COCENTER OF p-ADIC GROUPS, II: INDUCTION MAP

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ABSTRACT. In this paper, we study some relation between the cocenter $\bar{H}(G)$ of the Hecke algebra $H(G)$ of a connected reductive group G over an nonarchimedean local field and the cocenter $\bar{H}(M)$ of its Levi subgroups M.

Given any Newton component of $\bar{H}(G)$, we construct the induction map $\bar{i}$ from the corresponding Newton component of $\bar{H}(M)$ to it. We show that this map is surjective. This leads to the Bernstein-Lusztig type presentation of the cocenter $\bar{H}(G)$, which generalizes the work [13] on the affine Hecke algebras. We also show that the map $\bar{i}$ we constructed is adjoint to the Jacquet functor and in characteristic 0, the map $\bar{i}$ is an isomorphism.

INTRODUCTION

0.1. Let $\mathbb{G}$ be a connected reductive group over a nonarchimedean local field F of arbitrary characteristic and $G = \mathbb{G}(F)$. Let R be an algebraically closed field of characteristic not equal to p, where p is the characteristic of residue field of F. Let H_R be the Hecke algebra of G over R and $\bar{H}_R = H_R/[H_R, H_R]$ be its cocenter. Let $\mathfrak{R}(G)_R$ be the R-vector space with basis the isomorphism classes of irreducible smooth admissible representations of G over R. Then we have the trace map

$$\mathrm{Tr}_R : \bar{H}_R \longrightarrow \mathfrak{R}(G)^*_R.$$

On the representation side, we have the induction functor and the Jacquet functor

$$i_{M,R} : \mathfrak{R}(M)_R \longrightarrow \mathfrak{R}(G)_R, \qquad r_{M,R} : \mathfrak{R}(G)_R \longrightarrow \mathfrak{R}(M)_R,$$

where M is a Levi subgroup of G.

What happens on the cocenter side?

The functor adjoint to the induction functor i_M is the restriction map $\bar{r}_{M,R}$: $\bar{H}(G)_R \to \bar{H}(M)_R$. It can be expressed explicitly via the Van Dijk's formula. In this paper, we investigate the functor $\bar{i}_{M,R} : \bar{H}_R(M) \to \bar{H}_R(G)$, which is adjoint to the Jacquet functor $r_{M,R} : \mathfrak{R}(G)_R \to \mathfrak{R}(M)_R$.

0.2. We first describe the properties we expect for the map $\bar{i}_{M,R}$ and then discuss the approach toward it.

First, instead of working over various algebraically closed fields R, it is desirable to have the map $\bar{i}_M$ defined on the integral form $\bar{H}$ (the cocenter of the Hecke algebra of $\mathbb{Z}[\frac{1}{p}]$-valued functions). Such map, if exists, provides not only a uniform approach to the map $\bar{i}_{M,R}$ for all R, but also some useful information on the mod-l

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representations (see Theorem D in the introduction and a future work [6] for some results in this direction).

Second, in [11], we introduced the Newton decomposition. Roughly speaking,

$$G = \sqcup G(v) \quad \text{and} \quad \bar{H} = \oplus \bar{H}(v),$$

where v runs over the set of dominant rational coweights of G. Such description is expected to play an important role in the representation theory of p-adic groups. In order to relate the Newton decomposition with the representations, we would like to know that the Newton decomposition is compatible with the map $\bar{i}_M$.

0.3. Now we discuss several approaches in the literature towards the understanding of the map $\bar{i}_M$.

Over $\mathbb{C}$, the spectral density Theorem of Kazhdan [14] asserts that the trace map $\mathrm{Tr}_{\mathbb{C}} : \bar{H}_{\mathbb{C}} \to \mathfrak{R}(G)^*_{\mathbb{C}}$ is injective. Hence the map $\bar{i}_{M,\mathbb{C}}$ is uniquely determined by the adjunction formula

$$\mathrm{Tr}^M_{\mathbb{C}}(f, r_{M,\mathbb{C}}(\pi)) = \mathrm{Tr}^G_{\mathbb{C}}(\bar{i}_{M,\mathbb{C}}(f), \pi).$$

However, if R is of positive characteristic, the trace map Tr_R may not be injective and thus the map $\bar{i}_{M,R}$ is not uniquely determined by the adjunction formula.

In those cases, one may use the categorical description of the cocenter to give a definition of $\bar{i}_{M,R}$. Bernstein's second adjointness theorem implies that the map $\bar{i}_{M,R}$ defined in this way is adjoint to the Jacquet functor (see [7, (1.8)]). However, it is not clear that this map preserves the integral structure (see some discussion in [7, §4.27]). Also it is not clear if this description is compatible with the Newton decomposition.

0.4. A different, but more explicit approach is given by Bushnell in [2].

Note that the induction functor $i_{M,R}$ on the representations of M depends not only on the Levi subgroup M, but also on the parabolic subgroup P with Levi factor M. However, when passing to the Grothendieck group of the representations, the dependence of P disappears. On the other hand, the Jacquet functor $r_{M,R}$, even if one passes to the Grothendieck groups of the representations, still depend on the choice of parabolic subgroup.

Let v be a rational coweight. Then v determines a Levi subgroup $M = M_v$ and the parabolic subgroup $P_v = MN_v$. Let $\mathcal{K}$ be a "nice" open compact subgroup of G (e.g. the n-th congruent subgroup $\mathcal{I}_n$ of an Iwahori subgroup) and $\mathcal{K}_M = \mathcal{K} \cap M$. Bushnell introduced the P_v-positive elements of M and the subalgebra $H^v(M, \mathcal{K}_M)$ of $H(M, \mathcal{K}_M)$, consisting of compactly supported $\mathcal{K}_M$-biinvariant functions supported in the P_v-positive elements. Then he proves that

(a) The algebra $H(M, \mathcal{K}_M)$ is isomorphic to the localization of $H^v(M, \mathcal{K}_M)$ at a strongly positive element f_z.

(b) The map

$$j_{v,\mathcal{K}} : H^v(M, \mathcal{K}_M) \longrightarrow H(G, \mathcal{K}), \delta_{\mathcal{K}_M m \mathcal{K}_M} \longmapsto \delta_{P_v}(m)^{-\frac{1}{2}} \frac{\mu_G(\mathcal{K})}{\mu_M(\mathcal{K}_M)} \delta_{\mathcal{K} m \mathcal{K}}$$

is an injective algebra homomorphism.

(c) The map $j_{v,\mathcal{K}}$ is adjoint to the Jacquet functor $r_{M,\mathcal{K},R} : \mathfrak{R}_{\mathcal{K}}(G)_R \to \mathfrak{R}_{\mathcal{K} \cap M}(M)_R$ relative to P_v. Here $\mathfrak{R}_{\mathcal{K}}(G)_R \subset \mathfrak{R}(G)_R$ consists of representations generated by their $\mathcal{K}$-fixed vectors.

Moreover, Bushnell's map $j_{v,\mathcal{K}}$ also preserves the integral structure of the Hecke algebra.

0.5. It is tempting to apply Bushnell's result to the cocenter of Hecke algebras. However, there are several obstacles.

If $\mathcal{K}$ is the Iwahori or pro-p Iwahori subgroup, then the map $j_{v,\mathcal{K}}$ extends to an algebra homomorphism $H(M, \mathcal{K} \cap M) \to H(G, \mathcal{K})$. In this case, the localization of Hecke algebra $H^v(M, \mathcal{K} \cap M)$ is consistent with the Bernstein-Lusztig presentation ([10] and [18]). However, as pointed out in [2], these are essentially the only cases of this kind. Thus one may only use $j_{v,\mathcal{K}}$ to deduce the induction map from part of the cocenter of $H(M)$ to the cocenter of $H(G)$.

The Newton strata of M with integral dominant Newton points are positive, but the strata with rational (but not integral) Newton point may not be positive for any parabolic P. Those strata are not in the domain of the maps $j_{v,\mathcal{K}}$.

Also if one fixes M and P, the maps $j_{v,\mathcal{K}}$ are not compatible with the change of open compact subgroups $\mathcal{K}$, even at the cocenter level (see §2.5). Thus the maps $j_{v,\mathcal{K}}$ does not induce a well-defined map $\bar{H}^v(M) \to \bar{H}$.

0.6. The idea behind Bushnell's map $j_{v,\mathcal{K}}$ is to enlarge the open compact subset $\mathcal{K}_M m \mathcal{K}_M$ of M to the open compact subset $\mathcal{K} m \mathcal{K}$ of G by multiplying the open compact subgroup $\mathcal{K}$. Inspired by it, we have the following construction.

Let v be a rational coweight and $P = MN_v$ be the associated parabolic subgroup. The elements in the Newton stratum $M(v)$ may not be P_v-positive, but a sufficiently large power of it is P_v-positive. One may enlarge an open compact subset inside $M(v)$ by multiplying a suitable open compact subgroup of G to obtain an open compact subset of G. Unlike the situation in [2], the lack of P_v-positivity condition prevents us to give an explicit open compact subgroup of G that works in our situation. We have to use sufficiently small open compact subgroup of G. Since v is strictly positive with respect to N_v, we finally show that our construction is independent of the choice of such open compact subgroups. We have

Theorem A. *Let v be a rational coweight and $M = M_v$. Let $\bar{v}$ be the G-dominant coweight associated to v. Then*

(1) [Theorem 3.1] The map

$$\delta_{m\mathcal{K}_M} \longmapsto \delta_{P_v}(m)^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M \mathcal{K})} \delta_{m\mathcal{K}_M \mathcal{K}} + [H, H]$$

for sufficiently small open compact subgroup $\mathcal{K}$ of G gives a well-defined map

$$\bar{i}_v : \bar{H}(M; v) \longrightarrow \bar{H}.$$

(2) [Theorem 4.1] The image of $\bar{i}_v$ equals $\bar{H}(G; \bar{v})$.

(3) [Theorem 6.5] If moreover, char$(F) = 0$, then the map $\bar{i}_v$ gives a bijection between $\bar{H}(M; v)$ and $\bar{H}(G; \bar{v})$.

Theorem B (Theorem 5.2)**.** *Let v be a rational coweight and $M = M_v$. Then for any $f \in \bar{H}_R(M; v)$ and $\pi \in \mathfrak{R}(G)_R$, we have the following adjunction formula*

$$\mathrm{Tr}_R^M(f, r_{v,R}(\pi)) = \mathrm{Tr}_R^G(\bar{i}_v(f), \pi).$$

Here $r_{v,R} : \mathfrak{R}(G)_R \to \mathfrak{R}(M)_R$ is the Jacquet functor relative to P_v.

0.7. Now we discuss some applications. In [11], we introduced the rigid cocenter $\bar{H}^{\text{rig}} = \oplus \bar{H}(v)$, where v runs over rational central coweights.

Now for any standard Levi subgroup M, we introduce the $+$-rigid part $\bar{H}(M)^{+,\text{rig}} = \oplus \bar{H}(M; v)$, where v runs over rational dominant coweights with $M = M_v$. We then have the well-defined map

$$\bar{i}_M^+ = \oplus_v \bar{i}_v : \bar{H}(M)^{+,\text{rig}} \longrightarrow \bar{H}.$$

As an application of Theorem A and the Newton decomposition of $\bar{H}$ (see [11, Theorem 3.1]), we have

Theorem C. *We have the decomposition of the cocenter $\bar{H}$ into $+$-rigid parts:*

$$\bar{H} = \oplus_{M \text{ is a standard Levi subgroup}} \bar{i}_M^+(\bar{H}(M)^{+,rig}).$$

For affine Hecke algebras, such decomposition is first obtained in [13] via an elaborate analysis on the minimal length elements in the affine Weyl groups of G and its Levi subgroups M. In loc.cit., such decomposition is called the Bernstein-Lusztig presentation of the cocenter of affine Hecke algebras, since the explicit expression of $\bar{i}_M^+$ there is given in terms of the Bernstein-Lusztig presentation. Although there is no Bernstein-Lusztig type presentation for H, we follow [13] and still call the decomposition in Theorem C the Bernstein-Lusztig presentation of the cocenter $\bar{H}$. It is also worth mentioning that the proof in this paper does not involve the elaborate analysis on the minimal length elements as in [13], but based on the compatibility between the change of different open compact subgroups $\mathcal{K}$ of G.

Theorem C asserts that the rigid cocenters of Levi subgroups form the "building blocks" of the whole cocenter $\bar{H}$. We also show that that they are compatible with the trace map in the following way.

Theorem D (Theorem 6.1)**.** *Let R be an algebraically closed field of characteristic not equal to p. Then we have*

$$\ker \text{Tr}_R = \oplus_{M \text{ is a standard Levi subgroup}} \bar{i}_M^+(\ker \text{Tr}_R^M \cap \bar{H}_R(M)^{+,rig}).$$

If $R = \mathbb{C}$, we have the spectral density theorem and the kernel of the trace map is zero. Theorem D is trivial in this case. However, if R is of positive characteristic, especially when the spectral density theorem fails, then Theorem D would provide useful information toward the understanding of those representations.

0.8. The outline of the proof is as follows. In §2, we introduce the notion of quasi-positive elements and we use some remarkable properties on the minimal length elements established in [12] to show that any element in the Newton stratum $M(v)$ is quasi-positive. Then in §3, we use the quasi-positivity to show that the map in Theorem A (1) is well-defined and factors through $\bar{H}(M; v)$. This proves part (1) of Theorem A.

As to part (2) of Theorem A, we first prove in Proposition 4.2 that $M(v) \subset G(\bar{v})$. Then by the admissibility of Newton strata ([11, Theorem 3.2]), any open compact subset X of $M(v)$ enlarged by a sufficiently small open compact subgroup is still contained in $G(\bar{v})$. This shows that the image of $\bar{i}_v$ is contained in $\bar{H}(G; \bar{v})$. The key ingredients in the proof of surjectivity are

- The notation of P-alcove elements introduced in [8].

- The Iwahori-Matsumoto presentation of $\bar{H}(G;\bar{v})$ ([11, Theorem 4.1]).

By the quasi-positivity, for any $f \in H(M;v)$, $f^l \in H^v(M)$ for sufficiently large l. Theorem B follows from the adjunction formula proved in [2], the comparison between $i_v(f)^l$ with $j_{v,*}(f^l)$ and a trick of Casselman [4].

Finally, the injectivity in part (3) of Theorem A follows from the adjunction formula (Theorem B), the spectral density theorem and the freeness of the cocenter $\bar{H}$ (which is only known in the case of $\mathrm{char}(F)=0$).

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1. Preliminary

1.1. Let $\mathbb{G}$ be a connected reductive group over a nonarchimedean local field F of arbitrary characteristic. Let $G=\mathbb{G}(F)$. We fix a maximal F-split torus A and an alcove $\mathfrak{a}_C$ in the corresponding apartment, and denote by $\mathcal{I}$ the associated Iwahori subgroup.

Let $Z=Z_G(A)$. We denote by $W_0=N_GA(F)/Z(F)$ the *relative Weyl group* and $\tilde{W}=N_GA(F)/Z_0$ the *Iwahori-Weyl group*, where Z_0 is the unique parahoric subgroup of $Z(F)$.

We fix a special vertex of $\mathfrak{a}_C$ and identify $\tilde{W}$ as

$$\tilde{W} \cong X_*(Z)_{\mathrm{Gal}(\bar{F}/F)} \rtimes W_0 = \{t^\lambda w; \lambda \in X_*(Z)_{\mathrm{Gal}(\bar{F}/F)}, w \in W_0\}.$$

We have a semidirect product

$$\tilde{W} = W_a \rtimes \Omega,$$

where W_a is the affine Weyl group associated to $\tilde{W}$ and Ω is the stabilizer of the alcove $\mathfrak{a}_C$ in $\tilde{W}$. Let $\tilde{\mathbb{S}}$ be the set of affine simple reflections of W_a determined by the fundamental alcove $\mathfrak{a}_C$. The groups W_a and $\tilde{W}$ are equipped with a Bruhat order $\leqslant$ and a length function ℓ. The subgroup Ω of $\tilde{W}$ is the subgroup consisting of length-zero elements.

1.2. For any $K \subset \tilde{\mathbb{S}}$, let W_K be the subgroup of $\tilde{W}$ generated by $s \in K$. Let ${}^K\tilde{W}$ be the set of elements $w \in \tilde{W}$ of minimal length in the cosets $W_K w$.

Let $\Phi=\Phi(G,A)$ be the set of roots of G relative to A and Φ^+ be the set of positive roots so that $\mathfrak{a}_C$ is contained in the antidominant chamber of V determined by Φ^+. Let $\mathscr{R}=\{\alpha\}$ be the set of affine roots on $\mathscr{A}$. We choose a normalization of the valuation on F so that if $\alpha \in \mathscr{R}$, then so is $\alpha \pm 1$ (see [1, §5.2.23]). For any $n \in \mathbb{N}$, let $\mathcal{I}_n$ be the n-th Moy-Prasad subgroup associated to the barycenter of $\mathfrak{a}_C$ [15]. This is the subgroup of G generated by the n-th congruence subgroup of $Z(F)$ and the affine root subgroup $X_{\alpha+n}$ for $\alpha \in \mathscr{R}_+$.

For any $n \in \mathbb{N}$ and a subgroup G' of G, we set $G'_n = G' \cap \mathcal{I}_n$. We write $\mathcal{I}_{G'}$ for $G' \cap \mathcal{I}$.

1.3. Let μ_G be the Haar measure on G such that the pro-p Iwahori subgroup $\mathcal{I}'$ has volume 1. As in [11, Section 1], we denote by $H = H(G)$ the Hecke algebra of locally constant, compactly supported $\mathbb{Z}[\frac{1}{p}]$-valued functions on G. We have

$$H = \varinjlim_{\mathcal{K}} H(G, \mathcal{K}),$$

where $\mathcal{K}$ runs over open compact subgroups of G and $H(G,\mathcal{K})$ is the space of compactly supported, $\mathcal{K} \times \mathcal{K}$-invariant $\mathbb{Z}[\frac{1}{p}]$-valued functions on G, i.e., $H(G,\mathcal{K}) = \oplus_{g \in \mathcal{K}\backslash G/\mathcal{K}} \mathbb{Z}[\frac{1}{p}]\delta_{\mathcal{K}g\mathcal{K}}$, where $\delta_{\mathcal{K}g\mathcal{K}}$ is the characteristic function on $\mathcal{K}g\mathcal{K}$.

We define the action of G on H by ${}^x f(g) = f(x^{-1}gx)$ for $f \in H$, $x, g \in G$. By [11, Proposition 1.1], the commutator $[H, H]$ of H is the $\mathbb{Z}[\frac{1}{p}]$-submodule of H spanned by $f - {}^x f$ for $f \in H$ and $x \in G$. Let $\bar{H} = H/[H, H]$ be the cocenter of H.

1.4. Now we recall the Newton decomposition introduced in [11].

Set $V = X_*(Z)_{\mathrm{Gal}(\bar{F}/F)} \otimes \mathbb{R}$ and V_+ be the set of dominant elements in V. For any $w \in \tilde{W}$, there exists a positive integer l such that $w^l = t^\lambda$ for some $\lambda \in X_*(Z)_{\mathrm{Gal}(\bar{F}/F)}$. We set $\nu_w = \lambda/l \in V$ and $\bar{\nu}_w$ to be the unique dominant in the W_0-orbit of ν_w. The element ν_w and $\bar{\nu}_w$ are independent of the choice of l.

Let $\aleph = \Omega \times V_+$. We have a map (see [11, §2.1])

$$\pi = (\kappa, \bar{\nu}) : \tilde{W} \longrightarrow \aleph, \qquad w \longmapsto (wW_a, \bar{\nu}_w).$$

We denote by $\tilde{W}_{\min}$ be the subset of $\tilde{W}$ consisting of elements of minimal length in their conjugacy classes. For any $\nu \in \aleph$, we set

$$X_\nu = \cup_{w \in \tilde{W}_{\min}; \pi(w) = \nu} \mathcal{I}\dot{w}\mathcal{I} \quad \text{and} \quad G(\nu) = G \cdot_\theta X_\nu.$$

Here $\cdot$ means the conjugation action of G. Let $H(\nu)$ be the submodule of H consisting of functions supported in $G(\nu)$ and let $\bar{H}(\nu)$ be the image of $H(\nu)$ in the cocenter $\bar{H}$. The Newton decomposition of $\bar{H}$ is established in [11, Theorem 3.1 (2)].

Theorem 1.1. *We have that*

$$\bar{H} = \bigoplus_{\nu \in \aleph} \bar{H}(\nu).$$

In this paper, we are mainly interested in the V-factor of $\aleph$. For any $v \in V_+$, we also set $G(v) = \sqcup_{\nu = (\tau, v) \text{ for some } \tau \in \Omega} G(\nu)$, $H(v) = \oplus_{\nu = (\tau, v) \text{ for some } \tau \in \Omega} H(\nu)$ and $\bar{H}(v) = \oplus_{\nu = (\tau, v) \text{ for some } \tau \in \Omega} \bar{H}(\nu)$.

1.5. Let M be a semistandard Levi subgroup of G, i.e., a Levi subgroup of some parabolic subgroup of G that contains Z. Let $\mathcal{I}_M = \mathcal{I} \cap M$ be the Iwahori subgroup of M and $\tilde{W}(M)$ be the Iwahori-Weyl group of M. We denote by $\tilde{\mathbb{S}}(M)$ the set of affine simple reflections of $\tilde{W}(M)$ determined by the Iwahori subgroup $\mathcal{I}_M$.

We may regard $\tilde{W}(M)$ as a subgroup of $\tilde{W}$ in a natural way. However, the length function ℓ_M on $\tilde{W}(M)$ does not equal to the restriction ot $\tilde{W}$ of the length function ℓ on $\tilde{W}$.

Let Ω_M be the subgroup of $\tilde{W}(M)$ consisting of length-zero elements with respect to the length function ℓ_M. We have $\Omega_M \cong \tilde{W}(M)/W_a(M)$, where $W_a(M)$ is the affine Weyl group of the subgroup of $\tilde{W}(M)$. We have $W_a(M) \subset W_a$ and thus a natural map $\Omega_M \cong \tilde{W}(M)/W_a(M) \to \tilde{W}/W_a \cong \Omega$. Let V_+^M be the set of M-dominant elements in V. We set $\aleph_M = \Omega_M \times V_+^M$ and we have a map $\pi_M = (\kappa_M, \bar{\nu}_M) : \tilde{W}(M) \to \aleph_M$.

We also have a natural map $\aleph_M \to \aleph$ sending (τ, v) to $(\tau', \bar{v})$, where τ' is the image of τ in Ω and $\bar{v}$ is the unique (G-)dominant element in the W_0-orbit of v.

Let μ_M be the Haar measure on M such that the pro-p Iwahori subgroup of M has volume 1. Let $H(M)$ be the Hecke algebra of M and $\bar{H}(M)$ be its cocenter. For any $\nu_M \in \aleph_M$, we denote by $\bar{H}(M; \nu_M)$ the corresponding Newton component of $\bar{H}(M)$. By Theorem 1.1, we have

$$\bar{H}(M) = \oplus_{\nu_M \in \aleph_M} \bar{H}(M; \nu_M).$$

2. Quasi-positive elements

2.1. The semistandard Levi may be described as the centralizer of elements in V. For any $v \in V$, we set $\Phi_{v,0} = \{a \in \Phi; \langle a, v \rangle = 0\}$ and $\Phi_{v,+} = \{a \in \Phi; \langle a, v \rangle > 0\}$. Let $M_v \subset G$ be the Levi subgroup generated by Z and $U_a(F)$ for $a \in \Phi_{v,0}$ and $N_v \subset G$ be the unipotent subgroup generated by $U_a(F)$ for $a \in \Phi_{v,+}$. Set $P_v = M_v N_v$. Then P_v is a semistandard parabolic subgroup and M_v is a Levi subgroup of P_v. We denote by $P_v^- = M_v N_v^-$ the opposite parabolic. Let $\mu_{N_v}, \mu_{N_v^-}$ be the Haar measures on N_v and N_v^- respectively such that $\mu_G(nmn^-) = \mu_{N_v}(n)\mu_{M_v}(m)\mu_{N_v^-}(n^-)$ for $n \in N_v, m \in M_v, n^- \in N_v^-$. For $m \in M_v$, set $\delta_v(m) = \frac{\mu_{N_v}(mN_{v,0}m^{-1})}{\mu_{N_v}(N_{v,0})}$. For $\nu = (\tau, v) \in \aleph$, we may also write M_ν for M_v, $N(\nu)$ for N_v and $N^-(\nu)$ for N_v^-.

If v is dominant, then P_v is a standard parabolic subgroup of G and M_v is a standard Levi subgroup of G.

2.2. Let $v \in V$. Following [3, Definition 6.5 & Definition 6.14], we call an element $m \in M_v$ a $(P_v, \mathcal{I}_n)$-*positive* element if

$$mN_{v,n}m^{-1} \subset N_{v,n}, \text{ and } m^{-1}N_{v,n}^- m \subset N_{v,n}^-.$$

We call an element z in the center of M_v a *strongly P_v-positive* element if the sequences $z^n N_{v,0} z^{-n}, z^{-n} N_{v,0}^- z^n$ both tend monotonically to 1 as $n \to \infty$.

Following [2, §3.1], let $H^v(M_v, M_{v,n})$ be the subalgebra of $H(M_v, M_{v,n})$ of functions with support consisting of $(P_v, \mathcal{I}_n)$-positive elements. The following result is proved in [2, Proposition 5].

Proposition 2.1. *The map* $\delta_{M_{v,n}mM_{v,n}} \mapsto \delta_v(m)^{-\frac{1}{2}} \frac{\mu_{M_v}(M_{v,n})}{\mu_G(\mathcal{I}_n)} \delta_{\mathcal{I}_n m \mathcal{I}_n}$ *defines an injective algebra homomorphism*

$$j_{v,n} : H^v(M_v, M_{v,n}) \hookrightarrow H(G, \mathcal{I}_n).$$

The formula we have here differs from [2] by the factor $\delta_v(m)^{-\frac{1}{2}}$, since in [2] the map is adjoint to the (unnormalized) Jacquet functor while we consider the (normalized) Jacquet functor.

By [2, §3.2], $H(M_v, M_{v,n}) = S^{-1}H^v(M_v, M_{v,n})$ is the localization of $H^v(M_v, M_{v,n})$, where $S = \langle \delta_{M_{v,n}zM_{v,n}} \rangle$ is the the multiplicative closed set of the function $\delta_{M_{v,n}zM_{v,n}}$ with a strongly P_v-positive element z. It is pointed out in [2, Remark 5] that the map $j_{v,n}$ does not extend to an algebra homomorphism $H(M_v, M_{v,n}) \to H(G, \mathcal{I}_n)$ for $n > 0$.

2.3. Let $v \in V$ be a rational coweight and $M = M_v$. For any $l \in \mathbb{N}$ with $lv \in X_*(Z)$, the element t^{lv} is strongly P_v-positive. However, in general, the element in $M(v)$ may not be $(P_v, *)$-positive. Therefore, one can not deduce a map from $\bar{H}(M; v)$ to $\bar{H}$ via the map $j_{v,n}$.

Example 2.2. Let G be split GL_5 and $M = GL_3 \times GL_2$. Let $v = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2})$. Then $M = M_v$. The element $w = t^{(1,1,0,1,0)}(132)(45)$ of $\tilde{W}$ has Newton point v. However, $w(e_4 - e_3) = e_5 - e_2 - 1$ is a negative affine root. Therefore the element $\dot{w}$ is not $(P_v, *)$-positive.

2.4. To overcome the difficulty, we introduce the quasi-positive elements.

An element $m \in M_v$ is called P_v *quasi-positive* if there exists $l \in \mathbb{N}$ such that

(a) $$m^l N_{v,n} m^{-l} \subset N_{v,n+1}, \text{ and } m^{-l} N^-_{v,n} m^l \subset N^-_{v,n+1} \text{ for any } n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, $w \in \tilde{W}$ and $g \in \mathcal{I}\dot{w}\mathcal{I}$, we have $g\mathcal{I}_{n+\ell(w)}g^{-1} \subset \mathcal{I}_n$. So

(b) Let $w \in \tilde{W}(M)$ and $m \in \mathcal{I}_M \dot{w} \mathcal{I}_M$. If m satisfies (a), then we have

$$m^n N_{v,n'+(l-1)\ell(w)} m^{-n} \subset N_{v,n'}, \text{ and } m^{-n} N^-_{v,n'+(l-1)\ell(w)} m^n \subset N^-_{v,n'} \text{ for any } n, n' \in \mathbb{N}.$$

We first discuss some properties on the quasi-positive elements.

Proposition 2.3. *Let $v \in V$ and $M = M_v$. Let $w \in \tilde{W}(M)$ and $m \in \mathcal{I}_M \dot{w} \mathcal{I}_M$. Suppose that m satisfying the inclusion relation §2.4 (a).*

(1) For any $n \in \mathbb{N}$, any element in $m\mathcal{I}_{n+(l-1)\ell(w)}$ is conjugate by an element in $\mathcal{I}_n$ to an element in $mM_{n+(l-1)\ell(w)}$.

(2) For any $n, n' \in \mathbb{N}$ and $g \in \mathcal{I}_{n+(l-1)\ell(w)}$, we have

$$\delta_{mgM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+n'}} \equiv \delta_{mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+n'}} \mod [H, H].$$

Proof. (1) We first show that

(a) For any $i \in \mathbb{N}$, any element in $mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+i}$ is conjugate by $\mathcal{I}_{n+i}$ to an element in $mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+i+1}$.

Note that any element in $mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+i}$ is conjugate by $\mathcal{I}_{n+(l-1)\ell(w)+i}$ to an element of the form $u'gu$ with $u' \in N^-_{v,n+(l-1)\ell(w)+i}$, $g \in mM_{n+(l-1)\ell(w)}$ and $u \in N_{v,n+(l-1)\ell(w)+i}$. By §2.4 (b), $gug^{-1} \in N_{v,n+i}$. We have $(u', gug^{-1}) \in (\mathcal{I}_{n+(l-1)\ell(w)+i}, \mathcal{I}_{n+i}) \subset \mathcal{I}_{n+(l-1)\ell(w)+i+1}$. Now we have

$$u'gu = u'(gug^{-1})g \in (gug^{-1})u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}g.$$

So $u'gu$ is conjugate by $\mathcal{I}_{n+i}$ to an element in

$$\begin{aligned} u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}g(gug^{-1}) &= u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}(g^2u(g^2)^{-1})g \\ &= u'(g^2u(g^2)^{-1})\mathcal{I}_{n+(l-1)\ell(w)+i+1}g \\ &= (g^2u(g^2)^{-1})u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}g. \end{aligned}$$

By the same procedure, for any $l \in \mathbb{N}$, $u'gu$ is conjugate by $\mathcal{I}_{n+i}$ to an element in $(g^l u(g^l)^{-1})u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}g$. By §2.4 (a), $g^l ug^{-l} \in \mathcal{I}_{n+(l-1)\ell(w)+i+1}$. Hence $u'gu$ is conjugate by $\mathcal{I}_{n+i}$ to an element in $u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}g = u'g\mathcal{I}_{n+(l-1)\ell(w)+i+1}$. By the same argument, any element in $u'g\mathcal{I}_{n+(l-1)\ell(w)+i+1}$ is conjugate by $\mathcal{I}_{n+i}$ to an element in $g\mathcal{I}_{n+(l-1)\ell(w)+i+1}$.

(a) is proved.

Let $g_0 \in mM_n\mathcal{I}_{n+(l-1)\ell(w)}$. By (a), we may construct inductively an element $z_i \in \mathcal{I}_{n+i}$ for $i \in \mathbb{N}$ such that $g_{i+1} := z_i^{-1} g_i z_i$ is contained in $mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+i}$.

The convergent product $z := z_1z_2\cdots$ is a well-defined element in $\mathcal{I}_n$ and $z^{-1}gz \in mM_{n+(l-1)\ell(w)}$.

(2) By part (1), there exists $h \in \mathcal{I}_{n+n'}$ such that $hmgh^{-1} \in mM_{n+(l-1)\ell(w)}$. We have $(\mathcal{I}_{n+n'}, M_{n+(l-1)\ell(w)}) \subset \mathcal{I}_{n+n'+(l-1)\ell(w)}$. Therefore $M_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+n'}$ is a subgroup of $\mathcal{I}$ and is stable under the conjugation action of $\mathcal{I}_{n+n'}$. Thus $hmgM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+n'}h^{-1} = mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+n'}$. The statement is proved. □

2.5. We say that $m \in M$ is P_v *strictly positive* if for any $n \in \mathbb{N}$, we have

$$mN_{v,n}m^{-1} \subset N_{v,n+1}, \text{ and } m^{-1}N^-_{v,n}m \subset N^-_{v,n+1}.$$

We denote by $H^{v^\sharp}(M)$ the subalgebra of $H(M)$ consisting of functions with support consisting of P_v strictly positive elements. Note that the limit of the support of $j_{v,n}(\delta_{Z_0})$ for v dominant regular, as n goes to infinite, is just Z_0 itself, but the support of $j_{v,n}(\delta_{Z_0})$ for each n contains of nonsplit regular semisimple elements. Thus the maps $\{j_{v,n}\}$ are not compatible with the natural maps $\bar{H}^v(M, M_n) \to \bar{H}^v(M, M_{n+1})$.

However, we have the following compatibility result for P_v strictly positive part.

Corollary 2.4. *Let $n \in \mathbb{N}$. Then the following diagram commutes*

$$\begin{array}{ccc} \bar{H}^{v^\sharp}(M, M_n) & \xrightarrow{j_{v,n}} & \bar{H}(G, \mathcal{I}_n) \\ \big\downarrow & & \big\downarrow \\ \bar{H}^{v^\sharp}(M, M_{n+1}) & \xrightarrow{j_{v,n+1}} & \bar{H}(G, \mathcal{I}_{n+1}). \end{array}$$

Proof. Let $m \in M$ be P_v strictly positive. Then $\delta_{M_nmM_n} \in H^{v^\sharp}(M, M_n) \subset H^{v^\sharp}(M, M_{n+1})$. By definition,

$$j_{v,n+1}(\delta_{M_nmM_n}) = \delta_v(m)^{-\frac{1}{2}}\frac{\mu_M(M_{n+1})}{\mu_G(\mathcal{I}_{n+1})}\delta_{\mathcal{I}_{n+1}M_nmM_n\mathcal{I}_{n+1}}.$$

Note that $\mathcal{I}_{n+1}M_n = M_n\mathcal{I}_{n+1}$ is a subgroup of $\mathcal{I}$. We have

$$\mathcal{I}_nm\mathcal{I}_n = \sqcup_{(i_1,i_2,i'_1,i'_2)} i_1i'_1\mathcal{I}_{n+1}M_nmM_n\mathcal{I}_{n+1}i'_2i_2,$$

where $\{(i_1, i_2, i'_1, i'_2)\} \subset N_n \times N_n \times N^-_n \times N^-_n$ is a finite subset. By Proposition 2.3 (2), for $i_1, i_2 \in N_n$ and $i'_1, i'_2 \in N^-_n$, we have

$$\delta_{i_1i'_1\mathcal{I}_{n+1}M_nmM_n\mathcal{I}_{n+1}i'_2i_2} \equiv \delta_{\mathcal{I}_{n+1}M_nmM_n\mathcal{I}_{n+1}} \mod [H, H].$$

Thus

$$j_{v,n}(\delta_{M_nmM_n}) \equiv \delta_v(m)^{-\frac{1}{2}}\frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)}\frac{\mu_G(\mathcal{I}_nm\mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1}M_nmM_n\mathcal{I}_{n+1})}\delta_{\mathcal{I}_{n+1}M_nmM_n\mathcal{I}_{n+1}} \mod [H, H].$$

It remains to show that $\frac{\mu_M(M_{n+1})}{\mu_G(\mathcal{I}_{n+1})} = \frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)}\frac{\mu_G(\mathcal{I}_nm\mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1}M_nmM_n\mathcal{I}_{n+1})}$.

Suppose that $m \in \mathcal{I}_M\dot{w}\mathcal{I}_M$ for some $w \in \tilde{W}(M)$. By [11, Lemma 4.6],

$$\frac{\mu_G(\mathcal{I}_nm\mathcal{I}_n)}{\mu_G(\mathcal{I}_n)} = \frac{\mu_G(\mathcal{I}_{n+1}m\mathcal{I}_{n+1})}{\mu_G(\mathcal{I}_{n+1})} = q^{\ell(w)}, \frac{\mu_M(M_nmM_n)}{\mu_M(M_n)} = \frac{\mu_M(M_{n+1}mM_{n+1})}{\mu_M(M_{n+1})} = q^{\ell_M(w)}.$$

Now we have

$$\begin{aligned}\frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)}\frac{\mu_G(\mathcal{I}_n m \mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1})} &= \frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)}\frac{\mu_G(\mathcal{I}_n m \mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1} m \mathcal{I}_{n+1})}\frac{\mu_G(\mathcal{I}_{n+1} m \mathcal{I}_{n+1})}{\mu_G(\mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1})} \\ &= \frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)}\frac{\mu_G(\mathcal{I}_n m \mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1} m \mathcal{I}_{n+1})}\frac{\mu_M(M_{n+1} m M_{n+1})}{\mu_M(M_n m M_n)} \\ &= \frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)}\frac{\mu_G(\mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1})}\frac{\mu_M(M_{n+1})}{\mu_M(M_n)} = \frac{\mu_M(M_{n+1})}{\mu_G(\mathcal{I}_{n+1})}.\end{aligned}$$

The statement is proved. □

Finally we show that the elements in $M_\nu(\nu)$ are P_ν quasi-positive.

Proposition 2.5. *Let $v \in V$ be a rational coweight and $M = M_v$. Let $w \in \tilde{W}(M)$. Then there exists a positive integer $i_{\nu,w}$ such that for any $m \in \mathcal{I}_M \dot{w} \mathcal{I}_M \cap M(v)$ and $n \geqslant i_{v,w}$, we have*

$$m^{i_{v,w}} N_{v,n} (m^{i_{v,w}})^{-1} \subset N_{v,n+1}, \quad (m^{i_{v,w}})^{-1} N^-_{v,n} m^{i_{v,w}} \subset N^-_{v,n+1}.$$

2.6. The proof relies on some remarkable properties of the Iwahori-Weyl group, which we recall here.

For $w, w' \in \tilde{W}$ and $s \in \tilde{\mathbb{S}}$, we write $w \xrightarrow{s} w'$ if $w' = sws$ and $\ell(w') \leqslant \ell(w)$. We write $w \to w'$ if there is a sequence $w = w_0, w_1, \cdots, w_n = w'$ of elements in $\tilde{W}$ such that for any $1 \leqslant k \leqslant n$, $w_{k-1} \xrightarrow{s_k} w_k$ for some $s_k \in \tilde{\mathbb{S}}$. We write $w \approx w'$ if $w \to w'$ and $w' \to w$. It is easy to see that if $w \to w'$ and $\ell(w) = \ell(w')$, then $w \approx w'$. We have that

(a) If $w \xrightarrow{s} w'$ and $\ell(w) = \ell(w')$, then for any $g \in \mathcal{I}\dot{w}\mathcal{I}$, there exists $g' \in \mathcal{I}\dot{s}\mathcal{I}$ such that $g' g (g')^{-1} \in \mathcal{I}\dot{w}'\mathcal{I}$.

(b) If $w \xrightarrow{s} w'$ and $\ell(w') < \ell(w)$, then for any $g \in \mathcal{I}\dot{w}\mathcal{I}$, there exists $g' \in \mathcal{I}\dot{s}\mathcal{I}$ such that $g' g (g')^{-1} \in \mathcal{I}\dot{w}'\mathcal{I} \sqcup \mathcal{I}\dot{s}\dot{w}\mathcal{I}$.

An element $w \in \tilde{W}$ is called *straight* if $\ell(w^n) = n\ell(w)$ for any $n \in \mathbb{N}$. A triple (x, K, u) is called a *standard triple* if $x \in \tilde{W}$ is straight, $K \subset \tilde{\mathbb{S}}$ with W_K finite, $x \in {}^K\tilde{W}$ and $\mathrm{Ad}(x)(K) = K$, and $u \in W_K$. By definition,

(c) For any $n \in \mathbb{N}$ and $g_1, \cdots, g_n \in \mathcal{I}\dot{u}\dot{x}\mathcal{I}$, we have $g_1 g_2 \cdots g_n \in (\mathcal{I} W_K \mathcal{I})(\mathcal{I}\dot{x}^n\mathcal{I})$.

It is proved in [12, Theorem A & Proposition 2.7] that

Theorem 2.6. *For any $w \in \tilde{W}$, there exists a standard triple (x, K, u) such that $ux \in \tilde{W}_{\min}$ and $w \to ux$. In this case, $\pi(w) = \pi(x)$.*

Following [9, §4.3], we write $w \xrightharpoonup{s} w'$ if either $w \xrightarrow{s} w'$ or $w' = sw$ and $\ell(w) > \ell(sws)$, and we write $w \rightharpoonup w'$ if there exists a sequence $w = w_0, w_1, \cdots, w_n = w'$ of elements in $\tilde{W}$ such that for any $1 \leqslant k \leqslant n$, $w_{k-1} \xrightharpoonup{s_k} w_k$ for some $s_k \in \tilde{\mathbb{S}}$. It is easy to see that if $w \in \tilde{W}_{\min}$ and $w \rightharpoonup w'$, then $w \approx w'$.

We show that

Lemma 2.7. *Let $w \in \tilde{W}$ and $g \in \mathcal{I}\dot{w}\mathcal{I}$. Then there exists a standard triple (x, K, u), a sequence $w = w_0, w_1, \cdots, w_n = ux$ of distinct elements in $\tilde{W}$ and a sequence $g = g_0, g_1, \cdots, g_n$ of elements in G such that*

(1) $ux \in \tilde{W}_{\min}$;

(2) for any $0 \leqslant k \leqslant n$, $g_k \in \mathcal{I}\dot{w}_k\mathcal{I}$;

(3) for any $1 \leqslant k \leqslant n$, there exists $s_k \in \tilde{\mathbb{S}}$ and $h_k \in \mathcal{I}\dot{s}_k\mathcal{I}$ such that $w_{k-1} \xrightarrow{s_k} w_k$ and $g_k = h_k g_{k-1} h_k^{-1}$.

Remark 2.8. By definition, if $w \rightharpoonup w'$, then $w' \in wW_a$ and $\ell(w') \leqslant \ell(w)$. In particular, the length of the sequence is at most $\sharp\{x \in W_a; \ell(x) \leqslant \ell(w)\}$.

Proof. We argue by induction on $\ell(w)$.

If $w \in \tilde{W}_{\min}$, by Theorem 2.6, there exists a standard triple (x, K, u) with $ux \in \tilde{W}_{\min}$ and a sequence $w = w_0, w_1, \cdots, w_n = ux$ of distinct elements in $\tilde{W}$ such that for any $1 \leqslant k \leqslant n$, $w_{k-1} \xrightarrow{s_k} w_k$ for some $s_k \in \tilde{\mathbb{S}}$. Since $w \in \tilde{W}_{\min}$, we have $\ell(w_k) = \ell(w)$ for all k. Now the statement follows from §2.6 (a).

If $w \notin \tilde{W}_{\min}$, then by Theorem 2.6, there exists a sequence $w = w_0, w_1, \cdots, w_n$ of distinct elements in $\tilde{W}$ such that $\ell(w) = \ell(w_n)$, for any $1 \leqslant k \leqslant n$, $w_{k-1} \xrightarrow{s_k} w_k$ for some $s_k \in \tilde{\mathbb{S}}$ and there exists $s \in \tilde{\mathbb{S}}$ with $sw_ns < w_n$. Then we have $\ell(w_k) = \ell(w)$ for all k. By §2.6 (a), for any $1 \leqslant k \leqslant n$, there exists $h_k \in \mathcal{I}\dot{s}_k\mathcal{I}$ such that $g_k = h_k g_{k-1} h_k^{-1}$. By §2.6 (b), there exists $h_{n+1} \in \mathcal{I}\dot{s}\mathcal{I}$ such that $h_{n+1} g_n h_{n+1}^{-1} \in \mathcal{I}\dot{w}_{n+1}\mathcal{I}$ with $w_{n+1} \in \{sw_n, sw_ns\}$. Now the statement follows from inductive hypothesis on w_{n+1}. ☐

2.7. **Proof of Proposition 2.5.** Let $N_0 = \sharp\{w' \in W_a(M); \ell_M(w') \leqslant \ell_M(w)\}$. By Lemma 2.7 and remark 2.8, there exists a standard triple (x, K, u) of $\tilde{W}(M)$ and an element $h \in \cup_{z \in W_a(M); \ell(z) \leqslant N_0} \mathcal{I}_M \dot{z} \mathcal{I}_M$ such that $ux \in \tilde{W}(M)_{\min}$, $w \rightharpoonup ux$ and $hmh^{-1} \in \mathcal{I}_M \dot{u}\dot{x} \mathcal{I}_M$.

Let $\mathfrak{i}$ be a positive integer with $\mathfrak{i}v \in X_*(Z)$. Then $x^{\mathfrak{i}} = t^{\mathfrak{i}v} \in \tilde{W}$ represents a central element in M. By §2.6 (c), for any $l \in \mathbb{N}$,

$$(hmh^{-1})^{l\mathfrak{i}} \in (\mathcal{I}_M W_K \mathcal{I}_M)(\mathcal{I}_M t^{l\mathfrak{i}v} \mathcal{I}_M).$$

Let $N_1 = \max_{K \subset \tilde{\mathbb{S}}(M); W_K \text{ is finite}} \sharp W_K$. Let $i_{v,w} = (2N_0 + N_1 + 1)\mathfrak{i}$. Then for any $\alpha \in \Phi_{v,+}$, $\langle i_{v,w} v, \alpha \rangle \geqslant 2N_0 + N_1 + 1$. Note that $m^{i_{v,w}} = h^{-1}(g_1 g_2) h$ with $h \in \cup_{w' \in \tilde{W}(M); \ell(w') \leqslant N_0} \mathcal{I}_M \dot{w}' \mathcal{I}_M$, $g_1 \in \cup_{u' \in \tilde{W}(M); \ell_M(u') \leqslant N_1} \mathcal{I}_M \dot{u}' \mathcal{I}_M$ and $g_2 \in \mathcal{I}_M t^{i_{v,w} v} \mathcal{I}_M$. So

$$\begin{aligned} m^{i_{v,w}} N_{v,n} (m^{i_{v,w}})^{-1} &= h^{-1} g_1 g_2 h N_{v,n} h^{-1} g_2^{-1} g_1^{-1} h \\ &\subset h^{-1} g_1 g_2 N_{v,n-N_0} g_2^{-1} g_1^{-1} h \\ &\subset h^{-1} g_1 N_{v,n-N_0+(2N_0+N_1+1)} g_1^{-1} h \\ &\subset h^{-1} N_{v,n-N_0+(2N_0+N_1+1)-N_1} h \\ &\subset N_{v,v,n-N_0+(2N_0+N_1+1)-N_1-N_0} = N_{v,n+1}. \end{aligned}$$

Similarly, $m^{-i_{v,w}} N_{v,n}^- m^{i_{v,w}} \subset N_{v,n+1}^-$.

3. The map $\bar{i}_\nu$

We define the induction map $\bar{i}_\nu$, which is the main object in this paper.

Theorem 3.1. *Let M be a semistandard Levi subgroup of G and $\nu \in \aleph_M$ with $M = M_\nu$. Then*

(1) For $m \in M$ and an open compact subgroup $\mathcal{K}_M$ of $\mathcal{I}_M$ with $m\mathcal{K}_M \subset M(\nu)$, the map

$$\delta_{m\mathcal{K}_M} \longmapsto \delta_\nu(m)^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M \mathcal{K})} \delta_{m\mathcal{K}_M\mathcal{K}} + [H, H]$$

from $H(M;\nu)$ to $\bar{H}(\bar{\nu})$ is independent the choice of sufficiently small open compact subgroup $\mathcal{K}$ of G

(2) The map $i_\nu : H(M;\nu) \to \bar{H}$ defined above induces a map

$$\bar{i}_\nu : \bar{H}(M;\nu) \longrightarrow \bar{H}.$$

Remark 3.2. Unlike the map $j_{v,n}$, the map $\bar{i}_\nu$ does not send $\bar{H}(M, M_n;\nu)$ to $\bar{H}(G, \mathcal{I}_n;\bar{\nu})$. One needs to replace $\mathcal{I}_n$ by a smaller open compact subgroup of G. However, by the Iwahori-Matsumoto presentation of $\bar{H}(M, M_n;\nu)$ ([11, Theorem 4.1]) and Proposition 2.5, there exists a positive integer n' (depending on ν) such that $\bar{i}_\nu : \bar{H}(M, M_n;\nu) \to \bar{H}(G, \mathcal{I}_{n+n'};\bar{\nu})$ for any $n \in \mathbb{N}$.

Proof. (1) Let v be the V-factor of ν. Let $w \in \tilde{W}(M)$ with $m \in \mathcal{I}_M \dot{w} \mathcal{I}_M$. Let $i_{v,w}$ be an positive integer in Proposition 2.5. Let l be a multiple of $i_{v,w}\ell(w)$ with $M_l \subset \mathcal{K}_M$. By Proposition 2.3 (2), for any $n \in \mathbb{N}$ and $g \in \mathcal{I}_l$, we have

$$\delta_{m'gM_l\mathcal{I}_{l+n}} \equiv \delta_{m'M_l\mathcal{I}_{l+n}} \mod [H, H].$$

Let $\mathcal{K}, \mathcal{K}'$ be open compact subgroups of G with $\mathcal{K}, \mathcal{K}' \subset \mathcal{I}_l$. Let $n \in \mathbb{N}$ with $\mathcal{I}_{l+n} \subset \mathcal{K}, \mathcal{K}'$. Now we have

$$\begin{aligned}\delta_{m\mathcal{K}_M\mathcal{K}} &= \sum_{m' \in m\mathcal{K}_M/M_l} \delta_{m'M_l\mathcal{K}} \equiv \sum_{m' \in m\mathcal{K}_M/M_l} \frac{\mu_G(M_l\mathcal{K})}{\mu_G(M_l\mathcal{I}_{l+n})}\delta_{m'M_l\mathcal{I}_{l+n}} \\ &= \frac{\mu_G(M_l\mathcal{K})}{\mu_G(M_l\mathcal{I}_{l+n})}\delta_{m\mathcal{K}_M\mathcal{I}_{l+n}} \mod [H, H].\end{aligned}$$

As $\mathcal{K}_M$ is stable under the right multiplication of M_l, we have $\mu_G(\mathcal{K}_M\mathcal{I}_{l+n}) = \sharp(\mathcal{K}_M/M_l)\mu_G(M_l\mathcal{I}_{l+n})$ and $\frac{\mu_G(M_l\mathcal{K})}{\mu_G(M_l\mathcal{I}_{l+n})} = \frac{\mu_G(\mathcal{K}_M\mathcal{K})}{\mu_G(\mathcal{K}_M\mathcal{I}_{l+n})}$. Thus for any $n \in \mathbb{N}$, we have

$$\frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K})}\delta_{m\mathcal{K}_M\mathcal{K}} \equiv \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{I}_{l+n})}\delta_{m\mathcal{K}_M\mathcal{I}_{l+n}} \mod [H, H].$$

Similarly, $\frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K}')}\delta_{m\mathcal{K}_M\mathcal{K}'} \equiv \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{I}_{l+n})}\delta_{m\mathcal{K}_M\mathcal{I}_{l+n}} \mod [H, H]$. Part (1) is proved.

(2) By [11, §3.3 (2)], $[H(M), H(M)] = \oplus_{\nu \in \aleph_M}([H(M), H(M)] \cap H(M)_\nu)$, the kernel of the map $H(M)_\nu \to \bar{H}(M)_\nu$ is spanned by $\delta_{m\mathcal{K}_M} - {}^h\delta_{m\mathcal{K}_M}$ for $h, m \in M$ and open compact subgroup $\mathcal{K}_M$ of $\mathcal{I}_M$ such that $m\mathcal{K}_m \subset M_\nu$. It remains to prove that $i_\nu(\delta_{m\mathcal{K}_M}) = i_\nu({}^h\delta_{m\mathcal{K}_M})$.

Set $m' = hmh^{-1}$ and $\mathcal{K}'_M = h\mathcal{K}_M h^{-1}$. By part (1), there exists a sufficiently small open compact subgroup $\mathcal{K}$ of G such that

$$i_\nu(\delta_{m\mathcal{K}_M}) \equiv \delta_\nu(m)^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K})}\delta_{m\mathcal{K}_M\mathcal{K}} \mod [H, H],$$

$$i_\nu(\delta_{m'\mathcal{K}'_M}) \equiv \delta_\nu(m')^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}'_M)}{\mu_G(\mathcal{K}'_M\mathcal{K}')}\delta_{m'\mathcal{K}'_M\mathcal{K}'} \mod [H, H].$$

Here $\mathcal{K}' = h\mathcal{K}h^{-1}$.

We have $\delta_{m'\mathcal{K}'_M\mathcal{K}'} = \delta_{h(m\mathcal{K}_M\mathcal{K})h^{-1}} \equiv \delta_{m\mathcal{K}_M\mathcal{K}} \mod [H, H]$. Part (2) is proved. □

3.1. In the rest of this section, we show that the maps $\bar{i}_*$ are compatible with conjugating the Levi subgroups.

For any semistandard Levi subgroup M, we have a natural projection

$$X_*(Z)_{\mathrm{Gal}(\bar{F}/F)}/\mathbb{Z}\Phi_M^\vee \cong \Omega_M$$

and a natural map $V \mapsto V_+^M$. The natural action of W_0 on $X_*(Z)_{\mathrm{Gal}(\bar F/F)} \times V$ induces the following commutative diagram for any $w \in W_0$

$$\begin{array}{ccc} X_*(Z)_{\mathrm{Gal}(\bar F/F)} \times V & \xrightarrow{w\cdot} & X_*(Z)_{\mathrm{Gal}(\bar F/F)} \times V \\ \downarrow & & \downarrow \\ \aleph_M & \longrightarrow & \aleph_{\dot w M \dot w^{-1}}. \end{array}$$

We denote the induced map $\aleph_M \to \aleph_{\dot w M \dot w^{-1}}$ still by $w\cdot$. If moreover, $w \in W^M$, i.e. w sends the positive roots of M to the positive roots of $\dot w M \dot w^{-1}$, then we have $\dot w \mathcal{I}_M \dot w^{-1} = \mathcal{I}_{\dot w M \dot w^{-1}}$. By definition, the M-fundamental alcove is the unique M-alcove that contains the G-fundamental alcove. Since the conjugation by $\dot w$ sends the Iwahori-subgroup of M to the Iwahori-subgroup of $\dot w M \dot w^{-1}$, it also sends the M-fundamental alcove to the $\dot w M \dot w^{-1}$-fundamental alcove, and thus induces a length-preserving map from $\tilde W(M)$ to $\tilde W(\dot w M \dot w^{-1})$. In particular, the conjugation by w sends the minimal length elements of $\tilde W(M)$ (with respect to ℓ_M) to the minimal length elements of $\tilde W(\dot w M \dot w^{-1})$ (with respect to $\ell_{\dot w M \dot w^{-1}}$). Therefore, by the definition of Newton strata, we have that

(a) Let M be a semistandard Levi subgroup M and $\nu \in \aleph_M$. Let $w \in W_0$ and $M' = \dot w M \dot w^{-1}$, then

$$\dot w M(\nu) \dot w^{-1} = M'(w(\nu)).$$

Proposition 3.3. *Let M be a semistandard Levi subgroup and $\nu \in \aleph_M$ and $w \in W_0$. Then for any $m \in M$, and an open compact subgroup $\mathcal{K}_M$ of $\mathcal{I}_M$ with $m\mathcal{K}_M \subset M_\nu$ and $\dot w \mathcal{K}_M \dot w^{-1} \subset \mathcal{I}_{\dot w M \dot w^{-1}}$, we have*

$$i_\nu(\delta_{m\mathcal{K}_M}) = i_{w(\nu)}(\delta_{\dot w m \mathcal{K}_M \dot w^{-1}}) \in \bar H.$$

Proof. The proof is similar to the proof of Theorem 3.1 (2).

Set $M' = \dot w M \dot w^{-1}$, $m' = \dot w m \dot w^{-1}$ and $\mathcal{K}_{M'} = \dot w \mathcal{K}_M \dot w^{-1}$. By Theorem 3.1 (1), there exists a sufficiently small open compact subgroup $\mathcal{K}$ of G such that

$$i_\nu(\delta_{m\mathcal{K}_M}) \equiv \delta_\nu(m)^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K})} \delta_{m\mathcal{K}_M\mathcal{K}} \mod [H,H],$$

$$i_{w(\nu)}(\delta_{m'\mathcal{K}_{M'}}) \equiv \delta_{w(\nu)}(m')^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}_{M'})}{\mu_G(\mathcal{K}_{M'}\mathcal{K}')} \delta_{m'\mathcal{K}_{M'}\mathcal{K}'} \mod [H,H].$$

Here $\mathcal{K}' = \dot w \mathcal{K} \dot w^{-1}$.

We have $\delta_{m'\mathcal{K}'_M\mathcal{K}'} = \delta_{\dot w (m\mathcal{K}_M\mathcal{K}) \dot w^{-1}} \equiv \delta_{m\mathcal{K}_M\mathcal{K}} \mod [H,H]$. The statement is proved. □

Corollary 3.4. *Let M be a semistandard Levi subgroup of G and $\nu \in \aleph_M$ with $M = M_\nu$. Then for any $w \in W_0$,*

$$\mathrm{Im}(\bar i_\nu : \bar H(M;\nu) \longrightarrow \bar H) = \mathrm{Im}(\bar i_{w(\nu)} : \bar H(\dot w M \dot w^{-1}; w(\nu)) \longrightarrow \bar H).$$

4. The image of the map $\bar i_\nu$

The main result of this section is

Theorem 4.1. *Let M be a semistandard Levi subgroup and $\nu \in \aleph_M$ with $M = M_\nu$. Then the image of the the map $\bar i_\nu : \bar H(M;\nu) \to \bar H$ equals $\bar H(\bar\nu)$.*

We first compare the Newton strata of G and its Levi subgroups.

Proposition 4.2. *Let M be a semistandard Levi subgroup and $\nu \in \aleph_M$ with $M_\nu = M$. Then we have $M(\nu) \subset G(\bar{\nu})$.*

Proof. The idea is similar to the proof of [11, Theorem 2.1].

By §3.1 (a), after conjugating by a suitable element in W_0, we may assume that M is a standard Levi subgroup. Since $M = M_\nu$, the V-factor of ν is G-dominant. By the Newton decomposition of G ([11, Theorem 2.1]), it suffices to prove that $M(\nu) \cap G(\nu') = \emptyset$ for any $\nu' \in \aleph$ with $\nu' \neq \bar{\nu}$.

Let $\nu = (\tau, v)$ and $\nu' = (\tau', v')$. If the image of τ in Ω does not equal to τ', then $M(\nu) \cap G(\nu') = \emptyset$. Now we assume that the Ω-factor matches. Since $\nu' \neq \bar{\nu}$, we have $v' \neq v$.

By [11, Remark 2.6],

$$M(\nu) = \cup_{(x,K,u)} M \cdot \mathcal{I}_M \dot{u}\dot{x}\mathcal{I}_M, \quad G(\nu') = \cup_{(x',K',u')} G \cdot \mathcal{I}\dot{u}'\dot{x}'\mathcal{I},$$

where (x, K, u) runs over standard triples of $\tilde{W}(M)$ such that $ux \in \tilde{W}(M)_{\min}$ and $\pi_M(x) = \nu$, (x', K', u') runs over standard triples of $\tilde{W}$ such that $u'x' \in \tilde{W}_{\min}$ and $\pi(x') = \nu'$.

If $M(\nu) \cap G(\nu') \neq \emptyset$, then there exists standard triples (x, K, u) and (x', K', u') as above and $h \in \mathcal{I}_M \dot{u}\dot{x}\mathcal{I}_M, h' \in \mathcal{I}\dot{u}'\dot{x}'\mathcal{I}, g \in G$ such that $ghg^{-1} = h'$. For any $n \in \mathbb{N}$, we have $gh^n g^{-1} = (h')^n$. By §2.6 (c), we have

$$h^n \in (\mathcal{I}_M W_K \mathcal{I}_M)(\mathcal{I}_M \dot{x}^n \mathcal{I}_M), \quad (h')^n \in (\mathcal{I} W_{K'} \mathcal{I})(\mathcal{I}(\dot{x}')^n \mathcal{I}).$$

Let $l > 0$ with $lv, lv' \in X_*(Z)$. Suppose that $g \in \mathcal{I}\dot{z}\mathcal{I}$ for some $z \in \tilde{W}$. Then for any $n \in \mathbb{N}$, we have

$$\mathcal{I}\dot{z}\mathcal{I}t^{nlv}\mathcal{I}(\mathcal{I}W_K\mathcal{I})\mathcal{I}\dot{z}^{-1}\mathcal{I}(\mathcal{I}\dot{W}_{K'}\mathcal{I}) \cap \mathcal{I}t^{nlv'}\mathcal{I} \neq \emptyset.$$

Similar to the argument in [11, §2.6], this is impossible for $n \gg 0$. The statement is proved. □

Corollary 4.3. *The image of the map $\bar{i}_\nu$ is contained in $\bar{H}(\bar{\nu})$.*

Proof. Let $m \in M$ and $\mathcal{K}_M$ be an open compact subgroup of $\mathcal{I}_M$ with $m\mathcal{K}_M \subset M(\nu)$. By Proposition 4.2, $m\mathcal{K}_M \subset G(\bar{\nu})$. Let X be an open compact subset of G with $m\mathcal{K}_M \subset X$. By [11, Theorem 3.2], there exists $n \in \mathbb{N}$ such that $X \cap G(\bar{\nu})$ is stable under the right multiplication by $\mathcal{I}_n$. In particular, $m\mathcal{K}_M\mathcal{I}_n \subset G(\bar{\nu})$. Thus $\bar{i}_\nu(\delta_{m\mathcal{K}_M}) \in \bar{H}(\bar{\nu})$. □

4.1. In order to prove the other direction, we use the notion of alcove elements in [8] and [9].

Let $w \in \tilde{W}$. We may regard $w \in \text{Aff}(V)$ as an affine transformation. Let $p : \text{Aff}(V) = V \rtimes GL(V) \to GL(V)$ be the natural projection map. Let $v \in V$. We say that w is a v-alcove element if

- $p(w)(v) = v$;
- $N_v \cap \dot{w}\mathcal{I}\dot{w}^{-1} \subset N_v \cap \mathcal{I}$.

Note that the first condition implies that $\dot{w}M_v\dot{w}^{-1} = M_v$. We have the following result.

Theorem 4.4. *Let $w \in \tilde{W}$. If w is a ν_w-alcove element, then any element in $\mathcal{I}\dot{w}\mathcal{I}$ is conjugate by $\mathcal{I}$ to an element in $\dot{w}\mathcal{I}_{M_{\nu_w}}$.*

Proof. The basic idea is similar to the proof of [8, Theorem 2.1.2].

Write M for M_{ν_w} and N for N_{ν_w}. We start with the generic Moy-Prasad filtration $\mathcal{I} = \mathcal{I}[0] \supset \mathcal{I}[1] \supset \cdots$. As explained in [8, §6.2], it is a filtration satisfying the following conditions:

(1) Each $\mathcal{I}[r]$ is normal in $\mathcal{I}$;

(2) For each r, either $\mathcal{I}[r] \subset \mathcal{I}_M \mathcal{I}[r+1]$ or there exists a root $a \in \Phi - \Phi(M)$ and $s \in \mathrm{R}$ such that $\mathcal{I}[r] = X_{a+s}\mathcal{I}[r+1]$ and $X_{a+s+\epsilon} \subset \mathcal{I}[r+1]$ for any $\epsilon > 0$.

We show that each element $\dot{w} i_M i[r]$ with $i_M \in \mathcal{I}_M$ and $i[r] \in I[r]$ is conjugate by an element in $\mathcal{I}$ to an element in $\dot{w}\mathcal{I}_M \mathcal{I}[r+1]$ (and that the conjugator can be taken to be small when r is large).

If $\mathcal{I}[r] \subset \mathcal{I}_M \mathcal{I}[r+1]$, then we may absorb the $\mathcal{I}_M$ part into i_M. Otherwise, there exists a root a outside M such that $\mathcal{I}[r] = X_{a+s}\mathcal{I}[r+1]$ and $X_{a+s+\epsilon} \subset \mathcal{I}[r+1]$ for any $\epsilon > 0$. We prove the case where a is a root in N. The case where a is a root in N^- can be proved in the same way.

We have $i[r] \in u\mathcal{I}[r+1]$ for some $u \in X_{a+s} \subset N_s$. Set $m = \dot{w} i_M$. By the definition of P-alcove elements, $m^i u (m^i)^{-1} \subset N_s$ for all $i \in \mathbb{N}$. As in the proof of Proposition 2.3 (1), $\dot{w} i_M i[r]$ is conjugate by elements in N_s to elements in

$$\begin{aligned} mu\mathcal{I}[r+1] &= (mum^{-1}) m\mathcal{I}[r+1] \sim m\mathcal{I}[r+1](mum^{-1}) = m(mum^{-1})\mathcal{I}[r+1] \\ &= (m^2 u (m^2)^{-1}) m\mathcal{I}[r+1] \sim \cdots \sim (m^i u (m^i)^{-1}) m\mathcal{I}[r+1] \sim \cdots . \end{aligned}$$

Here $\sim$ means conjugation by elements in N_s.

By Proposition 2.5, there exist $i \in \mathbb{N}$ such that $m^i u (m^i)^{-1} \subset \mathcal{I}[r+1]$. Thus $\dot{w} i_M i[r]$ is conjugate by an element in N_s to an element in $\dot{w}\mathcal{I}_M \mathcal{I}[r+1]$.

Now we start with an element in $\dot{w}\mathcal{I}$. The convergent product of the conjugators (for all r) is an element in $\mathcal{I}$ and conjugates the given element to an element in $\dot{w}\mathcal{I}_M$. □

4.2. **Proof of Theorem 4.1.** By Corollary 4.3, the image of $\bar{i}_\nu$ is contained in $\bar{H}(\bar{\nu})$. Now we prove the other direction. By [11, Corollary 4.2],

$$\bar{H} = \sum_{w \in \tilde{W}_{\min}; \pi(w) = \bar{\nu}} \bar{H}_w,$$

where H_w is the submodule of H consisting of functions supported in $\mathcal{I}\dot{w}\mathcal{I}$ and $\bar{H}_w$ is the image of H_w in $\bar{H}$.

Let $w \in \tilde{W}_{\min}$ with $\pi(w) = \bar{\nu}$. By [9, Lemma 4.4.3 and Proposition 4.4.6], w is a ν_w-alcove element. Set $M' = M_{\nu_w}$ and $\nu' = \pi_{M'}(w) \in \aleph_{M'}$.

Let $i_{\nu', w}$ be a positive integer in Proposition 2.5. By definition, H_w is spanned by $\delta_{g\mathcal{I}_n}$ for $g \in \mathcal{I}\dot{w}\mathcal{I}$ and $n > i(\nu', w)\ell(w)$. By the proof of Theorem 3.1 (1), for any $n > i(\nu', w)\ell(w)$ and $g \in \dot{w}\mathcal{I}_{M'}$, $\delta_{g\mathcal{I}_n} + [H, H]$ is contained in the image of $\bar{i}_{\nu'}$.

Let $g \in \mathcal{I}\dot{w}\mathcal{I}$. By Theorem 4.4, there exists $i \in \mathcal{I}$ and $g' \in \dot{w}\mathcal{I}_{M'}$ such that $g = i g' i^{-1}$. Then

$$\delta_{g\mathcal{I}_n} = \delta_{i g' \mathcal{I}_n i^{-1}} \equiv \delta_{g'\mathcal{I}_n} \mod [H, H].$$

Therefore $\bar{H}_w$ is contained in the image of $\bar{i}_{\nu'}$. By Proposition 3.3, $\bar{H}_w$ is also contained in the image of $\bar{i}_\nu$.

5. Adjunction with the Jacquet functor

5.1. Let R be an algebraically closed field of characteristic $\neq p$. Set $H_R = H \otimes_{\mathbb{Z}[\frac{1}{p}]} R$, $\bar{H}_R = \bar{H} \otimes_{\mathbb{Z}[\frac{1}{p}]} R$ and $\bar{H}_R(\nu) = \bar{H}(\nu) \otimes_{\mathbb{Z}[\frac{1}{p}]} R$. Recall that $\mathfrak{R}(G)_R$ is the R-vector space with basis the isomorphism classes of irreducible smooth admissible representations of G over R. We consider the trace map

$$\mathrm{Tr}_R^G : \bar{H}_R \longrightarrow \mathfrak{R}(G)_R^*.$$

Similarly, for any semistandard Levi subgroup M, we have

$$\mathrm{Tr}_R^M : \bar{H}_R(M) \longrightarrow \mathfrak{R}(M)_R^*.$$

Let $v \in V$ and $M = M_v$. Let $r_{v,R} : \mathfrak{R}(G)_R \to \mathfrak{R}(M)_R$ be the (normalized) Jacquet functor. Note that the Jacquet functor does not only depend on the Levi M, but also depends on the direction v (or equivalently, the parabolic subgroup P_v with Levi factor M). The following result is proved by Bushnell in [2, Corollary 1].

Proposition 5.1. *Let $n \in \mathbb{N}$. Let $v \in V$ and $M = M_v$. Then for any $f \in H_R^v(M, M_n)$, and $\pi \in \mathfrak{R}_{\mathcal{I}_n}(G)_R$, we have*

$$\mathrm{Tr}_R^M(f, r_{v,R}(\pi)) = \mathrm{Tr}_R^G(j_{v,n}(f), \pi).$$

The main result of this section is the following adjunction formula.

Theorem 5.2. *Let M be a semistandard Levi subgroup and $\nu \in \aleph_M$. Suppose that $M = M_\nu$. Then for any $f \in \bar{H}_R(M; \nu)$ and $\pi \in \mathfrak{R}(G)_R$, we have*

$$\mathrm{Tr}_R^M(f, r_{\nu,R}(\pi)) = \mathrm{Tr}_R^G(\bar{i}_\nu(f), \pi).$$

5.2. Let (x, K, u) be a standard triple of $\tilde{W}(M)$ such that the Newton point of x is v. Let $\mathfrak{i}$ be the smallest positive integer with $\mathfrak{i}v \in X_*(Z)$. Let $i \in \mathbb{N}$ such that for any $\alpha \in \Phi_{v,+}$, $\langle i\mathfrak{i}v, \alpha\rangle \geqslant \sharp W_K + (\mathfrak{i}-1)\ell(x) + 1$. Let $l \geqslant i\mathfrak{i}$. Then $l = i'\mathfrak{i} + j$ for some $i' \geqslant i$ and $0 \leqslant j < \mathfrak{i}$. Then for any $m_1, \cdots, m_l \in \mathcal{I}_M \dot{u}\dot{x}\mathcal{I}_M$, by §2.6 (c), we have

$$m_1 m_2 \cdots m_l \in (\mathcal{I}_M W_K \mathcal{I}_M)(\mathcal{I}_M \dot{x}^j \mathcal{I}_M)(\mathcal{I}_M t^{i'\mathfrak{i}v} \mathcal{I}_M).$$

Note that for $g \in \mathcal{I} t^{i'\mathfrak{i}v} \mathcal{I}$, $g N_n g^{-1} \subset N_{n+\sharp W_K + (\mathfrak{i}-1)\ell(x)+1}$. Also $(\mathcal{I} W_K \mathcal{I})(\mathcal{I}\dot{x}^j\mathcal{I}) \subset \cup_{w \in \tilde{W}; \ell(w) \leqslant \sharp W_K + (\mathfrak{i}-1)\ell(x)} \mathcal{I}\dot{w}\mathcal{I}$. Thus $(m_1 \cdots m_l) N_n (m_1 \cdots m_l)^{-1} \subset N_{n+1}$. Similarly $(m_1 \cdots m_l)^{-1} N_n^- (m_1 \cdots m_l) \subset N_{n+1}^-$. Therefore,

(a) Let $l \geqslant i\mathfrak{i}$ and $m_1, \cdots, m_l \in \mathcal{I}_M \dot{u}\dot{x}\mathcal{I}_M$, then $m_1 m_2 \cdots m_l$ is a P_v strictly positive element.

Moreover, for any $n, l' \in \mathbb{N}$ and $m_1, \cdots, m_{l'} \in \mathcal{I}_M \dot{u}\dot{x}\mathcal{I}_M$, we have

$$(m_1 \cdots m_{l'}) N_{n+\sharp W_K + (\mathfrak{i}-1)\ell(x)} (m_1 \cdots m_{l'})^{-1} \subset N_n,$$
$$(m_1 \cdots m_{l'})^{-1} N^-_{n+\sharp W_K + (\mathfrak{i}-1)\ell(x)} (m_1 \cdots m_{l'}) \subset N_n^-.$$

One deduces that

(b) Let $n, l' \in \mathbb{N}$, and $g_1, \cdots g_{l'} \in N_{n+\sharp W_K + (\mathfrak{i}-1)\ell(x)} \mathcal{I}_M \dot{u}\dot{x} \mathcal{I}_M N^-_{n+\sharp W_K + (\mathfrak{i}-1)\ell(x)}$. Then $g_1 \cdots g_{l'} \in N_n M N_n^-$.

5.3. **Proof of Theorem 5.2.** By [11, Theorem 4.1 & §4.6], it suffices to prove it for locally constant functions on M, supported in $M\dot{u}\dot{x}M$, where (x, K, u) is a standard triple of $\tilde{W}(M)$ and the Newton point of x is v.

Let $n > \sharp W_K + (\mathfrak{i}-1)\ell(x)$ such that $\pi \in \mathfrak{R}_{\mathcal{I}_n}(G)_R$. It is enough to consider the function $f = \delta_{M_n m M_n}$, where $m \in M\dot{u}\dot{x}M$.

Let $n' \gg n$ and $\tilde{f} = \frac{\delta_v(m)^{-\frac{1}{2}}}{\mu_N(N_{n'})\mu_{N^-}(N^-_{n'})}\delta_{N_{n'}M_n m M_n N^-_{n'}}$. By Theorem 3.1 (1), $\tilde{f}$ represents the element $\bar{i}_v(f) \in \bar{H}$. By Casselman's trick [4, Corollary 4.2], it suffices to prove that for $l \gg 0$, $\mathrm{Tr}^M_R(f^l, r_v(\pi)) = \mathrm{Tr}^G_R(\tilde{f}^l, \pi)$.

Let $p_M : (M_n m M_n)^l \to M$ and $p_G : (N_{n'}M_n m M_n N^-_{n'})^l \to G$ be the multiplication map. Since $l \gg 0$, by §5.2 (a) and (b), any element in $\mathrm{Im}(p_M)$ is P_ν strictly positive and

$$\mathrm{Im}(p_G) \subset N_n \,\mathrm{Im}(p_M) N^-_n \cong N \times \mathrm{Im}(p_M) \times N^-.$$

We have the following commutative diagram

$$\begin{array}{ccc} (N_{n'}M_n m M_n N^-_{n'})^l & \xrightarrow{p_G} & \mathrm{Im}(p_G) \\ \Big\downarrow{\scriptstyle pr^l} & & \Big\downarrow{\scriptstyle pr_1} \\ (M_n m M_n)^l & \xrightarrow{p_M} & \mathrm{Im}(p_M), \end{array}$$

where $pr : N \times M \times N^- \to M$ is the projection map and pr_1 is the restriction of pr to $\mathrm{Im}(p_G)$.

Let $m' \in \mathrm{Im}(p_M)$. Then

$$\begin{aligned} \mu_{G^l}(p_G^{-1} pr_1^{-1}(M_n m' M_n)) &= \mu_{G^l}((pr^l)^{-1} p_M^{-1}(M_n m' M_n)) \\ &= \mu_N(N_{n'})^l \mu_{N^-}(N^-_{n'})^l \mu_{M^l}(p_M^{-1}(M_n m' M_n)). \end{aligned}$$

By Proposition 2.3 (2), $\delta_{i\mathcal{I}_{n'}M_n m' M_n \mathcal{I}_{n'} i'} \equiv \delta_{\mathcal{I}_{n'}M_n m' M_n \mathcal{I}_{n'}} \mod [H, H]$ for any $i \in N_n$ and $i' \in N^-_n$. Thus

$$\begin{aligned} \tilde{f}^l &\equiv \frac{\delta_v(m)^{-\frac{l}{2}}}{\mu_N(N_{n'})^l \mu_{N^-}(N^-_{n'})^l} \sum_{m' \in M_n \backslash M / M_n} \frac{\mu_{G^l}(p_G^{-1} pr_1^{-1}(M_n m' M_n))}{\mu_G(pr_1^{-1}(M_n m M_n))} \delta_{pr_1^{-1}(M_n m M_n)} \\ &\equiv \sum_{m' \in M_n \backslash M / M_n} \delta_v(m)^{-\frac{l}{2}} \frac{\mu_{M^l}(p_M^{-1}(M_n m' M_n))}{\mu_G(pr_1^{-1}(M_n m M_n))} \delta_{pr_1^{-1}(M_n m M_n)} \\ &\equiv \sum_{m' \in M_n \backslash M / M_n} \delta_v(m)^{-\frac{l}{2}} \frac{\mu_{M^l}(p_M^{-1}(M_n m' M_n))}{\mu_G(\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'})} \delta_{\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'}} \mod [H, H]. \end{aligned}$$

On the other hand,

$$f^l = \sum_{m' \in M_n \backslash M / M_n} \frac{\mu_{M^l}(p_M^{-1}(M_n m' M_n))}{\mu_M(M_n m' M_n)} \delta_{M_n m' M_n}.$$

By Corollary 2.4, we have

$$\begin{aligned} &j_{v,n}(f^l) \equiv j_{v,n'}(f^l) \\ &\quad = \sum_{m' \in M_n \backslash M / M_n} \delta_v(m)^{-\frac{l}{2}} \frac{\mu_{M^l}(p_M^{-1}(M_n m' M_n))}{\mu_M(M_n m' M_n)} \frac{\mu_M(M_{n'})}{\mu_G(\mathcal{I}_{n'})} \delta_{\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'}} \mod [H, H]. \end{aligned}$$

Since the elements in $M_n m' M_n$ are P_v strictly positive, we have $\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'} = N_{n'}(M_n m' M_n) N^-_{n'}$ and

$$\mu_G(\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'}) = \mu_N(N_{n'})\mu_{N^-}(N^-_{n'})\mu_M(M_n m' M_n) = \frac{\mu_G(\mathcal{I}_{n'})}{\mu_M(M_{n'})}\mu_M(M_n m' M_n).$$

So $\tilde{f}^l \equiv j_{v,n}(f^l) \mod [H,H]$ and $\mathrm{Tr}^M_R(f^l, r_v(\pi)) = \mathrm{Tr}^G_R(\tilde{f}^l, \pi)$.

6. The kernel of the trace map

6.1. Let M be a semistandard Levi subgroup of G. Let M^0 be the subgroup of G generated by the parahoric subgroups of M. Then we have $M/M^0 \cong \Omega_M$. Let $\Psi(M)_R = \mathrm{Hom}_{\mathbb{Z}}(M/M^0, R^\times)$ be the torus of unramified characters of M.

Let $i_{M,R} : \mathfrak{R}(M)_R \to \mathfrak{R}(G)_R$ be the induction functor. Then for any $\sigma \in \mathfrak{R}(M)_R$ and $f \in \bar{H}_R$, the map

$$\Psi(M)_R \longrightarrow R, \qquad \chi \longmapsto \mathrm{Tr}_R(f, i_{M,R}(\sigma \circ \chi))$$

is an algebraic function over $\Psi(M)_R$.

6.2. Let $v \in V$ and $M = M_v$. Recall that

(a) $$\bar{H}(M; v) = \oplus_{\nu_M \in \aleph_M; \nu = (\tau_M, v) \text{ for some } \tau_M \in \Omega_M} \bar{H}(M; \nu),$$

(b) $$\bar{H}(\bar{v}) = \oplus_{\nu \in \aleph; \nu = (\tau, \bar{v}) \text{ for some } \tau \in \Omega} \bar{H}(\nu).$$

Note that if $\tau_M, \tau'_M \in \Omega_M$ are mapped under κ to the same element in Ω, then they differ by a central cocharacter of M. By the definition of the map $\pi = (\kappa, \bar{\nu})$, if both (τ_M, v) and (τ'_M, v) are in the image of π_M and that $\kappa(\tau_M) = \kappa(\tau'_M)$, then $\tau_M = \tau'_M$. In other words, there is a natural bijection between the components appear on the right hand sides of (a) and (b). We define

$$\bar{i}_v = \oplus_{\nu_M \in \aleph_M; \nu = (\tau_M, v) \text{ for some } \tau \in \Omega_M} \bar{i}_\nu : \bar{H}(M; v) \longmapsto \bar{H}(\bar{v}).$$

Theorem 6.1. *Let $v \in V$ and $M = M_v$. Let $f \in \bar{H}(\bar{v})$. If $\mathrm{Tr}^G_R(f, i_{M,R}(\sigma)) = 0$ for all $\sigma \in \mathfrak{R}(M)_R$, then $f \in \bar{i}_v(\ker \mathrm{Tr}^M_R)$.*

Proof. For $\sigma \in \mathfrak{R}(M)_R$ and $\chi \in \Psi(M)_R$, the map

$$\chi \longmapsto \mathrm{Tr}^G(\bar{i}_v(f), i_{M,R}(\sigma \circ \chi))$$

is an algebraic function on χ. We consider its "positive part", i.e. the linear combination of the terms $\langle \chi, \lambda \rangle$ for dominant coweight λ. It is obvious that if an algebraic function is zero, then its "positive part" is also zero.

By the Mackey formula [16, §5.5], we have

$$\begin{aligned} \mathrm{Tr}^G_R(\bar{i}_v(f), i_{M,R}(\sigma \circ \chi)) &= \mathrm{Tr}^M_R(f, r_{M,R} \circ i_{M,R}(\sigma \circ \chi)) \\ &= \sum_{w \in {}^M W^M} \mathrm{Tr}^M_R(f, i^M_{M \cap {}^w M, R} \circ \dot{w} \circ r^M_{M \cap {}^{w^{-1}} M, R}(\sigma \circ \chi) \\ &= \sum_{w \in {}^M W^M} \mathrm{Tr}^M_R(f, i^M_{M \cap {}^w M, R}(\dot{w} \circ r^M_{M \cap {}^{w^{-1}} M, R}(\sigma) \circ {}^{\dot{w}}\chi)). \end{aligned}$$

As $w \in {}^M W^M$ and $M = M_v$, $w(v)$ is dominant if and only if $w = 1$. Therefore the "positive part" of $\mathrm{Tr}^G(\bar{i}_v(f), i_{M,R}(\sigma \circ \chi))$ is $\mathrm{Tr}^M_R(f, \sigma \circ \chi)$.

Therefore if $\mathrm{Tr}^G_R(f, i_{M,R}(\sigma)) = 0$ for any $\sigma \in \mathfrak{R}(M)_R$ and $\chi \in \Psi(M)_R$, then $\mathrm{Tr}^M_R(f, \sigma \circ \chi) = 0$ for any $\sigma \in \mathfrak{R}(M)_R$ and $\chi \in \Psi(M)_R$. Hence $f \in \ker \mathrm{Tr}^M_R$. □

Corollary 6.2. *Let $v \in V$ and $M = M_v$. Then*

$$\bar{i}_v^{-1}(\ker \mathrm{Tr}_R^G |_{\bar{H}_R(\bar{v})}) = \ker \mathrm{Tr}_R^M |_{\bar{H}_R(M;v)} .$$

Proof. If $f \in \ker \mathrm{Tr}_R^M$, then $\mathrm{Tr}_R^M(f, r_{M,R}(\pi)) = 0$ for any $\pi \in \mathfrak{R}(G)_R$. By Theorem 5.2, $\mathrm{Tr}_R^G(\bar{i}_v(f), \pi) = 0$. Thus $\bar{i}_v(f) \in \ker \mathrm{Tr}_R^G$. The other direction follows from Theorem 6.1. □

Theorem 6.3. *We have* $\ker \mathrm{Tr}_R^G = \bigoplus_{v \in V_+}(\ker \mathrm{Tr}_R^G \cap \bar{H}_R(v))$.

Remark 6.4. In general, $\oplus_{\nu \in \aleph}(\ker \mathrm{Tr}_R^G \cap \bar{H}_R(\nu)) \subset \ker \mathrm{Tr}_R^G$. However, the equality may not hold. For example, if $\Omega = \{1, \tau\}$ is finite of order 2 and characteristic of R is also 2, then for any $\lambda \in X_*(Z)_+$ and $f \in \bar{H}(\lambda)$, we have $f + \tau f \in \ker \mathrm{Tr}_R^G$.

Proof. The idea is similar to the proof of [5, Theorem 7.1].

Let $f = \sum_{v \in V_+} a_v f_v \in \ker \mathrm{Tr}_R^G$, where $f_v \in \bar{H}_v$ and $a_v \in R$. Let M be a minimal standard Levi subgroup such that $a_v \neq 0$ for some $v \in V_+$ with $M = M_v$. Then for $\sigma \in R(M)$ and $\chi \in \Psi(M)_R$, we have

(a)

$$\mathrm{Tr}_R^G(f, i_{M,R}(\sigma \circ \chi)) = \sum_{v \in V_+; M = M_v} a_v \mathrm{Tr}_R^G(f_v, i_{M,R}(\sigma \circ \chi)) + \sum_{v \in V_+; M \neq M_v} a_v \mathrm{Tr}_R^G(f_v, i_{M,R}(\sigma \circ \chi)).$$

This is an algebraic function on $\Psi(M)_R$. Note that in (a), the first part is more regular in $\Psi(M)_R$ than the second part. Therefore we have

$$\sum_{v \in V_+; M = M_v} a_v \mathrm{Tr}_R^G(f_v, i_{M,R}(\sigma \circ \chi)) = 0$$

for all $\sigma \in R(M)$ and $\chi \in \Psi(M)_R$. As an algebraic function on $\Psi(M)_R$, the "leading term" of $\mathrm{Tr}_R^G(f_v, i_{M,R}(\sigma \circ \chi))$ is a multiple of $\langle v, \chi \rangle$. Hence $a_v \mathrm{Tr}_R^G(f_v, i_{M,R}(\sigma \circ \chi)) = 0$ for every $v \in V_+$ with $M = M_v$. By Theorem 6.1,

$$a_v f_v \in \bar{i}_v(\ker \mathrm{Tr}_R^M |_{\bar{H}(M;v)}). \quad \square$$

Finally, we have

Theorem 6.5. *Assume that char$(F) = 0$. Let M be a semistandard Levi subgroup and $\nu \in \aleph_M$ with $M = M_\nu$. Then the map*

$$\bar{i}_\nu : \bar{H}(M;\nu) \xrightarrow{\cong} \bar{H}(\bar{\nu})$$

is an isomorphism.

Proof. Let $f \in \ker \bar{i}_\nu$. Set $\tilde{f} = f \otimes 1 \in \bar{H}_{\mathbb{C}}(M;\nu)$. By Theorem 6.1 (2), we have $\tilde{f} \in \ker \mathrm{Tr}_{\mathbb{C}}^M$. By the spectral density theorem [14, Theorem 0], $\tilde{f} = 0 \in \bar{H}(M)_{\mathbb{C}}$. By [17], $\bar{H}(M)$ is free. Hence $f = 0 \in \bar{H}(M)$. □

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