TOTAL POSITIVITY IN THE DE CONCINI-PROCESI COMPACTIFICATION

Xuhua He

ABSTRACT. We study the nonnegative part $\overline{G_{>0}}$ of the De Concini-Procesi compactification of a semisimple algebraic group G, as defined by Lusztig. Using positivity properties of the canonical basis and parametrization of flag varieties, we will give an explicit description of $\overline{G_{>0}}$. This answers the question of Lusztig in [L4]. We will also prove that $\overline{G_{>0}}$ has a cell decomposition which was conjectured by Lusztig.

0. Introduction

Let G be a connected split semisimple algebraic group of adjoint type over $\mathbf{R}$. We identify G with the group of its $\mathbf{R}$-points. In [DP], De Concini and Procesi defined a compactification $\bar{G}$ of G and decomposed it into strata indexed by the subsets of a finite set I. We will denote these strata by $\{Z_J \mid J \subset I\}$. Let $G_{>0}$ be the set of strictly totally positive elements of G and $G_{\geqslant 0}$ be the set of totally positive elements of G (see [L1]). We denote by $\overline{G_{>0}}$ the closure of $G_{>0}$ in $\bar{G}$. The main goal of this paper is to give an explicit description of $\overline{G_{>0}}$ (see 3.14). This answers the question in [L4, 9.4]. As a consequence, I will prove in 3.17 that $\overline{G_{>0}}$ has a cell decomposition which was conjectured by Lusztig.

To achieve our goal, it is enough to understand the intersection of $\overline{G_{>0}}$ with each stratum. We set $Z_{J,\geqslant 0} = \overline{G_{>0}} \bigcap Z_J$. Note that $Z_I = G$ and $Z_{I,\geqslant 0} = G_{\geqslant 0}$. We define $Z_{J,>0}$ as a certain subset of $Z_{J,\geqslant 0}$ analogous to $G_{>0}$ for $G_{\geqslant 0}$ (see 2.6). When G is simply-laced, we will prove in 2.7 a criterion for $Z_{J,>0}$ in terms of its image in certain representations of G, which is analogous to the criterion for $G_{>0}$ in [L4, 5.4]. As Lusztig pointed out in [L2], although the definition of total positivity was elementary, many of the properties were proved in a non-elementary way, using canonical bases and their positivity properties. Our theorem 2.7 is an example of this phenomenon. As a consequence, we will see in 2.9 that $Z_{J,\geqslant 0}$ is the closure of $Z_{J,>0}$ in Z_J.

Note that Z_J is a fiber bundle over the product of two flag manifolds. Then understanding $Z_{J,\geqslant 0}$ is equivalent to understanding the intersection of $Z_{J,\geqslant 0}$ with

2000 *Mathematics Subject Classification.* Primary 20G20; Secondary 14M15.

each fiber. In 3.5, we will give a characterization of $Z_{J,\geqslant 0}$ which is analogous to the elementary fact that $G_{\geqslant 0} = \bigcap_{g \in G_{>0}} g^{-1} G_{>0}$. It allows us to reduce our problem to the problem of understanding certain subsets of some unipotent groups. Using the parametrization of the totally positive part of the flag varieties (see [MR]), we will give an explicit description of the subsets of G(see 3.7). Thus our main theorem can be proved.

1. Preliminaries

1.1. We will often identify a real algebraic variety with the set of its $\mathbf{R}$-rational points. Let G be a connected semisimple adjoint algebraic group defined and split over $\mathbf{R}$, with a fixed épinglage $(T, B^+, B^-, x_i, y_i; i \in I)$ (see [L1, 1.1]). Let U^+, U^- be the unipotent radicals of B^+, B^-. Let X (resp. Y) be the free abelian group of all homomorphism of algebraic groups $T \longrightarrow \mathbf{R}^*$ (resp. $\mathbf{R}^* \longrightarrow T$) and $<,>$: $Y \times X \longrightarrow \mathbf{Z}$ be the standard pairing. We write the operation in these groups as addition. For $i \in I$, let $\alpha_i \in X$ be the simple root such that $t x_i(a) t^{-1} = x_i(a)^{\alpha_i(t)}$ for all $a \in \mathbf{R}, t \in T$ and $\alpha_i^\vee \in Y$ be the simple coroot corresponding to α_i. For any root α, we denote by U_α the root subgroup corresponding to α.

There is a unique isomorphisms $\psi : G \xrightarrow{\sim} G^{\text{opp}}$ (the opposite group structure) such that

$\psi\big(x_i(a)\big) = y_i(a)$, $\psi\big(y_i(a)\big) = x_i(a)$ for all $i \in I$, $a \in \mathbf{R}$ and $\psi(t) = t$, for all $t \in T$.

If P is a subgroup of G and $g \in G$, we write ${}^g P$ instead of gPg^{-1}.

For any algebraic group H, we denote the Lie algebra of H by $\text{Lie}(H)$ and the center of H by $Z(H)$.

For any variety X and an automorphism σ of X, we denote the fixed point set of σ on X by X^σ.

For any group, We will write 1 for the identity element of the group.

For any finite set X, we will write $|X|$ for the cardinal of X.

1.2. Let $N(T)$ be the normalizer of T in G and $\dot{s_i} = x_i(-1) y_i(1) x_i(-1) \in N(T)$ for $i \in I$. Set $W = N(T)/T$ and s_i to be the image of $\dot{s_i}$ in W. Then W together with $(s_i)_{i \in I}$ is a Coxeter group.

Define an expression for $w \in W$ to be a sequence $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ in W, such that $w_{(0)} = 1$, $w_{(n)} = w$ and for any $j = 1, 2, \dots, n$, $w_{(j-1)}^{-1} w_{(j)} = 1$ or s_i for some $i \in I$. An expression $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ is called reduced if $w_{(j-1)} < w_{(j)}$ for all $j = 1, 2, \dots, n$. In this case, we will set $l(w) = n$. It is known that $l(w)$ is independent of the choice of the reduced expression. Note that if $\mathbf{w}$ is a reduced expression of w, then for all $j = 1, 2, \dots, n$, $w_{(j-1)}^{-1} w_{(j)} = s_{i_j}$ for some $i_j \in I$. Sometimes we will simply say that $s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression of w.

For $w \in W$, set $\dot{w} = \dot{s_{i_1}} \dot{s_{i_1}} \cdots \dot{s_{i_n}}$ where $s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression of w. It is well known that $\dot{w}$ is independent of the choice of the reduced expression $s_{i_1} s_{i_2} \cdots s_{i_n}$ of w.

Assume that $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ is a reduced expression of w and $w_{(j)} = w_{(j-1)} s_{i_j}$ for all $j = 1, 2, \dots, n$. Suppose that $v \leqslant w$ for the standard partial order in W. Then there is a unique sequence $\mathbf{v}_+ = (v_{(0)}, v_{(1)}, \dots, v_{(n)})$ such that $v_{(0)} = 1, v_{(n)} = v, v_{(j)} \in \{v_{(j-1)}, v_{(j-1)} s_{i_j}\}$ and $v_{(j-1)} < v_{(j-1)} s_{i_j}$ for all $j = 1, 2, \dots, n$ (see [MR, 3.5]). $\mathbf{v}_+$ is called the positive subexpression of $\mathbf{w}$. We define

$$J^+_{\mathbf{v}_+} = \{j \in \{1, 2, \dots, n\} \mid v_{(j-1)} < v_{(j)}\},$$
$$J^\circ_{\mathbf{v}_+} = \{j \in \{1, 2, \dots, n\} \mid v_{(j-1)} = v_{(j)}\}.$$

Then by the definition of $\mathbf{v}_+$, we have $\{1, 2, \dots, n\} = J^+_{\mathbf{v}_+} \sqcup J^\circ_{\mathbf{v}_+}$.

1.3. Let $\mathcal{B}$ be the variety of all Borel subgroups of G. For B, B' in $\mathcal{B}$, there is a unique $w \in W$, such that (B, B') is in the G-orbit on $\mathcal{B} \times \mathcal{B}$ (diagonal action) that contains $(B^+, {}^{\dot{w}} B^+)$. Then we write $\operatorname{pos}(B, B') = w$. By the definition of pos, $\operatorname{pos}(B, B') = \operatorname{pos}({}^g B, {}^g B')$ for any $B, B' \in \mathcal{B}$ and $g \in G$.

For any subset J of I, let W_J be the subgroup of W generated by $\{s_j \mid j \in J\}$ and w_0^J be the unique element of maximal length in W_J. (We will simply write w_0^I as w_0.) We denote by P_J the subgroup of G generated by B^+ and by $\{y_j(a) \mid j \in J, a \in \mathbf{R}\}$ and denote by $\mathcal{P}^J$ the variety of all parabolic subgroups of G conjugated to P_J. It is easy to see that for any parabolic subgroup P, $P \in \mathcal{P}^J$ if and only if $\{\operatorname{pos}(B_1, B_2) \mid B_1, B_2 \text{ are Borel subgroups of } P\} = W_J$.

1.4. For any parabolic subgroup P of G, define U_P to be the unipotent radical of P and H_P to be the inverse image of the connected center of P/U_P under $P \to P/U_P$. If B is a Borel subgroup of G, then so is

$$P^B = (P \cap B) U_P.$$

It is easy to see that for any $g \in H_P$, we have ${}^g(P^B) = P^B$. Moreover, P^B is the unique Borel subgroup B' in P such that $\operatorname{pos}(B, B') \in W^J$, where W^J is the set of minimal length coset representatives of W/W_J (see [L5, 3.2(a)]).

Let P, Q be parabolic subgroups of G. We say that P, Q are opposed if their intersection is a common Levi of P, Q. (We then write $P \bowtie Q$.) It is easy to see that if $P \bowtie Q$, then for any Borel subgroup B of P and B' of Q, we have $\operatorname{pos}(B, B') \in W_J w_0$.

For any subset J of I, define $J^* \subset I$ by $\{Q \mid Q \bowtie P \text{ for some } P \in \mathcal{P}^J\} = \mathcal{P}^{J^*}$. Then we have $(J^*)^* = J$. Let Q_J be the subgroup of G generated by B^- and by $\{x_j(a) \mid j \in J, a \in \mathbf{R}\}$. We have $Q_J \in \mathcal{P}^{J^*}$ and $P_J \bowtie Q_J$. Moreover, for any $P \in \mathcal{P}^J$, we have $P = {}^g P_J$ for some $g \in G$. Thus $\psi(P) = {}^{\psi(g)^{-1}} Q_J \in \mathcal{P}^{J^*}$.

1.5. Recall the following definitions from [L1].

For any $w \in W$, assume that $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression of w. Define $\phi^\pm : R^n_{\geqslant 0} \to U^\pm$ by

$$\phi^+(a_1, a_2, \dots, a_n) = x_{i_1}(a_1) x_{i_2}(a_2) \cdots x_{i_n}(a_n),$$
$$\phi^-(a_1, a_2, \dots, a_n) = y_{i_1}(a_1) y_{i_2}(a_2) \cdots y_{i_n}(a_n).$$

Let $U^{\pm}_{w,\geqslant 0} = \phi^{\pm}(R^n_{\geqslant 0}) \subset U^{\pm}$, $U^{\pm}_{w,>0} = \phi^{\pm}(R^n_{>0}) \subset U^{\pm}$. Then $U^{\pm}_{w,\geqslant 0}$ and $U^{\pm}_{w,>0}$ are independent of the choice of the reduced expression of w. We will simply write $U^{\pm}_{w_0,\geqslant 0}$ as $U^{\pm}_{\geqslant 0}$ and $U^{\pm}_{w_0,>0}$ as $U^{\pm}_{>0}$.

$T_{>0}$ is the submonoid of T generated by the elements $\chi(a)$ for $\chi \in Y$ and $a \in \mathbf{R}_{>0}$.

$G_{\geqslant 0}$ is the submonoid $U^+_{\geqslant 0} T_{>0} U^-_{\geqslant 0} = U^-_{\geqslant 0} T_{>0} U^+_{\geqslant 0}$ of G.

$G_{>0}$ is the submonoid $U^+_{>0} T_{>0} U^-_{>0} = U^-_{>0} T_{>0} U^+_{>0}$ of $G_{\geqslant 0}$.

$\mathcal{B}_{>0}$ is the subset $\{{}^u B^- \mid u \in U^+_{>0}\} = \{{}^u B^+ \mid u \in U^-_{>0}\}$ of $\mathcal{B}$ and $\mathcal{B}_{\geqslant 0}$ is the closure of $\mathcal{B}_{>0}$ in the manifold $\mathcal{B}$.

For any subset J of I, $\mathcal{P}^J_{>0} = \{P \in \mathcal{P}^J \mid \exists B \in \mathcal{B}_{>0}, \text{ such that } B \subset P\}$ and $\mathcal{P}^J_{\geqslant 0} = \{P \in \mathcal{P}^J \mid \exists B \in \mathcal{B}_{\geqslant 0}, \text{ such that } B \subset P\}$ are subsets of $\mathcal{P}^J$.

1.6. For any $w, w' \in W$, define

$$\mathcal{R}_{w,w'} = \{B \in \mathcal{B} \mid \text{pos}(B^+, B) = w', \text{pos}(B^-, B) = w_0 w\}.$$

It is known that $\mathcal{R}_{w,w'}$ is nonempty if and only if $w \leqslant w'$ for the standard partial order in W(see [KL]). Now set

$$\mathcal{R}_{w,w',>0} = \mathcal{B}_{\geqslant 0} \cap \mathcal{R}_{w,w'}.$$

Then $\mathcal{R}_{w,w',>0}$ is a connected component of $\mathcal{R}_{w,w'}$ and is a semi-algebraic cell(see [R2, 2.8]). Furthermore, $\mathcal{B} = \bigsqcup_{w \leqslant w'} \mathcal{R}_{w,w'}$ and $\mathcal{B}_{\geqslant 0} = \bigsqcup_{w \leqslant w'} \mathcal{R}_{w,w',>0}$. Moreover, for any $u \in U^+_{w^{-1},>0}$, we have ${}^u\mathcal{R}_{w,w',>0} \subset \mathcal{R}_{1,w',>0}$ (see [R2, 2.2]).

Let J be a subset of I. Define $\pi^J : \mathcal{B} \longrightarrow \mathcal{P}^J$ to be the map which sends a Borel subgroup to the unique parabolic subgroup in $\mathcal{P}^J$ that contains the Borel subgroup. For any $w, w' \in W$ such that $w \leqslant w'$ and $w' \in W^J$, set $\mathcal{P}^J_{w,w'} = \pi^J(\mathcal{R}_{w,w'})$ and $\mathcal{P}^J_{w,w',>0} = \pi^J(\mathcal{R}_{w,w',>0})$. We have $\mathcal{P}^J_{\geqslant 0} = \bigsqcup_{w \leqslant w', w' \in W^J} \mathcal{P}^J_{w,w',>0}$ and $\pi^J \mid_{\mathcal{R}_{w,w',>0}}$ maps $\mathcal{R}_{w,w',>0}$ bijectively onto $\mathcal{P}^J_{w,w',>0}$ (see [R1, chapter 4, 3.2]). Hence, for any $u \in U^+_{w^{-1},>0}$, we have ${}^u\mathcal{P}^J_{w,w',>0} = \pi^J({}^u\mathcal{R}_{w,w',>0}) \subset \pi^J(\mathcal{P}^J_{1,w',>0})$.

1.7. Define $\pi_T : B^- B^+ \longrightarrow T$ by $\pi_T(utu') = t$ for $u \in U^-, t \in T, u' \in U^+$. Then for $b_1 \in B^-, b_2 \in B^- B^+, b_3 \in B^+$, we have $\pi_T(b_1 b_2 b_3) = \pi_T(b_1)\pi_T(b_2)\pi_T(b_3)$.

Let J be a subset of I. We denote by Φ^+_J the set of roots that are linear combination of $\{\alpha_j \mid j \in J\}$ with nonnegative coefficients. We will simply write Φ^+_I as Φ^+ and we will call a root α positive if $\alpha \in \Phi^+$. In this case, we will simply write $\alpha > 0$. Define U^+_J to be the subgroup of U^+ generated by $\{U_\alpha \mid \alpha \in \Phi^+_J\}$ and ${}'U^+_J$ to be the subgroup of U^+ generated by $\{U_\alpha \mid \alpha \in \Phi^+ - \Phi^+_J\}$. Then $U^- \times T \times' U^+_J \times U^+_J$ is isomorphic to $B^- B^+$ via $(u, t, u_1, u_2) \mapsto utu_1u_2$. Now define $\pi_{U^+_J} : B^- B^+ \longrightarrow U^+_J$ by $\pi_{U^+_J}(utu_1u_2) = u_2$ for $u \in U^-, t \in T, u_1 \in' U^+_J$ and $u_2 \in U^+_J$. (We will simply write $\pi_{U^+_I}$ as π_{U^+}.) Note that $U^- T \cdot U^- T' U^+_J = U^- T' U^+_J$. Thus it is easy to see that for any $a, b \in G$ such that $a, ab \in B^- B^+$, we have $\pi_{U_J^+}(ab) = \pi_{U_J^+}(\pi_{U^+}(a)b)$. Since $'U_J^+$ is a normal subgroup of U^+, $\pi_{U_J^+}|_{U^+}$ is a homomorphism of U^+ onto U_J^+. Moreover, we have

$$\pi_{U_J^+}\big(x_i(a)\big) = \begin{cases} x_i(a), & \text{if } i \in J; \\ 1, & \text{otherwise .} \end{cases}$$

Thus $\pi_{U_J^+}(U^+_{>0}) = U^+_{w_0^J,>0}$ and $\pi_{U_J^+}(U^+_{\geqslant 0}) = U^+_{w_0^J,\geqslant 0}$.

Let U_J^- be the subgroup of U^- generated by $\{U_{-\alpha} \mid \alpha \in \Phi_J^+\}$ and $'U_J^-$ to be the subgroup of U^- generated by $\{U_{-\alpha} \mid \alpha \in \Phi^+ - \Phi_J^+\}$. Then we define $\pi_{U_J^-} : U^- \to U_J^-$ by $\pi_{U_J^-}(u_1u_2) = u_1$ for $u_1 \in U_J^-, u_2 \in' U_J^-$. (We will simply write $\pi_{U_I^-}$ as π_{U^-}.) We have $\pi_{U_J^-}(U^-_{>0}) = U^-_{w_0^J,>0}$ and $\pi_{U_J^-}(U^-_{\geqslant 0}) = U^-_{w_0^J,\geqslant 0}$.

1.8. For any vector space V and a nonzero element v of V, we denote the image of v in $P(V)$ by $[v]$.

If (V, ρ) is a representation of G, we denote by (V^*, ρ^*) the dual representation of G. Then we have the standard isomorphism $St_V : V \otimes V^* \xrightarrow{\simeq} \text{End}(V)$ defined by $St_V(v \otimes v^*)(v') = v^*(v')v$ for all $v, v' \in V, v^* \in V^*$. Now we have the $G \times G$ action on $V \otimes V^*$ by $(g_1, g_2) \cdot (v \otimes v^*) = (g_1 v) \otimes (g_2 v^*)$ for all $g_1, g_2 \in G, v \in V, v^* \in V^*$ and the $G \times G$ action on $\text{End}(V)$ by $\big((g_1, g_2) \cdot f\big)(v) = g_1\big(f(g_2^{-1}v)\big)$ for all $g_1, g_2 \in G, f \in \text{End}(V), v \in V$. The standard isomorphism between $V \otimes V^*$ and $\text{End}(V)$ commutes with the $G \times G$ action. We will identify $\text{End}(V)$ with $V \otimes V^*$ via the standard isomorphism.

2. The strata of the De Concini-Procesi Compactification

2.1. Let $\mathcal{V}_G$ be the projective variety whose points are the $\dim(G)$-dimensional Lie subalgebras of $\text{Lie}(G \times G)$. For any subset J of I, define

$$Z_J = \{(P, Q, \gamma) \mid P \in \mathcal{P}^J, Q \in \mathcal{P}^{J^*}, \gamma = H_P g U_Q, P \bowtie^g Q\}$$

with the $G \times G$ action by $(g_1, g_2) \cdot (P, Q, H_P g U_Q) = \big({}^{g_1}P, {}^{g_2}Q, H_{{}^{g_1}P}(g_1 g g_2^{-1})U_{{}^{g_2}Q}\big)$.

For $(P, Q, \gamma) \in Z_J$ and $g \in \gamma$, we set

$$H_{P,Q,\gamma} = \{(l + u_1, \text{Ad}(g^{-1})l + u_2) \mid l \in \text{Lie}(P \cap {}^gQ), u_1 \in \text{Lie}(U_P), u_2 \in \text{Lie}(U_Q)\}.$$

Then $H_{P,Q,\gamma}$ is independent of the choice of g (see [L6, 12.2]) and is an element of $\mathcal{V}_G$ (see [L6, 12.1]). Moreover, $(P, Q, \gamma) \to H_{P,Q,\gamma}$ is an embedding of $Z_J \subset \mathcal{V}_G$ (see [L6, 12.2]). We will identify Z_J with the subvariety of $\mathcal{V}_G$ defined above. Then we have $\bar{G} = \bigsqcup_{J \subset I} Z_J$, where $\bar{G}$ is the the De Concini-Procesi compactification of G (see [L6, 12.3]). We will call $\{Z_J \mid J \subset I\}$ the strata of $\bar{G}$ and Z_I (resp. $Z_\varnothing$) the highest (resp. lowest) stratum of $\bar{G}$. It is easy to see that Z_I is isomorphic to G and $Z_\varnothing$ is isomorphic to $\mathcal{B} \times \mathcal{B}$.

Set $z_J^\circ = (P_J, Q_J, H_{P_J} U_{Q_J})$. Then $z_J^\circ \in Z_J$ (see 1.4) and $Z_J = (G \times G) \cdot z_J^\circ$.

Since G is adjoint, we have an isomorphism $\chi : T \xrightarrow{\simeq} (\mathbf{R}^*)^I$ defined by $\chi(t) = \big(\alpha_i(t)^{-1}\big)_{i \in I}$. We denote the closure of T in $\bar{G}$ by $\bar{T}$. We have $H_{P_J, Q_J, H_{P_J} U_{Q_J}} = \{(l + u_1, l + u_2) \mid l \in \mathrm{Lie}(P_J \cap Q_J), u_1 \in U_{P_J}, u_2 \in U_{Q_J}\}$. Moreover, for any $t \in Z(P_J \cap Q_J)$, H_t is the subspace of $\mathrm{Lie}(G) \times \mathrm{Lie}(G)$ spanned by the elements $(l, l), (u_1, \mathrm{Ad}(t^{-1}) u_1), (\mathrm{Ad}(t) u_2, u_2)$, where $l \in \mathrm{Lie}(P_J \cap Q_J), u_1 \in U_{P_J}, u_2 \in U_{Q_J}$. Thus it is easy to see that $z_J^\circ = \lim\limits_{\substack{t_j = 1, \forall j \in J \\ t_j \longrightarrow 0, \forall j \notin J}} \chi^{-1}\big((t_i)_{i \in I}\big) \in \bar{T}$.

Proposition 2.2. *The automorphism ψ of the variety G (see 1.1) can be extended in a unique way to an automorphism $\bar{\psi}$ of $\bar{G}$. Moreover, $\bar{\psi}(P, Q, \gamma) = \big(\psi(Q), \psi(P), \psi(\gamma)\big) \in Z_J$ for $J \subset I$ and $(P, Q, \gamma) \in Z_J$.*

Proof. The map $\psi : G \longrightarrow G$ induces a bijective map $\psi : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(G)$. Moreover, we have $\psi(\mathrm{Ad}(g) v) = \mathrm{Ad}\big(\psi(g)^{-1}\big) \psi(v)$ and $\psi(v + v') = \psi(v) + \psi(v')$ for $g \in G, v, v' \in \mathrm{Lie}(G)$. Now define $\delta : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(G) \times \mathrm{Lie}(G)$ by $\delta(v, v') = \big(\psi(v'), \psi(v)\big)$ for $v, v' \in \mathrm{Lie}(G)$. Then δ induces a bijection $\bar{\psi} : \mathcal{V}_G \longrightarrow \mathcal{V}_G$.

Note that for any $g \in G$, we have $H_g = \{(v, \mathrm{Ad}(g) v) \mid v \in \mathrm{Lie} G\}$ and $\bar{\psi}(H_g) = \{(\mathrm{Ad}(\psi(g)^{-1}) \psi(v), \psi(v)) \mid v \in \mathrm{Lie}(G)\} = H_{\psi(g)}$. Thus $\bar{\psi}$ is an extension of the automorphism ψ of G into $\mathcal{V}_G$.

Now for any $(P, Q, \gamma) \in Z_J$ and $g \in \gamma$, we have $\psi(P) \in \mathcal{P}^{J^*}, \psi(Q) \in \mathcal{P}^J$ and $\psi(Q) \bowtie^{\psi(g)} \psi(P)$ (see 1.4). Thus $\big(\psi(Q), \psi(P), \psi(\gamma)\big) \in Z_J$. Moreover,

$$\begin{aligned} \bar{\psi}(H_{P,Q,\gamma}) &= \{(\mathrm{Ad}(\psi(g)) \psi(l) + \psi(u_2), \psi(l) + \psi(u_1)) \mid l \in \mathrm{Lie}(P \cap {}^g Q), \\ &\qquad\qquad u_1 \in \mathrm{Lie}(U_P), u_2 \in \mathrm{Lie}(U_Q)\} \\ &= \{(l + u_2, \mathrm{Ad}(\psi(g)^{-1}) l + u_1) \mid l \in \mathrm{Lie}(\psi(Q) \cap^{\psi(g)} \psi(P)), \\ &\qquad\qquad u_1 \in \mathrm{Lie}(\psi(U_P)), u_2 \in \mathrm{Lie}(\psi(U_Q))\} \\ &= H_{\psi(Q), \psi(P), \psi(\gamma)}. \end{aligned}$$

Thus $\bar{\psi} \mid_{\bar{G}}$ is an automorphism of $\bar{G}$. Moreover, since $\bar{G}$ is the closure of G, $\bar{\psi} \mid_{\bar{G}}$ is the unique automorphism of $\bar{G}$ that extends the automorphism ψ of G.

The proposition is proved. $\square$

2.3. For any $\lambda \in X$, set $\mathrm{supp}(\lambda) = \{i \in I \mid < \alpha_i^\vee, \lambda > \neq 0\}$.

In the rest of the section, I will fix a subset J of I and $\lambda_1, \lambda_2 \in X^+$ with $\mathrm{supp}(\lambda_1) = I - J, \mathrm{supp}(\lambda_2) = J$. Let (V_{λ_1}, ρ_1) (resp. (V_{λ_2}, ρ_2)) be the irreducible representation of G with the highest weight λ_1 (resp. λ_2). Assume that $\dim V_{\lambda_1} = n_1, \dim V_{\lambda_2} = n_2$ and $\{v_1, v_2, \dots, v_{n_1}\}$ (resp. $\{v'_1, v'_2, \dots, v'_{n_2}\}$) is the canonical basis of (V_{λ_1}, ρ_1) (resp. (V_{λ_2}, ρ_2)), where v_1 and v'_1 are the highest weight vectors. Moreover, after reordering $\{2, 3, \dots, n_2\}$, we could assume that there exists some integer $n_0 \in \{1, 2, \dots, n_2\}$ such that for any $i \in \{1, 2, \dots, n_2\}$, the weight of v'_i is of the form $\lambda_2 - \sum_{j \in J} a_j \alpha_j$ if and only if $i \leqslant n_0$.

Define $i_J : G \longrightarrow P\big(\mathrm{End}(V_{\lambda_1})\big) \times P\big(\mathrm{End}(V_{\lambda_2})\big)$ by $i_J(g) = \Big([\rho_1(g)], [\rho_2(g)]\Big)$. Then since $\lambda_1 + \lambda_2$ is a dominant and regular weight, the closure of the image of i_J in $P\big(\mathrm{End}(V_{\lambda_1})\big) \times P\big(\mathrm{End}(V_{\lambda_2})\big)$ is isomorphic to the De Concini-Procesi compactification of G (See [DP, 4.1]). We will use i_J as the embedding of $\bar{G}$ into $P\big(\mathrm{End}(V_{\lambda_1})\big) \times P\big(\mathrm{End}(V_{\lambda_2})\big)$. We will also identify $\bar{G}$ with its image under i_J.

2.4. Now with respect to the canonical basis of V_{λ_1} and V_{λ_2}, we will identify $\mathrm{End}(V_{\lambda_1})$ with $gl(n_1)$ and $\mathrm{End}(V_{\lambda_2})$ with $gl(n_2)$. Thus we will regard $\rho_1(g), \rho_1^*(g)$ as $n_1 \times n_1$ matrices and $\rho_2(g), \rho_2^*(g)$ as $n_2 \times n_2$ matrices. It is easy to see that (in terms of matrices) for any $g \in G, \rho_1^*(g) =^t \rho_1(g^{-1})$ and $\rho_2^*(g) =^t \rho_2(g^{-1})$, where ${}^tM$ is the transpose of the matrix M. Now for any $g_1, g_2 \in G$, $M_1 \in gl(n_1)$, $M_2 \in gl(n_2)$, $(g_1, g_2) \cdot M_1 = \rho_1(g_1) M_1 \rho_1(g_2^{-1})$ and $(g_1, g_2) \cdot M_2 = \rho_2(g_1) M_2 \rho_2(g_2^{-1})$.

Set $L = P_J \cap Q_J$. Then L is a reductive algebraic group with the épinglage $(T, B^+ \cap L, B^- \cap L, x_j, y_j; j \in J)$. Now let V_L be the subspace of V_{λ_2} spanned by $\{v'_1, v'_2, \dots, v'_{n_0}\}$ and $I_L = (a_{ij}) \in gl(n_2)$, where

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \in \{1, 2, \dots, n_0\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then V_L is an irreducible representation of L with the highest weight λ_2 and canonical basis $\{v'_1, v'_2, \dots, v'_{n_0}\}$. Moreover, λ_2 is a dominant and regular weight for L. Now set $I_1 = \mathrm{diag}(1, 0, 0, \dots, 0) \in gl(n_1), I_2 = \mathrm{diag}(1, 0, 0, \dots, 0) \in gl(n_2)$. Then $i_J(z_J^\circ) = \lim\limits_{\substack{t_j = 1, \forall j \in J \\ t_j \longrightarrow 0, \forall j \notin J}} i_J\Big(\chi^{-1}\big((t_i)_{i \in I}\big)\Big) = \Big([v_1 \otimes v_1^*], [\sum_{i=1}^{n_0} v'_i \otimes {v'_i}^*]\Big) = \Big([I_1], [I_L]\Big)$, where $\{{v_1}^*, {v_2}^*, \dots, {v_{n_1}}^*\}$ (resp. $\{{v'_1}^*, {v'_2}^*, \dots, {v'_{n_2}}^*\}$) is the dual basis in $(V_{\lambda_1})^*$ (resp. $(V_{\lambda_2})^*$).

2.5. Recall that $\mathrm{supp}(\lambda_1) = I - J$. Thus for any $P \in \mathcal{P}^J$, there is a unique P-stable line $L_{\rho_1(P)}$ in (V_{λ_1}, ρ_1) and $P \mapsto L_{\rho_1(P)}$ is an embedding of $\mathcal{P}^J$ into $P(V_{\lambda_1})$. Similarly, for any $Q \in \mathcal{P}^{J^*}$, there is a unique Q-stable line $L_{\rho_1^*(Q)}$ in $(V_{\lambda_1}^*, \rho_1^*)$ and $Q \mapsto L_{\rho_1^*(Q)}$ is an embedding of $\mathcal{P}^{J^*}$ into $P(V_{\lambda_1}^*)$. It is easy to see $L_{\rho_1(P_J)} = [v_1]$, $L_{\rho_1^*(Q_J)} = [{v_1}^*]$ and $L_{\rho_1({}^gP)} = \rho_1(g) L_{\rho_1(P)}, L_{\rho_1^*({}^gQ)} = \rho_1^*(g) L_{\rho_1^*(Q)}$ for $P \in \mathcal{P}^J, Q \in \mathcal{P}^{J^*}, g \in G$.

There are projections $p_1 : P\big(\mathrm{End}(V_{\lambda_1})\big) \times P\big(\mathrm{End}(V_{\lambda_2})\big) \longrightarrow P\big(\mathrm{End}(V_{\lambda_1})\big)$ and $p_2 : P\big(\mathrm{End}(V_{\lambda_1})\big) \times P\big(\mathrm{End}(V_{\lambda_2})\big) \longrightarrow P\big(\mathrm{End}(V_{\lambda_2})\big)$. It is easy to see that $p_1 \mid_{Z_J}$, $p_2 \mid_{Z_J}$ commute with the $G \times G$ action and $p_1(z_J^\circ) = [v_1 \otimes {v_1}^*] = [L_{\rho_1(P_J)} \otimes L_{\rho_1^*(Q_J)}]$. Now for any $g_1, g_2 \in G$, we have

$$p_1\big((g_1, g_2) \cdot z_J^\circ\big) = [\rho_1(g_1) L_{\rho_1(P_J)} \otimes \rho_1^*(g_2) L_{\rho_1^*(Q_J)}] = [L_{\rho_1({}^{g_1}P)} \otimes L_{\rho_1^*({}^{g_2}Q)}].$$

In other words, $p_1(z) = [L_{\rho_1(P)} \otimes L_{\rho_1^*(Q)}]$ for $z = (P, Q, \gamma) \in Z_J$.

2.6. Let $\overline{G_{>0}}$ be the closure of $G_{>0}$ in $\bar{G}$. Then $\overline{G_{>0}}$ is also the closure of $G_{\geqslant 0}$ in $\bar{G}$. We have $z_J^\circ \in \overline{G_{>0}}$ (see 2.1). Now set

$$Z_{J, \geqslant 0} = Z_J \cap \overline{G_{>0}},$$

$$Z_{J,>0} = \{(g_1, g_2^{-1}) \cdot z_J^\circ \mid g_1, g_2 \in G_{>0}\}.$$

Since $\psi(G_{>0}) = G_{>0}$, we have $\bar{\psi}(\overline{G_{>0}}) = \overline{G_{>0}}$. Moreover, $\bar{\psi}(Z_J) = Z_J$ (see 2.2). Therefore $\bar{\psi}(Z_{J,\geqslant 0}) = Z_{J,\geqslant 0}$. Similarly, $(g_1, g_2^{-1}) \cdot Z_{J,\geqslant 0} \subset Z_{J,\geqslant 0}$ for any $g_1, g_2 \in G_{>0}$. Thus $Z_{J,>0} \subset Z_{J,\geqslant 0}$. Moreover, it is easy to see that $\bar{\psi}(Z_{J,>0}) = Z_{J,>0}$.

Note that for any $u_1, u_4 \in U^-_{>0}, u_2, u_3 \in U^+_{>0}, t, t' \in T_{>0}$, we have

$$\begin{aligned}(u_1 u_2 t, u_3^{-1} u_4^{-1} t') \cdot z_J^\circ &= (u_1 u_2, u_3^{-1} u_4^{-1}) \cdot (P_J, Q_J, H_{P_J} t t' U_{Q_J}) \\ &= (u_1, u_3^{-1}) \cdot \big(P_J, Q_J, H_{P_J} \pi_{U_J^+}(u_2) t t' \pi_{U_J^-}(u_4) U_{Q_J}\big).\end{aligned}$$

Thus

$$\begin{aligned}Z_{J,>0} &= \{(u_1, u_2^{-1}) \cdot (P_J, Q_J, H_{P_J} l U_{Q_J}) \mid u_1 \in U^-_{>0}, u_2 \in U^+_{>0}, l \in L_{>0}\} \\ &= \{(u_1' t, {u_2'}^{-1}) \cdot z_J^\circ \mid u_1' \in U^-_{>0}, u_2' \in U^+_{>0}, t \in T_{>0}\}.\end{aligned}$$

Moreover, for any $u_1, u_1' \in U^-, u_2, u_2' \in U^+$ and $t, t' \in T$, it is easy to see that $(u_1 t, u_2) \cdot z_J^\circ = (u_1' t', u_2') \cdot z_J^\circ$ if and only if $(u_1 t)^{-1} u_1' t' \in l H_{P_J} \bigcap B^- \subset l Z(L)$ and $u_2^{-1} u_2' \in l^{-1} H_{Q_J} \bigcap U^+ \subset l Z(L)$ for some $l \in L$, that is, $l \in Z(L)$, $u_1 = u_1', u_2 = u_2'$ and $t \in t' Z(L)$. Thus, $Z_{J,>0} \cong U^-_{>0} \times U^+_{>0} \times T_{>0} / \big(T_{>0} \bigcap Z(L)\big) \cong R_{>0}^{2l(w_0)+|J|}$.

Now I will prove a criterion for $Z_{J,>0}$.

Theorem 2.7. *Assume that G is simply-laced. Let $z \in Z_{J,\geqslant 0}$. Then $z \in Z_{J,>0}$ if and only if z satisfies the condition (*): $i_J(z) = \Big([M_1], [M_2]\Big)$ and $i_J\big(\bar{\psi}(z)\big) = \Big([M_3], [M_4]\Big)$ for some matrices $M_1, M_3 \in gl(n_1)$ and $M_2, M_4 \in gl(n_2)$ with all the entries in $\mathbf{R}_{>0}$.*

Proof. If $z \in Z_{J,>0}$, then $z = (g_1, g_2^{-1}) \cdot z_J^\circ$, for some $g_1, g_2 \in G_{>0}$. Assume that $g_1 \cdot v_1 = \sum_{i=1}^{n_1} a_i v_i$ and $g_2^{-1} \cdot v_1^* = \sum_{i=1}^{n_1} b_i v_i^*$. Then for any $i = 1, 2, \dots, n_1$, $a_i, b_i > 0$. Set $a_{ij} = a_i b_j$. Then $p_1(z) = [\rho_1(g_1) I_1 \rho_1(g_2)] = [(a_{ij})]$ is a matrix with all the entries in $\mathbf{R}_{>0}$.

We have $p_2(z) = [\rho_2(g_1) I_L \rho_2(g_2)] = [\rho_2(g_1) I_2 \rho_2(g_2) + \rho_2(g_1)(I_L - I_2)\rho_2(g_2)]$. Note that $\rho_2(g_1) I_2 \rho_2(g_2)$ is a matrix with all the entries in $\mathbf{R}_{>0}$ and $\rho_2(g_1)$, $\rho_2(g_2)$, $(I_L - I_2)$ are matrices with all the entries in $\mathbf{R}_{\geqslant 0}$. Thus $\rho_2(g_1)(I_L - I_2)\rho_2(g_2)$ is a matrix with all its entries in $\mathbf{R}_{\geqslant 0}$. So $\rho_2(g_1) I_L \rho_2(g_2)$ is a matrix with all the entries in $\mathbf{R}_{>0}$.

Similarly, $i_J\big(\bar{\psi}(z)\big) = \Big([M_3], [M_4]\Big)$ for some matrices M_3, M_4 with all their entries in $\mathbf{R}_{>0}$.

On the other hand, assume that z satisfies the condition (*). Suppose that $z = (P, Q, \gamma)$ and $L_{\rho_1(P)} = [\sum_{i=1}^{n_1} a_i v_i]$, $L_{\rho_1^*(Q)} = [\sum_{i=1}^{n_1} b_i v_i^*]$. We may also assume that $a_{i_0} = b_{i_1} = 1$ for some integers $i_0, i_1 \in \{1, 2, \dots, n_1\}$.

Set $M = (a_{ij}) \in gL(n_1)$, where $a_{ij} = a_i b_j$ for $i, j \in \{1, 2, \dots, n_1\}$. Then $p_1(z) = [L_{\rho_1(P)} \otimes L_{\rho_1^*(Q)}] = [M]$. By the condition (*) and since $a_{i_0, i_1} = a_{i_0} b_{i_1} =$

1, we have that M is a matrix with all its entries in $\mathbf{R}_{>0}$. In particular, for any $i \in \{1, 2, \dots, n_1\}, a_{i,i_1} = a_i > 0$. Therefore $L_{\rho_1(P)} = [\sum_{i=1}^{n_1} a_i v_i]$, where $a_i > 0$ for all $i \in \{1, 2, \dots, n_1\}$. By [R1, 5.1] (see also [L3, 3.4]), $P \in \mathcal{P}^J_{>0}$. Similarly, $\psi(Q) \in \mathcal{P}^J_{>0}$. Thus there exist $u_1 \in U^-_{>0}, u_2 \in U^+_{>0}$ and $l \in L$, such that $z = (u_1, u_2^{-1}) \cdot (P_J, Q_J, H_{P_J} l U_{Q_J})$.

We can express u_1, u_2 in a unique way as $u_1 = u_1' u_1''$, for some $u_1' \in' U^-_J$, $u_1'' \in U^-_J$ and $u_2 = u_2'' u_2'$, for some $u_2' \in' U^+_J$, $u_2'' \in U^+_J$ (see 1.7).

Recall that V_L is the subspace of V_{λ_2} spanned by $\{v_1', v_2', \dots, v_{n_0}'\}$. Let V_L' be the subspace of V_{λ_2} spanned by $\{v_{n_0+1}', v_{n_0+2}', \dots, v_{n_2}'\}$. Then $u \cdot v - v \in V_L'$ and $u \cdot V_L' \subset V_L'$, for all $v \in V_L$, $\alpha \notin \Phi^+_J$ and $u \in U_{-\alpha}$. Thus $u \cdot v - v \in V_L'$ and $u \cdot V_L' \subset V_L'$, for all $v \in V_L$ and $u \in' U^-_J$.

Similarly, let V_L^* be the subspace of $V_{\lambda_2}^*$ spanned by $\{{v_1'}^*, {v_2'}^*, \dots, {v_{n_0}'}^*\}$ and ${V_L'}^*$ be the subspace of $V_{\lambda_2}^*$ spanned by $\{{v_{n_0+1}'}^*, {v_{n_0+2}'}^*, \dots, {v_{n_2}'}^*\}$. Then for any $v^* \in V_L^*$ and $u \in' U^+_J$, we have $u \cdot v - v \in {V_L'}^*$ and $u {V_L'}^* \subset {V_L'}^*$.

We define a map $\pi_L : gl(n_2) \to gl(n_0)$ by

$$\pi_L\big((a_{ij})_{i,j \in \{1,2,\dots,n_2\}}\big) = (a_{ij})_{i,j \in \{1,2,\dots,n_0\}}$$

Then for any $u \in' U^-_J, u' \in' U^+_J$ and $M \in gl(n_2)$, we have $\pi_L\big((u, u') \cdot M\big) = \pi_L(M)$. Set $M_2 = \rho_2(u_1 l) I_L \rho_2(u_2)$ and $l' = u_1'' l u_2'' \in L$. Then

$$\begin{aligned} \pi_L(M_2) &= \pi_L\Big((u_1, u_2^{-1}) \cdot \big(\rho_2(l) I_L\big)\Big) = \pi_L\bigg((u_1', {u_2'}^{-1}) \cdot \Big(({u_1''}, {u_2''}^{-1}) \cdot \big(\rho_2(l) I_L\big)\Big)\bigg) \\ &= \pi_L\Big((u_1'', {u_2''}^{-1}) \cdot \big(\rho_2(l) I_L\big)\Big) = \pi_L\big(\rho_2(l') I_L\big) = \rho_L(l'). \end{aligned}$$

Since $p_2(z) = [M_2]$, M_2 is a matrix with all its entries nonzero. Therefore $\rho_L(l') = \pi_L(M_2)$ is a matrix with all its entries nonzero. Thus $l' = l_1 t_1 l_2$, for some $l_1 \in U^- \cap L, l_2 \in U^+ \cap L, t_1 \in T$.

Set $\widetilde{u_1} = u_1' l_1$ and $\widetilde{u_2} = u_2' l_2$. Then ${}^{\widetilde{u_1}} P_J = {}^{u_1 ({u_1''}^{-1} l_1)} P_J = {}^{u_1} P_J$. Similarly, we have ${}^{\widetilde{u_2}^{-1}} Q_J = {}^{u_2^{-1}} Q_J$. So $z = (\widetilde{u_1}, \widetilde{u_2}^{-1}) \cdot (P_J, Q_J, H_{P_J} t_1 U_{Q_J})$.

Now for any $i_0, j_0 \in \{1, 2, \dots, n_1\}$, define a map $\pi^1_{i_0,j_0} : gl(n_1) \to \mathbf{R}$ by $\pi^1_{i_0,j_0}\big((a_{ij})_{i,j \in \{1,2,\dots,n_1\}}\big) = a_{i_0,j_0}$ and for any $i_0, j_0 \in \{1, 2, \dots, n_2\}$, define a map $\pi^2_{i_0,j_0} : gl(n_2) \to \mathbf{R}$ by $\pi^2_{i_0,j_0}\big((a_{ij})_{i,j \in \{1,2,\dots,n_2\}}\big) = a_{i_0,j_0}$.

Now $z = (\widetilde{u_1} t_1, \widetilde{u_2}^{-1}) \cdot z_J^\circ$ and $\bar{\psi}(z) = \big(\psi(\widetilde{u_2}) t_1, \psi(\widetilde{u_1})^{-1}\big) \cdot z_J^\circ$.

Set

$$\begin{aligned} \tilde{M}_1 &= \rho_1(\widetilde{u_1} t_1) I_1 \rho_1(\widetilde{u_2}), \quad & \tilde{M}_3 &= \rho_1\big(\psi(\widetilde{u_2}) t_1\big) I_1 \rho_1\big(\psi(\widetilde{u_1})\big), \\ \tilde{M}_2 &= \rho_2(\widetilde{u_1} t_1) I_L \rho_2(\widetilde{u_2}), \quad & \tilde{M}_4 &= \rho_2\big(\psi(\widetilde{u_2}) t_1\big) I_1 \rho_2\big(\psi(\widetilde{u_1})\big). \end{aligned}$$

We have $\widetilde{u_1} \cdot v_1 = \sum_{i=1}^{n_1} \frac{\pi^1_{i,1}(\tilde{M}_1)}{\pi^1_{1,1}(\tilde{M}_1)} v_i$ and $\psi(\widetilde{u_2}) \cdot v_1 = \sum_{i=1}^{n_1} \frac{\pi^1_{i,1}(\tilde{M}_3)}{\pi^1_{1,1}(\tilde{M}_3)} v_i$.

Moreover, let V_0 be the subspace of V_{λ_2} spanned by $\{v'_2, v'_3, \dots, v'_{n_2}\}$ and ${V_0}^*$ be the subspace of $V^*_{\lambda_2}$ spanned by $\{{v'_2}^*, {v'_3}^*, \dots, {v'_{n_2}}^*\}$. Then we have $u \cdot V_0 \subset V_0$, for all $u \in U^-$ and $u' \cdot V^*_0 \subset V^*_0$, for all $u' \in U^+$.

Thus for all $i = 1, 2, \dots, n_2$,

$$\begin{aligned}\pi^2_{i,1}(M_2) &= \pi^2_{i,1}\big(\rho_2(\widetilde{u_1}t_1)I_2\rho_2(\widetilde{u_2})\big) + \pi^2_{i,1}\big(\rho_2(\widetilde{u_1}t_1)(I_L - I_2)\rho_2(\widetilde{u_2})\big) \\ &= \pi^2_{i,1}\big(\rho_2(\widetilde{u_1}t_1)I_2\rho_2(\widetilde{u_2})\big).\end{aligned}$$

So $\widetilde{u_1} \cdot v'_1 = \sum_{i=1}^{n_2} \frac{\pi^2_{i,1}(\tilde{M_2})}{\pi^2_{1,1}(\tilde{M_2})} v'_i$ and $\psi(\widetilde{u_2}) \cdot v'_1 = \sum_{i=1}^{n_2} \frac{\pi^2_{i,1}(\tilde{M_4})}{\pi^2_{1,1}(\tilde{M_4})} v'_i$. By [L2, 5.4], we have $\widetilde{u_1}, \psi(\widetilde{u_2}) \in U^-_{>0}$. Therefore to prove that $z \in Z_{J,>0}$, it is enough to prove that $t_1 \in T_{>0}Z(L)$, where $Z(L)$ is the center of L.

For any $g \in (U^-, U^+) \cdot \bar{T}$, g can be expressed in a unique way as $g = (u_1, u_2) \cdot t$, for some $u_1 \in U^-$, $u_2 \in U^+$, $t \in \bar{T}$. Now define $\pi_{\bar{T}} : (U^-, U^+) \cdot \bar{T} \to \bar{T}$ by $\pi_{\bar{T}}\big((u_1, u_2) \cdot t\big) = t$ for all $u_1 \in U^-, u_2 \in U^+, t \in \bar{T}$. Note that $(U^-, U^+) \cdot \bar{T} \cap \overline{G_{>0}}$ is the closure of $G_{>0}$ in $(U^-, U^+) \cdot \bar{T}$. Then $\pi_{\bar{T}}\big((U^-, U^+) \cdot \bar{T} \cap \overline{G_{>0}}\big)$ is contained in the closure of $T_{>0}$ in $\bar{T}$. In particular, $\pi_{\bar{T}}(z) = t_1 t_J$ is contained in the closure of $T_{>0}$ in $\bar{T}$. Therefore for any $j \in J$, $\alpha_j(t_1) > 0$. Now let t_2 be the unique element in T such that

$$\alpha_j(t_2) = \begin{cases} \alpha_j(t_1), & \text{if } j \in J; \\ \alpha_j(t_1)^2, & \text{if } j \notin J. \end{cases}$$

Then $t_2 \in T_{>0}$ and $t_2^{-1} t_1 \in Z(L)$. The theorem is proved. □

Remark. Theorem 2.7 is analogous to the following statement in [L4, 5.4]: Assume that G is simply laced and V is the irreducible representation of G with the highest weight λ, where λ is a dominant and regular weight of G. For any $g \in G$, let $M(g)$ be the matrix of $g : V \to V$ with respect to the canonical basis of V. Then for any $g \in G$, $g \in G_{>0}$ if and only if $M(g)$ and $M\big(\psi(g)\big)$ are matrices with all the entries in $\mathbf{R}_{>0}$.

2.8. Before proving corollary 2.9, I will introduce some technical tools.

Since G is adjoint, there exists (in an essentially unique way) $\tilde{G}$ with the épinglage $(\tilde{T}, \tilde{B}^+, \tilde{B}^-, \tilde{x}_{\tilde{i}}, \tilde{y}_{\tilde{i}}; \tilde{i} \in \tilde{I})$ and an automorphism $\sigma : \tilde{G} \to \tilde{G}$ (over $\mathbf{R}$) such that the following conditions are satisfied.

(a) $\tilde{G}$ is connected semisimple adjoint algebraic group defined and split over $\mathbf{R}$.

(b) $\tilde{G}$ is simply laced.

(c) σ preserves the épinglage, that is, $\sigma(\tilde{T}) = \tilde{T}$ and there exists a permutation $\tilde{i} \to \sigma(\tilde{i})$ of $\tilde{I}$, such that $\sigma\big(\tilde{x}_{\tilde{i}}(a)\big) = \tilde{x}_{\sigma(\tilde{i})}(a), \sigma\big(\tilde{y}_{\tilde{i}}(a)\big) = \tilde{y}_{\sigma(\tilde{i})}(a)$ for all $\tilde{i} \in \tilde{I}$ and $a \in \mathbf{R}$.

(d) If $\tilde{i}_1 \neq \tilde{i}_2$ are in the same orbit of $\sigma : \tilde{I} \to \tilde{I}$, then $\tilde{i}_1, \tilde{i}_2$ do not form an edge of the Coxeter graph.

(e) $\tilde{i}$ and $\sigma(\tilde{i})$ are in the same connected component of the Coxeter graph, for any $\tilde{i} \in \tilde{I}$.

(f) There exists an isomorphism $\phi : \tilde{G}^\sigma \longrightarrow G$ (as algebraic groups over $\mathbf{R}$) which is compatible with the épinglage of G and the épinglage $(\tilde{T}^\sigma, \tilde{B}^{+\sigma}, \tilde{B}^{-\sigma}, \tilde{x}_p, \tilde{y}_p; p \in \bar{I})$ of $\tilde{G}^\sigma$, where $\bar{I}$ is the set of orbit of $\sigma : \tilde{I} \longrightarrow \tilde{I}$ and $\tilde{x}_p(a) = \prod_{\tilde{i} \in p} \tilde{x}_{\tilde{i}}(a), \tilde{y}_p(a) = \prod_{\tilde{i} \in p} \tilde{y}_{\tilde{i}}(a)$ for all $p \in \bar{I}$ and $a \in \mathbf{R}$.

Let λ be a dominant and regular weight of $\tilde{G}$ and (V, ρ) be the irreducible representation of $\tilde{G}$ with highest weight λ. Let $\bar{\tilde{G}}$ be the closure of $\{[\rho(\tilde{g})] \mid \tilde{g} \in \tilde{G}\}$ in $P\big(\mathrm{End}(V)\big)$ and $\overline{\tilde{G}^\sigma}$ be the closure of $\{[\rho(\tilde{g})] \mid \tilde{g} \in \tilde{G}^\sigma\}$ in $P\big(\mathrm{End}(V)\big)$. Then since λ is a dominant and regular weight of $\tilde{G}$ and $\lambda \mid_{\tilde{T}^\sigma}$ is a dominant and regular weight of $\tilde{G}^\sigma$, we have that $\bar{\tilde{G}}$ is the De Concini-Procesi compactification of $\tilde{G}$ and $\overline{\tilde{G}^\sigma}$ is the De Concini-Procesi compactification of $\tilde{G}^\sigma$. Since $\bar{\tilde{G}}$ is closed in $P\big(\mathrm{End}(V)\big)$, $\overline{\tilde{G}^\sigma}$ is the closure of $\{[\rho(\tilde{g})] \mid \tilde{g} \in \tilde{G}^\sigma\}$ in $\bar{\tilde{G}}$.

We have $\bar{\tilde{G}} = \bigsqcup_{\tilde{J} \subset \tilde{I}} \tilde{Z}_{\tilde{J}} = \bigsqcup_{\tilde{J} \subset \tilde{I}} (\tilde{G} \times \tilde{G}) \cdot \tilde{z}^\circ_{\tilde{J}}$ and $\overline{\tilde{G}^\sigma} = \bigsqcup_{\tilde{J} \subset \tilde{I}, \sigma \tilde{J} = \tilde{J}} (\tilde{G}^\sigma \times \tilde{G}^\sigma) \cdot \tilde{z}^\circ_{\tilde{J}}$. Moreover, σ can be extended in a unique way to an automorphism $\bar{\sigma}$ of $\bar{\tilde{G}}$. Since $\bar{\tilde{G}}^{\bar{\sigma}} = \bigsqcup_{\tilde{J} \subset \tilde{I}, \sigma \tilde{J} = \tilde{J}} (\tilde{Z}_{\tilde{J}})^{\bar{\sigma}}$ is a closed subset of $\bar{\tilde{G}}$ containing $\tilde{G}^\sigma$, we have $\overline{\tilde{G}^\sigma} \subset \bigsqcup_{\tilde{J} \subset \tilde{I}, \sigma \tilde{J} = \tilde{J}} (\tilde{Z}_{\tilde{J}})^{\bar{\sigma}}$.

By the condition (f), there exists a bijection ϕ between $\bar{I}$ and I, such that $\phi\big(\tilde{x}_p(a)\big) = x_{\phi(p)}(a)$, for all $p \in \bar{I}, a \in \mathbf{R}$. Moreover, the isomorphism ϕ from $\tilde{G}^\sigma$ to G can be extended in a unique way to an isomorphism $\bar{\phi} : \overline{\tilde{G}^\sigma} \longrightarrow \bar{G}$. It is easy to see that for any $\tilde{J} \subset \tilde{I}$ with $\sigma \tilde{J} = \tilde{J}$, we have $\bar{\phi}\big((\tilde{G}^\sigma \times \tilde{G}^\sigma) \cdot \tilde{z}^\circ_{\tilde{J}}\big) = Z_{\phi \circ \pi(\tilde{J})}$, where $\pi : \tilde{I} \longrightarrow \bar{I}$ is the map sending element of $\tilde{I}$ into the σ-orbit that contains it.

Corollary 2.9. *$Z_{J, \geqslant 0} = \bigcap_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J, >0}$ is the closure of $Z_{J, >0}$ in Z_J. As a consequence, $Z_{J, \geqslant 0}$ and $\overline{G_{>0}}$ are contractible.*

Proof. I will prove that $Z_{J, \geqslant 0} \subset \bigcap_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J, >0}$.

First, assume that G is simply laced.

For any $g \in G_{>0}$, $i_J(g) = \Big([\rho_1(g)], [\rho_2(g)]\Big)$, where $\rho_1(g)$ and $\rho_2(g)$ are matrices with all the entries in $\mathbf{R}_{>0}$. Then for any $z \in Z_{J, \geqslant 0}$, we have $i_J(z) = \Big([M_1], [M_2]\Big)$ for some matrices with all the entries in $\mathbf{R}_{\geqslant 0}$. Similarly, $i_J\big(\bar{\psi}(z)\big) = \Big([M_3], [M_4]\Big)$ for some matrices with all their entries in $\mathbf{R}_{\geqslant 0}$.

Note that for any $M'_1, M'_2, M'_3 \in gl(n)$ such that M'_1, M'_3 are matrices with all their entries in $\mathbf{R}_{>0}$ and M'_2 is a nonzero matrix with all the entries in $\mathbf{R}_{\geqslant 0}$, we have that $M'_1 M'_2 M'_3$ is a matrix with all the entries in $\mathbf{R}_{>0}$. Thus for any $g_1, g_2 \in G_{>0}$, we have that $(g_1, g_2^{-1}) \cdot z$ satisfies the condition (*) in 2.7. Moreover,

$(g_1, g_2^{-1}) \cdot z \in Z_{J,\geqslant 0}$. Therefore by 2.7, $(g_1, g_2^{-1}) \cdot z \in Z_{J,>0}$ for all $g_1, g_2 \in G_{>0}$.

In the general case, we will keep the notation of 2.8. Since the isomorphism $\phi : \tilde{G}^\sigma \to G$ is compatible with the épinglages, we have $\phi\big((\tilde{U}^\pm_{>0})^\sigma\big) = U^\pm_{>0}$, $\phi\big((\tilde{T}_{>0})^\sigma\big) = T_{>0}$ and $\phi\big((\tilde{G}_{>0})^\sigma\big) = G_{>0}$. Now for any $z \in Z_{J,\geqslant 0}$, z is contained in the closure of $G_{>0}$ in $\bar{G}$. Thus $\bar{\phi}^{-1}(z)$ is contained in the closure of $(\tilde{G}_{>0})^\sigma$ in $\overline{\tilde{G}^\sigma}$, hence contained in the closure of $(\tilde{G}_{>0})^\sigma$ in $\bar{\tilde{G}}$. Therefore, $\bar{\phi}^{-1}(z) \in \tilde{Z}_{\tilde{J},\geqslant 0}$, where $\tilde{J} = \pi^{-1} \circ \phi^{-1}(J)$.

For any $\widetilde{g_1}, \widetilde{g_2} \in (\tilde{G}_{>0})^\sigma$, we have $(\widetilde{g_1}, \widetilde{g_2}^{-1}) \cdot \bar{\phi}^{-1}(z) = (\widetilde{u_1}\tilde{t}, \widetilde{u_2}^{-1}) \cdot \tilde{z}^\circ_{\tilde{J}}$ for some $\widetilde{u_1} \in \tilde{U}^-_{>0}, \widetilde{u_2} \in \tilde{U}^+_{>0}, \tilde{t} \in \tilde{T}_{>0}$. Since $\bar{\phi}^{-1}(z) \in (\bar{\tilde{G}})^{\bar{\sigma}}$, we have $(\widetilde{g_1}, \widetilde{g_2}^{-1}) \cdot \bar{\phi}^{-1}(z) \in (\tilde{Z}_{\tilde{J},>0})^{\bar{\sigma}}$. Then

$$\begin{aligned}\bar{\sigma}\big((\widetilde{u_1}\tilde{t}, \widetilde{u_2}^{-1}) \cdot \tilde{z}^\circ_{\tilde{J}}\big) &= \big(\sigma(\widetilde{u_1}\tilde{t}), \sigma(\widetilde{u_2}^{-1})\big) \cdot \bar{\sigma}(\tilde{z}^\circ_{\tilde{J}}) = \big(\sigma(\widetilde{u_1})\sigma(\tilde{t}), \sigma(\widetilde{u_2}^{-1})\big) \cdot \tilde{z}^\circ_{\tilde{J}} \\ &= (\widetilde{u_1}\tilde{t}, \widetilde{u_2}^{-1}) \cdot \tilde{z}^\circ_{\tilde{J}}.\end{aligned}$$

Thus $\sigma(\widetilde{u_1}) = \widetilde{u_1}$ and $\sigma(\widetilde{u_2}) = \widetilde{u_2}$. Moreover, $(\tilde{t}, 1) \cdot \tilde{z}^\circ_{\tilde{J}} = \big(\sigma(\tilde{t}), 1\big) \cdot \tilde{z}^\circ_{\tilde{J}}$, that is, $\tilde{\alpha}_{\tilde{j}}(\tilde{t}) = \tilde{\alpha}_{\tilde{j}}\big(\sigma((\tilde{t})\big) = \tilde{\alpha}_{\sigma(\tilde{j})}(\tilde{t})$ for all $\tilde{j} \in \tilde{J}$, where $\{\tilde{\alpha}_{\tilde{i}} \mid \tilde{i} \in \tilde{I}\}$ is the set of simple roots of $\tilde{G}$. Let $\tilde{t}'$ be the unique element in $\tilde{T}$ such that

$$\tilde{\alpha}_{\tilde{j}}(\tilde{t}') = \begin{cases} \tilde{\alpha}_{\tilde{j}}(\tilde{t}), & \text{if } \tilde{j} \in \tilde{J}; \\ 1, & \text{otherwise .} \end{cases}$$

Then $\tilde{t}' \in (\tilde{T}_{>0})^\sigma$ and $(\tilde{t}, 1) \cdot \tilde{z}^\circ_{\tilde{J}} = (\tilde{t}', 1) \cdot \tilde{z}^\circ_{\tilde{J}}$. Thus $(\widetilde{g_1}, \widetilde{g_2}^{-1}) \cdot \bar{\phi}^{-1}(z) = (\widetilde{u_1}\tilde{t}', \widetilde{u_2}^{-1}) \cdot \tilde{z}^\circ_{\tilde{J}}$. We have

$$\begin{aligned}\big(\phi(\widetilde{g_1}), \phi(\widetilde{g_2})^{-1}\big) \cdot z &= \bar{\phi}\big((\widetilde{g_1}, \widetilde{g_2}^{-1}) \cdot \bar{\phi}^{-1}(z)\big) = \bar{\phi}\big((\widetilde{u_1}\tilde{t}', \widetilde{u_2}^{-1}) \cdot \tilde{z}^\circ_{\tilde{J}}\big) \\ &= \big(\phi(\widetilde{u_1})\phi(\tilde{t}'), \phi(\widetilde{u_2}^{-1})\big) \cdot z^\circ_J \in Z_{J,>0}.\end{aligned}$$

Since $\phi\big((\tilde{G}_{>0})^\sigma\big) = G_{>0}$, we have $Z_{J,\geqslant 0} \subset \bigcap\limits_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J,>0}$.

Note that $(1,1)$ is contained in the closure of $\{(g_1, g_2^{-1}) \mid g_1, g_2 \in G_{>0}\}$. Hence, for any $z \in \bigcap\limits_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J,>0}$, z is contained in the closure of $Z_{J,>0}$. On the other hand, $Z_{J,\geqslant 0}$ is a closed subset in Z_J. $Z_{J,\geqslant 0}$ contains $Z_{J,>0}$, hence contains the closure of $Z_{J,>0}$ in Z_J. Therefore, $Z_{J,\geqslant 0} = \bigcap\limits_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J,>0}$ is the closure of $Z_{J,>0}$ in Z_J.

Now set $g_r = \exp\big(r \sum\limits_{i \in I} (e_i + f_i)\big)$, where e_i and f_i are the Chevalley generators related to our épinglage by $x_i(1) = \exp(e_i)$ and $y_i(1) = \exp(f_i)$. Then $g_r \in G_{>0}$ for $r \in \mathbf{R}_{>0}$ (see [L1, 5.9]). Define $f : R_{\geqslant 0} \times Z_{J,\geqslant 0} \to Z_{J,\geqslant 0}$ by $f(r, z) = (g_r, g_r^{-1}) \cdot z$

for $r \in R_{\geqslant 0}$ and $z \in Z_{J,\geqslant 0}$. Then $f(0,z) = z$ and $f(1,z) \in Z_{J,>0}$ for all $z \in Z_{J,\geqslant 0}$. Using the fact that $Z_{J,>0}$ is a cell (see 2.6), it follows that $Z_{J,\geqslant 0}$ is contractible.

Similarly, define $f' : R_{\geqslant 0} \times \overline{G_{>0}} \longrightarrow \overline{G_{>0}}$ by $f'(r,z) = (g_r, g_r^{-1}) \cdot z$ for $r \in R_{\geqslant 0}$ and $z \in \overline{G_{>0}}$. Then $f'(0,z) = z$ and $f'(1,z) \in \bigsqcup_{K \subset I} Z_{K,>0}$ for all $z \in \overline{G_{>0}}$. Note that $\bigsqcup_{K \subset I} Z_{K,>0} = \big(U^-_{>0}, (U^+_{>0})^{-1}\big) \cdot \bigsqcup_{K \subset I} (T_{>0}, 1) \cdot z_K^\circ \cong U^-_{>0} \times U^+_{>0} \times \bigsqcup_{K \subset I} (T_{>0}, 1) \cdot z_K^\circ$ (see 2.6). Moreover, by [DP, 2.2], we have $\bigsqcup_{K \subset I} (T_{>0}, 1) \cdot z_K^\circ \cong R^I_{\geqslant 0}$. Thus $\bigsqcup_{K \subset I} Z_{K,>0} \cong R^{2l(w_0)}_{>0} \times R^I_{\geqslant 0}$ is contractible. Therefore $\overline{G_{>0}}$ is contractible. □

3. The cell decomposition of $Z_{J,\geqslant 0}$

3.1. For any $P \in \mathcal{P}^J, Q \in \mathcal{P}^{J^*}, B \in \mathcal{B}$ and $g_1 \in H_P, g_2 \in U_Q, g \in G$, we have $\mathrm{pos}\big(P^B, {}^{g_1 g g_2}(Q^B)\big) = \mathrm{pos}\big({}^{g_1^{-1}}(P^B), {}^{g g_2}(Q^B)\big) = \mathrm{pos}\big(P^B, {}^g(Q^B)\big)$. If moreover, $P \bowtie^g Q$, then $\mathrm{pos}\big(P^B, {}^g(Q^B)\big) = w w_0$ for some $w \in W_J$ (see 1.4). Therefore, for any $v, v' \in W$, $w, w' \in W^J$ and $y, y' \in W_J$ with $v \leqslant w$ and $v' \leqslant w'$, Lusztig introduced the subset $Z_J^{v,w,v',w';y,y'}$ and $Z_{J,>0}^{v,w,v',w';y,y'}$ of Z_J which are defined as follows.

$$\begin{aligned} Z_J^{v,w,v',w';y,y'} = \{&(P, Q, H_P g U_Q) \in Z_J \mid P \in \mathcal{P}^J_{v,w}, \psi(Q) \in \mathcal{P}^J_{v',w'}, \\ &\mathrm{pos}\big(P^{B^+}, {}^g(Q^{B^+})\big) = y w_0, \mathrm{pos}\big(P^{B^-}, {}^g(Q^{B^-})\big) = y' w_0\} \end{aligned}$$

and

$$Z_{J,>0}^{v,w,v',w';y,y'} = Z_J^{v,w,v',w';y,y'} \cap Z_{J,\geqslant 0}.$$

Then

$$Z_J = \bigsqcup_{\substack{v,v' \in W, w,w' \in W^J, y,y' \in W_J \\ v \leqslant w, v' \leqslant w'}} Z_J^{v,w,v',w';y,y'},$$

$$Z_{J,\geqslant 0} = \bigsqcup_{\substack{v,v' \in W, w,w' \in W^J, y,y' \in W_J \\ v \leqslant w, v' \leqslant w'}} Z_{J,>0}^{v,w,v',w';y,y'}.$$

Lusztig conjectured that for any $v, v' \in W, w, w' \in W^J, y, y' \in W_J$ such that $v \leqslant w, v' \leqslant w'$, $Z_{J,>0}^{v,w,v',w';y,y'}$ is either empty or a semi-algebraic cell. If it is nonempty, then it is also a connected component of $Z_J^{v,w,v',w';y,y'}$.

In this section, we will prove this conjecture. Moreover, we will show exactly when $Z_{J,>0}^{v,w,v',w';y,y'}$ is nonempty and we will give an explicit description of $Z_{J,>0}^{v,w,v',w';y,y'}$.

First, I will prove some elementary facts about the total positivity of G.

Proposition 3.2.

$$\bigcap_{u\in U^{\pm}_{>0}} u^{-1}U^{\pm}_{>0}=\bigcap_{u\in U^{\pm}_{>0}} U^{\pm}_{>0}u^{-1}=\bigcap_{u\in U^{\pm}_{>0}} u^{-1}U^{\pm}_{\geqslant 0}=\bigcap_{u\in U^{\pm}_{>0}} U^{\pm}_{\geqslant 0}u^{-1}=U^{\pm}_{\geqslant 0},$$

$$\bigcap_{g\in G_{>0}} g^{-1}G_{>0}=\bigcap_{g\in G_{>0}} G_{>0}g^{-1}=\bigcap_{g\in G_{>0}} g^{-1}G_{\geqslant 0}=\bigcap_{g\in G_{>0}} G_{\geqslant 0}g^{-1}=G_{\geqslant 0}.$$

Proof. I will only prove $\bigcap_{u\in U^+_{>0}} u^{-1}\cdot U^+_{>0}=U^+_{\geqslant 0}$. The rest of the equalities could be proved in the same way.

Note that $uu_1\in U^+_{>0}$ for all $u_1\in U^+_{\geqslant 0}, u\in U^+_{>0}$. Thus $u_1\in \bigcap_{u\in U^+_{>0}} u^{-1}\cdot U^+_{>0}$. On the other hand, assume that $u_1\in \bigcap_{u\in U^+_{>0}} u^{-1}\cdot U^+_{>0}$. Then $uu_1\in U^+_{>0}$ for all $u\in U^+_{>0}$. We have $u_1=\lim_{\substack{u\in U^+_{>0}\\ u\longrightarrow 1}} uu_1$ is contained in the closure of $U^+_{>0}$ in U^+, that is, $u_1\in U^+_{\geqslant 0}$. So $\bigcap_{u\in U^+_{>0}} u^{-1}\cdot U^+_{>0}=U^+_{\geqslant 0}$. $\square$

For any $v,v'\in W$, $w,w'\in W^J$ such that $v\leqslant w, v'\leqslant w'$, set $Z_J^{v,w,v',w'}=\bigsqcup_{y,y'\in W_J} Z_J^{v,w,v',w';y,y'}$ and $Z_{J,>0}^{v,w,v',w'}=\bigsqcup_{y,y'\in W_J} Z_{J,>0}^{v,w,v',w';y,y'}$. We will give a characterization of $z\in Z_{J,>0}^{v,w,v',w'}$ in 3.5.

Lemma 3.3. *For any $w\in W$, $u\in U^-_{\geqslant 0}$, $\{\pi_{U^+}(u_1u)\mid u_1\in U^+_{w,>0}\}=U^+_{w,>0}$.*

Proof. The following identities hold (see [L1, 1.3]):

(a) $tx_i(a)=x_i\big(\alpha_i(t)a\big)t, ty_i(a)=y_i\big(\alpha_i(t)^{-1}a\big)t$ for all $i\in I, t\in T, a\in \mathbf{R}$.

(b) $y_{i_1}(a)x_{i_2}(b)=x_{i_2}(b)y_{i_1}(a)$ for all $a,b\in\mathbf{R}$ and $i_1\neq i_2\in I$.

(c) $x_i(a)y_i(b)=y_i(\frac{b}{1+ab})\alpha_i^\vee(\frac{1}{1+ab})x_i(\frac{a}{1+ab})$ for all $a,b\in\mathbf{R}_{>0}, i\in I$.

Thus $U^+_{w,>0}U^-_{\geqslant 0}\subset U^-_{\geqslant 0}T_{>0}U^+_{w,>0}$ for $w\in W$. So we only need to prove that $U^+_{w,>0}\subset\{\pi_{U^+}(u_1u)\mid u_1\in U^+_{w,>0}\}$. Now I will prove the following statement:

$\{\pi_{U^+}\big(u_1y_i(a)\big)\mid u_1\in U^+_{w,>0}\}=U^+_{w,>0}$ for $i\in I, a\in\mathbf{R}_{>0}$.

We argue by induction on $l(w)$. It is easy to see that the statement holds for $w=1$. Now assume that $w\neq 1$. Then there exist $j\in I$ and $w_1\in W$ such that $w=s_jw_1$ and $l(w_1)=l(w)-1$. For any $u_1'\in U^+_{w,>0}$, we have $u_1'=u_2'u_3'$ for some $u_2'\in U^+_{s_j,>0}$ and $u_3'\in U^+_{w_1,>0}$. By induction hypothesis, there exists $u_3\in U^+_{w_1,>0}, u'\in U^-$ and $t\in T$ such that $u_3y_i(a)=u'tu_3'$. Since $U^+_{w,>0}U^-_{s_i,>0}\subset U^-_{s_i,>0}T_{>0}U^+_{w,>0}$, we have $u'\in U^-_{s_i,>0}$ and $t\in T_{>0}$.

Now by (a), we have $tu_2't^{-1}\in U^+_{s_j,>0}$. So by (b) and (c), there exists $u_2\in U^+_{s_j,>0}$ such that $\pi_{U^+}(u_2u')=tu_2't^{-1}$. Thus

$$\begin{aligned}\pi_{U^+}\big(u_2u_3y_i(a)\big)&=\pi_{U^+}\Big(\big(u_2u'\big)\big({u'}^{-1}u_3y_i(a)\big)\Big)=\pi_{U^+}\big(\pi_{U^+}(u_2u'){u'}^{-1}u_3y_i(a)\big)\\&=\pi_{U^+}(tu_2't^{-1}tu_3')=\pi_{U^+}(tu_2'u_3')=u_1'.\end{aligned}$$

So $u'_1 \in \{\pi_{U^+}(u_1 y_i(a)) \mid u_1 \in U^+_{w,>0}\}$. The statement is proved.

Now assume that $u \in U^-_{w',>0}$. I will prove the lemma by induction on $l(w')$. It is easy to see that the lemma holds for $w' = 1$. Now assume that $w' \neq 1$. Then there exist $i \in I$ and $w'_1 \in W$ such that $l(w'_1) = l(w') - 1$ and $w' = s_i w'_1$. We have $u = y_i(a) u'$ for some $a \in \mathbf{R}_{>0}$ and $u' \in U^-_{w'_1,>0}$. So

$$\begin{aligned}\{\pi_{U^+}(u_1 u) \mid u_1 \in U^+_{w,>0}\} &= \{\pi_{U^+}\big(u_1 y_i(a) u'\big) \mid u_1 \in U^+_{w,>0}\} \\ &= \{\pi_{U^+}\Big(\pi_{U^+}\big(u_1 y_i(a)\big) u\Big) \mid u_1 \in U^+_{w,>0}\} \\ &= \{\pi_{U^+}(u'_1 u') \mid u'_1 \in U^+_{w,>0}\}.\end{aligned}$$

By induction hypothesis, we have
$\{\pi_{U^+}(u_1 u) \mid u_1 \in U^+_{w,>0}\} = \{\pi_{U^+}(u'_1 u') \mid u'_1 \in U^+_{w,>0}\} = U^+_{w,>0}$. $\square$

Lemma 3.4. *Set* $Z^1_{J,>0} = \{(g_1, g_2^{-1}) \cdot z^\circ_J \mid g_1 \in U^-_{\geqslant 0} T_{>0}, g_2 \in U^+_{\geqslant 0}\}$. *Then*

(a) $Z_{J,\geqslant 0} = \bigcap_{u_1 \in U^+_{>0}, u_2^{-1} \in U^-_{>0}} (u_1^{-1}, u_2) \cdot Z^1_{J,>0}$.

(b)

$$\begin{aligned}Z^1_{J,>0} &= \bigsqcup_{w_1, w_2 \in W^J} \{({}^{u_1}P_J, {}^{u_2^{-1}}Q_J, u_1 H_{P_J} l U_{Q_J} u_2) \mid u_1 \in U^-_{w_1,>0}, u_2 \in U^+_{w_2,>0}, l \in L_{\geqslant 0}\} \\ &= \{(P, Q, \gamma) \in Z_{J,\geqslant 0} \mid P = {}^{u_1} P_J, \psi(Q) = {}^{u_2} P_J \text{ for some } u_1, u_2 \in U^-_{\geqslant 0}\}.\end{aligned}$$

Proof. (a) By 2.9 and 3.2, we have

$$\begin{aligned}Z_{J,\geqslant 0} &= \bigcap_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J,>0} = \bigcap_{\substack{t_1, t_2 \in T_{>0} \\ u_1, u_2 \in U^+_{>0}, u_3, u_4 \in U^-_{>0}}} (u_1^{-1} u_3^{-1} t_1^{-1}, u_4 u_2 t_2) \cdot Z_{J,>0} \\ &= \bigcap_{u_1 \in U^+_{>0}, u_4 \in U^-_{>0}} (u_1^{-1}, u_4) \cdot \bigcap_{u_2 \in U^+_{>0}, u_3 \in U^-_{>0}} (u_2^{-1}, u_3) \cdot \bigcap_{t_1, t_2 \in T_{>0}} (t_1^{-1}, t_2) \cdot Z_{J,>0} \\ &= \bigcap_{u_1 \in U^+_{>0}, u_4 \in U^-_{>0}} (u_1^{-1}, u_4) \cdot \bigcap_{u_2 \in U^+_{>0}, u_3 \in U^-_{>0}} (u_2^{-1}, u_3) \cdot Z_{J,>0} \\ &= \bigcap_{u_1 \in U^+_{>0}, u_4 \in U^-_{>0}} (u_1^{-1}, u_4) \cdot \bigcap_{u_2 \in U^+_{>0}, u_3 \in U^-_{>0}} \big(u_2^{-1} U^-_{>0} T_{>0}, (U^+_{>0} u_3^{-1})^{-1}\big) \cdot z^\circ_J \\ &= \bigcap_{u_1 \in U^+_{>0}, u_2^{-1} \in U^-_{>0}} (u_1^{-1}, u_2) \cdot \Big(\big(U^-_{\geqslant 0} T_{>0}, (U^+_{\geqslant 0})^{-1}\big) \cdot z^\circ_J\Big).\end{aligned}$$

(b) For any $u \in U^-_{\geqslant 0}, v \in U^+_{\geqslant 0}, t \in T_{>0}$, there exist $w_1, w_2 \in W^J, w_3, w_4 \in W_J$, such that $u = u_1 u_3$ for some $u_1 \in U^-_{w_1,>0}$, $u_3 \in U^-_{w_3,>0}$ and $v = u_4 u_2$ for some

$u_2 \in U^+_{w_2,>0}$, $u_4 \in U^+_{w_4,>0}$. Then $(ut, v^{-1}) \cdot z_J^\circ = ({}^{u_1}P_J, {}^{u_2^{-1}}Q_J, u_1 H_{P_J} u_3 t u_4 U_{Q_J} u_2)$. On the other hand, assume that $l \in L_{\geqslant 0}$, then $l = u_3 t u_4$ for some $u_3 \in U^-_{\geqslant 0}, u_4 \in U^+_{\geqslant 0}, t \in T_{>0}$. Thus for any $u_1 \in U^-_{\geqslant 0}, u_2 \in U^+_{\geqslant 0}$, we have

$({}^{u_1}P_J, {}^{u_2^{-1}}Q_J, u_1 H_{P_J} l U_{Q_J} u_2) = (u_1 u_3 t, u_2^{-1} u_4^{-1}) \cdot z_J^\circ \in Z^1_{J,>0}$.

Therefore

$$Z^1_{J,>0} = \bigsqcup_{w_1, w_2 \in W^J} \{({}^{u_1}P_J, {}^{u_2^{-1}}Q_J, u_1 H_{P_J} l U_{Q_J} u_2) \mid u_1 \in U^-_{w_1,>0}, u_2 \in U^+_{w_2,>0}, l \in L_{\geqslant 0}\}$$
$$\subset \{(P, Q, \gamma) \in Z_{J,\geqslant 0} \mid P = {}^{u_1} P_J, \psi(Q) = {}^{u_2} P_J \text{ for some } u_1, u_2 \in U^-_{\geqslant 0}\}.$$

Note that $\{{}^u P_J \mid u \in U^-_{\geqslant 0}\} = \bigsqcup_{w \in W^J} \{{}^u P_J \mid u \in U^-_{w,>0}\}$. Now assume that $z = \big({}^{u_1}P_J, {}^{\psi(u_2)^{-1}}Q_J, u_1 H_{P_J} l U_{Q_J} \psi(u_2)\big)$ for some $w_1, w_2 \in W^J$ and $u_1 \in U^-_{w_1,>0}$, $u_2 \in U^-_{w_2,>0}$, $l \in L$. To prove that $z \in Z^1_{J,>0}$, it is enough to prove that $l \in L_{\geqslant 0} Z(L)$. By part (a), for any $u_3, u_4 \in U^+_{>0}$,

$$\big(u_3, \psi(u_4)^{-1}\big) \cdot z = \big({}^{u_3 u_1}P_J, {}^{\psi(u_4 u_2)^{-1}}Q_J, u_3 u_1 H_{P_J} l U_{Q_J} \psi(u_4 u_2)\big) \in Z^1_{J,>0}.$$

Note that $u_3 u_1 = u_1' t_1 \pi_{U^+}(u_3 u_1)$ for some $u_1' \in U^-_{w_1,>0}, t_1 \in T_{>0}$ and $u_4 u_2 = u_2' t_2 \pi_{U^+}(u_4 u_2)$ for some $u_2' \in U^-_{w_2,>0}, t_2 \in T_{>0}$. So we have ${}^{u_3 u_1}P_J = {}^{u_1'} P_J$, ${}^{\psi(u_4 u_2)^{-1}}Q_J = {}^{\psi(u_2')^{-1}} Q_J$ and

$$\begin{aligned} u_3 u_1 H_{P_J} l U_{Q_J} \psi(u_4 u_2) &= u_1' t_1 \pi_{U^+}(u_3 u_1) H_{P_J} l U_{Q_J} \psi\big(\pi_{U^+}(u_4 u_2)\big) t_2 \psi(u_2') \\ &= u_1' H_{P_J} t_1 \pi_{U_J^+}(u_3 u_1) l \psi\big(\pi_{U_J^+}(u_4 u_2)\big) t_2 U_{Q_J} \psi(u_2'). \end{aligned}$$

Then $t_1 \pi_{U_J^+}(u_3 u_1) l \psi\big(\pi_{U_J^+}(u_4 u_2)\big) t_2 \in L_{\geqslant 0} Z(L)$. Since $t_1, t_2 \in T_{>0}$, we have $\pi_{U_J^+}(u_3 u_1) l \psi\big(\pi_{U_J^+}(u_4 u_2)\big) \in L_{\geqslant 0} Z(L)$ for all $u_3, u_4 \in U^+_{>0}$. By 1.8 and 3.3, $\pi_{U_J^+}(U^+_{>0} u_1) = \pi_{U_J^+}\big(\pi_{U^+}(U^+_{>0} u_1)\big) = \pi_{U_J^+}(U^+_{>0}) = U^+_{w_0^J,>0}$. Similarly, we have $\pi_{U_J^+}(U^+_{>0} u_2) = U^+_{w_0^J,>0}$. Thus

$$\begin{aligned} l &\in \bigcap_{u_3, u_4 \in U^+_{w_0^J,>0}} u_3^{-1} U^+_{w_0^J,\geqslant 0} T_{>0} Z(L) U^-_{w_0^J,\geqslant 0} \psi(u_4)^{-1} \\ &= U^+_{w_0^J,\geqslant 0} T_{>0} Z(L) U^-_{w_0^J,\geqslant 0} = L_{\geqslant 0} Z(L). \end{aligned}$$

The lemma is proved. □

Proposition 3.5. *Let $z \in Z_J^{v,w,v',w'}$, then $z \in Z_{J,>0}^{v,w,v',w'}$ if and only if for any $u_1 \in U^+_{v^{-1},>0}, u_2 \in U^+_{v'^{-1},>0}$, $\big(u_1, \psi(u_2^{-1})\big) \cdot z \in Z^1_{J,>0}$.*

Proof. Assume that $z \in \bigcap_{u_1 \in U^+_{v^{-1},>0}, u_2 \in U^+_{v'^{-1},>0}} \big(u_1^{-1}, \psi(u_2)\big) Z^1_{J,>0}$. Then we have $z = \lim_{u_1, u_2 \longrightarrow 1} \big(u_1, \psi(u_2)^{-1}\big) \cdot z$ is contained in the closure of $Z^1_{J,>0}$ in Z_J. Note that $Z_{J,>0} \subset Z^1_{J,>0} \subset Z_{J,\geqslant 0}$. Thus by 2.9, $Z_{J,\geqslant 0}$ is the closure of $Z^1_{J,>0}$ in Z_J. Therefore, z is contained in $Z_{J,\geqslant 0}$.

On the other hand, assume that $z = (P, Q, \gamma) \in Z^{v,w,v',w'}_{J,>0}$. By 3.4(a), for any $u_1 \in U^+_{v^{-1},>0}$, $u_2 \in U^+_{v'^{-1},>0}$, we have $\big(u_1, \psi(u_2^{-1})\big) \cdot z \in Z_{J,\geqslant 0}$. Moreover, we have ${}^{u_1}P = {}^{u_1'} P_J$ for some $u_1' \in U^-_{w,>0}$ (see 1.6). Similarly, we have $\psi({}^{\psi(u_2^{-1})}Q) = {}^{u_2}\psi(Q) = {}^{u_2'} P_J$ for some $u_2' \in U^-_{w',>0}$. By 3.4(b), $\big(u_1, \psi(u_2^{-1})\big) \cdot z \in Z^1_{J,>0}$. □

3.6. Now I will fix $w \in W^J$ and a reduced expression $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ of w. Assume that $w_{(j)} = w_{(j-1)} s_{i_j}$ for all $j = 1, 2, \dots, n$. Let $v \leqslant w$ and $\mathbf{v}_+ = (v_{(0)}, v_{(1)}, \dots, v_{(n)})$ the positive subexpression of $\mathbf{w}$.

Define

$$G_{\mathbf{v}_+,\mathbf{w}} = \left\{ g = g_1 g_2 \cdots g_k \,\middle|\, \begin{matrix} g_j = y_{i_j}(a_j) \text{ for } a_j \in \mathbf{R} - \{0\}, & \text{if } v_{(j-1)} = v_{(j)} \\ g_j = \dot{s}_{i_j}, & \text{if } v_{(j-1)} < v_{(j)} \end{matrix} \right\},$$

$$G_{\mathbf{v}_+,\mathbf{w},>0} = \left\{ g = g_1 g_2 \cdots g_k \,\middle|\, \begin{matrix} g_j = y_{i_j}(a_j) \text{ for } a_j \in \mathbf{R}_{>0}, & \text{if } v_{(j-1)} = v_{(j)} \\ g_j = \dot{s}_{i_j}, & \text{if } v_{(j-1)} < v_{(j)} \end{matrix} \right\}.$$

Marsh and Rietsch have proved that the morphism $g \mapsto {}^g B^+$ maps $G_{\mathbf{v}_+,\mathbf{w}}$ into $\mathcal{R}_{v,w}$ (see [MR, 5.2]) and $G_{\mathbf{v}_+,\mathbf{w},>0}$ bijectively onto $\mathcal{R}_{v,w,>0}$ (see [MR, 11.3]).

The following proposition is a technical tool needed in the proof of the main theorem.

Proposition 3.7. *For any $g \in G_{\mathbf{v}_+,\mathbf{w},>0}$, we have*

$$\bigcap_{u \in U^+_{v^{-1},>0}} \big(\pi_{U_J^+}(ug)\big)^{-1} \cdot U^+_{w_0^J,\geqslant 0} = \begin{cases} U^+_{w_0^J,\geqslant 0}, & \text{if } v \in W^J; \\ \varnothing, & \text{otherwise.} \end{cases}$$

The proof will be given in 3.13.

Lemma 3.8. *Suppose α_{i_0} is a simple root such that $v_1^{-1}\alpha_{i_0} > 0$ for $v \leqslant v_1 \leqslant w$. Then for all $g \in G_{\mathbf{v}_+,\mathbf{w},>0}$ and $a \in \mathbf{R}$, we have $x_{i_0}(a) g = g t g'$ for some $t \in T_{>0}$ and $g' \in \prod_{\alpha \in R(v)} U_\alpha \cdot \big(\dot{v}^{-1} x_{i_0}(a) \dot{v}\big)$, where $R(v) = \{\alpha \in \Phi^+ \mid v\alpha \in -\Phi^+\}$.*

Proof. Marsh and Rietsch proved in [MR, 11.8] that g is of the form $g = \big(\prod_{j \in J^\circ_{\mathbf{v}_+}} y_{v_{(j-1)}\alpha_{i_j}}(t_j)\big)\dot{v}$ and $v_{(j-1)}\alpha_{i_1} \neq \alpha_{i_0}$, for all $j = 1, 2, \dots, n$. Thus $g =$

$g_1\dot{v}$ for some $g_1 \in \prod_{\alpha\in\Phi^+-\{\alpha_{i_0}\}} U_{-\alpha}$. Set $T_1 = \{t \in T \mid \alpha_{i_0}(t) = 1\}$, then $T_1 \prod_{\alpha\in\Phi^+-\{\alpha_{i_0}\}} U_{-\alpha}$ is a normal subgroup of $\psi(P_{\{i_0\}})$. Now set $x = x_{i_0}(a)$, then $xg_1x^{-1} \in B^-$. We may assume that $xg_1x^{-1} = u_1t_1$ for some $u_1 \in U^-$ and $t_1 \in T$. Now $xg = xg_1\dot{v} = (xg_1x^{-1})x\dot{v} = u_1\dot{v}(\dot{v}^{-1}t_1\dot{v})(\dot{v}^{-1}x\dot{v})$. Moreover, by [MR, 11.8], $xg \in gB^+$. Thus $xg = g_1\dot{v}t_2g_2g_3 = g_1(\dot{v}t_2g_2t_2^{-1}\dot{v}^{-1})\dot{v}t_2g_3$, for some $t_2 \in T$, $g_2 \in \prod_{\alpha\in R(v)} U_\alpha$ and $g_3 \in \prod_{\alpha\in\Phi^+-R(v)} U_\alpha$. Note that $g_1(\dot{v}t_2g_2t_2^{-1}\dot{v}^{-1}), u_1 \in U^-$, $t_2, \dot{v}^{-1}t_1\dot{v} \in T$ and $g_3, \dot{v}^{-1}x\dot{v} \in \prod_{\alpha\in\Phi^+-R(v)} U_\alpha$. Thus $g_1(\dot{v}t_2g_2t_2^{-1}\dot{v}^{-1}) = u_1$, $t_2 = \dot{v}^{-1}t_1\dot{v}$ and $g_3 = \dot{v}^{-1}x\dot{v}$. Note that $g^{-1}x_{i_0}(b)g \in B^+$ for $b \in \mathbf{R}$ (see [MR, 11.8]). We have that $\{\pi_T(g^{-1}x_{i_0}(b)g) \mid b \in \mathbf{R}\}$ is connected and contains $\pi_T\big(g^{-1}x_{i_0}(0)g\big) = 1$. Hence $\pi_T(g^{-1}x_{i_0}(b)g) \in T_{>0}$ for $b \in \mathbf{R}$. In particular, $\pi_T(g^{-1}xg) = t_2 \in T_{>0}$. Therefore $xg = gt_2g'$ with $t_2 \in T_{>0}$ and $g' = g_2g_3 \in \prod_{\alpha\in R(v)} U_\alpha \cdot (\dot{v}^{-1}x\dot{v})$. □

Remark. In [MR, 11.9], Marsh and Rietsch pointed out that for any $j \in J^+_{\mathbf{v}_+}$, we have $u^{-1}\alpha_{i_j} > 0$ for all $v_{(j)}^{-1}v \leqslant u \leqslant w_{(j)}^{-1}w$.

3.9. Suppose that $J^+_{\mathbf{v}_+} = \{j_1, j_2, \dots, j_k\}$, where $j_1 < j_2 < \cdots < j_k$ and $g = g_1g_2\cdots g_n$, where

$$g_j = \begin{cases} y_{i_j}(a_j) \text{ for } a_j \in \mathbf{R}_{>0}, & \text{if } j \in J^\circ_{\mathbf{v}_+}; \\ \dot{s_{i_j}}, & \text{if } j \in J^+_{\mathbf{v}_+}. \end{cases}$$

For any $m = 1, \dots, k$, define $v_m = v_{(j_m)}^{-1}v$, $g_{(m)} = g_{j_m+1}g_{j_m+2}\cdots g_n$ and $f_m(a) = g_{(m)}^{-1}x_{i_{j_m}}(-a)g_{(m)} \in B^+$ (see [MR, 11.8]). Now I will prove the following lemma.

Lemma 3.10. *Keep the notation in 3.9. Then*

(a) For any $u \in U^+_{v^{-1},>0}$, $ug = g'tu'$ for some $g' \in U^-_{w,>0}, t \in T_{>0}$ and $u' \in U^+$.

(b)

$$\pi_{U^+}(U^+_{v^{-1},>0}g) = \{\pi_{U^+}\big(f_k(a_k)f_{k-1}(a_{k-1})\cdots f_1(a_1)\big) \mid a_1, a_2, \dots, a_k \in \mathbf{R}_{>0}\}.$$

Proof. I will prove the lemma by induction on $l(v)$. It is easy to see that the lemma holds when $v = 1$. Now assume that $v \neq 1$.

For any $u \in U^+_{v^{-1},>0}$, since ${}^gB^+ \in \mathcal{R}_{v,w,>0}$, we have ${}^{ug}B^+ \in \mathcal{R}_{1,w,>0}$. Thus $ug = g'tu'$ for some $g' \in U^-_{w,>0}, t \in T$ and $u' \in U^+$. Set $y = g_{i_1}g_{i_2}\cdots g_{i_{j_1-1}}$. Note that $y \in U^-_{\geqslant 0}$, we have $uy = y'tu'$ for some $y' \in U^-$, $u' \in U^+_{v^{-1},>0}$ and $t \in T_{>0}$. Hence $\pi_T(ug) = \pi_T(uy\dot{s_{i_{j_1}}}g_{(1)}) = \pi_T(y'tu'\dot{s_{i_{j_1}}}g_{(1)}) \in T_{>0}\pi_T(u'\dot{s_{i_{j_1}}}g_{(1)})$. To prove that $\pi_T(U^+_{v^{-1},>0}g) \subset T_{>0}$, it is enough to prove that $\pi_T(u\dot{s_{i_{j_1}}} g_{(1)}) \in T_{>0}$ for all $u \in U^+_{v^{-1},>0}$.

For any $u \in U^+_{v^{-1},>0}$, we have $u = u_1 x_{i_{j_1}}(a)$ for some $u_1 \in U^+_{v^{-1}s_{i_{j_1}},>0}$ and $a \in \mathbf{R}_{>0}$. It is easy to see that $x_{i_{j_1}}(a)\dot{s_{i_{j_1}}} g_{(1)} = \alpha^\vee_{i_{j_1}}(a) y_{i_{j_1}}(a) x_{i_{j_1}}(-a^{-1}) g_{(1)}$. Note that $\alpha^\vee_{i_{j_1}}(a) \in T_{>0}$ and by 3.8, ${g_{(1)}}^{-1} x_{i_{j_1}}(-a^{-1}) g_{(1)} \in T_{>0}U^+$. Hence by 1.7, we have

$$\begin{aligned}\pi_T(u\dot{s_{i_{j_1}}} g_{(1)}) &= \pi_T\Big(u_1 \alpha^\vee_{i_{j_1}}(a) y_{i_{j_1}}(a) g_{(1)} \big({g_{(1)}}^{-1} x_{i_{j_1}}(-a^{-1}) g_{(1)}\big)\Big) \\ &\in T_{>0}\pi_T\big(U^+_{v^{-1}s_{i_{j_1}},>0} y_{i_{j_1}}(a) g_{(1)}\big) T_{>0}.\end{aligned}$$

Set

$$\begin{aligned}\mathbf{w}' &= (1, w^{-1}_{(j_1-1)} w_{(j_1)}, \dots, w^{-1}_{(j_1-1)} w_{(n)}), \\ \mathbf{v}'_+ &= (1, s_{i_{j_1}} v_{(j_1)}, s_{i_{j_1}} v_{(j_1+1)}, \dots, s_{i_{j_1}} v_{(n)}).\end{aligned}$$

Then $\mathbf{w}'$ is a reduced expression of $w^{-1}_{(j_1-1)} w_{(n)}$ and $\mathbf{v}'_+$ is a positive subexpression of $\mathbf{w}'$. For any $a \in \mathbf{R}_{>0}$, $y_{i_{j_1}}(a) g_{(1)} \in G_{\mathbf{v}'_+, \mathbf{w}', >0}$. Thus by induction hypothesis, for any $a \in \mathbf{R}_{>0}$, $\pi_T(U^+_{v^{-1}s_{i_{j_1}},>0} y_{i_{j_1}}(a) g_{(1)}) \subset T_{>0}$. Therefore, $\pi_T(ug) \in T_{>0}$. Part (a) is proved.

We have

$$\begin{aligned}\pi_{U^+}(U^+_{v^{-1},>0}g) &= \pi_{U^+}(U^+_{v^{-1},>0} y \dot{s_{i_{j_1}}} g_{(1)}) = \pi_{U^+}(\pi_{U^+}\big(U^+_{v^{-1},>0} y) \dot{s_{i_{j_1}}} g_{(1)}\big) \\ &= \pi_{U^+}(U^+_{v^{-1},>0} \dot{s_{i_{j_1}}} g_{(1)}) = \bigcup_{a \in \mathbf{R}_{>0}} \pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0} x_{i_{j_1}}(a^{-1}) \dot{s_{i_{j_1}}} g_{(1)}\big) \\ &= \bigcup_{a \in \mathbf{R}_{>0}} \pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0} \alpha^\vee_{i_{j_1}}(a^{-1}) y_{i_{j_1}}(a^{-1}) g_{(1)} f_1(a)\big) \\ &= \bigcup_{a \in \mathbf{R}_{>0}} \pi_{U^+}\Big(\pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0} \alpha^\vee_{i_{j_1}}(a^{-1}) y_{i_{j_1}}(a^{-1})\big) g_{(1)} f_1(a)\Big) \\ &= \bigcup_{a \in \mathbf{R}_{>0}} \pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0} g_{(1)} f_1(a)\big) = \bigcup_{a \in \mathbf{R}_{>0}} \pi_{U^+}\Big(\pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0} g_{(1)}\big) f_1(a)\Big).\end{aligned}$$

By induction hypothesis,

$$\pi_{U^+}(U^+_{v^{-1}s_{i_{j_1}},>0} g_{(1)}) = \{\pi_{U^+}\big(f_k(a_k) f_{k-1}(a_{k-1}) \cdots f_2(a_2)\big) \mid a_2, a_3, \dots, a_k \in \mathbf{R}_{>0}\}.$$

Thus

$$\begin{aligned}\pi_{U^+}(U^+_{v^{-1},>0}g) &= \bigcup_{a \in \mathbf{R}_{>0}} \pi_{U^+}\Big(\pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0} g_{(1)}\big) f_1(a)\Big) \\ &= \{\pi_{U^+}\big(f_k(a_k) f_{k-1}(a_{k-1}) \cdots f_1(a_1)\big) \mid a_1, a_2, \dots, a_k \in \mathbf{R}_{>0}\}. \quad \square\end{aligned}$$

Remark.. The referee pointed out to me that the assertion $t \in T_{>0}$ of 3.10(a) could also been proved using generalized minors.

Lemma 3.11. *Assume that α is a positive root and $u \in U_\alpha$, $u' \in U^+$ such that $u^n u' \in U^+_{\geqslant 0}$ for all $n \in \mathbf{N}$. Then $u = x_i(a)$ for some $i \in I$ and $a \in \mathbf{R}_{\geqslant 0}$.*

Proof. There exists $t \in T_{>0}$, such that $\alpha_i(t) = 2$ for all $i \in I$. Then $tut^{-1} = u^{\alpha(t)} = u^m$ for some $m \in \mathbf{N}$. By assumption, $t^n u t^{-n} u' \in U^+_{\geqslant 0}$ for all $n \in \mathbf{N}$. Thus $u\big(t^{-n} u' t^n\big) = t^{-n}\big(t^n u t^{-n} u'\big) t^n \in U^+_{\geqslant 0}$. Moreover, it is easy to see that $\lim\limits_{n \to \infty} t^{-n} u' t^n = 1$. Since $U^+_{\geqslant 0}$ is a closed subset of U^+, $\lim\limits_{n \to \infty} u t^{-n} u' t^n = u \in U^+_{\geqslant 0}$. Thus $u = x_i(a)$ for some $i \in I$ and $a \in \mathbf{R}_{\geqslant 0}$. □

Lemma 3.12. *Assume that $w \in W$ and $i, j \in I$ such that $w^{-1}\alpha_i = \alpha_j$. Then there exists $c \in \mathbf{R}_{>0}$, such that $\dot{w}^{-1} x_i(a) \dot{w} = x_j(ca)$ for all $a \in \mathbf{R}$.*

Proof. There exist $c, c' \in \mathbf{R} - \{0\}$, such that $y_i(a)\dot{w} = \dot{w} y_j(c'a)$ and $x_i(a)\dot{w} = \dot{w} x_j(ca)$ for $a \in \mathbf{R}$. Since ${}^{\dot{w}}B^- \in \mathcal{B}_{\geqslant 0}$, we have ${}^{y_i(1)\dot{w}}B^+ = {}^{\dot{w} y_j(c')}B^+ \in \mathcal{B}_{\geqslant 0}$. By 3.6, $c' \geqslant 0$. Thus $c' > 0$. Moreover, since $w\alpha_j = \alpha_i > 0$, we have $w s_j w^{-1} = s_i$ and $l(ws_j) = l(s_i w) = l(w) + 1$. Hence, setting $w' = ws_j = s_i w$, we have $\dot{w}' = \dot{w}\dot{s_j} = \dot{s_i}\dot{w}$, that is $\dot{w} x_i(-1) y_i(1) x_i(-1) = x_j(-c) y_j(c') x_i(-c)\dot{w} = x_j(-1) y_j(1) x_j(-1)\dot{w}$. Therefore, $c = {c'}^{-1} > 0$. □

3.13. Proof of Proposition 3.7. If $v \in W^J$, then $v\alpha > 0$ for $\alpha \in \Phi^+_J$. So $\pi_{U^+_J}(\prod\limits_{\alpha \in R(v)} U_\alpha) = \{1\}$. By 3.8, $f_m(a) \in T(\prod\limits_{\alpha \in R(v_m)} U_\alpha) \cdot U_{v_m^{-1}\alpha_{i_{j_m}}}$ for all $m \in \{1, 2, \dots, k\}$. Note that $v\alpha \in -\Phi^+$ for all $a \in R(v_m)$ and $v v_m^{-1} \alpha_{i_{j_m}} = v_{(j_m)} \alpha_{i_{j_m}} \in -\Phi^+$. So $f_m(a) \in T \prod\limits_{\alpha \in R(v)} U_\alpha$ and $f_k(a_k) f_{k-1}(a_{k-1}) \cdots f_1(a_1) \in T \prod\limits_{\alpha \in R(v)} U_\alpha$. Hence by 3.10(b), $\pi_{U^+_J}(ug) = 1$ for all $u \in U^+_{v^{-1}, >0}$. Therefore

$$\bigcap_{u \in U^+_{v^{-1}, >0}} \big(\pi_{U^+_J}(ug)\big)^{-1} \cdot U^+_{w_0^J, \geqslant 0} = U^+_{w_0^J, \geqslant 0}.$$

If $v \notin W^J$, then there exists $\alpha \in \Phi^+_J$ such that $v\alpha \in -\Phi^+_J$, that is, $v_m^{-1}\alpha_{i_{j_m}} \in \Phi^+_J$ for some $m \in \{1, 2, \dots, k\}$. Set $k_0 = \max\{m \mid v_m^{-1}\alpha_{i_{j_m}} \in \Phi^+_J\}$. Then since $R(v_{k_0}) = \{v_m^{-1}\alpha_{i_{j_m}} \mid m > k_0\}$, we have that $v_{k_0}\alpha > 0$ for $\alpha \in \Phi^+_J$. Hence by 3.8, $\pi_{U^+_J}\big(f_{k_0}(a)\big) = \dot{v_{k_0}}^{-1} x_{i_{j_{k_0}}}(-a) \dot{v_{k_0}}$. If $u' \in \bigcap\limits_{u \in U^+_{v^{-1}, >0}} \big(\pi_{U^+_J}(ug)\big)^{-1} \cdot U^+_{w_0^J, \geqslant 0}$, then $\pi_{U^+_J}\big(f_k(a_k) f_{k-1}(a_{k-1}) \cdots f_1(a_1)\big) u' \in U^+_{w_0^J, \geqslant 0}$ for all $a_1, a_2, \dots, a_k \in \mathbf{R}_{>0}$. Since $U^+_{w_0^J, \geqslant 0}$ is a closed subset of G, $\pi_{U^+_J}\big(f_k(a_k) f_{k-1}(a_{k-1}) \cdots f_1(a_1)\big) u' \in U^+_{w_0^J, \geqslant 0}$ for all $a_1, a_2, \dots, a_k \in \mathbf{R}_{\geqslant 0}$. Now take $a_m = 0$ for $m \in \{1, 2, \dots, k\} - \{k_0\}$, then $\pi_{U^+_J}\big(f_{k_0}(a)\big) u' \in U^+_{w_0^J, \geqslant 0}$ for all $a \in \mathbf{R}_{>0}$. Set $u_1 = \dot{v_{k_0}}^{-1} x_{i_{j_{k_0}}}(-1) \dot{v_{k_0}}$. Then $u_1^n u' \in U^+_{w_0^J, \geqslant 0}$ for all $n \in N$. Thus by 3.11, $v_{k_0}^{-1} \alpha_{i_{j_{k_0}}} = \alpha_{j'}$ for some $j' \in J$ and $u_1 \in U^+_{w_0^J, \geqslant 0}$. By 3.12, $u_1 = x_{j'}(-c)$ for some $c \in \mathbf{R}_{>0}$. That is a contradiction. The proposition is proved. □

Let me recall that $L = P_J \bigcap Q_J$ (see 2.4). Now I will prove the main theorem.

Theorem 3.14. *For any $v, w, v', w' \in W^J$ such that $v \leqslant w, v' \leqslant w'$, set*

$$\tilde{Z}_{J,>0}^{v,w,v',w'} = \Big\{ \big({}^{g}P_J, {}^{\psi(g')^{-1}} Q_J, gH_{p_J} l U_{Q_J} \psi(g') \big) \Big| \begin{array}{l} g \in G_{\mathbf{v}_+,\mathbf{w},>0}, \quad g' \in G_{\mathbf{v}'_+,\mathbf{w}',>0} \\ \text{and } l \in L_{\geqslant 0} \end{array} \Big\}.$$

Then

$$Z_{J,>0}^{v,w,v',w'} = \begin{cases} \tilde{Z}_{J,>0}^{v,w,v',w'}, & \text{if } v,w,v',w' \in W^J, v \leqslant w, v' \leqslant w'; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Proof. Note that $\{(P, Q, \gamma) \in Z_J \mid P \in \mathcal{P}^J_{\geqslant 0}, \psi(Q) \in \mathcal{P}^J_{\geqslant 0}\}$ is a closed subset containing $Z_{J,>0}$. Hence it contains $Z_{J,\geqslant 0}$. Now fix $g \in G_{\mathbf{v}_+,\mathbf{w},>0}, g' \in G_{\mathbf{v}'_+,\mathbf{w}',>0}$ and $l \in L$. By 3.10 (a), for any $u \in U^+_{v^{-1},>0}$, $ug = at\pi_{U^+}(ug)$ for some $a \in U^-_{w,>0}$ and $t \in T_{>0}$. Similarly, for any $u' \in U^+_{v'^{-1},>0}$, $u'g' = a't'\pi_{U^+}(u'g')$ for some $a' \in U^-_{w',>0}$ and $t' \in T_{>0}$. Set $z = \big({}^{g}P_J, {}^{\psi(g')^{-1}} Q_J, gH_{p_J} l U_{Q_J} \psi(g') \big)$. We have

$$\begin{aligned} \big(u, \psi(u')^{-1}\big) \cdot z &= \Big({}^{a}P_J, {}^{\psi(a')^{-1}} Q_J, at\pi_{U^+}(ug) H_{P_J} l U_{Q_J} \psi\big(\pi_{U^+}(u'g')\big) t' \psi(a') \Big) \\ &= \Big({}^{a}P_J, {}^{\psi(a')^{-1}} Q_J, aH_{P_J} t \pi_{U_J^+}(ug) l \psi\big(\pi_{U_J^+}(u'g')\big) t' U_{Q_J} \psi(a') \Big). \end{aligned}$$

Then $\big(u, \psi(u')^{-1}\big) \cdot z \in Z^1_{J,>0}$ if and only if $t\pi_{U_J^+}(ug) l \psi\big(\pi_{U_J^+}(u'g')\big) t' \in L_{\geqslant 0} Z(L)$, that is,

$$\begin{aligned} l &\in \pi_{U_J^+}(ug)^{-1} L_{\geqslant 0} Z(L) \psi\big(\pi_{U_J^+}(u'g')\big)^{-1} \\ &= \big(\pi_{U_J^+}(ug)^{-1} U^+_{w_0^J,\geqslant 0}\big) T_{>0} Z(L) \psi\big(\pi_{U_J^+}(u'g')^{-1} U^+_{w_0^J,\geqslant 0}\big). \end{aligned}$$

So by 3.5, $z \in Z_{J,\geqslant 0}$ if and only if

$$\begin{aligned} l &\in \bigcap_{\substack{u \in U^+_{v^{-1},>0} \\ u' \in U^+_{v'^{-1},>0}}} \big(\pi_{U_J^+}(ug)^{-1} U^+_{w_0^J,\geqslant 0}\big) T_{>0} Z(L) \psi\big(\pi_{U_J^+}(u'g')^{-1} U^+_{w_0^J,\geqslant 0}\big) \\ &= \bigcap_{u \in U^+_{v^{-1},>0}} \big(\pi_{U_J^+}(ug)^{-1} U^+_{w_0^J,\geqslant 0}\big) T_{>0} Z(L) \psi\Big(\bigcap_{u' \in U^+_{v'^{-1},>0}} \pi_{U_J^+}(u'g')^{-1} U^+_{w_0^J,\geqslant 0}\Big). \end{aligned}$$

By 3.7, $z \in Z_{J,\geqslant 0}$ if and only if $v, v' \in W^J$ and $l \in L_{\geqslant 0} Z(L)$. The theorem is proved. □

3.15. It is known that $G_{\geqslant 0} = \bigsqcup_{w,w' \in W} U^-_{w,>0} T_{>0} U^+_{w',>0}$, where for any $w, w' \in W$, $U^-_{w,>0} T_{>0} U^+_{w',>0}$ is a semi-algebraic cell (see [L1, 2.11]) and is a connected component of $B^+ \dot{w} B^+ \cap B^- \dot{w}' B^-$ (see [FZ]). Moreover, Rietsch proved in [R2, 2.8] that $\mathcal{B}_{\geqslant 0} = \bigsqcup_{v \leqslant w} \mathcal{R}_{v,w,>0}$, where for any $v, w \in W$ such that $v \leqslant w$, $\mathcal{R}_{v,w,>0}$ is a semi-algebraic cell and is a connected component of $\mathcal{R}_{v,w}$.

The following result generalizes these facts.

Corollary 3.16. $\overline{G_{>0}} = \bigsqcup_{J \subset I} \bigsqcup_{\substack{v,w,v',w' \in W^J \\ v \leqslant w, v' \leqslant w'}} \bigsqcup_{y,y' \in W_J} Z^{v,w,v',w';y,y'}_{J,>0}$. *Moreover, for any* $v, w, v', w' \in W^J$, $y, y' \in W_J$ *with* $v \leqslant w$, $v' \leqslant w'$, $Z^{v,w,v',w';y,y'}_{J,>0}$ *is a connected component of* $Z^{v,w,v',w';y,y'}_J$ *and is a semi-algebraic cell which is isomorphic to* $\mathbf{R}^d_{>0}$, *where* $d = l(w) + l(w') + 2l(w^J_0) + \mid J \mid - l(v) - l(v') - l(y) - l(y')$.

Proof. $\mathcal{P}^J_{v,w,>0}$ (resp. $\mathcal{P}^J_{v',w',>0}$) is a connected component of $\mathcal{P}^J_{v,w}$ (resp. $\mathcal{P}^J_{v',w'}$) (see [L3]). Thus $\{(P, Q, \gamma) \in Z^{v,w,v',w';y,y'}_J \mid P \in \mathcal{P}^J_{v,w,>0}, \psi(Q) \in \mathcal{P}^J_{v',w',>0}\}$ is open and closed in $Z^{v,w,v',w';y,y'}_J$. To prove that $Z^{v,w,v',w';y,y'}_{J,>0}$ is a connected component of $Z^{v,w,v',w';y,y'}_J$, it is enough to prove that $Z^{v,w,v',w';y,y'}_{J,>0}$ is a connected component of $\{(P, Q, \gamma) \in Z^{v,w,v',w';y,y'}_J \mid P \in \mathcal{P}^J_{v,w,>0}, \psi(Q) \in \mathcal{P}^J_{v',w',>0}\}$.

Assume that $g \in G_{\mathbf{v}_+,\mathbf{w},>0}, g' \in G_{\mathbf{v}'_+,\mathbf{w}',>0}$ and $l \in L$. We have that $({}^g P_J)^{B^+}$ is the unique element $B \in \mathcal{R}_{v,w}$ that is contained in ${}^g P_J$(see 1.4). Therefore $({}^g P_J)^{B^+} = {}^g B^+$. Similarly, $({}^g P_J)^{B^-} = {}^{g \dot{w}^J_0} B^+$, $({}^{\psi(g'^{-1})} Q_J)^{B^+} = {}^{\psi(g'^{-1}) \dot{w}^J_0} B^-$ and $({}^{\psi(g'^{-1})} Q_J)^{B^-} = {}^{\psi(g')^{-1}} B^-$. Thus $\text{pos}\Big(({}^g P_J)^{B^+}, {}^{g l \psi(g')} \big(({}^{\psi(g'^{-1})} Q_J)^{B^+}\big)\Big) = \text{pos}(B^+, {}^{l \dot{w}^J_0} B^-)$ and $\text{pos}\Big(({}^g P_J)^{B^-}, {}^{g l \psi(g')} \big(({}^{\psi(g'^{-1})} Q_J)^{B^-}\big)\Big) = \text{pos}({}^{\dot{w}^J_0} B^+, {}^l B^-)$. Therefore we have that $({}^g P_J, {}^{\psi(g')^{-1}} Q_J, g H_{P_J} l U_{Q_J} \psi(g')) \in Z^{v,w,v',w';y,y'}_J$ if and only if $l \in B^+ \dot{y} \dot{w}_0 B^+ \dot{w}_0 \dot{w}^J_0 \cap \dot{w}^J_0 B^+ \dot{y}' \dot{w}_0 B^+ \dot{w}_0 = B^+ \dot{y} B^- \dot{w}^J_0 \cap \dot{w}^J_0 B^+ \dot{y}' B^-$.

Note that $L \cap B^+ \subset {}^{\dot{w}^J_0} B^-$. Thus for any $x \in W_J$, $(L \cap B^+) \dot{x} (L \cap B^+) \subset B^+ \dot{x} \dot{w}^J_0 B^- \dot{w}^J_0$. Therefore,

$$\begin{aligned} L \cap B^+ \dot{y} B^- \dot{w}^J_0 &= \bigsqcup_{x \in W_J} (L \cap B^+) \dot{x} (L \cap B^+) \cap B^+ \dot{y} B^- \dot{w}^J_0 \\ &= (L \cap B^+) \dot{y} \dot{w}^J_0 (L \cap B^+). \end{aligned}$$

Similarly, $L \cap \dot{w}^J_0 B^+ \dot{y}' B^- = (L \cap B^-) \dot{w}^J_0 \dot{y}' (L \cap B^-)$.

Then $\{(P, Q, \gamma) \in Z^{v,w,v',w';y,y'}_J \mid P \in \mathcal{P}^J_{v,w,>0}, \psi(Q) \in \mathcal{P}^J_{v',w',>0}\}$ is isomorphic to $G_{v,w,>0} \times G_{v',w',>0} \times \big((L \cap B^+) \dot{y} \dot{w}^J_0 (L \cap B^+) \cap (L \cap B^-) \dot{w}^J_0 \dot{y}' (L \cap B^-)\big)/Z(L)$. Note that $\big((L \cap B^+) \dot{y} \dot{w}^J_0 (L \cap B^+) \cap (L \cap B^-) \dot{w}^J_0 \dot{y}' (L \cap B^-)\big) \cap L_{\geqslant 0} = U^-_{y w^J_0,>0} T_{>0} U^+_{w^J_0 y',>0}$.

Therefore

$$Z_{J,>0}^{v,w,v',w';y,y'} \cong G_{v,w,>0} \times G_{v',w',>0} \times U^-_{yw_0^J,>0} T_{>0} U^+_{w_0^J y',>0} / \big(Z(L) \cap T_{>0}\big)$$
$$\cong \mathbf{R}_{>0}^{l(w)+l(w')+2l(w_0^J)+|J|-l(v)-l(v')-l(y)-l(y')}.$$

By 3.15, we have that $U^-_{yw_0^J,>0} T_{>0} U^+_{w_0^J y',>0} / \big(Z(L) \cap T_{>0}\big)$ is a connected component of $\big((L \cap B^+)\dot{y}\dot{w}_0^J(L \cap B^+) \cap (L \cap B^-)\dot{w}_0^J\dot{y'}(L \cap B^-)\big)/Z(L)$. The corollary is proved. □

Acknowledgements.

I thank George Lusztig for suggesting the problem and for many helpful discussions. I also thank the referee for pointing out several mistakes in the original manuscript and for some useful comments, especially concerning 3.8, 3.10 and 3.15.

References

[DP] De Concini, C., and C.Procesi, *Complete symmetric varieties*, Invariant theory (Montecatini 1982), Lect. Notes Math., vol. 996, Springer, 1983, pp. 1-44.

[FZ] Fomin, S., and A. Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), no. 2 335–380.

[MR] Marsh, R., and K. Rietsch, *Parametrizations of flag varieties*, RT/0307017.

[KL] Kazhdan, D., and G. Lusztig, *Representations of Coxeter groups and Hecke algebras,*, Invent. Math. **53** (1979), 165-184.

[L1] Lusztig, G., *Total positivity in reductive groups*, Lie Theory and Geometry: in honor of Bertram Kostant, Progress in Math. **123** (1994), Birkhäuser, Boston, 531-568.

[L2] Lusztig, G., *Total positivity and canonical bases*, Algebraic groups and Lie groups(ed. G.I.Lehrer), Cambridge Univ. Press, 1997, pp. 281-295.

[L3] Lusztig, G., *Total positivity in partial flag manifolds*, Represent.Theory **2** (1998), 70-78(electronic).

[L4] Lusztig, G., *Introduction to total positivity*, Positivity in Lie theory: open problems(ed. J.Hilgert,J.D.Lawson,K.H.Neeb,E.B.Vinberg), de Gruyter Berlin, 1998, pp. 133-145.

[L5] Lusztig, G., *Parabolic character sheaves, I*, RT/0302151.

[L6] Lusztig, G., *Parabolic character sheaves, II*, RT/0302317.

[R1] Rietsch, K., *Total positivity and real flag varieties*, MIT thesis (1998).

[R2] Rietsch, K., *An algebraic cell decomposition of the nonnegative part of a flag variety*, J. Algebra **213** (1999), no. 1 144-154.

Department of Mathematics, M.I.T., Cambridge, MA 02139

E-mail address: hugo@math.mit.edu