G-stable pieces and partial flag varieties

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Abstract. We will use the combinatorics of the G-stable pieces to describe the closure relation of the partition of partial flag varieties in [**L3**, section 4].

Introduction

In 1977, Lusztig introduced a finite partition of a (partial) flag variety Y. In the case where Y is the full flag variety, this partition is the partition into Deligne-Lusztig varieties (see [**DL**]). In this case, it follows easily from the Bruhat decomposition that the closure of a Deligne-Lusztig variety is the union of some other Deligne-Lusztig varieties and the closure relation is given by the Bruhat order on the Weyl group.

In this paper, we will use some combinatorial technique in [**H4**] to study the partition on a partial flag variety. We show that the partition is a stratification and the closure relation is given by the partial order introduced in [**H2**, 5.4] and [**H3**, 3.8 & 3.9]. We also study some other properties of the locally closed subvarieties that appear in the partition.

1. Some combinatorics

1.1. Let $\mathbf{k}$ be an algebraic closure of the finite field $\mathbf{F}_q$ and G be a connected reductive algebraic group defined over $\mathbf{F}_q$ with Frobenius map $F : G \to G$. We fix an F-stable Borel subgroup B of G and an F-stable maximal torus $T \subset B$. Let I be the set of simple roots determined by B and T. Then F induces an automorphism on the Weyl group W which we deonte by δ. The autmorphism restricts to a bijection on the set I of simple roots. By abusion notations, we also denote the bijection by δ.

For any $J \subset I$, let P_J be the standard parabolic subgroup corresponding to J and $\mathcal{P}_J$ be the set of parabolic subgroups that are G-conjugate to P_J. We simply write $\mathcal{P}_\emptyset$ as $\mathcal{B}$. Let L_J be the Levi subgroup of P_J that contains T.

For any parabolic subgroup P, let U_P be the unipotent radical of P. We simply write U for U_B.

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For $J \subset I$, we denote by W_J the standard parabolic subgroup of W generated by J and by W^J (resp. ${}^J W$) the set of minimal coset representatives in W/W_J (resp. $W_J \backslash W$). For $J, K \subset I$, we simply write $W^J \cap {}^K W$ as ${}^K W^J$.

For $P \in \mathcal{P}_J$ and $Q \in \mathcal{P}_K$, we write $\operatorname{pos}(P, Q) = w$ if $w \in {}^J W^K$ and there exists $g \in G$ such that $P = g P_J g^{-1}$, $Q = g \dot{w} P_K \dot{w}^{-1} g^{-1}$, where $\dot{w}$ is a representative of w in $N(T)$.

For $g \in G$ and $H \subset G$, we write ${}^g H$ for $g H g^{-1}$.

We first recall some combinatorial results.

1.2. For $J \subset I$, let $\mathcal{T}(J, \delta)$ be the set of sequences $(J_n, w_n)_{n \geq 0}$ such that
(a) $J_0 = J$,
(b) $J_n = J_{n-1} \cap \operatorname{Ad}(w_{n-1}) \delta(J_{n-1})$ for $n \geq 1$,
(c) $w_n \in {}^{J_n} W^{\delta(J_n)}$ for $n \geq 0$,
(d) $w_n \in W_{J_n} w_{n-1} W_{\delta(J_{n-1})}$ for $n \geq 1$.

Then for any sequence $(J_n, w_n)_{n \geq 0} \in \mathcal{T}(J, \delta)$, we have that $w_n = w_{n+1} = \cdots$ and $J_n = J_{n+1} = \cdots$ for $n \gg 0$. By [**Be**], the assignment $(J_n, w_n)_{n \geq 0} \mapsto w_m^{-1}$ for $m \gg 0$ defines a bijection $\mathcal{T}(J, \delta) \to W^J$.

Now we prove some result that will be used in the proof of Lemma 2.5.

LEMMA 1.1. *Let $(J_n, w_n)_{n \geq 0} \in \mathcal{T}(J, \delta)$ be the element that corresponds to w. Then*

(1) $w(L_J \cap U_{P_{J_1}}) w^{-1} \subset U_{P_{\delta(J)}}$.
(2) $w(L_{J_i} \cap U_{P_{J_{i+1}}}) w^{-1} \subset L_{\delta(J_{i-1})} \cap U_{P_{\delta(J_i)}}$ for $i \geq 1$.

PROOF. We only prove part (1). Part (2) can be proved in the same way.

Assume that part (1) is not true. Then there exists $\alpha \in \Phi_J^+ - \Phi_{J_1}^+$ such that $wa \in \Phi^+_{\delta(J)}$. Let $i \in J - J_1$ with $\alpha_i \leq \alpha$. Since $w \in W^J$, we have that $w \alpha_i \in \Phi^+_{\delta(J)}$. By definition, $w^{-1} = w_1 v$ for some $v \in W_{\delta(J}$. Then $\alpha_i \in w^{-1} \Phi^+_{\delta(J)} = w_1 v \Phi^+_{\delta(J)} = w_1 \Phi_{\delta(J)}$. Since $w_1 \in W^{\delta(J)}$, we must have $\alpha_i = w_1 \alpha_j$ for some $j \in \delta(J)$. Hence $i \in J_1$, which is a contradiction. Part (1) is proved. □

1.3. Define a W_J-action on W by $x \cdot y = \delta(x) y x^{-1}$. For $w \in W^J$, set

$$I(J, \delta; w) = \max\{K \subset J; \operatorname{Ad}(w)(K) = \delta(K)\}$$

and $[w]_J = W_J \cdot (w W_{I(J, \delta; w)})$. Then $W = \sqcup_{w \in W^J} [w]_J$. See [**H4**, Corollary 2.6].

Given $w, w' \in W$ and $j \in J$, we write $w \xrightarrow{s_j}_\delta w'$ if $w' = s_{\delta(j)} w s_j$ and $l(w') \leq l(w)$. If $w = w_0, w_1, \cdots, w_n = w'$ is a sequence of elements in W such that for all k, we have $w_{k-1} \xrightarrow{s_j}_\delta w_k$ for some $j \in J$, then we write $w \to_{J, \delta} w'$.

We call $w, w' \in W$ *elementarily strongly (J, δ)-conjugate* if $l(w) = l(w')$ and there exists $x \in W_J$ such that $w' = \delta(x) w x^{-1}$ and either $l(\delta(x) w) = l(x) + l(w)$ or $l(w x^{-1}) = l(x) + l(w)$. We call w, w' *strongly (J, δ)-conjugate* if there is a sequence $w = w_0, w_1, \cdots, w_n = w'$ such that w_{i-1} is elementarily strongly (J, δ)-conjugate to w_i for all i. We will write $w \sim_{J, \delta} w'$ if w and w' are strongly (J, δ)-conjugate. If $w \sim_{J, \delta} w'$ and $w \to_{J, \delta} w'$, then we say that w and w' are in the same (J, δ)-cyclic shift and write $w \approx_{J, \delta} w'$. Then it is easy to see that $w \approx_{J, \delta} w'$ if and only if $w \to_{J, \delta} w'$ and $w' \to_{J, \delta} w$.

By [**H4**, Proposition 3.4], we have the following properties:

(a) for any $w \in W$, there exists $w_1 \in W^J$ and $v \in W_{I(J, \delta; w_1)}$ such that $w \to_{J, \delta} w_1 v$.

(b) if w, w' are in the same W_J-orbit $\mathcal{O}$ of W and w, w' are of minimal length in $\mathcal{O}$, then $w \sim_{J,\delta} w'$. If moreover, $\mathcal{O} \cap W^J \neq \emptyset$, then $w \approx_{J,\delta} w'$.

1.4. By [**H4**, Corollary 4.5], for any W_J-orbit $\mathcal{O}$ and $v \in \mathcal{O}$, the following conditions are equivalent:

(1) v is a minimal element in $\mathcal{O}$ with respect to the restriction to $\mathcal{O}$ of the Bruhat order on W.

(2) v is an element of minimal length in $\mathcal{O}$.

We denote by $\mathcal{O}_{\min}$ the set of elements in $\mathcal{O}$ satisfy the above conditions. The elements in $(W_J \cdot w)_{\min}$ for some $w \in W^J$ are called *distinguished elements* (with respect to J and δ).

As in [**H4**, 4.7], we have a natural partial order $\leq_{J,\delta}$ on W^J defined as follows:

Let $w, w' \in W^J$. Then $w \leq_{J,\delta} w'$ if for some (or equivalently, any) $v' \in (W_J \cdot w')_{\min}$, there exists $v \in (W_J \cdot w)_{\min}$ such that $v \leq v'$.

In general, for $w \in W^J$ and $w' \in W$, we write $w \leq_{J,\delta} w'$ if there exists $v \in (W_J \cdot w)_{\min}$ such that $v \leq w'$.

2. G_F-stable pieces

2.1. For $J \subset I$, set $Z_J = \{(P, gU_P); P \in \mathcal{P}_J, g \in G\}$ with the $G \times G$-action defined by

$$(g_1, g_2) \cdot (P, gU_P) = ({}^{g_2}P, g_1 g U_P g_2^{-1}).$$

Set $h_J = (P_J, U_{P_J})$. Then the isotropic subgroup R_J of h_J is $\{(lu_1, lu_2); l \in L_J, u_1, u_2 \in U_{P_J}\}$. It is easy to see that

$$Z_J \cong (G \times G)/R_J.$$

Set $G_F = \{(g, F(g)); g \in G\} \subset G \times G$. For $w \in W^J$, set

$$Z_{J,F;w} = G_F(B, BwB) \cdot h_J.$$

We call $Z_{J,F;w}$ a G_F-stable piece of Z_J.

Lemma 2.1. *Let $w, w' \in W$.*

(1) If $w \to_{J,\delta} w'$, then

$$G_F(B, BwB) \cdot h_J \subset G_F(B, Bw'B) \cdot h_J \cup \cup_{v<w} G_F(B, BvB) \cdot h_J.$$

(2) If $w \approx_{J,\delta} w'$, then

$$G_F(B, BwB) \cdot h_J = G_F(B, Bw'B) \cdot h_J.$$

Proof. It suffices to prove the case where $w \xrightarrow{s_j}_\delta w'$ for some $j \in J$.

Notice that $F(Bs_iB) = Bs_{\delta(i)}B$ for $i \in I$.

If $ws_j < w$, then

$$\begin{aligned}(B, BwB) \cdot h_J &= (B, Bws_jB)(B, Bs_jB) \cdot h_J \\ &= (Bs_jB, Bws_jB) \cdot h_J \\ &\subset G_F(B, Bs_{\delta(j)}Bws_jB, B) \cdot h_J \\ &\subset G_F(B, Bw'B) \cdot h_J \cup G_F(B, Bws_jB) \cdot h_J.\end{aligned}$$

If moreover, $l(w') = l(w)$, then $Bs_{\delta(j)}Bws_jB = Bw'B$ and $G_F(B, BwB) \cdot h_J = G_F(B, Bw'B) \cdot h_J$.

If $s_{\delta(j)} w < w$, then

$$\begin{aligned}(B, BwB) \cdot h_J &= (B, Bs_{\delta(j)}B)(B, Bs_{\delta(j)}wB) \cdot h_J \\ &\subset G_F(Bs_jB, Bs_{\delta(j)}wB) \cdot h_J \\ &= G_F(B, Bs_{\delta(j)}wBs_jB) \cdot h_J \\ &\subset G_F(B, Bw'B) \cdot h_J \cup G_F(B, Bs_{\delta(j)}wB) \cdot h_J.\end{aligned}$$

If moreover, $l(w') = l(w)$, then $Bs_{\delta(j)}wBs_jB = Bw'B$ and $G_F(B, BwB) \cdot h_J = G_F(B, Bw'B) \cdot h_J$.

If $ws_{\delta(j)} > w$ and $s_j w > w$, then $l(w') = l(w)$. By [**L1**, Proposition 1.10], $w' = w$. The statements automatically hold in this case. □

Lemma 2.2. *We have that* $Z_J = \cup_{w \in W^J} Z_{J,F;w}$.

Remark. We will see in subsection 2.3 that Z_J is the disjoint union of $Z_{J,F;w}$ for $w \in W^J$.

Proof. Let $z \in Z_J$. Since $G \times G$ acts transitively on Z_J, z is contained in the G-orbit of an element $(1, g) \cdot h_J$ for some $g \in G$. By the Bruhat decomposition of G, we have that $z \in G_F(1, Bw_1B) \cdot h_J$ for some $w_1 \in W$. We may assume furthermore that w_1 is of minimal length among all the Weyl group elements w_1' with $z \in G_F(1, Bw_1'B) \cdot h_J$.

By part (1) of the previous lemma and 1.3 (a),

$$z \in G_F(B, BwvB) \cdot h_J \cup \cup_{l(w') < l(w_1)} G_F(B, Bw'B) \cdot h_J$$

for some $w \in W^J$ and $v \in W_{I(J,\delta;w)}$. By our assumption on w_1, we have that $z \in G_F(B, BwvB) \cdot h_J$ and $l(wv) = l(w_1)$. In particular, z is contained in the G_F-orbit of an element $(1, g'l) \cdot h_J$ for some $l \in L_K$ and $g' \in U_{P_{\delta(K)}} w U_{P_K}$, where $K = I(J, \delta; w)$.

Set $F' : L_K \to L_K$ by $F'(l_1) = w^{-1} F(l_1) w$. By Lang's theorem for F', we can find $l_1 \in L_K$ such that $F'(l_1) l l_1^{-1} = 1$. Then

$$\begin{aligned}(l_1, F(l_1))(1, g'l) \cdot h_J &= (1, F(l_1) g' l l_1^{-1}) \cdot h_J \\ &\in (1, U_{P_{\delta(K)}} w F'(l_1) l l_1^{-1} U_{P_K}) \cdot h_J \subset (1, BwB) \cdot h_J\end{aligned}$$

and $z \in Z_{J,F;w}$. □

2.2. For any parabolic subgroups P and Q of G, we set $P^Q = (P \cap Q) U_P$. It is known that P^Q is a parabolic subgroup of G. The following properties are easy to check.

(1) For any $g \in G$, $({}^g P)^{({}^g Q)} = {}^g (P^Q)$.

(2) If $P \in \mathcal{B}$, then $P^Q = P$ for any parabolic subgroup Q.

Lemma 2.3. *Let* $J, K \subset I$ *and* $w \in {}^J W$. *Set* $J_1 = J \cap \mathrm{Ad}(w_1) K$, *where* $w_1 = \min(w W_K)$. *Then for* $g \in BwB$, *we have that* $P_J^{({}^g P_K)} = P_{J_1}$.

Proof. By 2.2 (1), it suffices to prove the case where $g = \dot{w}$. Now

$$\begin{aligned}&(P_J)^{({}^{\dot{w}} P_K)} = (P_J)^{({}^{\dot{w}_1} P_K)} = (L_J \cap {}^{\dot{w}_1} L_K)(L_J \cap {}^{\dot{w}_1} U_{P_K})(U_{P_J} \cap {}^{\dot{w}_1} P_K) U_{P_J} \\ &= (L_J \cap {}^{\dot{w}_1} L_K)((L_J \cap U) \cap {}^{\dot{w}_1} L_K)(L_J \cap {}^{\dot{w}_1} U_{P_K})(U_{P_J} \cap {}^{\dot{w}_1} P_K) U_{P_J}.\end{aligned}$$

Since $w \in {}^J W$ and $w_1 = \min(wW_K)$, we have that $w_1 \in {}^J W^K$. Therefore $L_J \cap {}^{\dot w_1} L_K = L_{J_1}$ and

$$\begin{aligned}&((L_J \cap U) \cap {}^{\dot w_1} L_K)(L_J \cap {}^{\dot w_1} U_{P_K})\\ &= ((L_J \cap U) \cap {}^{\dot w_1} L_K)((L_J \cap U) \cap {}^{\dot w_1} U_{P_K})\\ &= (L_J \cap U) \cap {}^{\dot w_1} P_K = L_J \cap U.\end{aligned}$$

So $(P_J)^{({}^{\dot w} P_K)} = L_{J_1}(L_J \cap U) U_{P_J} = L_{J_1} U = P_{J_1}$. $\square$

Lemma 2.4. *To each $(P, gU_P) \in Z_J$, we associate a sequence $(P^n, J_n, w_n)_{n \geq 0}$ as follows*

$$P^0 = P, \quad P^n = (P^{n-1})^{F({}^g P^{n-1})} \qquad \textit{for } n \geq 1,$$
$$J_n \subset I \textit{ with } P^n \in \mathcal{P}_{J_n}, \quad w_n = \mathrm{pos}(P^n, F({}^g P^n)) \qquad \textit{for } n \geq 0.$$

Let $w \in W^J$. Let $(P, gU_P) \in Z_{J,F;w}$ and $(P^n, J_n, w_n)_{n \geq 0}$ be the sequence associated to (P, gU_P). Then $(J_n, w_n)_{n \geq 0} \in \mathcal{T}(J, \delta)$ and $w_m^{-1} = w$ for $m \gg 0$.

Proof. Using 2.2 (1), it is easy to see by induction on n that the sequence associated to $({}^{F(h)} P, hgU_P F(h)^{-1})$ is $({}^{F(h)} P^n, J_n, w_n)_{n \geq 0}$. Then it suffices to prove the case where $(P, gU_P) = (P_J, kU_{P_J})$ for some $k \in B\delta^{-1}(w)^{-1} B$.

Let $(J'_n, w'_n)_{n \geq 0} \in \mathcal{T}(J, \delta)$ be the element that corresponds to w. Then $w'_n = \min(w^{-1} W_{\delta(J_n)})$ for $n \geq 0$. By the previous lemma, we can show by induction on n that $P^n = P_{J'_n}$ for all $n \geq 0$. Then $J_n = J'_n$ for $n \geq 0$. Moreover, $w_n = \mathrm{pos}(P^n, F({}^k P^n)) = \mathrm{pos}(P_{J_n}, {}^{F(k)} P_{\delta(J_n)}) = w'_n$ since $k \in B\delta^{-1}(w)^{-1} B$. $\square$

(A similar result with a similar proof appears in [**H1**, Lemma 2.3].)

2.3. We can now define a map $\beta : Z_J \to W^J$ by $\beta(P, gU_P) = w_m^{-1}$ for $m \gg 0$, where $(P^n, J_n, w_n)_{n \geq 0}$ is the sequence associated to (P, gU_P). Then $Z_J = \sqcup_{w \in W^J} \beta^{-1}(w)$ is a partition of Z_J into locally closed subvarieties. Since $Z_{J,F;w} \subset \beta^{-1}(w)$ and $Z_J = \cup_{w \in W^J} Z_{J,F;w}$, we have that $Z_{J,F;w} = \beta^{-1}(w)$ and

$$Z_J = \sqcup_{w \in W^J} Z_{J,F;w}.$$

Fix $w \in W^J$ and let $(J_n, w_n)_{n \geq 0}$ be the element in $\mathcal{T}(J, \delta)$ that corresponds to w. Clearly, the map $(P, gU_P) \mapsto P^m$ for $m \gg 0$ is a morphism $\vartheta : Z_{J,F;w} \to \mathcal{P}_{I(J,\delta;w)}$.

Lemma 2.5. *Let $w \in W^J$. Set $x = \delta^{-1}(w)^{-1}$ and $K = \delta^{-1} I(J, \delta; w)$. Then*

$$(U_{P_K})_F (x, 1) \cdot h_J = (U_{P_K} x, U_{P_{\delta(K)}}) \cdot h_J.$$

Proof. Notice that

$$\begin{aligned}(U_{P_K} x, U_{P_{\delta(K)}}) \cdot h_J &= (U_{P_K})_F (x, U_{P_{\delta(K)}}) \cdot h_J\\ &= (U_{P_K})_F (x, U_{P_{\delta(K)}} \cap L_J) \cdot h_J\\ &= (U_{P_K})_F (x(U_{P_{\delta(K)}} \cap L_J), 1) \cdot h_J.\end{aligned}$$

So it suffices to show that for any $v \in U_{P_{\delta(K)}} \cap L_J$, there exists $u \in U_{P_K} \cap L_{\delta^{-1}(J)}$ such that $x^{-1} u x F(u)^{-1} \in v U_{P_J}$.

Let $(J_n, w_n)_{n \geq 0} \in \mathcal{T}(J, \delta)$ be the element that corresponds to w. By Lemma 1.1,

$$x^{-1}(L_{\delta^{-1}(J)} \cap U_{P_{\delta^{-1}(J_1)}})x \subset U_{P_J},$$
$$x^{-1}(L_{\delta^{-1}(J_i)} \cap U_{P_{\delta^{-1}(J_{i+1})}})x \subset L_{J_{i-1}} \cap U_{P_{J_i}} \text{ for } i \geq 1.$$

We have that $\delta(K) = J_m$ for some $m \in \mathbb{N}$. Now $v = v_m v_{m-1} \cdots v_0$ for some $v_i \in L_{J_i} \cap U_{P_{J_{i+1}}}$. We define $u_i \in L_{\delta^{-1}(J_i)} \cap U_{P_{\delta^{-1}(J_{i+1})}}$ as follows:

Let $u_m = 1$. Assume that $k < m$ and that $u_i \in L_{\delta^{-1}(J_i)} \cap U_{P_{\delta^{-1}(J_{i+1})}}$ are already defined for $k < i \leq m$ and that

$$(x^{-1}(u_m u_{m-1} \cdots u_{k+2})^{-1} x) F(u_m u_{m-1} \cdots u_{k+1})^{-1} = v_m v_{m-1} \cdots v_{k+1}.$$

Let u_k be the element with

$$\begin{aligned} F(u_k)^{-1} &= (x^{-1}(u_m u_{m-1} \cdots u_{k+1})x) v_m v_{m-1} \cdots v_k F(u_m u_{m-1} \cdots u_{k+1}) \\ &= (x^{-1} u_{k+1} x) F(u_m u_{m-1} \cdots u_{k+1})^{-1} v_k F(u_m u_{m-1} \cdots u_{k+1}) \\ &\in L_{J_K} \cap U_{P_{J_{k+1}}}. \end{aligned}$$

Thus $u_{k+1} \in L_{\delta^{-1}(J_k)} \cap U_{P_{\delta^{-1}(J_{k+1})}}$ and that

$$(x^{-1}(u_m u_{m-1} \cdots u_{k+1})x) F(u_m u_{m-1} \cdots u_k)^{-1} = v_m v_{m-1} \cdots v_k.$$

This completes the inductive definition.

Now set $u = u_m u_{m-1} \cdots u_0$. Then

$$\begin{aligned} &(x^{-1} u x) F(u)^{-1} \\ &= (x^{-1}(u_m u_{m-1} \cdots u_1)x) F(u)^{-1} (F(u)(x^{-1} u_0 x) F(u)^{-1}) \\ &\in v U_{P_{\delta(J)}}. \end{aligned}$$

The lemma is proved. □

By the proof of Lemma 2.2,

$$Z_{J,F;w} = G_F(U_{P_{\delta^{-1} I(J,\delta;w)}} \delta^{-1}(w)^{-1} U_{P_{\delta(I(J,\delta;w))}}, 1) \cdot h_J.$$

Then we have the following consequence.

Corollary 2.6. *Let $w \in W^J$. Then G_F acts transitively on $Z_{J,F;w}$.*

Remark. Therefore there are only finitely many G_F-orbits on Z_J and they are indexed by W^J. This is quite different from the set of G_Δ-orbits on Z_J.

Proposition 2.7. *Let $w \in W$. Then*

$$\overline{G_F(B, Bw) \cdot h_J} = \sqcup_{w' \in W^J, w' \leq_{J,\delta} w} Z_{J,F;w'}.$$

Remark. Similar results appear in [**H3**, Proposition 4.6], [**H2**, Corollary 5.5] and [**H4**, Proposition 5.8]. The following proof is similar to the proof of [**H4**, Proposition 5.8].

Proof. We prove by induction on $l(w)$.

Using the proper map $p : G_F \times_{B_F} Z_J \to Z_J$ defined by $(g, z) \mapsto (g, F(g)) \cdot z$, one can prove that

$$\overline{G_F(B, Bw) \cdot h_J} = G_F \overline{(B, Bw) \cdot h_J} = \cup_{v \leq w} G_F(B, Bv) \cdot h_J.$$

By 1.3 (a), $w \to_{J,\delta} w_1 v$ for some $w_1 \in W^J$ and $v \in W_{I(J,\delta;w)}$. By Lemma 2.1,

$$\begin{aligned}\overline{G_F(B,Bw)\cdot h_J} &= G_F(B,Bw)\cdot h_J \cup \cup_{w'<w} G_F(B,Bw')\cdot h_J\\ &= G_F(B,Bw_1v)\cdot h_J \cup \cup_{w'<w} G_F(B,Bw')\cdot h_J.\end{aligned}$$

By the proof of Lemma 2.2, $G_F(B,Bw_1v)\cdot h_J \subset G_F(B,Bw_1)\cdot h_J$. Thus by induction hypothesis,

$$\overline{G_F(B,Bw)\cdot h_J} \subset \cup_{w'\in W^J, w'\leq_{J,\delta} w} Z_{J,F;w'}.$$

On the other hand, if $w' \in W^J$ with $w' \leq_{J,\delta} w$, then there exists $w'' \approx_{J,\delta} w'$ with $w'' \leq w$. Then by Lemma 2.1,

$$Z_{J,F;w'} = G_F(B,Bw'')\cdot h_J \subset \overline{G_F(B,Bw)\cdot h_J}.$$

Therefore $\overline{G_F(Bw,B)\cdot h_J} = \cup_{w'\in W^J, w'\leq_{J,\delta} w} Z_{J,F;w'}$. By 2.3, $Z_{J,F;w_1} \cap Z_{J,F;w_2} = \emptyset$ if $w_1, w_2 \in W^J$ and $w_1 \neq w_2$. Thus $\overline{G_F(Bw,B)\cdot h_J} = \sqcup_{w'\in W^J, w'\leq_{J,\delta} w} Z_{J,F;w'}$. The proposition is proved. □

3. A stratification of partial flag varieties

3.1. It is easy to see that there is a canonical bijection between the G_F-orbits on Z_J and the R_J-orbits on $(G\times G)/G_F$ which sends $G_F(1,w)\cdot h_J$ to $R_J(1,w^{-1})G_F/G_F$. Notice that the map $(g_1,g_2)\mapsto g_2F(g_1)^{-1}$ gives an isomorphism of R_J-varieties $(G\times G)/G_F \cong G$, where the R_J-action on G is defined by

$$(lu_1, lu_2)\cdot g = lu_2 g F(lu_1)^{-1}.$$

Using the results of G_F-orbits on Z_J above, we have the following results.

(1) For $w \in {}^J W$, $R_J \cdot w = R_J \cdot (BwB)$. If moreover, $w' \in (W_J\cdot w)_{\min}$, then $R_J\cdot (Bw'B) = R_J\cdot(BwB)$.

(2) $G = \sqcup_{w\in {}^J W} R_J \cdot w$.

(3) For $w \in W$, $\overline{R_J\cdot w} = \sqcup_{w'\in {}^J W, (w')^{-1}\leq_{J,\delta} w^{-1}} R_J\cdot w'$.

Notice that if $J=\emptyset$, part (2) above follows easily from Bruhat decomposition. One may regard (2) as an extension of Bruhat's Lemma. We will also discuss a variation of (2) in section 4.

3.2. Now we review the partition on $\mathcal{P}_J$ introduced by Lusztig in [**L3**, section 4].

To each $P\in\mathcal{P}_J$, we associate a sequence $(P^n, J_n, w_n)_{n\geq 0}$ as follows

$$\begin{gathered}P^0 = P, \quad P^n = (P^{n-1})^{F(P^{n-1})} \text{ for } n\geq 1,\\ J_n \subset I \text{ with } P^n \in \mathcal{P}_{J_n}, \quad w_n = \operatorname{pos}(P^n, F(P^n)) \qquad \text{for } n \geq 0.\end{gathered}$$

By [**L3**, 4.2], $(J_n, w_n)_{n\geq 0} \in \mathcal{T}(J)$. Thus we have a map $i: \mathcal{P}_J \to {}^J W$. For $w\in {}^J W$, let

$$\mathcal{P}_{J,w} = \{P\in\mathcal{P}_J; w_m = w \text{ for } m \gg 0\}.$$

Then $\mathcal{P}_J = \sqcup_{w\in {}^J W}\mathcal{P}_{J,w}$.

It is easy to see that $\mathcal{P}_{J,w} = \{P\in\mathcal{P}_J; (P,U_P)\in Z_{J,F;w^{-1}}\}$.

Notice that Lie (G_Δ) + Lie (G_F) = Lie $(G)\oplus$ Lie (G). Then for any $x\in Z_J$, $G_\Delta\cdot x$ and $G_F\cdot x$ intersects transversally at x. In particular, $\mathcal{P}_{J,w}$ is the transversal intersection of $G_\Delta\cdot h_J$ and $Z_{J,F;w^{-1}}$.

We simply write $\mathcal{P}_{\emptyset,w}$ as $\mathcal{B}_w$. By 3.2 (3),

$$\mathcal{B}_w = \{B_1\in\mathcal{B}; \operatorname{pos}(B_1, F(B_1)) = w\} = \{{}^g B; g^{-1}F(g)\in B\dot{w}B\}.$$

Since the Lang isogeny $g^{-1}F(g)$ is an isomorphism $G^F \backslash G \to G$, we have that

(a) $$\overline{\mathcal{B}_w} = \sqcup_{v \leq w} \mathcal{B}_v.$$

Now we can prove our main theorem.

THEOREM 3.1. *Let $p : \mathcal{B} \to \mathcal{P}_J$ be the morphism which sends a Borel subgroup B' to the unique parabolic subgroup in $\mathcal{P}_J$ that contains B'. Then*

(1) For $w \in {}^J W$, $p(\mathcal{B}_w) = \mathcal{P}_{J,w}$. If moreover, $v \in (W_J \cdot w)_{\min}$, then $p(\mathcal{B}_v) = p(\mathcal{B}_w) = \mathcal{P}_{J,w}$.

(2) For $w \in W$, $\overline{\mathcal{P}_{J,w}} = p(\overline{\mathcal{B}_w}) = \sqcup_{w' \in {}^J W, (w')^{-1} \leq_{J,\delta} w^{-1}} \mathcal{P}_{J,w'}$.

REMARK. The closure relation of $\mathcal{P}_{J,w}$ was conjectured by G. Lusztig in private conversation.

PROOF. (1) Let $w \in {}^J W$ and $g \in G$ with ${}^g B \in \mathcal{B}_w$. Then $g^{-1}F(g) \in BwB$. Thus

$$\begin{aligned} ({}^g P_J, U_{P_J}) = (g, g) \cdot h_J &= (g, F(g))(1, F(g)^{-1} g) \cdot h_J \\ &\in G_F(B, Bw^{-1}) \cdot h_J = Z_{J,F;w^{-1}} \end{aligned}$$

and $p(\mathcal{B}_w) \subset \mathcal{P}_{J,w}$.

By 3.1, for any $g \in G$, there exists $l \in L_J$ such that $(gl)^{-1}F(gl) \in B\dot{w}'B$ for some $w' \in {}^J W$. Hence

(a) $$\begin{aligned} p(\mathcal{B}) &= \cup_{g \in G} p({}^g B) = \cup_{w' \in {}^J W} \cup_{g^{-1}F(g) \in B\dot{w}'B} \; p({}^g B) \\ &= \cup_{w' \in {}^J W} p(\mathcal{B}_{w'}) \subset \sqcup_{w' \in {}^J W} \mathcal{P}_{J,w} = \mathcal{P}_J. \end{aligned}$$

Since p is proper, we have that $p(\mathcal{B}) = \mathcal{P}_J$. Thus the inequality in (a) is actually an equality and $p(\mathcal{B}_{w'}) = \mathcal{P}_{J,w'}$ for all $w' \in {}^J W$.

If moreover, $v \in (W_J \cdot w)_{\min}$, then by 3.1, there exists $l \in L_J$ such that $(gl)^{-1}F(gl) = l^{-1}g^{-1}F(g)F(l) \in B\dot{w}B$. Thus $p({}^g B) = {}^g P_J = {}^{gl} P_J \in p(\mathcal{B}_w)$ and $p(\mathcal{B}_v) \subset p(\mathcal{B}_w)$. Similarly, we have that $p(\mathcal{B}_w) \subset p(\mathcal{B}_v)$. Then $p(\mathcal{B}_v) = p(\mathcal{B}_w)$.

Part (1) is proved.

(2) Since p is proper, we have that $\overline{\mathcal{P}_{J,w}} = p(\overline{\mathcal{B}_w})$. By 3.2 (a), $\overline{\mathcal{B}_w} = \sqcup_{v \leq w} \mathcal{B}_v$ and $p(\overline{\mathcal{B}_w}) = \cup_{v \leq w} p(\mathcal{B}_v) = \cup_{g \in G, g^{-1}F(g) \in \overline{B\dot{v}B}} p({}^g B)$. By 3.1 (3),

$$\begin{aligned} p(\overline{\mathcal{B}_w}) &= \cup_{w' \in {}^J W, (w')^{-1} \leq_{J,\delta} w^{-1}} \cup_{g^{-1}F(g) \in B\dot{w}'B} \; p({}^g B) \\ &= \cup_{w' \in {}^J W, (w')^{-1} \leq_{J,\delta} w^{-1}} p(\mathcal{B}'_w) \\ &= \cup_{w' \in {}^J W, (w')^{-1} \leq_{J,\delta} w^{-1}} \mathcal{P}_{J,w'}. \end{aligned}$$

Part (2) is proved. $\square$

Let us discuss some other properties of $\mathcal{P}_{J,w}$.

PROPOSITION 3.2. *Assume that G is quasi-simple and $J \neq I$. Then $\mathcal{P}_{J,w}$ is irreducible if and only if* $\operatorname{supp}_\delta(w) = I$.

PROOF. By [**L3**, 4.2 (d)], $\mathcal{P}_{J,w}$ is isomorphic to $\mathcal{P}_{K,w}$, where $K = I(J, \delta; w)$. By [**BR**, Theorem 2], $\mathcal{P}_{K,w}$ is irreducible if and only if wW_K is not contained in $W_{J'}$ for any δ-stable proper subset J' of I.

Let J' be the minimal δ-stable subset of I with $wW_K \subset W_{J'}$. It is easy to see that if $\operatorname{supp}_\delta(w) = I$, then $J' = I$. On the other hand, suppose that $\operatorname{supp}_\delta(w) \neq I$ and $J' = I$. Then for any $i \in K - \operatorname{supp}_\delta(w)$, we have that $w\alpha_i \in \delta(K)$. Since

$w\alpha_i \in \alpha_i + \sum_{j \in \mathrm{supp}(w)} \mathbb{Z}\alpha_j$, we must have that $w\alpha_i = \alpha_i$ and $i \in \delta(K)$. In particular, $K - \mathrm{supp}_\delta(w)$ is δ-stable, $w\alpha_i = \alpha_i$ for all $i \in K - \mathrm{supp}_\delta(w)$ and $K - \mathrm{supp}_\delta(w) = I - \mathrm{supp}_\delta(w)$. Since G is quasi-simple, there exists $i \in K - \mathrm{supp}_\delta(w)$ such that $(\alpha_i, \alpha_j^\vee) < 0$ for some $j \in \mathrm{supp}(w)$. Now assume that $w = s_{j_1} s_{j_2} \cdots s_{j_m}$ is a reduced expression and $m' = \max\{n; (\alpha_i, \alpha_{j_n}^\vee) \neq 0\}$. Then $s_{j_1} s_{j_2} \cdots s_{j_{m'}} \alpha_i = s_{j_1} s_{j_2} \cdots s_{j_m} \alpha_i = \alpha_i$. Thus

$$0 > (\alpha_i, \alpha_{j_{m'}}^\vee) = (s_{j_1} \cdots s_{j_{m'}} \alpha_i, s_{j_1} \cdots s_{j_{m'}} \alpha_{j_{m'}}^\vee) = (\alpha_i, s_{j_1} \cdots s_{j_{m'}} \alpha_{j_{m'}}^\vee).$$

However, $s_{j_1} \cdots s_{j_{m'}} \alpha_{j_{m'}}^\vee$ is a negative coroot. Thus

$$(\alpha_i, s_{j_1} \cdots s_{j_{m'}} \alpha_{j_{m'}}^\vee) \geq 0,$$

which is a contradiction. Therefore if $\mathrm{supp}_\delta(w) \neq I$, then $J' \neq I$. The proposition is proved. □

3.3. By [**L3**, 4.2 (d)], $\mathcal{P}_{J,w}$ is isomorphic to $\mathcal{P}_{K,w}$, where $K = I(J, \delta; w)$. Similar to [**DL**, 1.11], we have that

$$\begin{aligned}
\mathcal{P}_{K,w} &= \{g \in G; g^{-1}F(g) \in P_K \dot{w} P_K\}/P_K \\
&= \{g \in G; g^{-1}F(g) \in \dot{w} P_K\}/P_K \cap {}^{\dot{w}}P_K \\
&= \{g \in G; g^{-1}F(g) \in \dot{w} U_{P_K}\}/L_K^{\mathrm{Ad}(\dot{w}) \circ F}(U_{P_K} \cap {}^{\dot{w}}U_{P_K}).
\end{aligned}$$

Let $P \in \mathcal{P}_{K,w}$ such that there exists a F-stable Levi subgroup L of P. Then similar to [**DL**, 1.17], we have that

$$\begin{aligned}
\mathcal{P}_{K,w} &= \{g \in G; g^{-1}F(g) \in PF(P)\}/P \\
&= \{g \in G; g^{-1}F(G) \in F(P)\}/P \cap F(P) \\
&= \{g \in G; g^{-1}F(g) \in F(U_P)\}/L^F(U_P \cap F(U_P)).
\end{aligned}$$

4. An extension of Bruhat decomposition

After the paper was submitted, I learned from A. Vasiu about his conjecture in [**Va**, 2.2.1]. We state it in the following slightly stronger version.

Corollary 4.1. *Let P be a parabolic subgroup of G of type J with a Levi subgroup L. Let $R = \{(lu, lu'); l \in L, u, u' \in U_P\}$ and define the action of R on G by $(lu, lu') \cdot g = lugF(lu')^{-1}$. Then*

(1) There are only finitely many R-orbits on G, indexed by ${}^J W$.

(2) If moreover, there exists a maximal torus $T' \subset P$ such that $F(T') = T'$, then each R-orbit contains an element in $N_G(T')$.

Proof. We may assume that $P = {}^g P_J$ and $L = {}^g L_J$. For any $w \in {}^J W$, set $w^* = gwF(g)^{-1}$. Then it is to see that $R \cdot w^* = g(R_J \cdot w)F(g)^{-1}$. Now part (1) follows from 3.1 (2).

If moreover, $T' = {}^g T \subset P$ is F-stable, then we have that $g^{-1}F(g) \in N_G(T)$. Thus $w^* = gwF(g)^{-1} = g(wF(g)^{-1}g)g^{-1}$ and $wF(g)^{-1}g \in N_G(T)$. So $w^* \in N_G(T')$ and part (2) is proved. □

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