ELEMENTS WITH FINITE COXETER PART IN AN AFFINE WEYL GROUP

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Abstract. Let W_a be an affine Weyl group and $\eta : W_a \longrightarrow W_0$ be the natural projection to the corresponding finite Weyl group. We say that $w \in W_a$ has finite Coxeter part if $\eta(w)$ is conjugate to a Coxeter element of W_0. The elements with finite Coxeter part is a union of conjugacy classes of W_a. We show that for each conjugacy class $\mathcal{O}$ of W_a with finite Coxeter part there exits a unique maximal proper parabolic subgroup W_J of W_a, such that the set of minimal length elements in $\mathcal{O}$ is exactly the set of Coxeter elements in W_J. Similar results hold for twisted conjugacy classes.

Introduction

In [3], Geck and Pfeiffer showed that elements of minimal length in the conjugacy classes of finite Weyl groups play a quite special role. The results on minimal length elements have a lot of applications in representation theory of finite Hecke algebra and algebraic groups, as well as the geometry of unipotent classes.

Recently, the first author, joint with Nie [4], [6] studied minimal length elements in the conjugacy classes of affine Weyl groups and showed that these elements also play a special role. It is expected that the minimal length elements will have applications in representation theory of affine Hecke algebra and p-adic groups, as well as reduction of Shimura varieties.

Although the proof of [6] is case-free, it is still useful to have concrete data available for each conjugacy class. In this paper, we study some special conjugacy classes of affine Weyl groups and give an explicit description of the minimal length elements in these conjugacy classes. We show that the minimal length elements in a conjugacy class with finite Coxeter part (see §1.6 for the precise definition) are exactly the Coxeter elements for a unique maximal proper parabolic subgroup of the affine Weyl group. The precise statement for this "Coxeter=Coxeter" theorem is Theorem 1.1.

This result is also a necessary ingredient of in the study of dimension formula of affine Deligne-Lusztig varieties. See [2] and [5].

1. The main Theorem

1.1. Let S be a finite set and $(m_{ij})_{i,j\in S}$ be a matrix with entries in $\mathbb{N}\cup\{\infty\}$ such that $m_{ii}=1$ and $m_{ij}=m_{ji}\geqslant 2$ for all $i\neq j$. Let W be a group defined by generators s_i for $i\in S$ and relations $(s_is_j)^{m_{ij}}=1$ for $i,j\in S$ with $m_{ij}<\infty$. We say that (W,S) is a *Coxeter group*. Sometimes we just call W itself a Coxeter group.

Let H be a group of automorphisms of the group W that preserves S. Set $W'=W\rtimes H$. Then an element in W' is of the form $w\delta$ for some $w\in W$ and $\delta\in H$. We have that $(w\delta)(w'\delta')=w\delta(w')\delta\delta'\in W'$ with $\delta,\delta'\in H$.

For $w\in W$ and $\delta\in H$, we set $\ell(w\delta)=\ell(w)$, where $\ell(w)$ is the length of w in the Coxeter group (W,S). Thus H consists of length 0 elements in W'.

For $J\subset S$, we denote by W_J the standard parabolic subgroup of W generated by s_j for $j\in J$ and by W^J (resp. ${}^JW$) the set of minimal coset representatives in W/W_J (resp. $W_J\backslash W$).

For $J\subset S$ with W_J finite, we denote by w_0^J the maximal element in W_J.

1.2. For $w\in W$, we denote by $\operatorname{supp}(w)$ the set of $i\in S$ such that s_i appears in some (or equivalently, any) reduced expression of w. For $w\in W$ and $\delta\in H$, we set $\operatorname{supp}(w\delta)=\cup_{n\in\mathbb{Z}}\delta^n(\operatorname{supp}(w))$. Then $\operatorname{supp}(w\delta)$ is the minimal δ-stable subset J of S such that $w\delta\in W_J\rtimes\langle\delta\rangle\subset W'$.

We follow [8, 7.3]. Let $\delta\in H$. For each δ-orbit in S, we pick a simple reflection. Let g be the product of these simple reflections (in any order) and put $c=g\delta\in W'$. We call c a *Coxeter element* of W'. Then $\operatorname{supp}(c)=S$ for any Coxeter element c of W'.

1.3. Let Φ be an irreducible reduced root system and W_0 be the corresponding finite Weyl group. Then (W_0,S_0) is a Coxeter group, where $S_0=\{i:\ s_i \text{ is a simple reflection in } W_0\}$.

Let $P^\vee$ be the coweight lattice and $Q^\vee$ be the coroot lattice. Let

$$W_a=Q^\vee\rtimes W_0=\{t^\chi w;\chi\in Q^\vee,w\in W_0\}$$

be the associated affine Weyl group and

$$\tilde{W}=P^\vee\rtimes W_0=\{t^\chi w;\chi\in P^\vee,w\in W_0\}$$

be the associated extended affine Weyl group. The multiplication is given by the formula $(t^\chi w)(t^{\chi'}w')=t^{\chi+w\chi'}ww'$.

Set $\tilde{S}=S_0\cup\{0\}$ and $s_0=t^{\theta^\vee}s_\theta$, where θ is the corresponding largest positive root. Then W_a is a normal subgroup of $\tilde{W}$ and is a Coxeter group with generators s_i (for $i\in\tilde{S}$).

Following [7], we define the length function on $\tilde{W}$ by

$$\ell(t^\chi w)=\sum_{\alpha\in\Phi^+,w^{-1}(\alpha)\in\Phi^+}|\langle\chi,\alpha\rangle|+\sum_{\alpha\in\Phi^+,w^{-1}(\alpha)\in\Phi^-}|\langle\chi,\alpha\rangle-1|.$$

For any coset of W_a in $\tilde{W}$, there is a unique element of length 0. Moreover, there is a natural group isomorphism between $\Omega = \{\tau \in \tilde{W}; \ell(\tau) = 0\}$ and $\tilde{W}/W_a \cong P^\vee/Q^\vee$.

1.4. Let δ be a diagram automorphism of (W_0, S_0) and $\langle\delta\rangle$ be the group of automorphisms on W_0 generated by δ. Set

$$W_0' = W_0 \rtimes \langle\delta\rangle.$$

Notice that δ induces natural actions on $Q^\vee$, $P^\vee$, W_a and $\tilde{W}$, which we still denote by δ. It also gives a bijection on $\tilde{S}$ which sends S_0 to S_0 and sends $0 \in \tilde{S}$ to 0. Set

$$\tilde{W}' = P^\vee \rtimes W_0' = \tilde{W} \rtimes \langle\delta\rangle.$$

Then $\Omega' = \Omega \rtimes \langle\delta\rangle$ is the set of length 0 elements in $\tilde{W}'$ and $\tilde{W}' = W_a \rtimes \Omega'$.

1.5. Define the action of W_0 on W_0' by $w \cdot w' = ww'w^{-1}$. Each orbit of W_0 is called a W_0-*conjugacy class* of W_0'. We define W_a-conjugacy classes and $\tilde{W}$-conjugacy classes of $\tilde{W}'$ in the same way. Notice that W_a is a normal subgroup of $\tilde{W}'$. Thus each W_a-conjugacy class of $\tilde{W}'$ is contained in $W_a\tau$ for some $\tau \in \Omega'$.

Let $\eta : \tilde{W}' \to W_0'$ be the projection map, i.e., $\eta(t^\chi w) = w$ for any $\chi \in P^\vee$ and $w \in W_0'$. For any $\tilde{w} \in \tilde{W}'$, we call $\eta(\tilde{w})$ the *finite part* of $\tilde{w}$.

It is easy to see that η sends a $\tilde{W}$-conjugacy class of $\tilde{W}'$ to a W_0-conjugacy class of W_0'.

1.6. It is known that any two Coxeter elements of W_0' in the same coset W_0'/W_0 are conjugated by an element of W_0.

Let $\mathcal{O}$ be a W_a-conjugacy class of $\tilde{W}'$ and $\mathcal{O}'$ be a $\tilde{W}$-conjugacy class of $\tilde{W}'$. We say that $\mathcal{O}$ (resp. $\mathcal{O}'$) *has finite Coxeter part* if $\eta(\mathcal{O})$ (resp. $\eta(\mathcal{O}')$) contains a Coxeter element of W_0'. The purpose of this paper is to give an explicit description of the minimal length element in $\mathcal{O}$. We prove the following "Coxeter=Coxeter" theorem.

Theorem 1.1. *Let $\mathcal{O}$ be a W_a-conjugacy class of $\tilde{W}'$ with finite Coxeter part and $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Let $\tau \in \Omega'$ with $\mathcal{O} \subset W_a\tau$. Then there exists a unique maximal proper τ-stable subset J of $\tilde{S}$ such that $\mathcal{O}_{\min}$ is the set of Coxeter elements of $W_J \rtimes \langle\tau\rangle$ that are contained in $W_J\tau \subset W_J \rtimes \langle\tau\rangle$. Here we embed $W_J \rtimes \langle\tau\rangle$ into $\tilde{W}'$ in a natural way.*

Remark. For type A, it is first proved by the first author in [4].

1.7. Before proving the theorem, we first explain why a W_a-conjugacy class of $\tilde{W}'$ with finite Coxeter part does not contain a Coxeter element of $\tilde{W}'$ and hence why we need proper subset of $\tilde{S}$ in the theorem. Although it is not needed in the proof, it serves as a motivation for the theorem.

Let $t^\chi w \in \mathcal{O}$ with $\chi \in P^\vee$ and w a finite Coxeter element of W_0'. Let n be the order of w in W_0'. It is known that the action of $1-w$ on $P^\vee \otimes_{\mathbb{Q}} \mathbb{C}$ is invertible. Hence

$$(t^\chi w)^n = t^{\chi + w\chi + \cdots + w^{n-1}\chi} w^n = t^{\frac{1-w^n}{1-w}\chi} = 1.$$

Therefore $t^\chi w$ is of finite order and hence any element in $\mathcal{O}$ is of finite order.

On the other hand, it is proved in [9, Theorem 1] (for untwisted case) and [6, Proposition 3.1] that any Coxeter element of $\tilde{W}'$ is of infinite order. Hence $\mathcal{O}$ doesn't contain a Coxeter element of $\tilde{W}'$.

2. EXISTENCE OF J

2.1. Let $\mathcal{O}'$ be a $\tilde{W}$-conjugacy class of $\tilde{W}'$. Then $\mathcal{O}' = \bigsqcup_{i=1}^{r} \mathcal{O}_i$ is a disjoint union of W_a-conjugacy classes of $\tilde{W}'$. Since $\tilde{W} = W_a \rtimes \Omega$, Ω acts transitively on $\{\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_r\}$. Moreover, if $\mathcal{O}_i = \tau \mathcal{O}_j \tau^{-1}$ for some $\tau \in \Omega$, then $(\mathcal{O}_i)_{min} = \tau (\mathcal{O}_j)_{min} \tau^{-1}$.

2.2. Let $\mathcal{O}'$ be a $\tilde{W}$-conjugacy class of $\tilde{W}'$ with finite Coxeter part, and let $\mathcal{O}$ be a W_a-conjugacy class of $\tilde{W}'$ with $\mathcal{O} \subset \mathcal{O}'$. The main purpose of this section is to show the "existence" part of the Theorem 1.1 for $\mathcal{O}'$ instead of $\mathcal{O}$. More precisely, there exists $\tau \in \Omega'$ and a maximal proper τ-stable subset J of $\tilde{S}$ and a Coxeter element c_J of $W_J \rtimes \langle \tau \rangle$ such that $c_J \in \mathcal{O}'$.

By §2.1, there exists $\sigma \in \Omega$ such that $\sigma c_J \sigma^{-1} \in \mathcal{O}$. It is easy to see that $\sigma c_J \sigma^{-1}$ is a Coxeter element of $W_{\sigma(J)} \rtimes \langle \sigma \tau \sigma^{-1} \rangle$. Thus the "existence" part of the theorem for $\tilde{W}$-conjugacy class $\mathcal{O}'$ deduces the "existence" part of it for W_a-conjugacy class $\mathcal{O}$.

Compared with W_a-conjugacy classes, it is much easier to classify $\tilde{W}$-conjugacy classes with finite Coxeter part and to find representatives. This is the reason that we consider $\tilde{W}$-conjugacy classes instead of W_a-conjugacy classes in this section.

2.3. We identify $\tilde{W}/W_a$ with $P^\vee/Q^\vee$ in the natural way. Let δ be a diagram automorphism of (W_0, S_0). Then $\langle \delta \rangle$ acts on $\tilde{W}/W_a \cong P^\vee/Q^\vee$. Let $(P^\vee/Q^\vee)_\delta$ be the δ-coinvariant of $P^\vee/Q^\vee$. Let

$$\kappa_\delta : \tilde{W}\delta \to (P^\vee/Q^\vee)_\delta, \qquad w \mapsto w\delta^{-1}W_a$$

be the natural projection. We call κ_δ the *Kottwitz map*.

The following result classifies the $\tilde{W}$-conjugacy classes of $\tilde{W}'$ with finite Coxeter part.

Proposition 2.1. *We keep the assumption as above. Let $\mathcal{O}_0 \subset W_0\delta$ be a W_0-conjugacy class containing a Coxeter element of W'_0. Then for any $v \in (P^\vee/Q^\vee)_\delta$, $\eta^{-1}(\mathcal{O}_0) \bigcap \kappa_\delta^{-1}(v)$ is a single $\tilde{W}$-congugacy class of $\tilde{W}'$.*

Proof. Let $\mu \in P^\vee$ such that the image of μ under the map $P^\vee \to (P^\vee/Q^\vee)_\delta$ is v. Let $c\delta \in \mathcal{O}_0$. Then $t^\mu c\delta \in \eta^{-1}(\mathcal{O}_0) \cap \kappa_\delta^{-1}(v)$. It is easy to see that $\eta^{-1}(\mathcal{O}_0) \bigcap \kappa_\delta^{-1}(v)$ is a union of $\tilde{W}$-conjugacy classes. Now we prove that $\tilde{W}$ acts transitively on $\eta^{-1}(\mathcal{O}_0) \bigcap \kappa_\delta^{-1}(v)$.

Let $\mu' \in P^\vee$ and $c'\delta \in \mathcal{O}_0$ such that $t^{\mu'} c'\delta \in \eta^{-1}(\mathcal{O}_0) \bigcap \kappa_\delta^{-1}(v)$. Then after conjugating by a suitable element of W_0, we may assume that $c' = c$. By definition, $\mu' \in \mu + (1-\delta)P^\vee + Q^\vee$. Thus it suffices to show that

(a) $(1-\delta)P^\vee + Q^\vee = (1-c\delta)P^\vee$.

For any $\lambda \in P^\vee$, $(1-c\delta)\lambda = (1-\delta)\lambda + (1-c)\delta(\lambda) \in (1-\delta)P^\vee + Q^\vee$. Hence $(1-\delta)P^\vee + Q^\vee \supset (1-c\delta)P^\vee$.

We first prove that

(b) $Q^\vee \subset (1-c\delta)P^\vee$.

We may assume that $c = s_{i_1} s_{i_2} \cdots s_{i_k}$. Since $c\delta$ is a Coxeter element of W'_0, δ-orbits on S_0 are

$$\{i_1, \delta(i_1), \cdots, \delta^{r_1}(i_1)\}, \{i_2, \delta(i_2), \cdots, \delta^{r_2}(i_2)\}, \cdots, \{i_k, \delta(i_k), ..., \delta^{r_k}(i_k)\}.$$

For $1 \leqslant j \leqslant k$,

$$\begin{aligned}(1-c\delta)(\omega_{i_j}^\vee + \omega_{\delta(i_j)}^\vee + \cdots + \omega_{\delta^{r_j}(i_j)}^\vee) &= (1-c)(\omega_{i_j}^\vee + \omega_{\delta(i_j)}^\vee + \cdots + \omega_{\delta^{r_j}(i_j)}^\vee) \\ &= (1-c)\omega_{i_j}^\vee = s_{i_1} s_{i_2} ... s_{i_{j-1}} \alpha_{i_j}^\vee.\end{aligned}$$

Therefore $\{\alpha_{i_1}^\vee, \alpha_{i_2}^\vee, ..., \alpha_{i_k}^\vee\} \subset (1-c\delta)P^\vee$.

For any $m \in S_0$,

$$(1-c\delta)\alpha_m^\vee = \alpha_m^\vee - \alpha_{\delta(m)}^\vee + (1-c)\alpha_{\delta(m)}^\vee \in \alpha_m^\vee - \alpha_{\delta(m)}^\vee + \sum_{1 \leqslant j \leqslant k} \mathbb{Z}\alpha_{i_j}^\vee.$$

Thus $\alpha_m^\vee - \alpha_{\delta(m)}^\vee \in (1-c\delta)P^\vee$ for all $m \in S_0$. Hence for $1 \leqslant j \leqslant k$ and $n \in \mathbb{N}$, one may show by induction that $\alpha_{\delta^n(i_j)}^\vee \in (1-c\delta)P^\vee$.

(b) is proved.

Now $(1-c\delta)P^\vee/Q^\vee = (1-\delta)P^\vee/Q^\vee$. Thus (a) is proved. □

2.4. In order to prove the "existence" part of Theorem 1.1 for $\mathcal{O}'$, we need the following key lemma which will be proved in section 3 via a case-by-case analysis.

Lemma 2.2. *Let δ' be a diagram automorphism of (W_0, S_0) and $\tau = t^{\omega_i^\vee} w_0^{S_0 - \{i\}} w_0^{S_0}$, where $\omega_i^\vee$ is a minuscule coweight. Then there exists a maximal proper $\tau\delta'$-stable subset J of $\tilde{S}$ and $c \in W_0$ such that*

$\mathrm{supp}(\tau\delta' c) = J$ and $w_0^{S_0-\{i\}} w_0^{S_0}\delta'(c)\delta'$ is conjugate to a Coxeter element of W_0'.

2.5. Now we prove the "existence" part of Theorem 1.1 for $\mathcal{O}'$.

Let $\tau \in \Omega$ and $\delta' \in \langle\delta\rangle$ such that $\mathcal{O}' \cap W_a\tau\delta' \neq \emptyset$. If $\tau = 1$, then we may take $J = S_0$ and c_J be any Coxeter element of $W_0\delta' \subset W_0'$.

If $\tau \neq 1$, then $\tau = t^{\omega_i^\vee} w_0^{S_0-\{i\}} w_0^{S_0}$ for some minuscule coweight $\omega_i^\vee$. We take J and c from Lemma 2.2. Then $\tau\delta' c$ is a Coxeter element of $W_J \rtimes \langle\tau\delta'\rangle$ and $\eta(\tau\delta' c) = w_0^{S_0-\{i\}} w_0^{S_0}\delta'(c)\delta'$ is conjugate to a Coxeter element of W_0'. By Proposition 2.1, $\tau\delta' c \in \mathcal{O}'$.

3. THE KEY LEMMA

In this section, we verify Lemma 2.2. We use the same labeling of Dynkin diagram as in [1].

Type A_{n-1}

This case was proved by the first author in [4, Lemma 5.1].

Type ${}^2A_{n-1}$

We may regard δ' as the permutation $w_0^{S_0} = (1\ n)(2\ \ n-1)\cdots$ in S_n and regard $W_0\delta' \subset W_0'$ as S_n. Under this identification, the W_0 conjugacy class that contains a Coxeter element in $W_0\delta'$ is the set of n-cycles when n is odd and is the set of $n-1$ cycles when n is even.

Let $\tau = \tau_i$. Then $\tau\delta'$-orbits on $\tilde{S}$ are $\{0, i\}$, $\{j, i-j\}$ for $0 < j < i$, and $\{i+j, n-j\}$ for $0 < j < n-i$.

We have the following four different cases:

Case 1: n is odd and i is odd.

In this case, we take $J = \tilde{S} - \{\frac{n+i}{2}\}$ and $c = s_{\frac{i+1}{2}} s_{\frac{i+3}{2}} \cdots s_{\frac{n+i}{2}-1}$. Then $w_0^{S_0-\{i\}}c$ is an n-cycle. In other words, $w_0^{S_0-\{i\}} w_0^{S_0}\delta'(c)\delta'$ is conjugate to a Coxeter element in $W_0\delta'$.

Case 2: n is odd and i is even.

In this case, we take $J = \tilde{S} - \{\frac{i}{2}\}$ and $c = s_{\frac{i}{2}+1} s_{\frac{i}{2}+2} \cdots s_{\frac{n+i-1}{2}}$. Then $w_0^{S_0-\{i\}}c$ is an n-cycle. In other words, $w_0^{S_0-\{i\}} w_0^{S_0}\delta'(c)\delta'$ is conjugate to a Coxeter element in $W_0\delta'$.

Case 3: n is even and i is odd.

In this case, we take $J = \tilde{S} - \{\frac{i-1}{2}, \frac{i+1}{2}\}$ and $c = s_{\frac{i+3}{2}} s_{\frac{i+5}{2}} \cdots s_{\frac{n+i-1}{2}}$. Then $w_0^{S_0-\{i\}}c$ is an $n-1$ cycle. In other words, $w_0^{S_0-\{i\}} w_0^{S_0}\delta'(c)\delta'$ is conjugate to a Coxeter element in $W_0\delta'$.

Case 4: n is even and i is even.

In this case, we take $J = \tilde{S} - \{\frac{n+i}{2}\}$ and $c = s_{\frac{i}{2}} s_{\frac{i}{2}+1} \cdots s_{\frac{n+i}{2}-1}$. Then $w_0^{S_0-\{i\}}c$ is an $n-1$ cycle. In other words, $w_0^{S_0-\{i\}} w_0^{S_0}\delta'(c)\delta'$ is conjugate to a Coxeter element in $W_0\delta'$.

Type B_n

There is only one minuscule coweight: $\omega_1^\vee$. So $\tau = \tau_1$. Now τ-orbits on $\tilde{S}$ are $\{0, 1\}$ and $\{i\}$ for $2 \leqslant i \leqslant n$. We take $J = \tilde{S} - \{n\}$ and $c = s_1 s_2 \cdots s_{n-1}$. Then $w_0^{S_0 - \{1\}} w_0^{S_0} c$ is conjugate to a Coxeter element of W_0.

Type C_n

There is only one minuscule coweight: $\omega_n^\vee$. So $\tau = \tau_n$. Now τ-orbits on $\tilde{S}$ are $\{i, n-i\}$ for $0 \leqslant i \leqslant n$.

If n is odd, we take $J = \tilde{S} - \{0, n\}$ and $c = s_{\frac{n+1}{2}} s_{\frac{n+3}{2}} \cdots s_{n-1}$. Then $w_0^{S_0 - \{n\}} w_0^{S_0} c$ is conjugate to a Coxeter element of W_0.

If n is even, we take $J = \tilde{S} - \{\frac{n}{2}\}$ and $c = s_{\frac{n}{2}+1} s_{\frac{n}{2}+2} \cdots s_n$. Then $w_0^{S_0 - \{n\}} w_0^{S_0} c$ is conjugate to a Coxeter element of W_0.

Type D_n

There are three minuscule coweights: $\omega_1^\vee$, $\omega_{n-1}^\vee$, $\omega_n^\vee$. There is an outer diagram automorphism of D_n permuting the last two coweights. Thus it suffices to consider the case where $\tau = \tau_1$ or τ_n.

Case 1: $\tau = \tau_1$.

The τ-orbits on $\tilde{S}$ are $\{0, 1\}$, $\{n-1, n\}$ and $\{i\}$ for $2 \leqslant i \leqslant n-2$. We take $J = \tilde{S} - \{n-1, n\}$ and $c = s_1 s_2 \cdots s_{n-2}$. Then $w_0^{S_0 - \{1\}} w_0^{S_0} c = s_1 s_2 \cdots s_n$ is a Coxeter element of W_0.

Case 2: $\tau = \tau_n$ and n is odd.

The τ-orbits on $\tilde{S}$ are $\{0, n, 1, n-1\}$ and $\{i, n-i\}$ for $2 \leqslant i \leqslant \frac{n-1}{2}$. We take $J = \tilde{S} - \{\frac{n-1}{2}, \frac{n+1}{2}\}$ and $c = s_{\frac{n+3}{2}} s_{\frac{n+5}{2}} \cdots s_{n-2} s_n$. Then $w_0^{S_0 - \{n\}} w_0^{S_0} c$ is conjugate to a Coxeter element of W_0.

Case 3: $\tau = \tau_n$ and n is even.

The τ-orbits on $\tilde{S}$ are $\{i, n-i\}$ for $0 \leqslant i \leqslant \frac{n}{2}$. We take $J = \tilde{S} - \{0, n\}$ and $c = s_{\frac{n}{2}} s_{\frac{n}{2}+1} \cdots s_{n-1}$. Then $w_0^{S_0 - \{n\}} w_0^{S_0} c$ is conjugate to a Coxeter element of W_0.

Type ${}^2D_n$

As explained above, it suffices to consider the following three cases.

Case 1: $\tau = \tau_1$.

The $\tau\delta'$-orbits on $\tilde{S}$ are $\{0, 1\}$ and $\{i\}$ for $2 \leqslant i \leqslant n$. We take $J = \tilde{S} - \{n\}$ and $c = s_1 s_2 \cdots s_{n-2} s_{n-1}$. Then $w_0^{S_0 - \{1\}} w_0^{S_0} \delta'(c) \delta' = s_1 s_2 \cdots s_{n-2} s_{n-1} \delta'$ is a Coxeter element in $W_0 \delta'$.

Case 2: $\tau = \tau_n$ and n is odd.

The $\tau\delta'$-orbits on $\tilde{S}$ are $\{i, n-i\}$ for $0 \leqslant i \leqslant \frac{(n-1)}{2}$. We take $J = \tilde{S} - \{0, n\}$ and $c = s_{\frac{n+1}{2}} s_{\frac{n+3}{2}} \cdots s_{n-2} s_{n-1}$. Then $w_0^{S_0 - \{n\}} w_0^{S_0} \delta'(c) \delta'$ is conjugate to a Coxeter element in $W_0 \delta'$.

Case 3: $\tau = \tau_n$ and n is even.

The $\tau\delta'$-orbits on $\tilde{S}$ are $\{0, 1, n-1, n\}$ and $\{i, n-i\}$ for $2 \leqslant i \leqslant \frac{n}{2}$. We take $J = \tilde{S} - \{\frac{n}{2}\}$ and $c = s_{\frac{n}{2}+1} s_{\frac{n}{2}+2} \cdots s_{n-2} s_{n-1}$. Then $w_0^{S_0 - \{n\}} w_0^{S_0} \delta'(c)\delta'$ is conjugate to a Coxeter element in $W_0\delta'$.

Type $^3D_4$

Without loss of generality, we may assume that δ' is the outer diagram automorphism on D_4 sending s_1 to s_3, s_3 to s_4 and s_4 to s_1. As $\langle \delta' \rangle$ acts transitively on $\{1, 3, 4\}$, it suffices to consider the case where $\tau = \tau_1$.

In this case, the $\tau\delta'$-orbits on $\tilde{S}$ are $\{0, 1, 4\}, \{2\}, \{3\}$. We take $J = \tilde{S} - \{3\}$ and $c = s_2 s_1$, then $w_0^{S_0 - \{1\}} w_0^{S_0} \delta'(c)\delta'$ is conjugate to a Coxeter element in $W_0\delta'$.

Type E_6

There are two minuscule coweights: $\omega_1^\vee$ and $\omega_6^\vee$. The unique outer diagram automorphism of E_6 permutes these two coweights. Thus it suffices to consider the case where $\tau = \tau_1$. In this case, τ-orbits on $\tilde{S}$ are $\{0, 1, 6\}, \{2, 3, 5\}, \{4\}$. We take $J = \tilde{S} - \{0, 1, 6\}$ and $c = s_4 s_5$. Then $w_0^{S_0 - \{1\}} w_0^{S_0} c$ is conjugate to a Coxeter element of W_0.

Type $^2E_6$

As explained above, it suffices to consider the case where $\tau = \tau_1$. In this case, $\tau\delta'$-orbits on $\tilde{S}$ are $\{0, 1\}, \{2, 3\}, \{4\}, \{5\}, \{6\}$. We take $J = \tilde{S} - \{6\}$ and $c = s_5 s_4 s_3 s_1$. Then $w_0^{S_0 - \{1\}} w_0^{S_0} \delta'(c)\delta'$ is conjugate to a Coxeter element of W_0'.

Type E_7

There is a unique minuscule coweight: $\omega_7^\vee$. So $\tau = \tau_7$. In this case, τ-orbits on $\tilde{S}$ are $\{0, 7\}, \{1, 6\}, \{3, 5\}, \{2\}, \{4\}$. We take $J = \tilde{S} - \{0, 7\}$ and $c = s_2 s_4 s_5 s_6$. Then $w_0^{S_0 - \{7\}} w_0^{S_0} c$ is conjugate to a Coxeter element of W_0.

4. Proof of the main theorem

4.1. We keep the notation in section 1. For any $w, w' \in \tilde{W}'$ and $i \in \tilde{S}$, we write $w \xrightarrow{s_i} w'$ if $w' = s_i w s_i$ and $\ell(w') \leqslant \ell(w)$. We write $w \to w'$ if there is a sequence of $w = w_0, w_1, \cdots, w_n = w'$ of elements in $\tilde{W}'$ such that for any $k \in \{0, 1, \cdots, n-1\}$, $w_k \xrightarrow{s_i} w_{k+1}$ for some $i \in \tilde{S}$. We write $w \approx w'$ if $w \to w'$ and $w' \to w$.

The following result is proved in [6].

Theorem 4.1. *Let $\mathcal{O}$ be a W_a-conjugacy class of $\tilde{W}'$ with finite Coxeter part and $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then for any $w \in \mathcal{O}$ and $w' \in \mathcal{O}_{\min}$, $w \to w'$.*

4.2. Let $\mathcal{O}$ be a W_a-conjugacy class of $\tilde{W}'$ with finite Coxeter part and let $\tau \in \Omega'$ with $\mathcal{O} \subset W_a\tau$. In section 2, we have proved that there exists a maximal proper τ-stable subset J of $\tilde{S}$ and a Coxeter element c_J of $W_J \rtimes \langle \tau \rangle$ such that $c_J \in \mathcal{O}$.

Let w be a minimal length element in $\mathcal{O}$. By Theorem 4.1, $c_J \to w$. Since c_J is a Coxeter element of $W_J \rtimes \langle \tau \rangle$, w is also a Coxeter element of $W_J\tau \subset W_J \rtimes \langle \tau \rangle$ and $c_J \approx w$.

Since J is a proper subset of $\tilde{S}$, $W_J \rtimes \langle \tau \rangle$ is a finite group. Hence any two Coxeter element of $W_J\tau$ are conjugated by an element of W_J. Thus all the Coxeter elements of $W_J\tau$ are contained in $\mathcal{O}$.

Therefore $\mathcal{O}_{\min}$ is the set of Coxeter elements in $W_J\tau \subset W_J \rtimes \langle \tau \rangle$.

Moreover, $J = \mathrm{supp}(w)$ for any $w \in \mathcal{O}_{\min}$. This proves the uniqueness of J.

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