6.5.5}$$

as W-representations (or $O(V)$-representations), where $a_V = 1$ if $\dim V$ is even, and $a_V = 2$ if $\dim V$ is odd.

The group $\widetilde{W}$ admits a Coxeter-like presentation. Denote by $m(\alpha,\beta)$ the order in W of $s_\alpha s_\beta$. Then:

$$\widetilde{W} = \langle -1, \widetilde{s}_\alpha, \alpha \in \Pi \mid (-1)^2 = 1,\ (\widetilde{s}_\alpha \widetilde{s}_\beta)^{m(\alpha,\beta)} = -1 \rangle. \tag{6.5.6}$$

6.6. Let -1 be the automorphism of $O(V)$ induced by $\xi \mapsto -\xi$ on V and recall $W_\# = \langle W, \delta\rangle = \langle W, -1\rangle \subset O(V)$. Define $\widetilde{W}_\# = p^{-1}(W_\#) \subset \mathsf{Pin}(V)$.

We fix an orthonormal basis $\{\xi_1, \xi_2, \dots, \xi_n\}$ of V permuted by δ, and set

$$z = \xi_1\xi_2 \dots \xi_n \in \mathsf{Pin}(V). \tag{6.6.1}$$

Lemma 6.5. *The element z satisfies the following properties:*

(1) $z^2 = (-1)^{\frac{n(n+1)}{2}}$;
(2) $p(z) = -1 \in O(V)$;
(3) $z\xi = (-1)^{n-1}\xi z$, $\xi \in V$.

Proof. Straightforward. □

Therefore, z is a central element in $C(V)$ if $\dim V$ is odd, and z is a central element of $C(V)_{\mathsf{even}}$, when $\dim V$ is even. When $\dim V$ is even, denote by $S^\pm$ the two constituents in the restriction of spin module S to $C(V)_{\mathsf{even}}$. With this notation, z acts by scalars on $S^\pm$ in both cases $\dim V$ odd or even. Since the trace of $-1 \in O(V)$ on $\wedge V$ is zero, (6.5.5) implies that the trace of z in $S^+ + S^-$ is zero as well. Therefore, the scalars by which z acts on S^+, S^- differ by a sign, i.e.,

$$z|_{S^+} = -z|_{S^-} = c \in \mathbb{C}. \tag{6.6.2}$$

(We have $c^2 = (-1)^{n(n+1)/2}$, but we will not need to use this fact.)

We will use the formula ([1, Lemma 3.4])

$$\widetilde{w} \cdot \xi \cdot \widetilde{w}^{-1} = \mathsf{sgn}(\widetilde{w})w(\xi),\ \widetilde{w} \in \widetilde{W},\ \xi \in V. \tag{6.6.3}$$

Fix once for all $\widetilde{w}_0 \in p^{-1}(w_0)$ and set

$$\widetilde{\delta} = \widetilde{w}_0 z, \text{ so that } p(\widetilde{\delta}) = \delta.$$

It is easy to check that

$$\widetilde{\delta}\xi = (-1)^{n+\ell(w_0)}\delta(\xi)\widetilde{\delta},\ \xi \in V. \tag{6.6.4}$$

The group $\widetilde{W}_\#$ is generated by $\widetilde{W}$ and z. It is also generated by $\widetilde{W}$ and $\widetilde{\delta}$. Set

$$\widetilde{W}' = \begin{cases} \widetilde{W} \cap C(V)_{\mathsf{even}}, & \dim V \text{ even},\\ \widetilde{W}, & \dim V \text{ odd}, \end{cases} \tag{6.6.5}$$

and

$$\widetilde{W}'_\# = \begin{cases} \langle \widetilde{W} \cap C(V)_{\mathsf{even}}, z\rangle, & \dim V \text{ even},\\ \widetilde{W}_\#, & \dim V \text{ odd}. \end{cases} \tag{6.6.6}$$

The discussion above shows that $S^\pm$ are $\widetilde{W}'_\#$-representations.

6.7. Following [1, Definition 3.1], define the Dirac element

$$\mathcal{D} = \sum_{i=1}^{n} \widetilde{\xi}_i \otimes \xi_i \in \mathbb{H} \otimes C(V) \subset \mathbb{H}_\# \otimes C(V). \tag{6.7.1}$$

Let $\rho : \mathbb{C}[\widetilde{W}_\#] \to \mathbb{H}_\# \otimes C(V)$ be the linear map extending $\widetilde{w} \mapsto p(\widetilde{w}) \otimes \widetilde{w}$, $\widetilde{w} \in \widetilde{W}$ and $\widetilde{\delta} \mapsto \delta \otimes \widetilde{\delta}$. Notice that $\rho(z) = w_0\delta \otimes z \in \mathbb{H}\delta \otimes C(V)$.

The algebra $C(V)$ also has a $*$-operation defined by the transpose map ^{t}. On $\mathsf{Pin}(V)$ this corresponds to the inversion operation. With respect to this operation, the spin modules $S, S^\pm$ admit positive definite invariant forms (i.e., they are unitary).

Define the Casimir element of $\mathbb{H}$ ([1, Definition 2.3]):

$$\Omega = \sum_{i=1}^{n} \xi_i^2. \tag{6.7.2}$$

This is an element in $S(V)^{W_\#}$, thus central in $\mathbb{H}_\#$. The central characters of irreducible $\mathbb{H}_\#$-modules are parameterized by $W_\#$-orbits in $V_\mathbb{C}$. By [1, Lemma 2.5], if (π, X) is an irreducible $\mathbb{H}_\#$-module with central character $W_\# \cdot \nu$, then $\pi(\Omega)$ acts on X by the scalar (ν, ν).

Let $\Phi^\vee \subset V$ be corresponding coroots and $\Phi^{\vee,+}$ the positive coroots, $\Phi^{\vee,-}$, the negative coroots. Define the Casimir element of $\widetilde{W}$ ([1, section 3.4])

$$\Omega_{\widetilde{W}} = (-1) \sum_{\alpha,\beta \in \Phi^+, s_\alpha(\beta) \in \Phi^-} |\alpha^\vee||\beta^\vee| \widetilde{s}_\alpha \widetilde{s}_\beta \in \mathbb{C}[\widetilde{W}']^{\widetilde{W}}, \tag{6.7.3}$$

where (-1), $\widetilde{s}_\alpha$ are as in (6.5.6).

Proposition 6.6. *The element $\mathcal{D}$ has the following properties:*

(1) $\mathcal{D}^* = \mathcal{D}$.
(2) $\rho(\widetilde{w})\mathcal{D} = \mathsf{sgn}(\widetilde{w})\mathcal{D}\rho(\widetilde{w})$, *for all* $\widetilde{w} \in \widetilde{W}$;
(3) $\rho(z)\mathcal{D} = (-1)^{\ell(w_0)+n}\mathsf{sgn}(w_0)\mathcal{D}\rho(z)$.
(4) $\mathcal{D}^2 = -\Omega \otimes 1 + \mathbf{r}^2 \rho(\Omega_{\widetilde{W}}) \in \mathbb{H} \otimes C(V)$.

Proof. (1) Straightforward.

(2) This is [1, Lemma 3.4].

(3) We have $z = \widetilde{w}_0^{-1}\widetilde{\delta}$. From (6.6.4), we see that $\rho(\widetilde{\delta})\mathcal{D} = (-1)^{\ell(w_0)+n}\mathcal{D}\rho(\widetilde{\delta})$. The claim now follows from (2).

(4) This is [1, Theorem 3.5].

□

Definition 6.7. If X is an $\mathbb{H}_\#$-module, and S is a spin $C(V)$-module (when $\dim V$ is odd, there are two choices), left action by $\mathcal{D}$ defines the Dirac operator (of X and S)

$$D_\# : X \otimes S \to X \otimes S.$$

Define the extended Dirac cohomology of X (with respect to S)

$$H^D_\#(X) = \ker(D_\#)/(\ker D_\#) \cap (\operatorname{im} D_\#). \tag{6.7.4}$$

By Proposition 6.6(2),(3), $H^D_\#(X)$ is a $\widetilde{W}_\#$-representation. From Proposition 6.6(1), when X is a $*$-unitary $\mathbb{H}_\#$-module, $H^D_\#(X) = \ker D_\#$.

Definition 6.8. Suppose $\dim V$ is even. By restriction, $D_\#$ defines two operators

$$D^\pm_\# : X \otimes S^\pm \to X \otimes S^\mp. \tag{6.7.5}$$

Suppose $\dim V$ is odd. In this case, S^+ and S^- are realized on the same vector space U (coming from the unique simple module of $C(V)_{\mathsf{even}}$). Then, as in [8, section 2.9], $D_\# : X \otimes S^+ \to X \otimes S^+$ can be composed with the vector space identity map $S^+ \to S^-$ to yield $D^+_\# : X \otimes S^+ \to X \otimes S^-$. Similarly, define $D^-_\#$.

In both cases, set

$$H^{D^\pm}_\#(X) = \ker(D^\pm_\#)/(\ker D^\pm_\#) \cap (\operatorname{im} D^\mp_\#), \text{ and } I_\#(X) = H^{D^+}_\#(X) - H^{D^-}_\#(X). \tag{6.7.6}$$

We call $I_{\#}(X)$ the extended Dirac index. By Proposition 6.6(2),(3), $H^{D^{\pm}}_{\#}(X)$ are $\widetilde{W}'_{\#}$-representations, and $I_{\#}(X)$ is a virtual $\widetilde{W}'_{\#}$-module.

Notice that

$$D^{\pm}_{\#} \circ D^{\mp}_{\#} = -\Omega \otimes 1 + \mathbf{r}^2 \rho(\Omega_{\widetilde{W}}), \tag{6.7.7}$$

by Proposition 6.6(4), since $D^{\pm}_{\#}$ are given by the left action of $\mathcal{D}$.

Proposition 6.9. *For every $\mathbb{H}_{\#}$-module X, we have $I_{\#}(X) = X \otimes (S^+ - S^-)$ as virtual $\widetilde{W}'_{\#}$-modules. In particular, for $\widetilde{w} \in \widetilde{W}'$,*

$$\begin{aligned} \operatorname{tr}(\widetilde{w}, I_{\#}(X)) &= \operatorname{tr}(w, X)\operatorname{tr}(\widetilde{w}, S^+ - S^-), \\ \operatorname{tr}(\widetilde{w}z, I_{\#}(X)) &= c\operatorname{tr}(ww_0\delta, X)\operatorname{tr}(\widetilde{w}, \mathcal{S}), \end{aligned} \tag{6.7.8}$$

where $\mathcal{S}$ is as in (6.5.4), and c is the scalar by which z acts in S^+.

Proof. The first claim is proved identically with [7, Lemma 4.1]. The second claim is immediate from the first since z acts by c in S^+ and by $-c$ in S^-, and $\rho(z) = w_0\delta \otimes z$. □

6.8. We are now in position to compute the extended Dirac index of the simple tempered $\mathbb{H}_{\#}$-modules $E_{e,h,\psi}$.

Theorem 6.10. *Let $E_{e,h,\psi}$ be a simple tempered $\mathbb{H}_{\#}$-module as above.*

(1) *The δ-twisted trace of $E_{e,h,\psi}$ on W is given by:*

$$\operatorname{tr}(ww_0\delta, E_{e,h,\psi}) = (-1)^{d_e}\mathsf{sgn}(w_0)X_{-1}(e,\psi)(w), \quad \text{for all } w \in W. \tag{6.8.1}$$

(2) *The extended Dirac index of $E_{e,h,\psi}$ is given by the formula:*

$$\begin{aligned} \operatorname{tr}(\widetilde{w}, I_{\#}(E_{e,h,\psi})) &= \operatorname{tr}(\widetilde{w}, X_1(e,\psi) \otimes (S^+ - S^-)), \\ \operatorname{tr}(\widetilde{w}z, I_{\#}(E_{e,h,\psi})) &= c' \operatorname{tr}(\widetilde{w}, X_{-1}(e,\psi) \otimes \mathcal{S}), \end{aligned} \tag{6.8.2}$$

with $\widetilde{w} \in \widetilde{W}'$, z is as in (6.6.1), $c' = (-1)^{d_e}\mathsf{sgn}(w_0)c$, where c is the scalar from Proposition 6.9.

Proof. The above construction of the δ-action on $E_{e,h,\psi}$ gives $\operatorname{tr}(w\delta, E_{e,h,\psi}) = X_1(u,\psi)(w\delta)$. Then (1) follows from Corollary 4.2.

For (2), Proposition 6.9 implies that $\operatorname{tr}(\widetilde{w}, I_{\#}(E_{e,h,\psi})) = tr(w, E_{e,h,\psi})\operatorname{tr}(\widetilde{w}, S^+ - S^-)$ and $\operatorname{tr}(\widetilde{w}z, I_{\#}(E_{e,h,\psi})) = c\operatorname{tr}(ww_0\delta, E_{e,h,\psi})\operatorname{tr}(\widetilde{w}, \mathcal{S})$. The formula now follows from part (1) and (6.4.3).

□

6.9. The analogue of Vogan's conjecture from real reductive groups in the setting of the graded affine Hecke algebra was stated and proved in [1]. We need the following algebraic form.

Theorem 6.11 ([1, Theorem 4.2])**.** *For every $y \in Z(\mathbb{H})$, there exist a unique element $\zeta(y) \in Z(\mathbb{C}[\widetilde{W}])$ and an element $a \in \mathbb{H} \otimes C(V)$ such that*

$$y \otimes 1 = \rho(\zeta(y)) + \mathcal{D}a + a\mathcal{D}, \tag{6.9.1}$$

as elements of $\mathbb{H} \otimes C(V)$. Moreover, the map $\zeta : Z(\mathbb{H}) \to Z(\mathbb{C}[\widetilde{W}])$ is an algebra homomorphism.

In fact, as one can see from the proof of [1, Theorem 4.2] (cf. [7, Theorem 3.2]), the element a belongs to $\mathbb{H} \otimes C(V)_{\mathsf{odd}}$. It is also noticed in [7, Corollary 3.3] that the image of the map ζ lies in $\mathbb{C}[\widetilde{W}']^{\widetilde{W}}$. Following [7, Definition 4.5] (compare also with [1, Definition 4.3]), if $\widetilde{\sigma}$ is an irreducible $\widetilde{W}'$-representation, one can attach canonically a homomorphism $\chi^{\widetilde{\sigma}} : Z(\mathbb{H}) \to \mathbb{C}$ (i.e., a central character of $\mathbb{H}$-modules) by the requirement

$$\chi^{\widetilde{\sigma}}(y) = \widetilde{\sigma}(\zeta(y)), \text{ for all } y \in Z(\mathbb{H}).$$

Notice that if $\widetilde{\sigma}$ is an irreducible $\widetilde{W}$-representation, the central character $\chi^{\widetilde{\sigma}_1}$ is the same for all irreducible $\widetilde{W}'$-representations that appear in the restriction of $\widetilde{\sigma}$ to $\widetilde{W}'$. Thus we can also denote $\chi^{\widetilde{\sigma}}$ for an irreducible $\widetilde{W}$-representation $\widetilde{\sigma}$.

Let $W \cdot \nu_{\widetilde{\sigma}}$ denote the W-orbit in $V_{\mathbb{C}}$ corresponding to $\chi^{\widetilde{\sigma}}$.

We following is a slight sharpenning of [1, Theorem 4.4].

Corollary 6.12. *Let $\epsilon \in \{+, -\}$. Suppose (π, X) is an irreducible $\mathbb{H}_{\#}$-module with central character $W_{\#} \cdot \nu$ and that $\widetilde{\sigma}$ is an irreducible $\widetilde{W}'$-representation such that*

$$\mathrm{Hom}_{\widetilde{W}'}[\widetilde{\sigma}, H_{\#}^{D^{\epsilon}}(X)] \neq 0,$$

where $H_{\#}^{D^{\epsilon}}(X)$ is as in Definition 6.8. Then $W_{\#} \cdot \nu = W_{\#} \cdot \nu_{\widetilde{\sigma}}$.

Proof. We show how the claim follows from Theorem 6.11. This is analogous with the proof of [1, Theorem 4.4], but we need some minor modification because we are considering $D^{\pm}$ (rather than operators D).

When $\dim V$ is odd, S^{ϵ} and $S^{-\epsilon}$ are realized on the same vector space $U^{\epsilon} = U^{-\epsilon}$. When $\dim V$ is even, S^{ϵ} and $S^{-\epsilon}$ are realized on the vector spaces U^{ϵ} and $U^{-\epsilon}$, respectively.

Let γ denote Clifford multiplication by elements in $C(V)$ on the spin module S^{ϵ}. When $\dim V$ is odd, γ is a $C(V)$-action on U^{ϵ}. When $\dim V$ is even, $\gamma(\xi)$, where $\xi \in C(V)_{\mathsf{odd}}$ take U^{ϵ} to $U^{-\epsilon}$.

Let $y \in Z(\mathbb{H}_{\#})$ and $\widetilde{x} \in X \otimes U^{\epsilon}$ be an element in the $\widetilde{W}'$-isotypic component of $\widetilde{\sigma}$ in $H_{\#}^{D^{\epsilon}}(X)$. Then $(\pi(y) \otimes \gamma(1))\widetilde{x} = \chi_{\nu}(y)\widetilde{x}$ and $(\pi \otimes \gamma^{\epsilon})(\rho(\zeta(y)))\widetilde{x} = \widetilde{\sigma}(\rho(\zeta(y))\widetilde{x} = \chi^{\widetilde{\sigma}}(y)\widetilde{x}$. Together with (6.9.1), it follows that:

$$\begin{aligned}
(\chi_{\nu}(y) - \chi^{\widetilde{\sigma}}(y))\widetilde{x} &= (\pi \otimes \gamma)(y \otimes 1 - \rho(\zeta(y)))\widetilde{x} \\
&= (\pi \otimes \gamma)(\mathcal{D}a + a\mathcal{D})\widetilde{x}, \\
&= (\pi \otimes \gamma)(\mathcal{D}a\widetilde{x}).
\end{aligned}$$

By the discussion above $a \in \mathbb{H} \otimes C(V)_{\mathsf{odd}}$, and so when $\dim V$ is even, $ax \in X \otimes U^{-\epsilon}$. Therefore, regardless of the parity of $\dim V$, the right hand side is in $\operatorname{im} D_{\#}^{-\epsilon}$, and it follows by the definition of $H_{\#}^{D^{\epsilon}}(X)$ that it must be zero. In conclusion, $\chi_{\nu} = \chi^{\widetilde{\sigma}}$. □

7. Spin Weyl group representations

7.1. We denote by $\widehat{\widetilde{W}}_{\mathsf{gen}}$ the set of (isomorphism classes of) irreducible genuine representations of $\widetilde{W}$, i.e., the irreducible representations of $\widetilde{W}$ which do not factor through W. Let $R(\widetilde{W})_{\mathsf{gen}}$ be the subspace of $R(\widetilde{W})$ spanned by $\widehat{\widetilde{W}}_{\mathsf{gen}}$. Denote

$$\mathsf{Sg} : R(\widetilde{W}) \to R(\widetilde{W}), \qquad \mathsf{Sg}(\sigma) = \sigma \otimes \mathsf{sgn}.$$

Let $R(\widetilde{W})^{\mathsf{Sg}}$ denote the $(+1)$-eigenspace of Sg. Notice that $\mathcal{S}\otimes\mathsf{sgn}=\mathcal{S}$. Define the linear map

$$\iota: R(W)\to R(\widetilde{W})^{\mathsf{Sg}}_{\mathsf{gen}},\quad \iota(\sigma)=\sigma\otimes\mathcal{S}. \tag{7.1.1}$$

Proposition 7.1. *The map ι induces an injective linear map $\iota:\overline{R}_{-1}(W)\to R(\widetilde{W})^{\mathsf{Sg}}_{\mathsf{gen}}$ such that for $\sigma,\sigma'\in\overline{R}(W)$,*

$$\langle\iota(\sigma),\iota(\sigma')\rangle_{\widetilde{W}}=a_V\langle\sigma,\sigma'\rangle^{-1}_{W}.$$

Moreover, $\iota(X_{-1}(e,\phi))\neq 0$ if and only if $e\in\mathcal{N}^{\mathsf{sol}}$ and

$$\langle\iota(X_{-1}(e,\phi)),\iota(X_{-1}(e',\phi'))\rangle_{\widetilde{W}}=\begin{cases}a_V\langle\phi,\phi'\rangle^{-1}_{A(e)}, & \text{if } e=e'\in\mathcal{N}^{\mathsf{sol}}\\ 0, & \text{otherwise.}\end{cases} \tag{7.1.2}$$

Proof. Since $\mathcal{S}^*\cong\mathcal{S}$, the first claim is immediate from (6.5.5) and the definition of $\langle\ ,\ \rangle^{-1}_{W}$. The second claim follows from Proposition 5.1(2).

By Theorem 2.7 in the case $q=-1$, the map $\phi\to X(e,\phi)$ induces an isometric isomorphism $\overline{R}_{-1}(A(e))_0\to R^e_{-1}(W)$. Composing with ι, this implies that $\iota(X_{-1}(e,\phi))\in R(\widetilde{W})_{\mathsf{gen}}$, $e\in\mathcal{N}^{\mathsf{sol}}$, is nonzero and that (7.1.2) holds. ☐

We relate these facts with the extended Dirac index from section 6.

Lemma 7.2. $W_{(\text{-}1)\text{-}ell}\subset\ker\mathsf{sgn}$.

Proof. Suppose $w\in W$ is such that $\det_V(1+w)\neq 0$. This means that w acting on V does not have the eigenvalue -1. Since V is a real representation, the only real eigenvalue of w is 1 and the complex eigenvalues come in pairs. Thus $\mathsf{sgn}(w)=\det_V(w)=1$. ☐

Proposition 7.3. *Let $E_{e,h,\phi}$ be a simple tempered $\mathbb{H}_\#$-module as in section 6.4. The extended Dirac index $I_\#(E_{e,h,\phi})\neq 0$ if and only if $e\in\mathcal{N}^{\mathsf{sol}}$.*

Proof. In one direction, suppose $e\in\mathcal{N}^{\mathsf{sol}}$. By Proposition 7.1, $\iota(X_{-1}(e,\phi))\neq 0$. By (6.5.5), $\mathcal{S}$ is supported on $W_{(\text{-}1)\text{-ell}}$ and Lemma 7.2 implies that the support of $\iota(X_{-1}(e,\phi))$ is in $\widetilde{W}'$. Therefore, (6.8.2) says that the restriction of $I_\#(E_{e,h,\phi})$ to $\widetilde{W}'z$ is nonzero.

For the converse, suppose that $I_\#(E_{e,h,\phi})\neq 0$. Again by (6.8.2), there are two cases. If there exists $\widetilde{w}\in\widetilde{W}'$ such that $\operatorname{tr}(\widetilde{w},I_\#(E_{e,h,\phi}))\neq 0$, then $\operatorname{tr}(\widetilde{w},X_1(e,\phi)\otimes(S^+-S^-))\neq 0$. By [8, Proposition 3.1], which is the analogue of Proposition 7.1 above, it follows that

$$\langle X_1(e,\phi),X_1(e,\phi)\rangle^1_{A(u)}\neq 0.$$

Then Proposition 5.1(1) implies that e is quasidistinguished and so $e\in\mathcal{N}^{\mathsf{sol}}$.

If, on the other case, $\operatorname{tr}(\widetilde{w}z,I_\#(E_{e,h,\phi}))\neq 0$, then $\operatorname{tr}(\widetilde{w},\iota(X_{-1}(e,\phi))\neq 0$, and by Proposition 7.1, $e\in\mathcal{N}^{\mathsf{sol}}$. ☐

7.2. For $e\in\mathcal{N}^{\mathsf{sol}}$ and $\phi\in\widehat{A(e)}_0$, let $[\phi]$ be the image of ϕ in $\overline{R}_{-1}(A(e))_0$. Denote $m_{e,[\phi]}(\widetilde{\sigma})=\langle\widetilde{\sigma},\iota(X_{-1}(e,\phi))\rangle_{\widetilde{W}}$, for every $\widetilde{\sigma}\in\widehat{\widetilde{W}}_{\mathsf{gen}}$, and set

$$\mathsf{Irr}_{e,[\phi]}\widetilde{W}=\{\widetilde{\sigma}\in\widehat{\widetilde{W}}_{\mathsf{gen}}:m_{e,[\phi]}(\widetilde{\sigma})\neq 0\},\quad \mathsf{Irr}_e\widetilde{W}=\cup_\phi\mathsf{Irr}_{e,[\phi]}\widetilde{W}. \tag{7.2.1}$$

It is clear that $\mathsf{Irr}_{e,[\phi]}\widetilde{W}$ is closed under tensoring with sgn, in fact $m_{e,[\phi]}(\widetilde{\sigma})=m_{e,[\phi]}(\widetilde{\sigma}\otimes\mathsf{sgn})$.

Proposition 7.3 and Corollary 6.12 yield the following remarkable fact.

Corollary 7.4. *Let $\{e,h,f\}$ be a Lie triple in $\mathfrak{g}$, $h\in V^\delta$, such that $e\in\mathcal{N}^{\mathsf{sol}}$, and $\widetilde\sigma\in\mathsf{Irr}_e\widetilde W$. Then $W\cdot\nu_{\widetilde\sigma}=W\cdot h$, where $\nu_{\widetilde\sigma}$ is the central character of $\mathbb{H}$ defined by $\widetilde\sigma$ in section 6.9.*

In particular, $\widetilde\sigma(\Omega_{\widetilde W})=(h,h)$, where $\Omega_{\widetilde W}$ is as in (6.7.3).

Proof. Let $\widetilde\sigma\in\mathsf{Irr}_e\widetilde W$. There exists $\phi\in\widehat{A(e)}_0$ such that $\mathrm{Hom}_{\widetilde W}[\widetilde\sigma,\iota(X_{-1}(e,\phi))]\neq 0$. Let $\widetilde\sigma_1$ be an irreducible $\widetilde{W}'$-representation such that $\mathrm{Hom}_{\widetilde{W}'}[\widetilde\sigma_1,\widetilde\sigma]\neq 0$ and $\langle\widetilde\sigma_1,\iota(X_{-1}(e,\phi))\rangle_{\widetilde{W}'}\neq 0$.

Write $I_\#(E_{e,h,\phi})=\sum_j a^j\widetilde\sigma^j_\#$, for $\widetilde\sigma^j_\#$ irreducible distinct $\widetilde{W}'_\#$-representations, and $a^j\in\mathbb{Z}^*$. Suppose further that $\widetilde\sigma^j_\#=\sum_i\widetilde\sigma^j_i$ as $\widetilde{W}'$-representations, where $\widetilde\sigma^j_i$ are irreducible $\widetilde{W}'$-representations. Since z commutes with $\widetilde{W}'$, it acts by a nonzero scalar u^j_i (in fact, a fourth root of 1) on $\widetilde\sigma^i_j$. Thus

$$\mathrm{tr}(\widetilde wz,I_\#(E_{e,h,\phi}))=\sum_j\sum_i a^ju^j_i\,\mathrm{tr}(\widetilde w,\widetilde\sigma^j_i).$$

On the other hand, by Proposition 7.3, the left hand side equals (up to a nonzero scalar) $\mathrm{tr}(\widetilde w,\iota(X_{-1}(e,\phi))$, and so $\mathrm{tr}(\widetilde w,\widetilde\sigma_1)$ appears in the linear combination. By the linear independence of irreducible $\widetilde{W}'$-characters, it follows that there exist i,j such that $\widetilde\sigma^j_i=\widetilde\sigma_1$. In other words, there exists j such that $\widetilde\sigma^j_\#$ contains $\widetilde\sigma_1$.

Since $\widetilde\sigma^j_\#$ occurs in $I_\#(E_{e,h,\phi})$, it must occur in one of the spaces $H^{D^\pm}_\#(E_{e,h,\phi})$ for a choice of sign $\pm$. This implies that $\mathrm{Hom}_{\widetilde{W}'}[\widetilde\sigma_1,H^{D^\pm}_\#(E_{e,h,\phi})]\neq 0$. Corollary 6.12 says that $W_\#\cdot h=W_\#\cdot\nu_{\widetilde\sigma_1}=W_\#\cdot\nu_{\widetilde\sigma}$.

Since h is δ-stable, the first claim of the corollary is proved. For the second claim, it is sufficient to notice that by definition $(\nu_{\widetilde\sigma},\nu_{\widetilde\sigma})=\widetilde\sigma(\Omega_{\widetilde W})$. □

7.3. We state the main results of this section.

Theorem 7.5. $\widehat{\widetilde W}_{\mathsf{gen}}=\sqcup_{e\in G\backslash\mathcal{N}^{\mathsf{sol}}}\mathsf{Irr}_e\widetilde W$.

Proof. Let $\widetilde\sigma$ be an irreducible genuine $\widetilde W$-representation. Then $\widetilde\sigma\otimes\mathcal{S}$ is a W-representation. Since $\{X_{-1}(e,\phi):e\in G\backslash\mathcal{N},\phi\in\widehat{A(e)}_0\}$ is a basis of $R(W)$, there exists (e,ϕ) such that $\langle\widetilde\sigma\otimes\mathcal{S},X_{-1}(e,\phi)\rangle_W\neq 0$, or equivalently $\langle\widetilde\sigma,\iota(X_{-1}(e,\phi))\rangle_{\widetilde W}\neq 0$. In particular, $\iota(X_{-1}(e,\phi))\neq 0$, thus by Proposition 7.1, $e\in\mathcal{N}^{\mathsf{sol}}$. The disjointness follows from Corollary 7.4. □

Theorem 7.6. *Let $e\in\mathcal{N}^{\mathsf{sol}}$ and $\phi\in\widehat{A(e)}_0$ be given.*

1. *For every $\widetilde\sigma\in\mathsf{Irr}_{e,[\phi]}\widetilde W$, $m_{e,\phi}(\widetilde\sigma)=\langle\sigma(e,\phi)\otimes\mathcal{S},\widetilde\sigma\rangle_{\widetilde W}$, and in particular, $\iota(X_{-1}(e,\phi))=X_{-1}(e,\phi)\otimes\mathcal{S}$ is the character of a genuine representation of $\widetilde W$.*
2. *If $\langle\phi,\phi\rangle^{-1}_{A(e)}=1$, in particular, when e is distinguished, then $\mathsf{Irr}_{e,[\phi]}\widetilde W=\{\widetilde\sigma(e,[\phi])\}$, where $\widetilde\sigma(e,[\phi])$ is irreducible sgn self dual if $\dim V$ is even, and $\mathsf{Irr}_{e,[\phi]}\widetilde W=\{\widetilde\sigma(e,[\phi])^+,\widetilde\sigma(e,[\phi])^-\}$, where $\widetilde\sigma(e,[\phi])^\pm$ are irreducible sgn dual representations when $\dim V$ is odd.*
3. *If $\langle\phi,\phi'\rangle^{-1}_{A(e)}=0$, $\phi\neq\phi'$, in particular, when e is distinguished, then $\mathsf{Irr}_{e,[\phi]}\widetilde W\cap\mathsf{Irr}_{e,[\phi']}\widetilde W=\emptyset$.*

Proof. Since the matrix of Green polynomials $K(q)$ is upper triangular with 1 on the diagonal, $K(q)^{-1}$ is again upper triangular with 1 on the diagonal and polynomials in q above the diagonal. It is well-known that the W-types in $X_q(e,\phi)$, other than $\sigma(e,\phi)$ are all of the form $\sigma(e',\phi')$ with $e'>e$. Thus, in $R_q(W)$, we have:

$$X_q(e,\phi)=\sigma(e,\phi)-\sum_{e'>e}K(q)^{-1}_{(e,\phi),(e',\phi')}X_q(e',\phi'). \tag{7.3.1}$$

Apply this identity when $q=-1$ and tensor with $\mathcal{S}$:

$$\iota(X_{-1}(e,\phi))=\sigma(e,\phi)\otimes\mathcal{S}-\sum_{e<e',e'\in\mathcal{N}^{\mathsf{sol}}}K(-1)^{-1}_{(e,\phi),(e',\phi')}\iota(X_{-1}(e',\phi')), \tag{7.3.2}$$

an identity in $R(\widetilde{W})_{\mathsf{gen}}$. Suppose that $\widetilde{\sigma}$ occurs in the LHS. By Corollary 7.4, $W\cdot\nu_{\widetilde{\sigma}}=W\cdot h$, where h is a neutral element for a Lie triple of e. In particular, $\widetilde{\sigma}$ cannot occur in any $\iota(X_{-1}(e',\phi'))$ with $G\cdot e'\neq G\cdot e$, since h determines $G\cdot e$. Thus $\widetilde{\sigma}$ can only occur in $\sigma(e,\phi)\otimes\mathcal{S}$. This proves (1).

For (2), suppose $\langle\phi,\phi\rangle^{-1}_{A(e)}=1$. (The fact that this is always the case when u is distinguished is immediate.) By Proposition 7.1, $\langle\iota(X_{-1}(e,\phi)),\iota(X_{-1}(e,\phi))\rangle_{\widetilde{W}}=a_V$. Since $\iota(X_{-1}(e,\phi))$ is also sgn-dual, the only possibility when $\dim V$ is even is the one stated in (3).

If $\dim V$ is odd, then $\iota(X_{-1}(e,\phi))=\widetilde{\sigma}(u,[\phi])^{+}+\widetilde{\sigma}(u,[\phi])^{-}$. It remains to verify that $\widetilde{\sigma}(u,[\phi])^{-}=\widetilde{\sigma}(u,[\phi])^{+}\otimes\mathsf{sgn}$. Suppose this is not the case, so that $\widetilde{\sigma}(u,[\phi])^{\pm}$ are sgn self-dual. By (7.3.2), they occur in $\sigma(e,\phi)\otimes\mathcal{S}$ with multiplicity 1. But $\mathcal{S}=S^{+}+S^{-}$ and since $S^{+}=S^{-}\otimes\mathsf{sgn}$, this is impossible.

Claim (3) follows similarly from Proposition 7.1.

□

7.4. As mentioned in the introduction, Theorem 7.6 can be used to obtain character formulas for $X_{-1}(e,\phi)$ and $H^*(\mathcal{B}_e)^{\phi}$. Recall the notation from the introduction

$$\widetilde{\Sigma}(e,\phi)=\iota(X_{-1}(e,\phi))=X_{-1}(e,\phi)\otimes\mathcal{S}.$$

For every $w\in W_{\text{(-1)-ell}}$, we have then

$$\operatorname{tr}(w,X_{-1}(e,\phi))=\sum_{i=0}^{d_e}(-1)^{d_e-i}\operatorname{tr}(w,H^{2i}(\mathcal{B}_e)^{\phi})=\frac{\operatorname{tr}(\widetilde{w},\widetilde{\Sigma}(e,\phi))}{\operatorname{tr}(\widetilde{w},\mathcal{S})}, \tag{7.4.1}$$

where $\widetilde{w}$ is a representative of the preimage of w in $\widetilde{W}$, and $d_e=\dim\mathcal{B}_e$. In particular, when $w=1$, we find

$$\sum_{i=0}^{d_e}(-1)^i\dim H^{2i}(\mathcal{B}_e)^{\phi}=(-1)^{d_e}\frac{\dim\widetilde{\Sigma}(e,\phi)}{\dim\mathcal{S}}. \tag{7.4.2}$$

Using Corollary 4.2, (7.4.1) can also be interpreted as a character formula of $H^*(\mathcal{B}_e)^{\phi}$ on δ-twisted elliptic conjugacy classes.

Corollary 7.7. *Let w be a δ-elliptic element of W. Then*

$$\operatorname{tr}(w\delta,H^*(\mathcal{B}_e)^{\phi})=(-1)^{d_e}\mathsf{sgn}(w_0)\frac{\operatorname{tr}(\widetilde{w}\widetilde{w}_0,\widetilde{\Sigma}(e,\phi))}{\operatorname{tr}(\widetilde{w}\widetilde{w}_0,\mathcal{S})},$$

where $\widetilde{w}\widetilde{w}_0$ is a representative of the preimage of ww_0 in $\widetilde{W}$.

The calculations in Appendix A allow us to describe $\widetilde{\Sigma}(e,\phi)$ explicitly and by comparison to [6], to identify $\widetilde{\Sigma}(e,\phi)$ in terms of the known classifications of irreducible $\widetilde{W}$-representations. Therefore, Corollary 7.7 can be effectively used as a character formula for $H^*(\mathcal{B}_e)^\phi$ on δ-elliptic classes.

Example 7.8. In $GL(n)$, the class of the element $e\in\mathcal{N}^{\mathsf{sol}}$ is parameterized, via the Jordan canonical form, by a partition λ of n into distinct parts, and $A(e)=\{1\}$. A partition λ of n is called even if $\ell(\lambda)\equiv n \pmod 2$, otherwise it is called odd, where $\ell(\lambda)$ is the number of parts of λ. Then

$$\widetilde{\Sigma}(u_\lambda)=a_\lambda\begin{cases}\widetilde{\sigma}_\lambda, & \lambda \text{ even},\\ \widetilde{\sigma}_\lambda^+ + \widetilde{\sigma}_\lambda^-, & \lambda \text{ odd },\end{cases}$$

where a_λ is as in Proposition A.3, and σ_λ, $\sigma_\lambda^\pm$ are the irreducible $\widetilde{S}_n$-representations constructed by Schur (cf. [34]). The dimension of $\widetilde{\sigma}_\lambda$, or of each of $\widetilde{\sigma}_\lambda^\pm$, $\lambda=(\lambda_1,\dots,\lambda_\ell)$, is given by the formula

$$2^{\lfloor\frac{n-\ell(\lambda)}{2}\rfloor}g^\lambda,\quad \text{where } g^\lambda=\frac{n!}{\lambda_1!\dots\lambda_\ell!}\prod_{i<j}\frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}.$$

Since $\dim\mathcal{S}=2^{\lfloor\frac{n}{2}\rfloor}$, (7.4.2) becomes in this case:

$$\sum_{i=0}^{d_{e_\lambda}}(-1)^i\dim H^{2i}(\mathcal{B}_{e_\lambda})=(-1)^{d_{e_\lambda}}\ g^\lambda.$$

7.5. We end with an application to the decomposition of tensor products $\sigma\otimes\mathcal{S}$, $\sigma\in\widehat{W}$.

Let σ be an irreducible W-representation. By Springer's correspondence, write $\sigma=\sigma(e,\phi)$ for $e\in\mathcal{N}$, $\phi\in\widehat{A(e)}_0$. Since every $\widetilde{\sigma}$ occurs in a $\widetilde{\Sigma}(e',\phi')=\iota(X_{-1}(e',\phi'))$, with $e'\in\mathcal{N}^{\mathsf{sol}}$, we consider:

$$\langle\sigma(e,\phi)\otimes\mathcal{S},\iota(X_{-1}(e',\phi'))\rangle_{\widetilde{W}}. \tag{7.5.1}$$

By (7.3.2),

$$\sigma(e,\phi)\otimes\mathcal{S}=\sum_{e''\geqslant e}K(-1)^{-1}_{(e,\phi),(e'',\phi'')}\ \iota(X_{-1}(e'',\phi'')), \tag{7.5.2}$$

with $K(-1)^{-1}_{(e,\phi),(e,\phi'')}=1$ if $\phi''=\phi$, and 0 if $\phi''\neq\phi$. Using Proposition 7.1, we find

$$\langle\sigma(e,\phi)\otimes\mathcal{S},\iota(X_{-1}(e',\phi'))\rangle_{\widetilde{W}}=a_V\sum_{\phi''\in\widehat{A(e')}_0}K(-1)^{-1}_{(e,\phi),(e',\phi'')}\ \langle\phi',\phi''\rangle^{-1}_{A(e)}. \tag{7.5.3}$$

In particular, this is zero unless $e'\geqslant e$.

Corollary 7.9. *Let $e\in\mathcal{N}$ and $\phi\in\widehat{A(e)}_0$. If $\langle\sigma(e,\phi)\otimes\mathcal{S},\widetilde{\sigma}\rangle_{\widetilde{W}}\neq 0$, then $\widetilde{\sigma}\in\mathsf{Irr}_{e'}\widetilde{W}$ for some $e'\in\mathcal{N}^{\mathsf{sol}}$ such that $e'\geqslant e$.*

Appendix A. Component groups and spin representations

In this appendix, we use a case by case analysis to determine the structure of the spaces $\overline{R}_{-1}(A(e))$, the image of the map ι, and the explicit decompositions of representations $\iota(X_{-1}(e,\phi))$. These calculations can also be used to relate our realization of irreducible $\widetilde{W}$-representations with the known case by case classifications in [23, 25, 34], and also [6].

Remark A.1. The dimension of $R(\widetilde{W})^{\mathsf{Sg}}_{\mathsf{gen}}$ equals $|\{\sigma \in \widehat{\widetilde{W}}_{\mathsf{gen}} : \sigma \cong \sigma \otimes \mathsf{sgn}\}| + \frac{1}{2}|\{\sigma \in \widehat{\widetilde{W}}_{\mathsf{gen}} : \sigma \ncong \sigma \otimes \mathsf{sgn}\}|$. In particular, the classification of irreducible $\widetilde{W}$-representations [23, 25, 34] gives the following dimensions for $R(\widetilde{W})^{\mathsf{Sg}}_{\mathsf{gen}}$:

1. A_{n-1}: the number of partitions of n into distinct parts;
2. B_n: the number of partitions of n;
3. D_n, n odd: $\frac{1}{2}|\{\lambda \vdash n : \lambda \neq \lambda^t\}| + |\{\lambda \vdash n : \lambda = \lambda^t|$;
4. D_n, n even: $\frac{1}{2}|\{\lambda \vdash n : \lambda \neq \lambda^t\}| + 2|\{\lambda \vdash n : \lambda = \lambda^t|$;
5. G_2: 3; F_4: 9; E_6: 9; E_7: 13; E_8: 30.

Comparing with the dimensions in Remark 5.7, we conclude that the map $\iota : \overline{R}_{-1}(W) \to R(\widetilde{W})^{\mathsf{Sg}}_{\mathsf{gen}}$ from Proposition 7.1 is an isomorphism for all irreducible W, except when $W = D_{2n}$ or E_7.

A.1. We first investigate type A.

Lemma A.2. *Suppose $G = PGL(n)$ and $e_\lambda \in \mathcal{N}^{\mathsf{sol}}$ is a nilpotent element given in the Jordan form by the partition λ of n with distinct parts. Then $A(e_\lambda) = \{1\}$ and*

$$\langle \mathsf{triv}, \mathsf{triv}\rangle^{-1}_{A(e_\lambda)} = 2^{\ell(\lambda)-1}.$$

Proof. Straightforward. □

Proposition A.3. *For every distinct partition λ of n, define*

$$\widetilde{\tau}_\lambda = \frac{1}{a_\lambda} X_{-1}(e_\lambda) \otimes \mathcal{S},$$

where $a_\lambda = 2^{\frac{\ell(\lambda)}{2}}$, if both n and λ are even, and $a_\lambda = 2^{\lfloor \frac{\ell(\lambda)-1}{2} \rfloor}$, otherwise. Then $\widetilde{\tau}_\lambda$ is irreducible sgn self dual, if λ is an even partition, while $\widetilde{\tau}_\lambda = \widetilde{\tau}^+_\lambda + \widetilde{\tau}^-_\lambda$ if λ is an odd partition, with $\widetilde{\tau}^\pm_\lambda$ irreducible sgn dual to each other.

Proof. The number of irreducible genuine $\widetilde{S}_n$-representations equals the number of conjugacy classes of S_n that split when pulled back to $\widetilde{S}_n$. A well-known result (going back to Schur, see [34, Theorem 2.1]) says that this number equals the number of partitions of n into odd parts plus the number of odd partitions of n into distinct parts. Denote $\mathsf{DP}(n)$ the set of distinct partitions of n and for every $\lambda \in \mathsf{DP}(n)$, set b_λ equal to 1 or 2 if λ is even or odd, respectively. Thus

$$|\widehat{(\widetilde{S}_n)}_{\mathsf{gen}}| = \sum_{\lambda \in \mathsf{DP}(n)} b_\lambda. \tag{A.1.1}$$

By Proposition 7.1 and Lemma A.2, we see that for $\lambda \in \mathsf{DP}(n)$,

$$\langle \iota(X_{-1}(e_\lambda)), \iota(X_{-1}(e_\lambda))\rangle_{\widetilde{W}} = \begin{cases} a_\lambda^2, & \text{if } \lambda \text{ is even,} \\ 2a_\lambda^2, & \text{if } \lambda \text{ is odd.} \end{cases}$$

This means that $\iota(X_{-1}(e_\lambda))$ contains at least two distinct irreducible $\widetilde{S}_n$-representations when $\lambda \in \mathsf{DP}(n)$ is odd. Since for $\lambda \neq \lambda'$, $\iota(X_{-1}(e_\lambda))$ and $\iota(X_{-1}(e_{\lambda'}))$ are orthogonal, the claim in the Proposition follows by comparison with (A.1.1). $\square$

A.2. Next, we prove a criterion which will cover most of the remaining cases when G is not type A and $e \in \mathcal{N}^{\mathsf{sol}}$, but e is not distinguished.

Lemma A.4. *Suppose $e \in \mathcal{N}^{\mathsf{sol}}$ is such that $A(e) = (\mathbb{Z}/2\mathbb{Z})^k \times (\mathbb{Z}/2\mathbb{Z})^l$, $k \neq 0$, acts on the k-dimensional space V_Z by the representation $\mathsf{refl}_k \boxtimes \mathsf{triv}_l$, where refl_k is the reflection representation of $(\mathbb{Z}/2\mathbb{Z})^k$ and triv_l is the trivial representation of $(\mathbb{Z}/2\mathbb{Z})^l$. Then $\dim \overline{R}_{-1}(A(e)) = 2^l$ and*

$$\langle \phi, \phi \rangle^{-1}_{A(e)} = 1, \text{ for all } \phi \in \widehat{A(e)}.$$

Moreover, if $[\phi_1] \neq [\phi_2]$ in $\overline{R}_{-1}(A(e))$, then

$$\langle [\phi_1], [\phi_2] \rangle^{-1}_{A(e)} = 0.$$

Proof. Let $\mathsf{sgn}^{(i)}$ denote the one dimensional $(\mathbb{Z}/2\mathbb{Z})^k$-representation, with sgn on the i-th position and triv everywhere else. Then $V_Z = \oplus_{i=1}^k \mathsf{sgn}^{(i)}$ as an $(\mathbb{Z}/2\mathbb{Z})^k$-representation. Thus $\det_{V_Z}(1+x) \neq 0$ if and only if $\mathrm{proj}_{(\mathbb{Z}/2\mathbb{Z})^k} x = 1$, hence the (-1)-elliptic element in $A(e)$ are the subgroup $(\mathbb{Z}/2\mathbb{Z})^l$. It follows that $\dim \overline{R}_{-1}(A(e)) = 2^l$.

For the second claim, notice that since $\phi \otimes \phi = \mathsf{triv}$, we have $\langle \phi, \phi \rangle^{-1}_{A(e)} = \langle \mathsf{triv}, \mathsf{triv} \rangle^{-1}_{A(e)}$. It is straightforward that $\langle \mathsf{triv}, \mathsf{triv} \rangle^{-1}_{A(e)} = \langle \mathsf{triv}, \wedge V_Z \rangle_{A(e)} = 1$.

For the last claim, let $[\phi_1] \neq [\phi_2]$ in $\overline{R}_{-1}(A(e))$. We can choose $\phi_i = \mathsf{triv}_k \otimes \phi_i'$, $i = 1, 2$, where triv_k is the trivial $(\mathbb{Z}/2\mathbb{Z})^k$-representation, and $\phi_1' \neq \phi_2'$ are one dimensional representations of $(\mathbb{Z}/2\mathbb{Z})^l$. Since $\phi_1' \otimes \phi_2' \neq \mathsf{triv}$, it does not occur in $\wedge V_Z$, so $\langle [\phi_1], [\phi_2] \rangle^{-1}_{A(e)} = 0$.

$\square$

Lemma A.5. *Let G be simple and adjoint and $e \in \mathcal{N}^{\mathsf{sol}}$, but not distinguished. Then $A(e)$ is as in Lemma A.4, except when:*

1. *G is of type D_n, n even, and $e = e_\lambda$ corresponds via the Jordan form to a partition $\lambda = (a_1, a_1, a_2, a_2, \dots, a_k, a_k)$ of $2n$ where a_i are distinct odd positive integers. In this case, $A(e_\lambda) = (\mathbb{Z}/2\mathbb{Z})^{k-1}$ acts on the k-dimensional space V_Z by twice the reflection representation. Then $\overline{R}_{-1}(A(e_\lambda))$ is one dimensional, and $\langle \mathsf{triv}, \mathsf{triv} \rangle^{-1}_{A(e_\lambda)} = 2$.*
2. *G is of type D_n, n odd, and $e = e_\lambda$ corresponds via the Jordan form to a partition $\lambda = (a_1, a_1, a_2, a_2, \dots, a_k, a_k)$ of $2n$ where a_i are distinct odd positive integers. In this case, $A(e_\lambda) = (\mathbb{Z}/2\mathbb{Z})^{k-1}$ acts on the k-dimensional space $V_Z = V_Z^{A(e)} \oplus V_Z'$, with $\dim V_Z^{A(e)} = 1$, by the reflection representation on V_Z'. Then $\overline{R}_{-1}(A(e_\lambda))$ is one dimensional, and $\langle \mathsf{triv}, \mathsf{triv} \rangle^{-1}_{A(e_\lambda)} = 2$.*
3. *G is of type E_7 and e is of type $A_4 + A_1$. The component group $A(e) = \mathbb{Z}/2\mathbb{Z}$ acts on the two dimensional space V_Z by twice the sgn representation. Then $\overline{R}_{-1}(A(e)$ is one dimensional, but $\langle \mathsf{triv}, \mathsf{triv} \rangle^{-1}_{A(e)} = 2$.*
4. *G is of type E_6 and e is of type $D_4(a_1)$. The component group $A(e) = S_3$ acts on the two dimensional space V_Z by the reflection representation. Then $\overline{R}_{-1}(A(e))$ is two dimensional, spanned by $[\mathsf{triv}]$ and $[\mathsf{refl}]$, and*

$$\langle \mathsf{triv}, \mathsf{triv} \rangle^{-1}_{A(e)} = 1,\ \langle \mathsf{refl}, \mathsf{refl} \rangle^{-1}_{A(e)} = 3,\ \langle \mathsf{triv}, \mathsf{refl} \rangle^{-1}_{A(e)} = 1. \tag{A.2.1}$$

Proof. The proof is a direct calculation based on the classification of nilpotent orbits and their component groups. □

A.3. Suppose W is of type B_n and $G = Sp(2n)$. The nilpotent orbits $e \in \mathcal{N}^{\mathsf{sol}}$ are in one to one correspondence with partitions μ of $2n$ such that μ has only even parts and the multiplicity of each part is at most 2. The distinguished nilpotent e_μ correspond to μ a partition with even distinct parts. Denote by $\mathsf{DP}(2n)_{\mathsf{even}}$ the set of distinct partitions with even parts of $2n$, and by $\mathsf{qDP}(2n)_{\mathsf{even}}$ the set of partitions with even parts of $2n$ where every part has multiplicity at most 2 and there is one part with multiplicity 2. By Remark A.1, the number of irreducible $\widetilde{W}$-representations, up to tensoring with sgn, equals $|P(n)|$, the number of partitions of n.

For every $\mu \in \mathsf{DP}(2n)_{\mathsf{even}} \cup \mathsf{qDP}(2n)_{\mathsf{even}}$ and $\phi \in \widehat{A(e_\mu)}_0$, set

$$\widetilde{\tau}(e_\mu, \phi) = X_{-1}(e_\mu, \phi) \otimes \mathcal{S}.$$

Then, Lemma A.4 yields:

Proposition A.6. (1) *If n is even, $\widetilde{\tau}(e_\mu, \phi)$ is an irreducible* sgn *self dual $\widetilde{W}$-representation.*

(2) *If n is odd, $\widetilde{\tau}(e_\mu, \phi) = \widetilde{\tau}(e_\mu, \phi)^+ + \widetilde{\tau}(e_\mu, \phi)^-$, where $\widetilde{\tau}(e_\mu, \phi)^\pm$ are irreducible $\widetilde{W}$-representations* sgn *dual to each other.*

A.4. If W (and G) is exceptional of type G_2, F_4, or E_8 for every $e \in \mathcal{N}$, either Theorem 7.6(2),(3) or Lemma A.4 applies. Since $\dim V$ is even, $a_V = 1$, and $\widetilde{\tau}(e, \phi) = \iota(X_{-1}(e, \phi))$ is an irreducible sgn self dual $\widetilde{W}$-representation.

A.5. Let W be of type E_6. There are seven orbits in $\mathcal{N}^{\mathsf{sol}}$, three of which are distinguished. From Lemma A.5, it follows that when u is of type $D_4(a_1)$, then $A(e) = S_3$ and

$$\widetilde{\tau}(D_4(a_1), \mathsf{triv}) := X_{-1}(D_4(a_1), \mathsf{triv}) \otimes \mathcal{S} \tag{A.5.1}$$

is an irreducible sgn self dual representation of $\widetilde{W}(E_6)$, while

$$\begin{aligned} \widetilde{\tau}(D_4(a_1), \mathsf{refl}) : &= X_{-1}(D_4(a_1), \mathsf{refl}) \otimes \mathcal{S} \\ &= \widetilde{\sigma}(D_4(a_1), \mathsf{triv}) + \widetilde{\sigma}(D_4(a_1), \mathsf{refl})^+ + \widetilde{\sigma}(D_4(a_1), \mathsf{refl})^-, \end{aligned} \tag{A.5.2}$$

where $\widetilde{\sigma}(D_4(a_1), \mathsf{refl})^\pm$ are irreducible sgn dual $\widetilde{W}(E_6)$-representations. A basis of $\overline{R}_{-1}(A(e))$ consisting of orthogonal elements is $\{[\mathsf{triv}], [\mathsf{refl}] - [\mathsf{triv}]\}$.

A.6. For type E_7, the interesting case is the nilpotent element e of type $A_4 + A_1$. Then $A(e) = \mathbb{Z}/2\mathbb{Z}$, and V_Z is two dimensional. By Lemma A.5, $\langle \iota(X_{-1}(A_4 + A_1, \mathsf{triv})), \iota(X_{-1}(A_4 + A_1, \mathsf{triv})) \rangle_{\widetilde{W}} = 4$ (since $\dim V$ is odd). Using 7.6(2),(3) and Lemma A.4, the classes in $\mathcal{N}^{\mathsf{sol}} \setminus \{A_4 + A_1\}$ account for 11 distinct irreducible $\widetilde{W}$-representations (modulo $\otimes\mathsf{sgn}$). This implies that $\iota(X_{-1}(A_4 + A_1, \mathsf{triv}))$ is either two copies of a sgn self-dual irreducible representation or a sum of two pairs of sgn dual irreducible representations. The latter is in fact the correct one, and this can be seen either by invoking the fact that $\widetilde{W}(E_7)$ does not have sgn self-dual irreducible representations [23], or by refining the argument used to prove Theorem 7.6(2) as follows. If $\iota(X_{-1}(A_4 + A_1, \mathsf{triv})) = 2\widetilde{\sigma}$, where $\widetilde{\sigma}$ is sgn self dual, then the only possibility is that $\sigma(A_4 + A_1, \mathsf{triv}) \otimes S^\pm$ each contain $\widetilde{\sigma}$ with multiplicity 1; moreover, there are no other $\widetilde{W}$-representations $\widetilde{\sigma}'$ such that $W \cdot \nu_{\widetilde{\sigma}'} = W \cdot \nu_{\widetilde{\sigma}} = W \cdot h_{A_4+A_1}$. By Theorem 6.12, only $\widetilde{\sigma}$ can occur in the Dirac cohomology spaces, and in particular, $X_1(A_4+A_1,\mathsf{triv})\otimes(S^+-S^-)=\widetilde{\sigma}-\widetilde{\sigma}=0$ (see [8, 7] for details about the Dirac index). Since $(S^+-S^-)\otimes(S^+-S^-)^*=2\wedge^{-1}V$, it follows that $\langle X_1(A_4+A_1,\mathsf{triv}),X_1(A_4+A_1,\mathsf{triv})\rangle^1_W=0$. But this is a contradiction with Proposition 5.1(1), since A_4+A_1 is quasidistinguished in E_7.

A.7. Suppose W is of type D_n and $G=PSO(2n)$.

When n is odd, all representatives of pairs (e,ϕ), $e\in\mathcal{N}^{\mathsf{sol}}$ are as in Theorem 7.6(2),(3) or as in Lemma A.4, except when $e=e_\mu$ corresponds to the partition $\mu=(a_1,a_1,a_2,a_2,\dots,a_k,a_k)$ of $2n$, where k is odd, a_i are all distinct and odd, see Lemma A.5. In this case, since $\dim V$ is odd, we have

$$\langle\iota(X_{-1}(e_\mu,\mathsf{triv})),\iota(X_{-1}(e_\mu,\mathsf{triv}))\rangle_{\widetilde{W}}=4.$$

One may resolve the ambiguity in the same way as for A_4+A_1 in E_7 and find that $\iota(X_{-1}(e_\mu,\mathsf{triv}))=2\widetilde{\sigma}(e_\mu,[\mathsf{triv}])$, for some irreducible, sgn self dual $\widetilde{W}$-representation $\sigma(e_\mu,[\mathsf{triv}])$. (In this case, e_μ is not quasidistinguished.)

When n is even, all representatives of pairs (e,ϕ), $e\in\mathcal{N}^{\mathsf{sol}}$ are as in Theorem 7.6(2),(3) or as in Lemma A.4, except when $e=e_\mu$ corresponds to the partition $\mu=(a_1,a_1,a_2,a_2,\dots,a_k,a_k)$ of $2n$, where k is even, a_i are all distinct and odd, see Lemma A.5. In this case, since $\dim V$ is even, we have

$$\langle\iota(X_{-1}(e_\mu,\mathsf{triv})),\iota(X_{-1}(e_\mu,\mathsf{triv}))\rangle_{\widetilde{W}}=2,$$

so $\iota(X_{-1}(e_\mu,\mathsf{triv})=\widetilde{\sigma}(e_\mu,[\mathsf{triv}])_1+\widetilde{\sigma}(e_\mu,[\mathsf{triv}])_2$. One can prove that $\widetilde{\sigma}(e_\mu,[\mathsf{triv}])_{1,2}$ are sgn self-dual [1] by invoking an argument similar to that for A_4+A_1 in E_7, using the Dirac index in the even case ([8, 7]) and the fact that e_μ is quasidistinguished in D_n, n even.

Remark A.7. The nilpotent element $u=A_4+A_1$ in E_7 can be realized as a regular nilpotent element in $Z_G(t)=A_3+A_3+A_1$, see the proof of Proposition 3.3. Similarly, the nilpotent element e_μ in D_n with $\mu=(a_1,a_1,a_2,a_2,\dots,a_k,a_k)$, a_i distinct and odd, can be realized as the pair (e_λ,e_λ), $\lambda=(a_1,a_2,\dots,a_k)$, in $Z_G(t)=D_{n/2}\times D_{n/2}$, when n is even, respectively $Z_G(t\delta)=B_{\frac{n+1}{2}}\times B_{\frac{n+1}{2}}$ when n is odd. Thus the automorphism θ coming from the symmetry of the affine Dynkin diagram interchanges the two factors of e_λ and the exceptions appear to be related to this phenomenon.

Appendix B. Relation with Kostka systems

We explain a relation between our approach and the results of Kato [16] about Kostka systems in the category of A_W-modules, where $A_W=\mathbb{C}[W]\ltimes S(V_{\mathbb{C}})$. The relevant homological properties of the category of A_W-modules are presented in [16, section 2].

B.1. Retain the notation from the previous section. Thus W is a finite Weyl group acting on the real reflection representation V. Define

$$A_W=\mathbb{C}[W]\ltimes S(V_{\mathbb{C}});\qquad\text{(B.1.1)}$$

in the language of section 6, this is the same as the graded affine Hecke algebra with zero parameters.

[1] Erratum: In [6, Theorem 3.8.1(2)], it is incorrectly stated that $(\widetilde{\sigma})_{1,2}$ are sgn dual to each other.

Let X, Y be A_W-modules. Then one may define a structure of A_W-modules on $X \otimes_{\mathbb{C}} Y$ and $\mathrm{Hom}_{\mathbb{C}}[X, Y]$ as follows:

(1) $X \otimes_{\mathbb{C}} Y$

$$\begin{aligned} w \cdot (x \otimes y) &= w \cdot x \otimes w \cdot y, \quad w \in W, \\ \xi \cdot (x \otimes y) &= \xi \cdot x \otimes y + x \otimes \xi \cdot y, \quad \xi \in V_{\mathbb{C}}; \end{aligned} \tag{B.1.2}$$

(2) $\mathrm{Hom}_{\mathbb{C}}[X, Y]$

$$\begin{aligned} (w \cdot \phi)(x) &= w \cdot \phi(w^{-1}x), \quad w \in W, \\ (\xi \cdot \phi)(x) &= \xi \cdot \phi(x) - \phi(\xi \cdot x), \quad \xi \in V_{\mathbb{C}}, \end{aligned} \tag{B.1.3}$$

for all $\phi \in \mathrm{Hom}_{\mathbb{C}}[X, Y]$.

Lemma B.1. (1) *Definitions (B.1.2) and (B.1.3) extend to actions of A_W.*

(2) *If $\mathbb{C}$ is the trivial A_W-module (w acts by 1 and ξ acts by 0), then $X \otimes_{\mathbb{C}} \mathbb{C} \cong X$, as A_W-modules.*

(3) $\mathrm{Hom}_{\mathbb{C}}[X, Y] \cong X^* \otimes_{\mathbb{C}} Y$.

Proof. Straightforward. □

B.2. Since $A_W \otimes_W M \cong S(V) \otimes_{\mathbb{C}} M$, for every W-module M, the usual Koszul complex of vector spaces admits an interpretation as a projective resolution of the trivial A_W-module. More precisely, let

$$\epsilon : A_W \otimes_{\mathbb{C}} \mathbb{C} \to \mathbb{C}, \ \epsilon(a \otimes \lambda) = a \cdot \lambda, \tag{B.2.1}$$

i.e., the A_W-action in the trivial module, and

$$\begin{aligned} &\partial_{n-1} : A_W \otimes_W \wedge^n V \to A_W \otimes_W \wedge^{n-1} V, \\ &a \otimes (\xi_1 \wedge \cdots \wedge \xi_n) \mapsto \sum_{\ell=1}^{n} (-1)^{\ell+1} a\xi_\ell \otimes (\xi_1 \wedge \cdots \wedge \hat{\xi}_\ell \wedge \cdots \wedge \xi_n). \end{aligned} \tag{B.2.2}$$

It is easy to check that ∂_{n-1} is a well-defined A_W-module homomorphism. Therefore

$$0 \leftarrow \mathbb{C} \xleftarrow{\epsilon} A_W \otimes_W \mathbb{C} \xleftarrow{\partial_0} A_W \otimes_W V \xleftarrow{\partial_1} A_W \otimes_W \wedge^2 V \xleftarrow{\partial_2} \dots \tag{B.2.3}$$

is a projective resolution of the trivial module in the category of A_W-modules. Since tensor products exist in this category, one may tensor this complex by $- \otimes_{\mathbb{C}} X$ to obtain a projective resolution for every finite dimensional A_W-module X:

$$0 \leftarrow X \xleftarrow{\epsilon} A_W \otimes_W X \xleftarrow{\partial_0} A_W \otimes_W V \otimes X \xleftarrow{\partial_1} A_W \otimes_W \wedge^2 V \otimes X \xleftarrow{\partial_2} \dots, \tag{B.2.4}$$

where we used the isomorphism $\mathbb{C} \otimes_{\mathbb{C}} X \cong X$, and therefore the morphism $\epsilon : A_W \otimes_W X \to X$ becomes the action of A_W on X.

B.3. We regard A_W as a graded algebra by assigning to w degree 0 and $\xi \in V_{\mathbb{C}}$ degree 1. This differs from the convention in [16], where the elements of $V_{\mathbb{C}}$ have degree 2, but it will be consistent with the results in section 2. We consider the category of graded A_W-modules: A_W-gmod. If X is a $\mathbb{Z}$-graded vector space, $X = \oplus_j X(j)$, let

$$\mathsf{gdim} X = \sum_j q^j \dim X(j)$$

denote the graded dimension of X.

Definition B.2 ([16])**.** If X, Y are finite dimensional graded A_W-modules, define the graded Euler-Poincaré pairing

$$\langle X, Y\rangle_{A_W}^{\mathsf{gEP}} := \sum_{i\geqslant 0}(-1)^i \mathsf{gdim}\,\mathrm{Ext}^i_{A_W}(X, Y) \in \mathbb{Z}[q]. \tag{B.3.1}$$

It is clear that the maps ∂_n in the Koszul complex are graded maps of degree 0, and this makes (B.2.3) and (B.2.4) graded complexes.

Define the graded W-character of $X \in A_W$-gmod:

$$\mathsf{gch}_W X = \sum_{\sigma\in\widehat{W}}\sum_{j\in\mathbb{Z}} q^j \dim \mathrm{Hom}_W[\sigma, X(j)].$$

Proposition B.3. *If X, Y are finite dimensional graded A_W-modules, then*

$$\langle X, Y\rangle_{A_W}^{\mathsf{gEP}} = \langle \mathsf{gch}_W X, \mathsf{gch}_W Y\rangle_W^q.$$

Proof. The proof is a simple application of the (graded) Euler-Poincaré principle using the resolution (B.2.4) and it is an immediate analogue of the proof for the group algebra of the affine Weyl in the non-graded setting [24, Theorem 3.2]:

$$\begin{aligned}\langle X, Y\rangle_{A_W}^{\mathsf{gEP}} &= \sum_{i\geqslant 0}(-1)^i \mathsf{gdim}\,\mathrm{Ext}^i_{A_W}(X, Y) = \sum_{i\geqslant 0}(-1)^i \mathsf{gdim} H^i(\mathrm{Hom}_{A_W}(A_W \otimes_W \wedge^* V \otimes X, Y))\\ &= \sum_{i\geqslant 0}(-1)^i \mathsf{gdim}\,\mathrm{Hom}_{A_W}(A_W \otimes_W \wedge^i V \otimes X, Y), \quad \text{by Euler-Poincaré principle}\\ &= \sum_{i\geqslant 0}(-1)^i \mathsf{gdim}\,\mathrm{Hom}_W(\mathsf{gch}_W(X \otimes \wedge^i V), \mathsf{gch}_W Y), \quad \text{by Frobenius reciprocity}\\ &= \langle X, Y\rangle_W^q, \quad \text{since } \mathsf{gch}_W \wedge^i V = q^i \wedge^i V.\end{aligned}$$

□

Example B.4. In [15], the Springer W-action on $H^*(\mathcal{B}_e)^\phi$ is upgraded to an action of the affine Weyl group. As a consequence, one can define a graded A_W-module structure $\mathcal{X}_q(e, \phi)$ on $H^*(\mathcal{B}_e)^\phi$ [16], such that $\mathsf{gch}_W \mathcal{X}_q(e, \phi) = X_q(e, \phi)$. These are particular examples of Kostka systems [16, Definition A]. Thus:

$$\langle \mathcal{X}_q(e, \phi), \mathcal{X}_q(e', \phi')\rangle_{A_W}^{\mathsf{gEP}} = \langle \mathsf{gch}_W X_q(e, \phi), \mathsf{gch}_W X_q(e', \phi')\rangle_W^q.$$

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