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Character sheaves on the semi-stable locus of a group compactification

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Abstract

We study the intermediate extension of the character sheaves on an adjoint group to the semi-stable locus of its wonderful compactification. We show that the intermediate extension can be described by a direct image construction. As a consequence, we show that the "ordinary" restriction of a character sheaf on the compactification to a "semi-stable stratum" is a shift of semisimple perverse sheaf and is closely related to Lusztig's restriction functor (from a character sheaf on a reductive group to a direct sum of character sheaves on a Levi subgroup). We also provide a (conjectural) formula for the boundary values inside the semi-stable locus of an irreducible character of a finite group of Lie type, which gives a partial answer to a question of Springer (2006) [21]. This formula holds for Steinberg character and characters coming from generic character sheaves. In the end, we verify Lusztig's conjecture Lusztig (2004) [16, 12.6] inside the semi-stable locus of the wonderful compactification.

Keywords: Perverse sheaf; Character sheaf; Group compactification

0. Introduction

0.1. Let G be a connected, semisimple algebraic group of adjoint type over an algebraically closed field k. In [16], Lusztig introduced a decomposition of the wonderful compactification $\bar{G}$ of G into G-stable pieces. The group G itself is a G-stable piece and each G-stable piece

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is a smooth, locally closed subvariety of $\bar{G}$ and the G-orbits on each piece (for the diagonal G-action) naturally correspond to the "twisted" conjugacy classes of a smaller group. Moreover, this correspondence leads to a natural equivalence between the bounded derived category of G-equivariant, constructible sheaves on that piece and the bounded derived category of certain constructible sheaves on the smaller group that are equivariant under the "twisted" conjugation action (see [16, 12.3]).

Character sheaves on a reductive group are some special simple perverse sheaves on the group that are equivariant under the ("twisted") conjugation action. The theory of character sheaves was developed by Lusztig in the series of papers [14] (for conjugation action) and [15] (for "twisted" conjugation action). Now using the natural equivalence we discussed above, one can define the character sheaves on each G-stable piece. The character sheaves on $\bar{G}$ are the intermediate extensions to $\bar{G}$ of the character sheaves on the G-stable pieces (see [16, 12.3]). The most interesting cases are the intermediate extension to $\bar{G}$ of the character sheaves on G. Roughly speaking, these sheaves can be regarded as the objects that describe the behavior at infinity of the character sheaves on G.

0.2. In order to understand the intermediate extensions to $\bar{G}$ of the character sheaves on a G-stable piece, in [7] we gave a second definition of character sheaves on $\bar{G}$ by imitating the definition of character sheaves on groups. This new definition coincides with Lusztig's definition we mentioned in the previous subsection (see [7, Corollary 4.6]). Moreover, using the new definition, one can show that the character sheaves on $\bar{G}$ have the following nice property (see [7, Section 4]):

Let i be the inclusion of a G-stable piece to $\bar{G}$, then

(1) for any character sheaf C on $\bar{G}$, any perverse constituent of $i^*(C)$ is a character sheaf on that piece;

(2) for any character sheaf C on that piece, any perverse constituent of $i_!(C)$ is a character sheaf on $\bar{G}$.

0.3. However, analyzing the intermediate extension of a character sheaf on a G-stable piece is still a challenging problem. In [21], Springer listed some interesting questions in this direction. One interesting question is to study the boundary values of an irreducible character of a finite group of Lie type.

A technical difficulty in analyzing the intermediate extension is as follows.

A character sheaf on G can be understood in terms of "admissible complex", which is obtained by pushing forward of some intersection cohomology complex under some small, proper map to the closure of a Lusztig's stratum of G.

Using the G-stable piece decomposition of $\bar{G}$ and the natural correspondence between the G-stable pieces and the smaller groups, one is able to generalize Lusztig's stratification on G to a decomposition on $\bar{G}$. However, an explicit description of the closure to $\bar{G}$ of a Lusztig's stratum is still unknown. A more serious problem is that the small map we used to construct "admissible complex" on G doesn't extend to a small map on $\bar{G}$.

0.4. In this paper, we will study the intermediate extension of a character sheaf on G, not to $\bar{G}$, but to the semi-stable locus $\bar{G}^{ss}$ of $\bar{G}$, an open smooth subvariety of $\bar{G}$ that contains G. In fact, $\bar{G}^{ss}$ is a union of some G-stable pieces. An explicit description of $\bar{G}^{ss}$ was obtained in a joint work with Starr [11]. We call the G-stable pieces inside $\bar{G}^{ss}$ *semi-stable strata*.

The idea of studying intermediate extension to $\bar{G}^{ss}$ instead of $\bar{G}$ comes from geometric invariant theory. Now we make a short digression from character sheaves and discuss about some basic ideas in the theory of geometric invariant theory.

Let H be a linear algebraic group and X be a H-variety. When considering the quotient space, a main problem is that the quotient X/H may not exist in the category of algebraic varieties. Geometric invariant theory suggests a method to distinguish "good" H-orbits from "bad" H-orbits in the sense that the union of "good" H-orbits forms an open subvariety U of X and U/H exists.

Motivated by this, one may wonder if the "good" G-orbits on $\bar{G}$ are still good in the study of character sheaves in the sense that the intermediate extension of a character sheaf on G to the union of "good" orbits can be analyzed. The answer is YES and this is what we are going to do in this paper.

0.5. Now let us consider the closure of a Lusztig's stratum in $\bar{G}$. If we take the limit in the direction of unipotent elements in the stratum, then by the results in [6] and [12], the boundary points are outside the semi-stable locus. On the other hand, taking the limit in the direction of semisimple elements in the stratum is more or less the same as calculating the closure of some subvariety in a toric variety. This naive thought suggests that the closure to $\bar{G}^{ss}$ of a Lusztig's stratum can be described explicitly.

The explicit description will be obtained in Section 3. Moreover, the small map we used to construct "admissible complex" on G extends to a small map on $\bar{G}^{ss}$. Based on this result, the intermediate extension of an "admissible complex" to $\bar{G}^{ss}$ can also be described by a direct image construction. This is a generalization of [15, Proposition 5.7].

Moreover, the restriction of the direct image to a semi-stable stratum can be calculated explicitly and is closely related to Lusztig's restriction functor introduced in [14, 3.8] and [15, 23.3]. The precise statement can be found in Theorem 4.4. Based on this, we give a (conjectural) formula for the boundary values inside the semi-stable locus of a character of a finite group of Lie type. The formula is true if the (virtual) character is obtained from the direct image construction. This gives a partial answer to a question of Springer [21, Problem 10].

0.6. There is a special character sheaf S on G that characterizes the semisimple elements of G. This sheaf is the alternating sum of the induced sheaves from the trivial local systems on the standard parabolic subgroups of G. In [16, 12.6], Lusztig generalized the notion of semisimple elements to $\bar{G}$ and conjectured that the intermediate extension to $\bar{G}$ of this sheaf characterizes the semisimple elements of $\bar{G}$.

It is known that the semisimple elements of $\bar{G}$ lie in the semi-stable locus. We will calculate the intermediate extension of S to $\bar{G}^{ss}$ and verify Lusztig's conjecture inside the semi-stable locus.

In order to do this, we will consider the intermediate extension of the induced sheaf from the trivial local system on a standard parabolic subgroup P. Therefore we need to understand the closure of P in $\bar{G}^{ss}$ and the intermediate extension of the trivial local system on P to this closure.

Let B be a Borel subgroup of P. Then P is stable under the action of $B \times B$ and the closure of P in $\bar{G}$ was obtained in [20, Corollary 2.5] in terms of the union of certain $B \times B$-orbits. However, $\bar{G}^{ss}$ is not stable under the action of $B \times B$. To describe the closure of P in $\bar{G}^{ss}$, we have to use the P-stable pieces, introduced by Lu and Yakimov as a generalization of the notions of $B \times B$-orbits and G-stable pieces. Although the closure of P in $\bar{G}$ is not smooth in general, the closure of P in $\bar{G}^{ss}$ is always smooth. Therefore, the intermediate extension of the trivial local system on P to the closure of P in $\bar{G}^{ss}$ is just the trivial local system on that closure. Now we can explicitly calculate the intermediate extension of S to $\bar{G}^{ss}$.

0.7. We now review the content of this paper in more detail.

In Section 1, we recall the definition and properties of P-stable pieces. In Section 2, we give an explicit description of the closure of a parabolic subgroup in $\bar{G}^{ss}$ and prove that the closure is smooth. In Section 3, we obtain the closure of a Lusztig's stratum of G in $\bar{G}^{ss}$. In Section 4, we study the intermediate extension of a character sheaf on G to $\bar{G}^{ss}$ and verify Lusztig's conjecture inside $\bar{G}^{ss}$.

1. $\mathcal{R}$-stable pieces on the wonderful compactification

1.1. Let G be a connected reductive algebraic group over an algebraically closed field k. Let B be a Borel subgroup of G, $T \subset B$ be a maximal torus and B^- be the opposite Borel subgroup. Let I be the set of simple roots and $W = N_G(T)/T$ be the corresponding Weyl group. For any $w \in W$, we choose a representative $\dot{w}$ of w in $N_G(T)$.

For $J \subset I$, let W_J be the subgroup of W corresponding to J and W^J (respectively ${}^J W$) be the set of minimal length coset representatives of W/W_J (respectively $W_J \backslash W$). Let w_0^J be the unique element of maximal length in W_J. (We simply write w_0 for w_0^I.) For $J, K \subset I$, we write ${}^J W^K$ for ${}^J W \cap W^K$.

For $J \subset I$, let Φ_J be the set of roots that are linear combination of simple roots in J. Let $P_J \supset B$ be the standard parabolic subgroup defined by J and $P_J^- \supset B^-$ be the opposite of P_J. Let $L_J = P_J \cap P_J^-$ and $G_J = L_J/Z(L_J)$. For any parabolic subgroup P, we denote by U_P its unipotent radical and H_P the inverse image of the connected center of P/U_P under $P \to P/U_P$. We simply write U for U_B and U^- for U_{B^-}.

For any $g \in G$ and subvariety $H \subset G$, we write ${}^g H$ for gHg^{-1}.

Now we will review the $\mathcal{R}$-stable pieces introduced in [18]. We will follow the approach in [9].

1.2. A triple $c = (J_1, J_2, \delta)$ consisting of $J_1, J_2 \subset I$ and an isomorphism $\delta: W_{J_1} \to W_{J_2}$ with $\delta(J_1) = J_2$ is called an *admissible triple* of $W \times W$. For an admissible triple $c = (J_1, J_2, \delta)$, set $W_c = \{(w, \delta(w));\ w \in W_{J_1}\} \subset W \times W$.

Let $c = (J_1, J_2, \delta)$ and $c' = (J'_1, J'_2, \delta')$ be admissible triples. For $w_1 \in W^{J_1}$ and $w_2 \in {}^{J'_2} W$, set

$$I(w_1, w_2, c, c') = \max\{K \subset J_1;\ w_1(K) \subset J'_1 \text{ and } \delta' w_1(K) = w_2 \delta(K)\},$$
$$[w_1, w_2, c, c'] = W_{c'}(w_1 W_{I(w_1,w_2,c,c')}, w_2) W_c \subset W \times W.$$

Then

$$W \times W = \bigsqcup_{w_1 \in W^{J_1}, w_2 \in {}^{J'_2} W} [w_1, w_2, c, c']. \tag{$*$}$$

See [9, Proposition 2.4 (1)].

Moreover, define an automorphism $\sigma : W_{I(w_1,w_2,c,c')} \to W_{I(w_1,w_2,c,c')}$ by $\sigma(w) = \delta^{-1}(w_2^{-1}\delta'(w_1 w w_1^{-1})w_2)$. Then map $W_{I(w_1,w_2,c,c')} \to W \times W$ defined by $w \to (w_1 w, w_2)$ induces a bijection from the σ-twisted conjugacy classes on $W_{I(w_1,w_2,c,c')}$ to the double cosets $W_{c'} \backslash [w_1, w_2, c, c'] / W_c$. See [9, Proposition 2.4(2)].

Let $\mathcal{O}$ be a double coset in $W_{c'} \backslash (W \times W)/W_c$. Then $\mathcal{O} \cap (W^{J_1} \times {}^{J'_2}W)$ contains at most one element (see [9, Corollary 2.5]). If $\mathcal{O} \cap (W^{J_1} \times {}^{J'_2}W) \neq \emptyset$, then we call $\mathcal{O}$ a distinguished double coset. We denote by $\mathcal{O}_{\min}$ the set of minimal length elements in $\mathcal{O}$. We have a natural partial order on the set of distinguished double cosets defined as follows: $\mathcal{O} \leqslant \mathcal{O}'$ if for some (or equivalently, any) $w' \in \mathcal{O}'_{\min}$, there exists $w \in \mathcal{O}_{\min}$ with $w \leqslant w'$. See [9, 4.7].

1.3. An *admissible triple* of $G \times G$ is by definition a triple $\mathcal{C} = (J_1, J_2, \theta_\delta)$ consisting of $J_1, J_2 \subset I$, an isomorphism $\delta : W_{J_1} \to W_{J_2}$ with $\delta(J_1) = J_2$ and an isomorphism $\theta_\delta : L_{J_1} \to L_{J_2}$ that maps T to T and the root subgroup U_{α_i} (for $i \in J_1$) to the root subgroup $U_{\alpha_{\delta(i)}}$. Then an admissible triple $\mathcal{C} = (J_1, J_2, \theta_\delta)$ of $G \times G$ determines an admissible triple $c = (J_1, J_2, \delta)$ of $W \times W$. For an admissible triple $\mathcal{C} = (J_1, J_2, \theta_\delta)$, define

$$\mathcal{R}_{\mathcal{C}} = \{(p, q);\ p \in P_{J_1},\ q \in P_{J_2},\ \theta_\delta(\bar{p}) = \bar{q}\},$$

where $\bar{p}$ is the image of p under the map $P_{J_1} \to L_{J_1}$ and $\bar{q}$ is the image of q under the map $P_{J_2} \to L_{J_2}$.

Let $\mathcal{C} = (J_1, J_2, \theta_\delta)$ and $\mathcal{C}' = (J'_1, J'_2, \theta_{\delta'})$ be admissible triples. For $w_1 \in W^{J_1}$ and $w_2 \in {}^{J'_2}W$, set

$$[w_1, w_2, \mathcal{C}, \mathcal{C}'] = \mathcal{R}_{\mathcal{C}'}(B\dot{w}_1 B, B\dot{w}_2 B)\mathcal{R}_{\mathcal{C}} \subset G \times G.$$

For any distinguished double coset $\mathcal{O} \in W_{c'} \backslash (W \times W)/W_c$, we also write $[\mathcal{O}, \mathcal{C}, \mathcal{C}']$ for $[w_1, w_2, \mathcal{C}, \mathcal{C}']$, where (w_1, w_2) is the unique element in $\mathcal{O} \cap (W^{J_1} \times {}^{J'_2}W)$. We call $[w_1, w_2, \mathcal{C}, \mathcal{C}']$ a $\mathcal{R}_{\mathcal{C}'} \times \mathcal{R}_{\mathcal{C}}$-stable piece of $G \times G$.

Now we list some properties of the $\mathcal{R}_{\mathcal{C}'} \times \mathcal{R}_{\mathcal{C}}$-stable pieces.

(1) The $\mathcal{R}_{\mathcal{C}'} \times \mathcal{R}_{\mathcal{C}}$-stable piece $[w_1, w_2, \mathcal{C}, \mathcal{C}']$ is a locally closed, smooth and irreducible subvariety of $G \times G$ of dimension equal to $\dim(G) + |I| + l(w_1) + l(w_2) + l(w_0^{J_1}) + l(w_0^{J_2})$. See [18, Theorem 1.1(i)]. See also [22, Theorem 2.6].

(2) $G \times G = \bigsqcup_{w_1 \in W^{J_1}, w_2 \in {}^{J'_2}W} [w_1, w_2, \mathcal{C}, \mathcal{C}']$. Lu and Yakimov [18, 1.3] and Springer [22, Theorem 2.6] gave two different proofs of this result. A different approach is sketched in [9, Proposition 5.6].

(3) Let $w_1 \in W^{J_1}$ and $w_2 \in {}^{J'_2}W$ and $\mathcal{O} = W_{c'}(w_1, w_2)W_c$. Then for any $(w'_1, w'_2) \in \mathcal{O}_{\min}$, $[\mathcal{O}, \mathcal{C}, \mathcal{C}'] = \mathcal{R}_{\mathcal{C}'}(B\dot{w}'_1 B, B\dot{w}'_2 B)\mathcal{R}_{\mathcal{C}}$. See [9, Proposition 5.3].

(4) Let $(w_1, w_2) \in W^{J_1} \times {}^{J'_2}W$. Define an automorphism $\theta_\sigma : L_{I(w_1,w_2,c,c')} \to L_{I(w_1,w_2,c,c')}$ by $\theta_\sigma(l) = \theta_\delta^{-1}(\dot{w}_2^{-1}\theta_\delta(\dot{w}_1 l \dot{w}_1^{-1})\dot{w}_2)$. Then map $L_{I(w_1,w_2,c,c')} \to G \times G$ defined by $l \to (\dot{w}_1 l, \dot{w}_2)$ induces a bijection between the θ_σ-twisted conjugacy classes on $L_{I(w_1,w_2,c,c')}$ and the double cosets $\mathcal{R}_{\mathcal{C}'} \backslash [w_1, w_2, \mathcal{C}, \mathcal{C}']/\mathcal{R}_{\mathcal{C}}$. See [18, 1.2] and [9, Proposition 5.6(2)].

(5) For any $(w_1, w_2) \in W \times W$, $\overline{\mathcal{R}_{\mathcal{C}'}(B\dot{w}_1 B, B\dot{w}_2 B)\mathcal{R}_{\mathcal{C}}} = \bigsqcup_{\mathcal{O}} [\mathcal{O}, \mathcal{C}, \mathcal{C}']$, where $\mathcal{O}$ runs over the distinguished double cosets in $W_{c'} \backslash (W \times W)/W_c$ that contains a minimal length element (w'_1, w'_2) with $w'_1 \leqslant w_1$ and $w'_2 \leqslant w_2$. See [9, Proposition 5.8]. A slightly more complicated description was obtained in [18, Theorem 4.1].

In particular,

(6) for any distinguished double coset $\mathcal{O} \in W_{c'} \backslash (W \times W)/W_c$, we have that $\overline{[\mathcal{O}, \mathcal{C}, \mathcal{C}']} = \bigsqcup_{\mathcal{O}' \leqslant \mathcal{O}} [\mathcal{O}', \mathcal{C}, \mathcal{C}']$. See [9, Corollary 5.9].

Now we will come to the wonderful compactifications and the P_K-stable-piece decompositions on the compactifications.

From now on, unless otherwise stated, we assume that G is adjoint and $\tilde{G}$ an algebraic group with identity component G. Let G^1 be a connected component of $\tilde{G}$. We fix an element $g_0 \in G^1$ with ${}^{g_0}B = B$ and ${}^{g_0}T = T$. If $G^1 = G$, then we choose $g_0 = 1$ and $\delta = id$. We denote by θ_δ the conjugation of g_0 on G. Then θ_δ gives automorphisms on I and W. We denote these automorphisms by δ.

1.4. We consider G as a $G \times G$-variety by left and right translation. Let $\bar{G}$ be the wonderful compactification of G. This compactification was first constructed by De Concini and Procesi [4] when $k = \mathbf{C}$ and later generalized by Strickland [24] to arbitrary algebraically closed field k. It is known that $\bar{G}$ is an irreducible, smooth projective $(G \times G)$-variety with finitely many $G \times G$-orbits Z_J indexed by the subsets J of I. Here Z_J is isomorphic to the quotient space $(G \times G) \times_{P_J^- \times P_J} G_J$ for the $P_J^- \times P_J$-action on $G \times G \times G_J$ defined by $(q, p) \cdot (g_1, g_2, z) = (g_1 q^{-1}, g_2 p^{-1}, \bar{q} z \bar{p}^{-1})$, where $\bar{q}$ is the image of q under the projection $P_J^- \to G_J$ and $\bar{p}$ is the image of p under the projection $P_J \to G_J$. Let h_J be the image of $(1, 1, 1)$ in Z_J under this isomorphism.

1.5. The wonderful compactification $\overline{G^1}$ of G^1 is the $(G \times G)$-variety which is isomorphic to $\bar{G}$ as a variety and where the $G \times G$-action is twisted by $(g, g') \mapsto (g, \theta_\delta(g'))$. The $G \times G$-orbits on $\overline{G^1}$ then coincide with the $G \times G$-orbits on $\bar{G}$. Let $Z_{J,\delta}$ be the orbit coinciding with $Z_{\delta(J)}$ and $h_{J,\delta} \in Z_{J,\delta}$ be the point identified with the base point $h_{\delta(J)} \in Z_{\delta(J)}$. Then G^1 is identified with the open $G \times G$-orbit $Z_{I,\delta}$ via $g g_0 \mapsto (g, 1) \cdot h_{I,\delta}$. Moreover, the isotropy subgroup of $h_{J,\delta}$ in $G \times G$ is

$$\big(U_{P^-_{\delta(J)}} \times U_{P_J} Z(L_J)\big)(L_J)_\delta,$$

where $(L_J)_\delta = \{(\theta_\delta(l), l);\ l \in L_J\}$.

In other words, we have the following commuting diagram

$$\begin{array}{ccc} G & \xrightarrow{\cdot g_0} & G^1 \\ \big\downarrow & & \big\downarrow \\ \bar{G} & \xrightarrow{r} & \overline{G^1}, \end{array}$$

where $r((g_1, g_2) \cdot h_{\delta(J)}) = (g_1, \theta_\delta^{-1}(g_2)) \cdot h_{J,\delta}$.

For any subvariety $X \subset \overline{G^1}$, we denote by $\bar{X}$ its closure.

1.6. For $J \subset I$, set $J_1 = w_0 w_0^{\delta(J)} \delta(J)$ and $\delta' = \delta^{-1} \circ \mathrm{Ad}(w_0 w_0^{\delta(J)})^{-1} : W_{J_1} \to W_J$. Then $c = (J_1, J, \delta')$ is an admissible triple on $W \times W$. Set $\theta_{\delta'} = \theta_\delta^{-1} \circ \mathrm{Ad}(\dot{w}_0 \dot{w}_0^{\delta(J)})^{-1} : L_{J_1} \to L_J$.

Then $\mathcal{C} = (J_1, J, \theta_{\delta'})$ is an admissible triple on $G \times G$. We may identify $(G \times G)/\mathcal{R}_{\mathcal{C}}(1, Z(L_J))$ with $Z_{J,\delta}$ as $G \times G$-variety via $(g_1, g_2) \mapsto (g_1 \dot{w}_0 \dot{w}_0^{\delta(J)}, g_2) \cdot h_{J,\delta}$.

Let $K \subset I$ and $\mathcal{C}' = (K, K, id)$. Then each $\mathcal{R}_{\mathcal{C}'} \times \mathcal{R}_{\mathcal{C}}$-stable piece of $G \times G$ is stable under the right action of $\mathcal{R}_{\mathcal{C}}(1, Z(L_J))$. For $w \in W^{\delta(J)}$ and $v \in {}^K W$, set

$$[J, w, v]_{K,\delta} = \left[w w_0^{\delta(J)} w_0, v, \mathcal{C}, \mathcal{C}'\right]/\mathcal{R}_{\mathcal{C}}\left(1, Z(L_J)\right) = (P_K)_\Delta(B\dot{w}, B\dot{v}) \cdot h_{J,\delta}.$$

We call $[J, w, v]_{K,\delta}$ a P_K-stable piece on $\overline{G^1}$. In the case where $K = \emptyset$, a P_K-stable piece is just a $B \times B$-orbit and we simply write $[J, w, v]_\delta$ for $[J, w, v]_{\emptyset,\delta}$. In the case where $K = I$, a P_K-stable piece is just Lusztig's G-stable piece introduced in [16, Section 12] and we simply write $Z_{J,w;\delta}$ for $[J, w, 1]_{I,\delta}$.

The following properties follow easily from the properties of $\mathcal{R}_{\mathcal{C}'} \times \mathcal{R}_{\mathcal{C}}$-stable pieces that we listed in Section 1.3.

(1) $[J, w, v]_{K,\delta}$ is an irreducible, locally closed subvariety of $\overline{G^1}$ of dimension $l(w_0) + |J| + l(v) - l(w) + l(w_0^K)$.

(2) $\overline{G^1} = \bigsqcup_{J \subset I, w \in W^{\delta(J)}, v \in {}^K W} [J, w, v]_{K,\delta}$.

(3) For any $J \subset I$, $x \in W^{\delta(J)}$ and $y \in W$, $\overline{(P_K)_\Delta \cdot [J, x, y]_\delta} \cap Z_{J,\delta} = \bigsqcup [J, w, v]_{K,\delta}$, where (w, v) runs over all elements in $W^{\delta(J)} \times {}^K W$ such that there exists $a \in W_K$ and $b \in W_J$ such that $a w \delta(b) w_0^{\delta(J)} w_0 \leqslant x w_0^{\delta(J)} w_0$, $avb \leqslant y$ and $l(a w \delta(b) w_0^{\delta(J)} w_0) + l(avb) = l(w w_0^{\delta(J)} w_0) + l(v)$.

(4) For $x \in W^{\delta(J)}$ and $y \in W$ with $l(y) - l(x) = l(v) - l(w)$ and there exists $a \in W_K$ and $b \in W_J$ such that $x = a w \delta(b)$ and $y = avb$, then we have that $(P_K)_\Delta \cdot [J, x, y]_\delta = [J, w, v]_{K,\delta}$.

The following explicit description of the closure of a P_K-stable piece in $\overline{G^1}$ was obtained in [18, Theorem 5.1], which generalized results on the $B \times B$-orbit closures in [20, Proposition 2.4] and [13, Proposition 6.3] and the G-stable-piece closures in [8, Theorem 4.5].

(5) $\overline{[J, w, v]_{K,\delta}}$ is a finite union of P_K-stable pieces. Moreover, $[J', w', v']_{K,\delta} \subset \overline{[J, w, v]_{K,\delta}}$ if and only if $J' \subset J$ and there exist $x \in W_K$ and $y \in W_J$ such that $x w' \geqslant w \delta(y)$ and $x v' \leqslant v y$.

We also need the following variation of Section 1.3 (4),

(6) $[J, w, v]_{K,\delta} = (P_K)_\Delta(L_{K_1} \dot{w}, \dot{v}) \cdot h_{J,\delta}$, where $K_1 = \max\{K' \subset K;\ w^{-1}(K') \subset J, w^{-1}(K') = \delta(v^{-1}(K'))\}$.

Moreover, we have an explicit description of the semi-stable locus $\overline{G^1}^{ss}$ for the diagonal G-action on $\overline{G^1}$ in terms of G-stable pieces (see [11]). The case where $G^1 = G$ was also studied by De Concini, Kannan and Maffei in [3].

(7) $\overline{G^1}^{ss} = \bigsqcup_{J \subset I} Z_{J,1;\delta}$.

We call $Z_{J,1;\delta}$ a *semi-stable stratum.*

The following consequence of (1) and (5) is also useful in this paper.

Corollary 1.1. *For $J' \subset J$, $\dim(\overline{[J, w, v]_{K,\delta}} \cap Z_{J',\delta}) = \dim([J, w, v]_{K,\delta}) - |J| + |J'|$.*

Proof. We have that $[J', w, v]_{K,\delta} \subset \overline{[J, w, v]_{K,\delta}} \cap Z_{J',\delta}$. By Section 1.6 (1), $\dim([J', w, v]_{K,\delta}) = l(w_0) + l(w_0^K) + |J'| + l(v) - l(w) = l(w_0) + l(w_0^K) + |J| + l(v) - l(w) + (|J'| - |J|) = \dim([J, w, v]_{K,\delta}) - |J| + |J'|$.

On the other hand, by Section 1.6 (5), $\overline{[J, w, v]_{K,\delta}} \cap Z_{J',\delta}$ is a finite union of P_K-stable pieces of the form $[J', w', v']_{K,\delta}$ with $x w' \geqslant w \delta(y)$ and $x v' \leqslant v y$ for some $x \in W_K$ and $y \in W_J$. For any such pair (w', v'),

$$\dim\left(\left[J', w', v'\right]_{K,\delta}\right) = l(w_0) + l\left(w_0^K\right) + \left|J'\right| + l\left(v'\right) - l\left(w'\right)$$

$$\begin{aligned}&\leqslant l(w_0)+l\big(w_0^K\big)+\big|J'\big|+l\big(xv'\big)-l\big(xw'\big)\\&\leqslant l(w_0)+l\big(w_0^K\big)+\big|J'\big|+l(vy)-l\big(w\delta(y)\big)\\&\leqslant l(w_0)+l\big(w_0^K\big)+\big|J'\big|+l(v)-l(w)\\&=\dim\big([J,w,v]_{K,\delta}\big)-|J|+\big|J'\big|.\end{aligned}$$

The corollary is proved. □

2. Closure of a parabolic subgroup in $\overline{G^1}^{ss}$

In this section, we will describe the closure of a standard parabolic subgroup P in the semi-stable locus of $\bar{G}$ and prove that the closure is smooth. The explicit description and smoothness of the closure will be used in Section 4 to study the intermediate extension of Steinberg character sheaf and to partially verify a conjecture of Lusztig. The explicit description of the closure will also be used in Section 3 to study the closure of a Lusztig's stratum in $\bar{G}^{ss}$.

Below are some notations.

For any $J \subset I$, set $J_\delta = \max\{J_1 \subset J;\ \delta(J_1) = J_1\}$.

For any $K \subset I$ with $\delta(K) = K$, we write $P_K^1 = P_K g_0 = N_{\tilde{G}} P_K \cap G^1$ and $G_K^1 = L_K g_0 / Z(L_K)$.

Now we give an explicit description of $\overline{P_K^1} \cap \overline{G^1}^{ss}$ using P_K-stable pieces.

Theorem 2.1. *For $K \subset I$ with $\delta(K) = K$, we have that*

$$\overline{P_K^1} \cap \overline{G^1}^{ß} = \bigsqcup_{J \subset I} \bigsqcup_{w \in {}^K W^J,\, wW_J \cap W^\delta \neq \emptyset} \big[J, \delta(w), w\big]_{K,\delta}.$$

Proof. By [23, Lemma 7.3], $P_K^1 = P_K g_0 = (P_K)_\Delta \cdot (B g_0) = [I, 1, 1]_{K,\delta}$. Thus by Section 1.6 (5),

$$\overline{P_K^1} \cap Z_{J,1;\delta} = \bigsqcup_{w \in W^{\delta(J)},\ v \in {}^K W,\ xw \geqslant \delta(xv) \text{ for some } x \in W_K} \big([J, w, v]_{K,\delta} \cap Z_{J,1;\delta}\big).$$

Let $w \in W^{\delta(J)},\ v \in {}^K W$ with $xw \geqslant \delta(xv)$ for $x \in W_K$. Since $v \in {}^K W$, we have that

$$l(w) \geqslant l(xw) - l(x) \geqslant l(xv) - l(x) = l(v).$$

By Section 1.6 (3), $\overline{G_\Delta \cdot [J, w, v]_{K,\delta}} \cap Z_{J,\delta} = \overline{G_\Delta \cdot [J, w, v]_\delta} \cap Z_{J,\delta}$ is a union of G-stable pieces. If $[J, w, v]_{K,\delta} \cap Z_{J,1;\delta} \neq \emptyset$, then $Z_{J,1;\delta} \subset \overline{G_\Delta \cdot [J, w, v]_\delta}$. Again by Section 1.6 (3), there exists $a \in W$ and $b \in W_J$ such that $a\delta(b) w_0^{\delta(J)} w_0 \leqslant w w_0^{\delta(J)} w_0$, $ab \leqslant v$ and $l(a\delta(b) w_0^{\delta(J)} w_0) + l(ab) = l(w_0^{\delta(J)} w_0)$. Therefore $a\delta(b) w_0^{\delta(J)} \geqslant w w_0^{\delta(J)}$ and

$$\begin{aligned}l\big(a\delta(b)w_0^{\delta(J)}\big) &\geqslant l\big(ww_0^{\delta(J)}\big) = l(w) + l\big(w_0^{\delta(J)}\big) \geqslant l(v) + l\big(w_0^{\delta(J)}\big)\\&\geqslant l(ab) + l\big(w_0^{\delta(J)}\big).\end{aligned}$$

Since $l(a\delta(b)w_0^{\delta(J)}w_0)+l(ab)=l(w_0^{\delta(J)}w_0)$, we have that

$$l(a\delta(b)w_0^{\delta(J)})=l(ab)+l(w_0^{\delta(J)}).$$

Therefore, $xw=\delta(xv)$, $a\delta(b)w_0^{\delta(J)}=ww_0^{\delta(J)}$ and $ab=v$. So $w\delta(b)^{-1}=vb^{-1}=a$ and $xvb^{-1}=xw\delta(b)^{-1}=\delta(xv)\delta(b)^{-1}=\delta(xvb^{-1})$.

We may write xvb^{-1} as $xvb^{-1}=z_1z_2$ for $z_1\in W_K$ and $z_2\in {}^KW$. Then $xvb^{-1}=\delta(xvb^{-1})=\delta(z_1)\delta(z_2)$ and $\delta(z_1)\in W_K$, $\delta(z_2)\in {}^KW$. Therefore $z_1=\delta(z_1)$ and $z_2=\delta(z_2)$. Write z_2 as $z_2=z_3z_4$, where $z_3\in W^J$ and $z_4\in W_J$. Then $z_3\in {}^KW^J$ and $xw\delta(b)^{-1}=xvb^{-1}=z_1z_3z_4=z_1\delta(z_3)\delta(z_4)$. By [9, Corollary 2.5], $(w,v)=(\delta(z_3),z_3)$ is the unique element in $(W^{\delta(J)}\times {}^KW)\cap\mathcal{O}$, where $\mathcal{O}=\{(x'w\delta(y'),\ x'vy');\ x'\in W_K,\ y'\in W_J\}=\{(x'\delta(z_3)\delta(y'),\ x'z_3y');\ x'\in W_K,\ y'\in W_J\}$.

Therefore $\overline{P_K^1}\cap Z_{J,1;\delta}\subset\bigsqcup_{z\in {}^KW^J,\ zW_J\cap W^\delta\neq\emptyset}[J,\delta(z),z]_{K,\delta}$.

Now for $z\in {}^KW^J$ such that $zu=\delta(zu)$ for some $u\in W_J$, we have that $G_\Delta\cdot[J,\delta(z),z]_{K,\delta}=G_\Delta\cdot[J,\delta(z),z]_\delta$. By Section 1.6 (4), $G_\Delta\cdot[J,\delta(z),z]_\delta=Z_{J,1;\delta}$. Hence $[J,\delta(z),z]_{K,\delta}\subset\overline{P_K^1}\cap Z_{J,1;\delta}$. The theorem is proved. □

Lemma 2.2. *Let $J,K\subset I$ with $\delta(K)=K$. Then the map $w\mapsto\min(wW_J)$ gives a bijection*

$$\epsilon: {}^KW^{J_\delta}\cap W^\delta\to\{x\in {}^KW^J,\ xW_J\cap W^\delta\neq\emptyset\}.$$

Moreover, for any $w\in {}^KW^{J_\delta}\cap W^\delta$,

$$\max\{K'\subset K;\ K'=\delta(K'),\ \epsilon(w)^{-1}(K')\subset J\}=K\cap w(J_\delta).$$

Proof. If $w\in {}^KW^{J_\delta}\cap W^\delta$ and $x=\min(wW_J)$. Then $x\in {}^KW^J$ and $w\in xW_J\cap W^\delta$. So the map is well-defined.

Now suppose that $x\in {}^KW^J$ with $xW_J\cap W^\delta\neq\emptyset$. Let $y\in xW_J\cap W^\delta$. Write y as $y=ab$ for $a\in W_K$ and $b\in {}^KW$. Since $\delta(K)=K$, we have that $\delta(a)\in W_K$ and $\delta(b)\in {}^KW$. Now $ab=y=\delta(y)=\delta(a)\delta(b)$. So $b=\delta(b)$. Since $x\in {}^KW^J$, $b\in W_KxW_J\cap {}^KW=x(W_J\cap {}^{K_1}W)$, where $K_1=K\cap x^{-1}(J)$.

Write b as $b=wc$ for $w\in {}^KW^{J_\delta}$ and $c\in W_{J_\delta}$. Then $wc=b=\delta(b)=\delta(w)\delta(c)$ and $\delta(w)\in {}^KW^{J_\delta}$, $\delta(c)\in W_{J_\delta}$. Thus $w=\delta(w)\in {}^KW^{J_\delta}\cap W^\delta$ and $\epsilon(w)=x$. The map is surjective.

If $w_1,w_2\in {}^KW^{J_\delta}\cap W^\delta$ with $\epsilon(w_1)=\epsilon(w_2)$. Then $w_2=w_1a$ for some $a\in W_J$. Thus $w_1a=w_2=\delta(w_2)=\delta(w_1)\delta(a)=w_1\delta(a)$ and $a=\delta(a)$. Let $\mathrm{supp}(a)$ be the set of simple roots whose associated simple reflections appear in a reduced expression of a. Then $\mathrm{supp}(a)=\delta(\mathrm{supp}(a))\subset J$. Hence $\mathrm{supp}(a)\subset J_\delta$ and $a\in W_{J_\delta}$. Since $w_1,w_2\in W^{J_\delta}$, we have that $a=1$ and $w_1=w_2$. The map is injective.

Let $w\in {}^KW^{J_\delta}\cap W^\delta$. Then $w=\epsilon(w)a$ for some $a\in W_J\cap W^{J_\delta}$. Let $K'\subset K$. If $a^{-1}\epsilon(w)^{-1}(K')=w^{-1}(K')\subset J_\delta\subset J$, then $\epsilon(w)^{-1}(K')\subset\Phi_J$. Since $\epsilon(w)\in W^J$, we must have that $\epsilon(w)^{-1}(K')\subset J$. Moreover, $\delta(K\cap wJ_\delta)=\delta(K)\cap\delta(w)\delta(J_\delta)=K\cap wJ_\delta$. Hence $K\cap w(J_\delta)\subset\max\{K'\subset K;\ \delta(K')=K',\ \epsilon(w)^{-1}(K')\subset J\}$. On the other hand, assume that

$K' \subset K$, $\delta(K') = K$ and $\epsilon(w)^{-1}(K') \subset J$. Then for any $i \in K'$, $w^{-1}(\alpha_i) = a^{-1}\epsilon(w)^{-1}(\alpha_i)$ is a root in Φ_J. Since $w \in {}^K W$, $w^{-1}(\alpha_i)$ is a positive root in Φ_J. Now

$$\delta\left(w^{-1}\sum_{i\in K'}\alpha_i\right) = \delta(w)^{-1}\sum_{i\in\delta(K')=K'}\alpha_i = w^{-1}\sum_{i\in K'}\alpha_i.$$

Hence, $w^{-1}(\alpha_i)$ is a positive root in Φ_{J_δ} for $i \in K'$. Notice that $w \in W^{J_\delta}$. Thus $w^{-1}(\alpha_i)$ is a simple root in Φ_{J_δ} for $i \in K'$ and $w^{-1}(K') \subset J_\delta$. □

Let $w \in {}^K W \cap W^\delta$ and $J \subset I$, $\min(wW_{J_\delta}) \in W^\delta$. Write w' for $\min(wW_{J_\delta})$. Then

$$\begin{aligned}(P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta} &= (P_K)_\Delta\big(B\dot{w}', B\dot{w}'\big) \cdot h_{J,\delta} \\ &= \big[J, \delta\big(w'\big), w'\big]_{K,\delta}.\end{aligned}$$

By Proposition 2.1 and the previous lemma, we have other descriptions of $\overline{P_K^1} \cap \overline{G^1}^{ss}$ which are sometimes more convenient to use.

Theorem 2.3. *For $K \subset I$ with $\delta(K) = K$, we have that*

$$\begin{aligned}\overline{P_K^1} \cap \overline{G^1}^{ss} &= \bigsqcup_{J\subset I}\ \bigsqcup_{w\in {}^K W^{J_\delta}\cap W^\delta} (P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta} \\ &= \bigsqcup_{J\subset I}\ \bigcup_{w\in {}^K W\cap W^\delta} (P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta} \\ &= \bigcup_{w\in {}^K W\cap W^\delta}\ \bigsqcup_{J\subset I} (P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta}.\end{aligned}$$

Theorem 2.4. *For any $K \subset I$ with $\delta(K) = K$, the variety $\overline{P_K^1} \cap \overline{G^1}^{ss}$ is smooth.*

The proof will be given in the rest of this section. The main idea of the proof is to find an open covering of $\overline{P_K^1} \cap \overline{G^1}^{ss}$ such that each open subvariety appearing in the covering is open in another smooth variety.

Lemma 2.5. *For any $K \subset I$ with $\delta(K) = K$ and $w \in {}^K W \cap W^\delta$, $\bigsqcup_{J\subset I}(B\dot{w}, B\dot{w}_0^K\dot{w}) \cdot h_{J,\delta}$ is a locally closed subvariety of $\overline{G^1}$ isomorphic to an affine space of dimension* $\dim(P_K)$.

Proof. Since $w \in {}^K W$ and $\delta(w) = w$, we have that

$$\begin{aligned}\theta_\delta^{-1}\big({}^{\dot{w}^{-1}}U \cap U^-\big) &= {}^{\dot{w}^{-1}}U \cap U^- = {}^{\dot{w}^{-1}}U_{P_K} \cap U^- \subset {}^{\dot{w}^{-1}\dot{w}_0^K}U \cap U^-, \\ \theta_\delta\big({}^{\dot{w}^{-1}\dot{w}_0^K}U \cap U\big) &= {}^{\dot{w}^{-1}\dot{w}_0^K}U \cap U = {}^{\dot{w}^{-1}}U_{P_K} \cap U \subset {}^{\dot{w}^{-1}}U \cap U.\end{aligned}$$

Notice that $(U_{P^-_{\delta(J)}}, U_{P_J}) \cdot h_{J,\delta} = h_{J,\delta}$ and $(\theta_\delta(l), 1) \cdot h_{J,\delta} = (1, l) \cdot h_{J,\delta}$ for all $l \in L_J$. For $J \subset I$, we have that

$$\begin{aligned}
&\left({}^{\dot w^{-1}}B, {}^{\dot w^{-1}\dot w_0^K}B\right)\cdot h_{J,\delta}\\
&\quad=\left({}^{\dot w^{-1}}U, {}^{\dot w^{-1}\dot w_0^K}B\right)\cdot h_{J,\delta}\\
&\quad=\left(\left({}^{\dot w^{-1}}U\cap U\right)\left({}^{\dot w^{-1}}U\cap U^-\right), {}^{\dot w^{-1}\dot w_0^K}B\right)\cdot h_{J,\delta}\\
&\quad=\left(\left({}^{\dot w^{-1}}U\cap U\right)\left({}^{\dot w^{-1}}U\cap U^-\cap L_{\delta(J)}\right), {}^{\dot w^{-1}\dot w_0^K}B\right)\cdot h_{J,\delta}\\
&\quad=\left({}^{\dot w^{-1}}U\cap U, {}^{\dot w^{-1}\dot w_0^K}B\theta_\delta^{-1}\left({}^{\dot w^{-1}}U\cap U^-\cap L_{\delta(J)}\right)\right)\cdot h_{J,\delta}\\
&\quad=\left({}^{\dot w^{-1}}U\cap U, {}^{\dot w^{-1}\dot w_0^K}B\right)\cdot h_{J,\delta}\\
&\quad=\left({}^{\dot w^{-1}}U\cap U, \left({}^{\dot w^{-1}\dot w_0^K}B\cap B^-\right)\left({}^{\dot w^{-1}\dot w_0^K}U\cap U\right)\right)\cdot h_{J,\delta}\\
&\quad=\left({}^{\dot w^{-1}}U\cap U, \left({}^{\dot w^{-1}\dot w_0^K}B\cap B^-\right)\left({}^{\dot w^{-1}\dot w_0^K}U\cap U\cap L_J\right)\right)\cdot h_{J,\delta}\\
&\quad=\left(\left({}^{\dot w^{-1}}U\cap U\right)\theta_\delta\left({}^{\dot w^{-1}\dot w_0^K}U\cap U\cap L_J\right), \left({}^{\dot w^{-1}\dot w_0^K}B\cap B^-\right)\right)\cdot h_{J,\delta}\\
&\quad=\left({}^{\dot w^{-1}}U\cap U, {}^{\dot w^{-1}\dot w_0^K}B\cap B^-\right)\cdot h_{J,\delta}.
\end{aligned}$$

Set $X=\bigsqcup_{J\subset I}(1,T)\cdot h_{J,\delta}$. Using the result of [5, 3.7 and 3.8], we see that

$$\left(\dot w^{-1}, \dot w^{-1}\dot w_0^K\right)\cdot\bigsqcup_{J\subset I}\left(B\dot w, B\dot w_0^K\dot w\right)h_{J,\delta}=\left({}^{\dot w^{-1}}U\cap U, {}^{\dot w^{-1}\dot w_0^K}U\cap U^-\right)\cdot X$$

is a closed subvariety of $(U,U^-)\cdot X$ isomorphic to an affine space of dimension $\dim(P_K)$.

Since $(U,U^-)\cdot X$ is open in $\overline{G^1}$, $(\dot w^{-1}, \dot w^{-1}\dot w_0^K)\cdot\bigsqcup_{J\subset I}(B\dot w, B\dot w_0^K\dot w)h_{J,\delta}$ is locally closed in $\overline{G^1}$. □

Lemma 2.6. *For any $K\subset I$ with $\delta(K)=K$ and $w\in{}^KW\cap W^\delta$, we have that $\bigsqcup_{J\subset I}(P_K\dot w, P_K\dot w)\cdot h_{J,\delta}$ is smooth.*

Proof. Set $X=\bigsqcup_{J\subset I}(B\dot w, B\dot w_0^K\dot w)\cdot h_{J,\delta}$. Then X is isomorphic to an affine space and

$$\bigsqcup_{J\subset I}(P_K\dot w, P_K\dot w)\cdot h_{J,\delta}=\bigcup_{p,q\in P_K}(p,q)\cdot X.$$

So it suffices to prove that X is open in $\bigsqcup_{J\subset I}(P_K\dot w, P_K\dot w)\cdot h_{J,\delta}$.

Suppose that X is not open in $\bigsqcup_{J\subset I}(P_K\dot w, P_K\dot w)\cdot h_{J,\delta}$. Notice that X and

$$\bigsqcup_{J\subset I}(P_K\dot w, P_K\dot w)\cdot h_{J,\delta}=\bigsqcup_{J\subset I}\bigcup_{x,y\in W_K}(B\dot x\dot w, B\dot y\dot w)\cdot h_{J,\delta}$$

are unions of some $B\times B$-orbits. Thus there exists a $B\times B$-orbit $\mathcal{O}$ in $\bigsqcup_{J\subset I}(P_K\dot w, P_K\dot w)\cdot h_{J,\delta}-X$ whose closure contains a $B\times B$-orbit $\mathcal{O}'$ in X.

We may assume that $\mathcal{O}\subset Z_{J,\delta}$ and $\mathcal{O}'\subset Z_{J',\delta}$. Set $w'=\min(wW_J)$. Then $w'\in{}^KW^J$ and

$$\begin{aligned}
\dim\left(\left(B\dot w, B\dot w_0^K\dot w\right)\cdot h_{J,\delta}\right)&=\dim\left(\left[J,\delta\left(w'\right), w_0^Kw'\right]_\delta\right)=l(w_0)+|J|+l\left(w_0^K\right)\\
&=\dim\left((P_K\dot w, P_K\dot w)\cdot h_{J,\delta}\right).
\end{aligned}$$

Thus $(B\dot{w}, B\dot{w}_0^K\dot{w})\cdot h_{J,\delta}$ is open in $(P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta}$ and $\dim(\mathcal{O}) < \dim((B\dot{w}, B\dot{w}_0^K\dot{w})\cdot h_{J,\delta})$. By Corollary 1.1,

$$\begin{aligned}\dim(\overline{\mathcal{O}} \cap Z_{J',\delta}) &< \dim\big((B\dot{w}, B\dot{w}_0^K\dot{w})\cdot h_{J,\delta}\big) - |J| + |J'| \\ &= \dim\big((B\dot{w}, B\dot{w}_0^K\dot{w})\cdot h_{J',\delta}\big) = \dim(\mathcal{O}').\end{aligned}$$

Therefore $\mathcal{O}' \nsubseteq \overline{\mathcal{O}}$, which is a contradiction. □

Lemma 2.7. *For any $K \subset I$ with $\delta(K) = K$ and $w \in {}^KW \cap W^\delta$, $\bigsqcup_{J\subset I}(P_K)_\Delta(B\dot{w}, B\dot{w})\cdot h_{J,\delta}$ is open in $\bigsqcup_{J\subset I}(P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta}$.*

Proof. By definition, $(P_K)_\Delta(B\dot{w}, B\dot{w})\cdot h_{J,\delta} = [J, \delta(\min(wW_J)), \min(wW_J)]_{K,\delta}$. Thus $\bigsqcup_{J\subset I}(P_K)_\Delta(B\dot{w}, B\dot{w})\cdot h_{J,\delta}$ is a union of P_K-stable pieces.

Notice that $(P_K)_\Delta \subset P_K \times P_K$ and $B \times B \subset P_K \times P_K$. Thus for any $J \subset I$ and $x, v \in W$, either

$$(P_K)_\Delta(B\dot{x}, B\dot{v})\cdot h_{J,\delta} \cap (P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta} = \emptyset$$

or

$$(P_K)_\Delta(B\dot{x}, B\dot{v})\cdot h_{J,\delta} \subset (P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta}.$$

In other words, $\bigsqcup_{J\subset I}(P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta}$ is a union of P_K-stable pieces.

Suppose that $\bigsqcup_{J\subset I}(P_K)_\Delta(B\dot{w}, B\dot{w})\cdot h_{J,\delta}$ is not open in $\bigsqcup_{J\subset I}(P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta}$. Then there exists a P_K-stable piece $\mathcal{O}$ in $\bigsqcup_{J\subset I}(P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta} - \bigsqcup_{J\subset I}(P_K)_\Delta(B\dot{w}, B\dot{w})\cdot h_{J,\delta}$ whose closure contains a P_K-stable piece $\mathcal{O}'$ in $\bigsqcup_{J\subset I}(P_K)_\Delta(B\dot{w}, B\dot{w})\cdot h_{J,\delta}$.

We may assume that $\mathcal{O} \subset Z_{J,\delta}$ and $\mathcal{O}' \subset Z_{J',\delta}$. Set $w' = \min(wW_J)$. By Section 1.6 (1),

$$\begin{aligned}\dim\big((P_K)_\Delta(B\dot{w}, B\dot{w})\cdot h_{J,\delta}\big) &= \dim\big(\big[J, \delta\big(w'\big), w'\big]_{K,\delta}\big) = l(w_0) + |J| + l\big(w_0^K\big) \\ &= \dim\big((P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta}\big).\end{aligned}$$

Thus $[J, \delta(w'), w']_{K,\delta}$ is open in $(P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta}$ and $\dim(\mathcal{O}) < \dim([J, \delta(w'), w']_{K,\delta})$. By Corollary 1.1,

$$\begin{aligned}\dim(\overline{\mathcal{O}} \cap Z_{J',\delta}) &< \dim\big(\big[J, \delta\big(w'\big), w'\big]_{K,\delta}\big) - |J| + |J'| \\ &= \dim\big((P_K)_\Delta(B\dot{w}, B\dot{w})\cdot h_{J',\delta}\big) = \dim(\mathcal{O}').\end{aligned}$$

Therefore $\mathcal{O}' \nsubseteq \overline{\mathcal{O}}$, which is a contradiction. □

Proof of Theorem 2.4. We showed in Lemma 2.7 that for $w \in {}^KW \cap W^\delta$, $\bigsqcup_{J\subset I}(P_K)_\Delta(B\dot{w}, B\dot{w})\cdot h_{J,\delta}$ is an open subvariety of a smooth variety $\bigsqcup_{J\subset I}(P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta}$. By Lemmas 2.5 and 2.6, $\bigsqcup_{J\subset I}(P_K\dot{w}, P_K\dot{w})\cdot h_{J,\delta}$ is a smooth variety of dimension $\dim(P_K)$. Hence $\bigsqcup_{J\subset I}(P_K)_\Delta(B\dot{w}, B\dot{w})\cdot h_{J,\delta}$ is a smooth variety of dimension $\dim(P_K)$. Now the theorem follows from Theorem 2.3. □

3. A stratification on $\overline{G^1}^{ss}$

3.1. In [14] and [15], Lusztig introduced a stratification of G. This stratification is a key ingredient for defining the notion of "admissible complex" on G. In this section, we will generalize the definition of Lusztig's stratum to $\bar{G}^{ss}$ and prove that the decomposition of $\bar{G}^{ss}$ into Lusztig's strata is a stratification. We will also prove that some maps are small. These maps will be used in Section 4 to study the intermediate extension of "admissible complex" to $\bar{G}^{ss}$.

First, we recall a stratification of G^1 introduced by Lusztig in [15].

3.2. An element $g \in G^1$ is called isolated if there is no proper parabolic subgroup P of G such that $h \in N_{\tilde{G}}(P)$ and $Z_G(h_s)^0 \subset P$, where h_s is the semisimple part of h [15, 2.2]. Then the set of isolated elements is closed in G^1 [15, Lemma 2.8] and the action of $Z(G) \times G$ on G^1 defined by $(z, g) \cdot g' = gzg'g^{-1}$ leaves stable the set of isolated elements in G^1 and there are finitely many orbits there [15, Lemma 2.7]. These orbits are called isolated strata of G^1 [15, 3.3].

3.3. Let P be a parabolic subgroup of G, L be a Levi subgroup of P and S be an isolated stratum of $N_{\tilde{G}}(L) \cap G^1$ such that $S \subset N_{\tilde{G}}(P)$. Set $S^* = \{g \in S;\ Z_G(g_s)^0 \subset L\}$ and $Y_{L,S} = \bigsqcup_{g \in G} gS^*g^{-1}$. We call $Y_{L,S}$ a stratum of G^1. It is known that $Y_{L,S}$ is smooth [15, 3.17] and $Y_{L,S}$ (for various (L, S)) form a stratification of G^1 [15, Propositions 3.12 and 3.15].

Moreover, let S' be the closure of S in $N_{\tilde{G}}(L) \cap G^1$ and $G \times_P (S'U_P)$ be the quotient space of $G \times (S'U_P)$ under the P-action defined by $p(g, z) = (gp^{-1}, pzp^{-1})$. Then the proper map $f : G \times_P (S'U_P) \to \overline{Y_{L,S}}$ defined by $(g, z) \mapsto gzg^{-1}$ is a small map. See the proof of [15, Proposition 5.7].

Now we generalize the definition of strata to $\overline{G^1}^{ss}$.

3.4. By [8, Proposition 1.10], the map $(g, z) \mapsto (g, g) \cdot z$ gives an isomorphism $G \times_{P_{J_\delta}} (P_{J_\delta}, P_{J_\delta}) \cdot h_{J,\delta} \to Z_{J,1;\delta}$.

Notice that the map $(g, z) \mapsto (g, 1)z$ gives an isomorphism from $U_{P_{J_\delta}} \times (L_{J_\delta}, 1) \cdot h_{J,\delta}$ to $(P_{J_\delta}, P_{J_\delta}) \cdot h_{J,\delta}$ and the action of $U_{P_{J_\delta}}$ on $(L_{J_\delta}, 1) \cdot h_{J,\delta}$ defined by $(g, z) \mapsto (1, g) \cdot z$ is trivial. Then $(g, z) \mapsto (g, g) \cdot z$ gives an isomorphism

$$U_{P_{J_\delta}} \times (L_{J_\delta}, 1) \cdot h_{J,\delta} \cong (P_{J_\delta}, P_{J_\delta}) \cdot h_{J,\delta}. \tag{a}$$

Therefore

$$\begin{aligned} P_{J_\delta} \times_{L_{J_\delta}} (L_{J_\delta}, 1) \cdot h_{J,\delta} &\cong (U_{P_{J_\delta}} \times L_{J_\delta}) \times_{L_{J_\delta}} (L_{J_\delta}, 1) \cdot h_{J,\delta} \\ &\cong U_{P_{J_\delta}} \times \big(L_{J_\delta} \times_{L_{J_\delta}} (L_{J_\delta}, 1) \cdot h_{J,\delta}\big) \\ &\cong (P_{J_\delta}, P_{J_\delta}) \cdot h_{J,\delta}, \end{aligned}$$

where $P_{J_\delta} \times_{L_{J_\delta}} (L_{J_\delta}, 1) \cdot h_{J,\delta}$ is the quotient space for the L_{J_δ} action on $P_{J_\delta} \times (L_{J_\delta}, 1) \cdot h_{J,\delta}$ defined by $l \cdot (p, z) = (pl^{-1}, (l, l) \cdot z)$.

Thus $Z_{J,1;\delta}$ is isomorphic to $G \times_{P_{J_\delta}} (P_{J_\delta} \times_{L_{J_\delta}} (L_{J_\delta}, 1) \cdot h_{J,\delta}) \cong G \times_{L_{J_\delta}} (L_{J_\delta}, 1) \cdot h_{J,\delta}$ via $(g, z) \mapsto (g, g) \cdot z$.

We may also identify $(L_{J_\delta}, 1) \cdot h_{J,\delta}$ with $L_{J_\delta} g_0/Z(L_J)$. Therefore we have an isomorphism

$$i_J\colon G \times_{L_{J_\delta}} L_{J_\delta} g_0/Z(L_J) \cong Z_{J,1;\delta}$$

via $(g, lg_0) \mapsto (gl, g) \cdot h_{J,\delta}$.

Notice that we have a stratification $G \times_{L_{J_\delta}} L_{J_\delta} g_0 = \bigsqcup G \times_{L_{J_\delta}} Y$, where Y runs over strata of $L_{J_\delta} g_0$. Moreover, each stratum Y of $L_{J_\delta} g_0$ is stable under the action of $Z(L_{J_\delta}) \supset Z(L_J)$. Then

$$\overline{G^1}^{ss} = \bigsqcup_{J \subset I} \bigsqcup_{Y \text{ is a stratum of } L_{J_\delta} g_0} i_J\bigl(G \times_{L_{J_\delta}} Y/Z(L_J)\bigr) \tag{b}$$

is a decomposition of $\overline{G^1}^{ss}$. We will see in the end of this section that (b) is in fact a stratification. For any $J \subset I$ and stratum Y of $L_{J_\delta} g_0$, we call $i_J(G \times_{L_{J_\delta}} Y/Z(L_J))$ a stratum of $\overline{G^1}^{ss}$.

We may define a decomposition for $\overline{G^1}$ in the same way. But it is very hard to give an explicit description of the closure of any subvariety appearing in the decomposition. However, [6, Theorem 4.3], [12, Theorem 7.4] and [8, Theorem 4.5] give some evidence that this decomposition for $\overline{G^1}$ may still be a stratification.

3.5. In this subsection, we assume that $G^1 = G$. It is known [4] that the map $(g, g', z) \mapsto (g, g') \cdot z$ gives an isomorphism

$$(G \times G) \times_{P_J^- \times P_J} \overline{G_J} \cong \overline{Z_J}.$$

Notice that any element in $\overline{Z_J} \cap \bar{G}^{ss}$ is of the form $(gl, g) \cdot h_K$ for some $K \subset J$, $g \in G$ and $l \in L_K$ and any element in $\overline{G_J}^{ss}$ is of the form $(g'l', g') \cdot h_K$ for some $K \subset J$, $g' \in L_J$ and $l' \in L_K$. Therefore $\overline{Z_J} \cap \bar{G}^{ss} = G_\Delta \cdot \overline{G_J}^{ss}$.

The morphism $(G \times G) \times_{P_J^- \times P_J} \overline{G_J} \to G/P_J^- \times G/P_J$, $(g, g', z) \mapsto (gP_J^-, g'P_J)$ sends $\overline{Z_J} \cap \bar{G}^{ss}$ to the open G_Δ orbit $\mathcal{O}$ in $G/P_J^- \times G/P_J$. It is easy to see that $\mathcal{O} \cong G/L_J$. Since each fiber of the G-equivariant morphism $\overline{Z_J} \cap \bar{G}^{ss} \to G/L_J$ is isomorphic to $\overline{G_J}^{ss}$, by [19, p. 26, Lemma 4], we have that

$$\overline{Z_J} \cap \bar{G}^{ss} \cong G \times_{L_J} \overline{G_J}^{ss}.$$

Here $G \times_{L_J} \overline{G_J}^{ss}$ is the quotient space for the L_J-action on $G \times \overline{G_J}^{ss}$ defined by $l \cdot (g, z) = (gl^{-1}, (l, l) \cdot z)$. This isomorphism extends the isomorphism $Z_{J,1;id} = Z_J \cap \bar{G}^{ss} \cong G \times_{L_J} G_J$ in the previous subsection.

Lemma 3.1. *Let $T_0 = \{t\theta_\delta(t^{-1});\ t \in T\}$. Let $J, K \subset I$ with $\delta(K) = K$ and $w \in W^\delta$. Then*

$$\overline{T_0 Z(L_K)} \cap (\dot{w}T, \dot{w}) \cdot h_J = \begin{cases} (T_0 Z(L_K)\dot{w}, \dot{w}) \cdot h_J, & \text{if } w^{-1}\Phi_K \subset \Phi_J; \\ \emptyset, & \text{otherwise}. \end{cases}$$

Proof. Let $X = \bigsqcup_{D \subset I} (T, 1) \cdot h_D$. Then for any positive root α, the morphism $T \to k$ defined by $t \mapsto \alpha(t)$ extends in a unique way to a morphism $X \to k$, which we denote by $\tilde{\alpha}$. It is easy to see that

$$\tilde{\alpha}_i\big((t, 1) \cdot h_J\big) = \begin{cases} \alpha_i(t), & \text{if } i \in J, \\ 0, & \text{if } i \notin J. \end{cases} \tag{a}$$

By definition, $T_0 Z(L_K) = \{t \in T;\ \prod_{i \in \mathcal{O}} \alpha_i(t) = 1,\ \forall \delta\text{-orbit } \mathcal{O} \text{ of } K\}$. So $(\dot{w}^{-1}, \dot{w}^{-1}) \cdot T_0 Z(L_K) = \{t \in T;\ \prod_{i \in \mathcal{O}} w^{-1}\alpha_i(t) = 1,\ \forall \delta\text{-orbit } \mathcal{O} \text{ of } K\}$. For any root α, set

$$\operatorname{sgn}(\alpha) = \begin{cases} 1, & \text{if } \alpha > 0; \\ -1, & \text{if } \alpha < 0. \end{cases}$$

Notice that $\delta(w^{-1}\alpha_i) = \delta(w)^{-1}\alpha_{\delta(i)} = w^{-1}\alpha_{\delta(i)}$. Thus for any δ-orbit $\mathcal{O}$ of K, either $w^{-1}\alpha_i > 0$ for all $i \in \mathcal{O}$ or $w^{-1}\alpha_i < 0$ for all $i \in \mathcal{O}$. So we may write $\operatorname{sgn}(w^{-1}\mathcal{O})$ for $\operatorname{sgn}(w^{-1}\alpha_i)$, where $i \in \mathcal{O}$. Now $(\prod_{i \in \mathcal{O}} \widetilde{w^{-1}(\alpha_i)} z)^{\operatorname{sgn}(w^{-1}\mathcal{O})}$ is a well-defined morphism from X to k and

$$\overline{\big(\dot{w}^{-1}, \dot{w}^{-1}\big) \cdot T_0 Z(L_K)} = \left\{ z \in X;\ \prod_{i \in \mathcal{O}} \widetilde{w^{-1}(\alpha_i)}(z)^{\operatorname{sgn}(w^{-1}\mathcal{O})} = 1 \right\}.$$

By (a), if $w^{-1}(\Phi_K) \not\subset \Phi_J$, then $\prod_{i \in K} \widetilde{w^{-1}(\alpha_i)}(z)^{\operatorname{sgn}(w^{-1}(\alpha_i))} = 0$ for all $z \in (T, 1) \cdot h_J$ and $\overline{(\dot{w}^{-1}, \dot{w}^{-1}) \cdot T_0 Z(L_K)} \cap (T, 1) \cdot h_J = \emptyset$. On the other hand, if $w^{-1}(\Phi_K) \subset \Phi_J$, then for any $z = (t, 1) \cdot h_J$ and $i \in K$, $\widetilde{w^{-1}(\alpha_i)}(z) = w^{-1}(\alpha_i)(t)$. Therefore $\overline{(\dot{w}^{-1}, \dot{w}^{-1}) \cdot T_0 Z(L_K)} \cap (T, 1) \cdot h_J = \{(t, 1) \cdot h_J;\ t \in (\dot{w}^{-1}, \dot{w}^{-1}) \cdot T_0 Z(L_K)\}$. The lemma is proved. □

Notice that $\theta_\delta(T_0) = T_0$ and $\theta_\delta Z(L_K) = Z(L_K)$ for $K \subset I$ with $\delta(K) = K$. By the identification of $\bar{G}$ with $\overline{G^1}$ in Section 1.5, we have the following variation of the previous lemma.

Lemma 3.2. *Let $J, K \subset I$ with $\delta(K) = K$ and $w \in W^\delta$. Then*

$$\overline{T_0 Z(L_K) g_0} \cap (\dot{w} T, \dot{w}) \cdot h_{J,\delta} = \begin{cases} (T_0 Z(L_K) \theta_\delta(\dot{w}), \dot{w}) \cdot h_{J,\delta}, & \text{if } w^{-1}\Phi_K \subset \Phi_J; \\ \emptyset, & \text{otherwise}. \end{cases}$$

Theorem 3.3. *Let $K \subset I$ with $\delta(K) = K$ and S be an isolated stratum of $L_K g_0$. Let S' be the closure of S in $L_K g_0$. Then*

$$\overline{U_{P_K} S} \cap \overline{G^1}^{ss} = \bigsqcup_{J \subset I, w \in {}^K W^{J_\delta} \cap W^\delta, w^{-1}(K) \subset J_\delta} \big(U_{P_K} S' \dot{w} g_0^{-1}, U_{P_K} \dot{w}\big) \cdot h_{J,\delta}.$$

Proof. Since $U_{P_K} S' \subset P_K^1$, then

$$\overline{U_{P_K} S'} \cap \overline{G^1}^{ss} \subset \overline{P_K^1} \cap \overline{G^1}^{ss} = \bigsqcup_{J \subset I} \bigsqcup_{w \in {}^K W^{J_\delta} \cap W^\delta} (P_K)_\Delta (B\dot{w}, B\dot{w}) \cdot h_{J,\delta}.$$

Since S is stable under the conjugation action of L_K, there exists $s \in (B \cap L_K)g_0$ such that $s \in S$. We may write s as $s = utg_0$ for some $t \in T$ and $u \in U \cap L_K$. Then

$$S = \{lutzg_0 l^{-1};\ l \in L_K, z \in Z(L_K)\} \subset (L_K)_\Delta\big((U \cap L_K)tT_0 Z(L_K)g_0\big).$$

By [15, Lemma 3.11], S^* is open dense in S. By [15, 3.13], $\dim((U_{P_K})_\Delta \cdot S^*) = \dim(U_{P_K}) + \dim(S^*) = \dim(U_{P_K} S')$. Since $(U_{P_K})_\Delta \cdot S \subset U_{P_K} S U_{P_K} = U_{P_K} S \subset U_{P_K} S'$, $(U_{P_K})_\Delta \cdot S^*$ is dense in $U_{P_K} S'$ and hence $(U_{P_K})_\Delta \cdot S$ is dense in $U_{P_K} S'$. Now consider the proper map $P_K \times_B \overline{UtT_0 Z(L_K)g_0} \to \overline{G^1}$ defined by $(g, z) \mapsto (g, g) \cdot z$. Since

$$(P_K)_\Delta\big(UtT_0 Z(L_K)g_0\big) = (U_{P_K})_\Delta (L_K)_\Delta \cdot \big(UtT_0 Z(L_K)g_0\big) \supset (U_{P_K})_\Delta \cdot S,$$

then

$$\overline{U_{P_K} S'} \subset (P_K)_\Delta \overline{UtT_0 Z(L_K)g_0}. \tag{a}$$

Let $J \subset I$ and $w \in {}^K W^{J_\delta} \cap W^\delta$. By definition,

$$\begin{aligned}
(P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta} &\subset \bigcup_{x \in W_K} (B\dot{x}B\dot{w}, B\dot{x}B\dot{w}) \cdot h_{J,\delta} \\
&= \bigcup_{x \in W_K} (B\dot{x}B\dot{w}, B\dot{x}\dot{w}) \cdot h_{J,\delta} \\
&\subset \bigcup_{x \in W_K} \Big((B\dot{x}\dot{w}, B\dot{x}\dot{w}) \cdot h_{J,\delta} \cup \bigcup_{y < xw} (B\dot{y}, B\dot{x}\dot{w}) \cdot h_{J,\delta}\Big).
\end{aligned}$$

If $(P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta} \cap \overline{U_{P_K} S'} \neq \emptyset$, by (a) we have that

$$\begin{aligned}
&\bigcup_{x \in W_K^\delta} (B\dot{x}\dot{w}, B\dot{x}\dot{w}) \cdot h_{J,\delta} \\
&\quad = (P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta} \cap \overline{B^1} \\
&\quad \supset (P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta} \cap \overline{UtT_0 Z(L_K)g_0} \neq \emptyset.
\end{aligned}$$

Therefore $\overline{UtT_0 Z(L_K)g_0} \cap (B\dot{x}\dot{w}, B\dot{x}\dot{w}) \cdot h_{J,\delta} \neq \emptyset$ for some $x \in W_K^\delta$. Set $X = \bigcup_{J' \subset I} (B\dot{x}\dot{w}, B\dot{x}\dot{w}) \cdot h_{J',\delta}$. Then the map $(u, u', z) \mapsto (u\dot{x}\dot{w}, u'\dot{x}\dot{w}) \cdot z$ defines an isomorphism

$$\big(U \cap {}^{\dot{x}\dot{w}}U\big) \times \big(U \cap {}^{\dot{x}\dot{w}}U^-\big) \times \bigcup_{J' \subset I} (T, 1) \cdot h_{J',\delta} \to X.$$

Notice that $UtT_0 Z(L_K)g_0 \subset X$. Then $\overline{UtT_0 Z(L_K)g_0} \cap X$ is the closure of $UtT_0 Z(L_K)g_0$ in X. Hence

$$\overline{UtT_0 Z(L_K)g_0} \cap X = \big(\big(U \cap {}^{\dot{x}\dot{w}}U\big)\dot{x}\dot{w}, \big(U \cap {}^{\dot{x}\dot{w}}U^-\big)\dot{x}\dot{w}\big) \cdot X', \tag{b}$$

where $X' = \overline{{}^{(\dot{x}\dot{w})^{-1}}T_0 Z(L_K)g_0} \cap \bigcup_{J' \subset I}(T,1) \cdot h_{J',\delta}$. Since $\overline{UtT_0 Z(L_K)g_0} \cap (B\dot{x}\dot{w}, B\dot{x}\dot{w}) \cdot h_{J,\delta} \neq \emptyset$, then $X' \cap (T,1) \cdot h_{J,\delta} \neq \emptyset$. By the previous lemma, $w^{-1}\Phi_K = w^{-1}x^{-1}\Phi_K \subset \Phi_J$. Since $w = \delta(w)$ and $K = \delta(K)$, we have that $\delta(w^{-1}\Phi_K) = w^{-1}\Phi_K \subset \Phi_{J_\delta}$. Notice that $w \in {}^K W^{J_\delta}$. Then $w^{-1}(K) \subset J_\delta$.

On the other hand, suppose that $w \in {}^K W \cap W^\delta$ with $w^{-1}(K) \subset I$. Set

$$Y = \bigsqcup_{w^{-1}(K) \subset D \subset I} (P_K)_\Delta (B\dot{w}, B\dot{w}) \cdot h_{D,\delta}.$$

If $w^{-1}(K) \subset D$, then $g_0 L_{w^{-1}(K)} g_0^{-1} = L_{w^{-1}(K)}$ and by [23, Lemma 7.3], $(L_{w^{-1}(K)})_\Delta \cdot ((B \cap L_{w^{-1}(K)})g_0) = L_{w^{-1}(K)} g_0$. Hence

$$\begin{aligned}
&(P_K)_\Delta (B\dot{w}, B\dot{w}) \cdot h_{D,\delta} \\
&\quad = (L_K)_\Delta \big(U_{P_K}\dot{w}, U_{P_K}\dot{w}(B \cap L_{w^{-1}(K)})\big) \cdot h_{D,\delta} \\
&\quad = (U_{P_K}\dot{w}, U_{P_K}\dot{w})\big(L_{w^{-1}(K)}\big)_\Delta \big(1, B \cap L_{w^{-1}(K)}\big) \cdot h_{D,\delta} \\
&\quad = (U_{P_K}\dot{w}, U_{P_K}\dot{w})(1, L_{w^{-1}(K)}) \cdot h_{D,\delta} \\
&\quad = (U_{P_K}\dot{w}, U_{P_K}\dot{w})(L_{w^{-1}(K)}, L_{w^{-1}(K)}) \cdot h_{D,\delta} = (P_K\dot{w}, P_K\dot{w}) \cdot h_{D,\delta}.
\end{aligned}$$

Therefore $Y = \bigsqcup_{w^{-1}(K) \subset D \subset I} (P_K\dot{w}, P_K\dot{w}) \cdot h_{D,\delta}$. Since $w^{-1}(K) \subset I$, $U_{P_K} \cap {}^{\dot{w}}U = U_{P_K} \cap {}^{\dot{w}}U_{P_{w^{-1}(K)}}$ and $U_{P_K} \cap {}^{\dot{w}}U^- = U_{P_K} \cap {}^{\dot{w}}U_{P^-_{w^{-1}(K)}}$. It is easy to see that the map $(u, u', z) \mapsto (u, u') \cdot z$ defines an isomorphism

$$\big(U_{P_K} \cap {}^{\dot{w}}U\big) \times \big(U_{P_K} \cap {}^{\dot{w}}U^-\big) \times \bigsqcup_{w^{-1}(K) \subset D \subset I} (L_K\dot{w}, L_K\dot{w}) \cdot h_{D,\delta} \to Y. \tag{c}$$

By the similar argument as we did for (b), one can show that the closure of $S' = (S'g_0^{-1}\theta_\delta(\dot{w}), \dot{w}) \cdot h_{I,\delta}$ in $\bigsqcup_{w^{-1}(K) \subset D \subset I} (L_K\dot{w}, L_K\dot{w}) \cdot h_{D,\delta}$ is $\bigsqcup_{w^{-1}(K) \subset D \subset I} (S'g_0^{-1}\theta_\delta(\dot{w}), \dot{w}) \cdot h_{D,\delta}$.

Hence the closure of $U_{P_K}S' = (U_{P_K} \cap {}^{\dot{w}}U, U_{P_K} \cap {}^{\dot{w}}U^-) \cdot (S'g_0^{-1}\theta_\delta(\dot{w}), \dot{w}) \cdot h_{I,\delta}$ in Y is

$$\begin{aligned}
&\big(U_{P_K} \cap {}^{\dot{w}}U, U_{P_K} \cap {}^{\dot{w}}U^-\big) \cdot \bigsqcup_{w^{-1}(K) \subset D \subset I} \big(S'g_0^{-1}\theta_\delta(\dot{w}), \dot{w}\big) \cdot h_{D,\delta} \\
&\quad = \bigsqcup_{w^{-1}(K) \subset D \subset I} \big(U_{P_K}S'g_0^{-1}\theta_\delta(\dot{w}), U_{P_K}\dot{w}\big) \cdot h_{D,\delta} \\
&\quad = \bigsqcup_{w^{-1}(K) \subset D \subset I} \big(U_{P_K}S'\dot{w}g_0^{-1}, U_{P_K}\dot{w}\big) \cdot h_{D,\delta}.
\end{aligned}$$

The theorem is proved. □

Now we state the result on the special case that $G^1 = G$.

Corollary 3.4. *Let $K \subset I$ and S be an isolated stratum of L_K. Let S' be the closure of S in L_K. Then for any $J \subset I$,*

$$\overline{U_{P_K} S} \cap \bar{G}^{ss} \cap \overline{Z_J} = \bigsqcup_{w \in {}^K W^J, w^{-1}(K) \subset J} (U_{P_K} \dot{w}, U_{P_K} \dot{w}) \cdot X_w,$$

where X_w is the closure of $(U_{P_{w^{-1}(K)}} \cap L_J)^{\dot{w}^{-1}} S / Z(L_J)$ in $\overline{G_J}^{ss}$.

3.6. Let $J, K \subset I$ with $\delta(K) = K$ and $w \in {}^K W^{J_\delta} \cap W^\delta$. Let $S \subset L_{J_\delta}$ be a subvariety. Since $Z_{J,1;\delta} \cong G \times_{P_{J_\delta}} (P_{J_\delta}, P_{J_\delta}) \cdot h_{J,\delta}$, we have a projection map $Z_{J,1;\delta} \to G/P_{J_\delta}$. Restricting the projection map to $({}^{\dot{w}^{-1}} P_K)_\Delta (S, 1) \cdot h_{J,\delta} \subset Z_{J,1;\delta}$, we obtain a morphism

$$\left({}^{\dot{w}^{-1}} P_K\right)_\Delta (S, 1) \cdot h_{J,\delta} \to {}^{\dot{w}^{-1}} P_K / {}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}.$$

By [19, p. 26, Lemma 4],

$$\left({}^{\dot{w}^{-1}} P_K\right)_\Delta (S, 1) \cdot h_{J,\delta} \cong {}^{\dot{w}^{-1}} P_K \times_{{}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}} \left({}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}\right)_\Delta (S, 1) \cdot h_{J,\delta}.$$

Notice that ${}^{\dot{w}^{-1}} P_K \cap P_{J_\delta} \cong {}^{\dot{w}^{-1}} P_K \cap U_{P_{J_\delta}} \times {}^{\dot{w}^{-1}} P_K \cap L_{J_\delta}$ and

$${}^{\dot{w}^{-1}} P_K \cap L_{J_\delta} = L_{K'} \left({}^{\dot{w}^{-1}} U_{P_K} \cap L_{J_\delta}\right) = L_{K'} (B \cap L_{J_\delta}) = P_{K'} \cap L_{J_\delta},$$

where $K' = w^{-1} K \cap J_\delta$. Therefore,

$$\left({}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}\right)_\Delta (S, 1) \cdot h_{J,\delta} = \left({}^{\dot{w}^{-1}} P_K \cap U_{P_{J_\delta}}\right)_\Delta (P_{K'} \cap L_{J_\delta})_\Delta (S, 1) \cdot h_{J,\delta}.$$

Set $X = (P_{K'} \cap L_{J_\delta})_\Delta (S, 1) \cdot h_{J,\delta}$. By Section 3.4 (a), the map $(g, z) \mapsto (g, g) \cdot z$ gives an isomorphism

$$\left({}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}\right) \times_{P_{K'} \cap L_{J_\delta}} X \cong \left({}^{\dot{w}^{-1}} P_K \cap U_{P_{J_\delta}}\right) \times X \cong \left({}^{\dot{w}^{-1}} P_K \cap U_{P_{J_\delta}}\right)_\Delta \cdot X.$$

Therefore, we have that

$$\begin{aligned}
&G \times_{{}^{\dot{w}^{-1}} P_K} \left({}^{\dot{w}^{-1}} P_K\right)_\Delta \cdot X \\
&\quad \cong G \times_{{}^{\dot{w}^{-1}} P_K} \left({}^{\dot{w}^{-1}} P_K \times_{{}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}} \left({}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}\right)_\Delta \cdot X\right) \\
&\quad \cong G \times_{{}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}} \left(\left({}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}\right)_\Delta \cdot X\right) \\
&\quad \cong G \times_{{}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}} \left(\left({}^{\dot{w}^{-1}} P_K \cap P_{J_\delta}\right) \times_{P_{K'} \cap L_{J_\delta}} X\right) \\
&\quad \cong G \times_{P_{K'} \cap L_{J_\delta}} X \cong G \times_{L_{J_\delta}} (L_{J_\delta} \times_{P_{K'} \cap L_{J_\delta}} X). \qquad \text{(a)}
\end{aligned}$$

Similarly, we may identify $G_\Delta \cdot X$ with $G \times_{L_{J_\delta}} (L_{J_\delta})_\Delta \cdot X$ and under these identifications, the map $(g, z) \mapsto (g, g) \cdot z$ from $G \times_{{}^{\dot{w}^{-1}} P_K} ({}^{\dot{w}^{-1}} P_K)_\Delta \cdot X$ to $G_\Delta \cdot X$ is induced from the map

$$G \times (L_{J_\delta} \times_{P_{K'} \cap L_{J_\delta}} X) \to G \times \big((L_{J_\delta})_\Delta \cdot X\big),$$

defined by $(g, l, z) \mapsto (g, (l, l) \cdot z)$.

3.7. Let $J, K \subset I$ with $\delta(K) = K$ and $w \in {}^K W^{J_\delta} \cap W^\delta$ with $w^{-1}(K) \subset J_\delta$. Let $S' \subset L_K g_0$ be the closure of an isolated stratum. We have that ${}^{\dot{w}^{-1}} P_K = {}^{\dot{w}^{-1}} U_{P_K} {}^{\dot{w}^{-1}} L_K = ({}^{\dot{w}^{-1}} U_{P_K} \cap U_{P_{J_\delta}})({}^{\dot{w}^{-1}} P_K \cap L_{J_\delta})({}^{\dot{w}^{-1}} U_{P_K} \cap U_{P^-_{J_\delta}})$. Since $w \in W^{J_\delta}$ and $w^{-1}(K) \subset J_\delta$, we have that ${}^{\dot{w}^{-1}} U_{P_K} \cap L_{J_\delta} = U_{P_{w^{-1}(K)}} \cap L_{J_\delta}$. Then

$$\begin{aligned}
&\big(\dot{w}^{-1}, \dot{w}^{-1}\big)(P_K)_\Delta\big(U_{P_K} S' \dot{w} g_0^{-1}, U_{P_K} \dot{w}\big) \cdot h_{J,\delta} \\
&\quad = \big({}^{\dot{w}^{-1}} P_K\big)_\Delta\big(\big({}^{\dot{w}^{-1}} U_{P_K} \cap L_{J_\delta}\big)\big({}^{\dot{w}^{-1}} U_{P_K} \cap U_{P^-_{J_\delta}}\big)^{\dot{w}^{-1}} S' g_0^{-1}, \big({}^{\dot{w}^{-1}} U_{P_K} \cap U_{P_{J_\delta}}\big)\big) \cdot h_{J,\delta} \\
&\quad = \big({}^{\dot{w}^{-1}} P_K\big)_\Delta\big(\big({}^{\dot{w}^{-1}} U_{P_K} \cap L_{J_\delta}\big)^{\dot{w}^{-1}} S' g_0^{-1}, 1\big) \cdot h_{J,\delta} \\
&\quad = \big({}^{\dot{w}^{-1}} P_K\big)_\Delta\big((U_{P_{w^{-1}K}} \cap L_{J_\delta})^{\dot{w}^{-1}} S' g_0^{-1}, 1\big) \cdot h_{J,\delta}.
\end{aligned}$$

The map $f : G \times_{P_K} (P_K)_\Delta(U_{P_K} S' \dot{w} g_0^{-1}, U_{P_K} \dot{w}) \cdot h_{J,\delta} \to G \times_{{}^{\dot{w}^{-1}} P_K} ({}^{\dot{w}^{-1}} P_K)_\Delta((U_{P_{w^{-1}K}} \cap L_{J_\delta})^{\dot{w}^{-1}} S' g_0^{-1}, 1) \cdot h_{J,\delta}$ defined by $(g, z) \mapsto (g\dot{w}, (\dot{w}^{-1}, \dot{w}^{-1})z)$ is an isomorphism. Moreover,

$$\pi = \pi' \circ f, \tag{$*$}$$

where

$$\begin{gathered}
\pi : G \times_{P_K} (P_K)_\Delta\big(U_{P_K} S' \dot{w} g_0^{-1}, U_{P_K} \dot{w}\big) \cdot h_{J,\delta} \to Z_{J,1;\delta}, \\
\pi' : G \times_{{}^{\dot{w}^{-1}} P_K} \big({}^{\dot{w}^{-1}} P_K\big)_\Delta\big((U_{P_{w^{-1}K}} \cap L_{J_\delta})^{\dot{w}^{-1}} S' g_0^{-1}, 1\big) \cdot h_{J,\delta} \to Z_{J,1;\delta},
\end{gathered}$$

are induced from the map $G \times Z_{J,1;\delta} \to Z_{J,1;\delta}$ defined by $(g, z) \mapsto (g, g) \cdot z$.

As in the previous subsection, the map π' is induced from the map $G \times (L_{J_\delta} \times_{P_{w^{-1}(K)} \cap L_{J_\delta}} X) \to G \times ((L_{J_\delta})_\Delta \cdot X)$ defined by $(g, l, z) \mapsto (g, (l, l) \cdot z)$, here

$$X = \big((U_{P_{w^{-1}K}} \cap L_{J_\delta})^{\dot{w}^{-1}} S' g_0^{-1}, 1\big) \cdot h_{J,\delta} \cong (U_{P_{w^{-1}K}} \cap L_{J_\delta})^{\dot{w}^{-1}} S' / Z(L_{J_\delta}).$$

Since ${}^{\dot{w}^{-1}} S' / Z(L_{J_\delta})$ is the closure of an isolated stratum in $L_{w^{-1}K} g_0 / Z(L_{J_\delta})$, by Section 3.3, $G_\Delta \cdot X$ is a union of strata in $Z_{J,1;\delta}$ and the map π' is a small map.

As a summary, we have the following result.

Theorem 3.5. *Let $K \subset I$ with $\delta(K) = K$ and S be an isolated stratum of $L_K g_0$. Then the proper map*

$$G \times_{P_K} \big(\overline{U_{P_K} S} \cap \overline{G^1}^{ss}\big) \to \overline{G_\Delta(U_{P_K} S)} \cap \overline{G^1}^{ss}$$

sending $(g, z) \to (g, g) \cdot z$ *is small and* $\overline{G_\Delta(U_{P_K} S)} \cap \overline{G^{1ss}}$ *is a union of strata in* $\overline{G^{1ss}}$.

3.8. Let $J \subset I$ and Y be a stratum of $L_{J_\delta} g_0$. By the same argument as above, we can show that $\overline{i_J(G \times_{L_{J_\delta}} Y/Z(L_J))} \cap \overline{G^{1ss}}$ is a union of strata of $\overline{G^{1ss}}$. Since we don't need this result in the rest of the paper, we skip the details.

4. Character sheaves on $\overline{G^{1ss}}$

4.1. In this section, we will use the results on the structure of the semi-stable locus established in the previous two sections to prove our main result (Theorem 4.4), which describes the restriction to a semi-stable stratum of the intermediate extension of an "admissible complex". We also give partial answers to the problem about "boundary value" of character of simple groups of Lie type (asked by Springer) and a conjecture about semisimple elements in $\bar{G}$ (due to Lusztig).

4.2. Fix a prime number l that is invertible in k. For any algebraic variety X over k, we write $\mathcal{D}(X)$ for $\mathcal{D}^b_c(X, \overline{\mathbb{Q}}_l)$, the bounded derived category of $\overline{\mathbb{Q}}_l$-constructible sheaves on X [1, 2.2.18].

For any subgroup H of G and an H-variety X, we define the H action on $G \times X$ by $h \cdot (g, x) = (gh^{-1}, h \cdot x)$ and denote by $G \times_H X$ the quotient space. For any perverse sheaf A on X that is equivariant for the H action, we denote by $i_H^G(A)$ the perverse sheaf on $G \times_H X$ such that $p^*(A)[\dim(H)] = \overline{\mathbb{Q}}_{l,G}[\dim(G)] \boxtimes A$, where $p : G \times X \to G \times_H X$ is the projection map.

4.3. In this subsection, we only assume that G is a connected reductive group.

Let $\mathcal{Z} = \{g \in Z(G);\ gg' = g'g \text{ for all } g' \in G^1\}$. For each isolated stratum S of G^1 and $n \in \mathbb{N}$ that is invertible in k, let $\mathcal{S}_n(S)$ be the set of local systems on S that are equivariant for the $\mathcal{Z}^0 \times G$-action defined by $(z, g) \cdot s = gz^n sg^{-1}$ [15, 5.2]. Let $\mathcal{S}(S)$ be the category whose objects are the local systems on S that are in $\mathcal{S}_n(S)$ for some n as above.

Now assume that $\mathcal{E}$ is an irreducible local system in $\mathcal{S}_n(S)$. For $y \in S$, let H_y be the isotropy subgroup of y for this $\mathcal{Z}^0 \times G$-action. Notice that for $(z, g) \in H_y$, $z^n = g^{-1} ygy^{-1}$. By [10, Lemma 1.1(2)], there are only finitely many possible choices for z. In particular, $H_y^0 = Z_G(y)^0$. Define a morphism $f : \mathcal{Z}^0 \times G/H_y^0 \to S$ by $(z, g) \mapsto (z, g) \cdot y$. Then $\mathcal{E}$ is a direct summand of $f_! \bar{\mathbf{Q}}_{l, \mathcal{Z}^0 \times G/H_y^0}$. Let C be the G-conjugacy class of y, then f factors through

$$\mathcal{Z}^0 \times G/H_y^0 \xrightarrow{f_1} \mathcal{Z}^0 \times C \xrightarrow{f_2} S,$$

where $f_1(z, g) = (z^n, gyg^{-1})$ is a principal $\mu_n \times Z_G(y)/Z_G(y)^0$-covering and $f_2(z, c) = zc$ is a A-covering. Here $A = \{z \in \mathcal{Z}^0;\ zC = C\}$ is a finite group. Therefore $\mathcal{E}$ is a direct summand of $(f_2)_!(\mathcal{F} \boxtimes \mathcal{E}')$, where $\mathcal{F}$ is an irreducible local system on $\mathcal{Z}^0$ which is a direct summand of $n_! \bar{\mathbf{Q}}_{l, \mathcal{Z}^0}$ for the n-th isogeny $n : \mathcal{Z}^0 \to \mathcal{Z}^0$ and $\mathcal{E}'$ is an irreducible local system on C which is a direct summand of $(f_1|_{\{1\} \times G/H_y^0})_! \bar{\mathbf{Q}}_{l, G/H_y^0}$.

Now let $\mathcal{Z}' = \{z\theta_\delta(z)^{-1};\ z \in Z(G)^0\}$. Since $Z(G)^0$ is Abelian, $\mathcal{Z}'$ is an Abelian subgroup of $Z(G)^0$. By [15, 1.2], $Z(G)^0 = \mathcal{Z}^0 \mathcal{Z}'$. It is easy to see that $\mathcal{Z}^0 \cap \mathcal{Z}'$ is finite. Since $C \subset G^1$

is stable under the conjugation action of $Z(G)^0$, we have that $\mathcal{Z}'C = C$. Therefore we have an isomorphism $Z(G)^0 \times_{\mathcal{Z}'} C \cong \mathcal{Z}^0 \times_{\mathcal{Z}^0 \cap \mathcal{Z}'} C$. Thus we have the following commuting diagram

$$\begin{array}{ccccc} \mathcal{Z}^0 \times \mathcal{Z}' \times C & \xrightarrow{b} & Z(G)^0 \times C & & \\ \downarrow a & & \downarrow c & & \\ \mathcal{Z}^0 \times C & \xrightarrow{f_3} & \mathcal{Z}^0 \times_{\mathcal{Z} \cap \mathcal{Z}'} C & \xrightarrow{f_4} & S, \end{array}$$

where a, f_3, c are projection maps, $b(z, z', c) = (zz', c)$ and $f_4(z, c) = zc$. The square (a, b, f_3, c) is a Cartesian square and $f_2 = f_4 \circ f_3$.

Thus $c^*(f_3)_!(\mathcal{F} \boxtimes \mathcal{E}') = b_! a^*(\mathcal{F} \boxtimes \mathcal{E}') = b_!(\mathcal{F} \boxtimes \bar{\mathbf{Q}}_{l,\mathcal{Z}'} \boxtimes \mathcal{E}')$. Any direct summand of $c^*(f_3)_!(\mathcal{F} \boxtimes \mathcal{E}')$ is of the form $\mathcal{F}' \boxtimes \mathcal{E}'$, where $\mathcal{F}'$ is an irreducible local system on $Z(G)^0$ which is a direct summand of $n_! \bar{\mathbf{Q}}_{l,Z(G)^0}$ for the n-th isogeny $n : Z(G)^0 \to Z(G)^0$.

As a summary,

(a) $\mathcal{E}$ is a direct summand of $(f_4)_! \mathcal{E}''$. Here $f_4 : Z(G)^0 \times_{\mathcal{Z}'} C \to S$, $(z, c) \mapsto zc$ and $\mathcal{E}''$ is a local system on $Z(G)^0 \times_{\mathcal{Z}'} C$ whose pull back to $Z(G)^0 \times C$ is of the form $\mathcal{F}' \boxtimes \mathcal{E}'$, where $\mathcal{F}'$ is an irreducible local system on $Z(G)^0$ which is a direct summand of $n_! \bar{\mathbf{Q}}_{l,Z(G)^0}$ for the n-th isogeny $n : Z(G)^0 \to Z(G)^0$.

Lemma 4.1. *Let $K \subset I$ with $\delta(K) = K$, $S \subset L_K g_0$ be an isolated stratum and S' the closure of S in $L_K g_0$. Let $w \in {}^K W \cap W^\delta$ with $w^{-1}(K) \subset I$. Set $Y = \bigsqcup_{w^{-1}(K) \subset D \subset I} (S \dot{w} g_0^{-1}, \dot{w}) \cdot h_{D,\delta}$ and $Y' = \bigsqcup_{w^{-1}(K) \subset D \subset I} (S' \dot{w} g_0^{-1}, \dot{w}) \cdot h_{D,\delta}$. For $J \subset I$ with $w^{-1}(K) \subset J$, let $\pi_J : S \to (S \dot{w} g_0^{-1}, \dot{w}) \cdot h_{J,\delta}$ be the map defined by $s \mapsto (s \dot{w} g_0^{-1}, \dot{w}) \cdot h_{J,\delta}$. Let $\mathcal{E} \in \mathcal{S}_n(S)$ be an irreducible local system. If $\mathcal{E} = \pi_J^* \mathcal{E}'$ for some local system on $Y \cap Z_{J,\delta}$, then $IC(Y', \mathcal{E})|_{Y' \cap Z_{J,\delta}} = IC(Y' \cap Z_{J,\delta}, \mathcal{E}')[|I - J|]$. Otherwise, $IC(Y', \mathcal{E})|_{Y' \cap Z_{J,\delta}} = 0$.*

Proof. Let $\tilde{Z} = \bigsqcup_{w^{-1}(K) \subset D \subset I} (g_0 Z(L_K) \dot{w} g_0^{-1}, \dot{w}) \cdot h_{D,\delta}$ be the closure of $g_0 Z(L_K)$ in Y'. Let $p : g_0 Z(L_K) \to \tilde{Z} \cap Z_{J,\delta} \cong g_0 Z(L_K)/\dot{w} Z(L_J) \dot{w}^{-1}$, $z \mapsto (z \dot{w} g_0^{-1}, \dot{w}) \cdot h_{J,\delta}$ be the projection map.

We show that

(a) Let $\mathcal{F}$ be an irreducible local system on $g_0 Z(L_K)$. If $\mathcal{F} = p^* \mathcal{F}'$ for some local system on $\tilde{Z} \cap Z_{J,\delta}$, then $IC(\tilde{Z}, \mathcal{F})|_{\tilde{Z} \cap Z_{J,\delta}} = \mathcal{F}'[|I - J|]$. Otherwise, $IC(\tilde{Z}, \mathcal{F})|_{\tilde{Z} \cap Z_{J,\delta}} = 0$.

For any $j \notin w^{-1}(K)$, let $\omega_j^\vee$ be the fundamental coweight. Then $f_j : k^* \to \tilde{Z}$, $a \mapsto g_0 \dot{w} \omega_j^\vee(a) \dot{w}^{-1}$ is a cross section to $\tilde{Z} \cap Z_{I - \{j\},\delta}$ in $\tilde{Z}$. Using [14, 1.6], $IC(\tilde{Z}, \mathcal{F})|_{\tilde{Z} \cap Z_{J,\delta}} \neq 0$ if and only if for any $j \notin J$, the monodromy of $\mathcal{F}$ around the divisor $\tilde{Z} \cap Z_{I - \{j\},\delta}$ is 0, i.e., $f_j^* \mathcal{F}$ is trivial. It is easy to see that $f_j^* \mathcal{F}$ is trivial for any $j \notin J$ if any only if $\mathcal{F} = p^* \mathcal{F}'$. In this case, one can show that $IC(\tilde{Z}, \mathcal{F})|_{\tilde{Z} \cap Z_{J,\delta}} = \mathcal{F}'$. Part (a) is proved.

Similarly,

(b) If $\mathcal{E} = \pi_J^* \mathcal{E}'$ for some local system on $Y \cap Z_{J,\delta}$, then $IC(Y, \mathcal{E})|_{Y \cap Z_{J,\delta}} = \mathcal{E}'[|I - J|]$. Otherwise, $IC(Y, \mathcal{E})|_{Y \cap Z_{J,\delta}} = 0$.

Let $\mathcal{Z}' = \{z \theta_\delta(z)^{-1};\ z \in Z(L_K)\}$. By Section 4.3 (a), $\mathcal{E}$ is a direct summand of $(f_4')_! \mathcal{E}''$. Here $f_4' : g_0 Z(L_K) \times_{\mathcal{Z}'} C \to S$, $(z, c) \mapsto g_0^{-1} zc = (g_0^{-1} z)(c g_0^{-1}) g_0 = c g_0^{-1}(g_0^{-1} z g_0)$ and $\mathcal{E}''$ is a local system on $g_0 Z(L_K) \times_{\mathcal{Z}'} C$ whose pull back to $g_0 Z(L_K) \times C$ is of the form $l_{g_0^{-1}}^* \mathcal{F}' \boxtimes \mathcal{E}'$, where

$\mathcal{F}'$ is an irreducible local system on $Z(L_K)$ which is a direct summand of $n_!\bar{\mathbf{Q}}_{l,Z(L_K)}$ for the n-th isogeny $n: Z(G)^0 \to Z(G)^0$ and $l_{g_0^{-1}}: g_0 Z(L_K) \to Z(L_K)$, $z \mapsto g_0^{-1} z$.

Let C' be the closure of C in $L_K g_0$. Then the map $f_4': g_0 Z(L_K) \times_{\mathcal{Z}'} C \to S$ extends in the natural way to a map $f_4'': \tilde{Z} \times_{\mathcal{Z}'} C' \to Y'$, $(z, c) \mapsto (cg_0^{-1}, 1) \cdot (g_0^{-1} z g_0)$. This is a surjective map and each fiber is finite. In particular, f_4'' is a small map and

$$IC\big(Y', (f_4')_! \mathcal{E}''\big) = (f_4'')_! IC\big(\tilde{Z} \times_{\mathcal{Z}'} C', \mathcal{E}''\big).$$

Consider the following diagram

$$\begin{array}{ccccc}
(\tilde{Z} \cap Z_{J,\delta}) \times_{\mathcal{Z}'} C & \hookrightarrow & (\tilde{Z} \cap Z_{J,\delta}) \times_{\mathcal{Z}'} C' & \hookrightarrow & \tilde{Z} \times_{\mathcal{Z}'} C' \\
\downarrow a & & \downarrow b & & \downarrow f_4'' \\
Y \cap Z_{J,\delta} & \hookrightarrow & Y' \cap Z_{J,\delta} & \hookrightarrow & Y',
\end{array}$$

where a, b are the restriction of f_4'' and are small maps. Both squares are Cartesian squares. So

$$IC\big(Y', (f_4')_! \mathcal{E}''\big)\big|_{Y' \cap Z_{J,\delta}} = \big((f_4'')_! IC\big(\tilde{Z} \times_{\mathcal{Z}'} C', \mathcal{E}''\big)\big)\big|_{Y' \cap Z_{J,\delta}} = b_! A,$$

where $A = IC(\tilde{Z} \times_{\mathcal{Z}'} C', \mathcal{E}'')|_{(\tilde{Z} \cap Z_{J,\delta}) \times_{\mathcal{Z}'} C'}$.

Notice that the pull back of A to $(\tilde{Z} \cap Z_{J,\delta}) \times C'$ is $IC(\tilde{Z}, l^*_{g_0^{-1}} \mathcal{F}')|_{\tilde{Z} \cap Z_{J,\delta}} \boxtimes IC(C', \mathcal{E}')$. The pull back is isomorphic to

$$IC\big((\tilde{Z} \cap Z_{J,\delta}) \times C', IC\big(\tilde{Z}, l^*_{g_0^{-1}} \mathcal{F}'\big)\big|_{\tilde{Z} \cap Z_{J,\delta}} \boxtimes \mathcal{E}'\big).$$

By (a), $IC(\tilde{Z}, l^*_{g_0^{-1}} \mathcal{F}')|_{\tilde{Z} \cap Z_{J,\delta}} \boxtimes \mathcal{E}'$ is a shift of an irreducible local system on $(\tilde{Z} \cap Z_{J,\delta}) \times C$ or 0. Hence $A = IC((\tilde{Z} \cap Z_{J,\delta}) \times_{\mathcal{Z}'} C', A|_{(\tilde{Z} \cap Z_{J,\delta}) \times_{\mathcal{Z}'} C})$ and

$$\begin{aligned}
IC\big(Y', (f_4')_! \mathcal{E}''\big)\big|_{Y' \cap Z_{J,\delta}} &= b_! A = IC\big(Y' \cap Z_{J,\delta}, a_!\big(A|_{(\tilde{Z} \cap Z_{J,\delta}) \times_{\mathcal{Z}'} C}\big)\big) \\
&= IC\big(Y' \cap Z_{J,\delta}, (b_! A)|_{Y \cap Z_{J,\delta}}\big).
\end{aligned}$$

Since $IC(Y', \mathcal{E})$ is a direct summand of $IC(Y', (f_4')_! \mathcal{E}'')$,

$$\begin{aligned}
IC\big(Y', \mathcal{E}\big)\big|_{Y' \cap Z_{J,\delta}} &= IC\big(Y' \cap Z_{J,\delta}, IC\big(Y', \mathcal{E}\big)\big|_{Y \cap Z_{J,\delta}}\big) \\
&= IC\big(Y' \cap Z_{J,\delta}, IC(Y, \mathcal{E})|_{Y \cap Z_{J,\delta}}\big).
\end{aligned}$$

Now the lemma follows from (b). □

From Section 4.4 to Lemma 4.2, we only assume that G is a connected reductive group. We first recall some results on character sheaves on disconnected groups. We follow the approach in [15].

4.4. Let P be a parabolic subgroup of G such that $N_{\tilde{G}}P \cap G^1 \neq \emptyset$. Let L be a Levi of P. Set $L^1 = N_{\tilde{G}}P \cap N_{\tilde{G}}L \cap G^1 \cong (N_{\tilde{G}}P \cap G^1)/U_P$. Consider the diagram

$$L^1 \xleftarrow{a} G \times \left(N_{\tilde{G}}P \cap G^1\right) \xrightarrow{b} G \times_P \left(N_{\tilde{G}}P \cap G^1\right) \xrightarrow{c} G^1,$$

where a, b are projection maps and $c(g,h) = ghg^{-1}$. To any simple perverse sheaf A on L^1 which is L-equivariant (for the conjugation action) we define $\mathrm{ind}_{G^1}^{L^1} A = c_! A_1$, where A_1 is the perverse sheaf on $G \times_P (N_{\tilde{G}}P \cap G^1)$ such that $a^* A[\dim(G) - \dim(P)] = b^* A_1$. We call $\mathrm{ind}_{G^1}^{L^1}$ an *induction functor*.

Consider the diagram

$$G^1 \xleftarrow{i} N_{\tilde{G}}P \cap G^1 \xrightarrow{\pi} L^1,$$

where i is the inclusion map and π is the projection. To any simple perverse sheaf B on G^1 which is G-equivariant (for the conjugation action), we define $\mathrm{res}_{G^1}^{L^1} B = \pi_! i^* B$. We call $\mathrm{res}_{G^1}^{L^1}$ a *restriction functor*.

4.5. For P, L and S as in Section 3.3, set

$$X_{L,S} = G \times_P S'U_P;$$
$$\tilde{Y}_{L,S} = G \times_L S^* \cong G \times_P \left(P_\Delta \cdot S^*\right)$$

where S' is the closure of S in G^1, L acts diagonally on S^* and P acts diagonally on $P_\Delta \cdot S^*$ and $S'U_P$.

We have the following commuting diagram

$$\begin{array}{ccccccc}
Y_{L,S} & \xleftarrow{\pi} & \tilde{Y}_{L,S} & \xleftarrow{a} & G \times S^* & \xrightarrow{b} & S \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y'_{L,S} & \xleftarrow{\pi'} & X_{L,S} & \xleftarrow{a'} & G \times S'U_P & \xrightarrow{b'} & S',
\end{array}$$

where $Y'_{L,S}$ is the closure of $Y_{L,S}$ in G^1, a, b, a', b' are projection maps and π, π' sends $(g,p) \to gpg^{-1}$.

Let $\mathcal{E} \in \mathcal{S}(S)$. Then there is a unique local system $\tilde{\mathcal{E}}$ on $\tilde{Y}_{L,S}$ with $a^*\tilde{\mathcal{E}} = b^*\mathcal{E}$ and the intersection cohomology complex $IC(S', \mathcal{E})$, $IC(X_{L,S}, \tilde{\mathcal{E}})$ are related by $(a')^* IC(X_{L,S}, \tilde{\mathcal{E}}) = (b')^* IC(S', \mathcal{E})$ (see [15, 5.6]). Moreover, $IC(Y'_{L,S}, \pi_! \tilde{\mathcal{E}})$ is canonically isomorphic to $\pi'_! IC(X_{L,S}, \tilde{\mathcal{E}}) = \mathrm{ind}_{G^1}^{L^1}(IC(S', \mathcal{E}))[-\dim(X_{L,S})]$ [15, Proposition 5.7].

A simple perverse sheaf on G^1 is called *admissible* if it is a direct summand of the perverse sheaf $IC(Y'_{L,S}, \pi_! \tilde{\mathcal{E}})[\dim(Y'_{L,S})]$ on G^1 (0 outside $Y'_{L,S}$) for some pair (L,S) as above and a cuspidal local system $\mathcal{E} \in \mathcal{S}(S)$ [15, 6.7]. The definition of cuspidal local system can be found in [15, 6.3].

Lemma 4.2. *We keep the notations as above. Let $\mathcal{E} \in \mathcal{S}(S)$ and A be a direct summand of $IC(Y'_{L,S}, \pi_! \tilde{\mathcal{E}})[\dim(Y'_{L,S})]$. Let Z be a connected subgroup of $Z(G)$. If A is equivariant for the right Z-action, then $\mathcal{E}$ is equivariant for the right Z-action on S.*

Proof. Consider the following diagram

$$\begin{array}{ccc} \tilde{Y}_{L,S} & \hookrightarrow & X_{L,S} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ Y_{L,S} & \hookrightarrow & Y'_{L,S}, \end{array}$$

where π and π' are defined in the previous subsection. By [15, Lemma 5.5] and [15, Proposition 5.7], this is a Cartesian square and π' is small. So

$$\begin{aligned} \big((\pi')^* (\pi')_! IC(X_{L,S}, \tilde{\mathcal{E}})\big)\big|_{\tilde{Y}_{L,S}} &= \pi^* \big((\pi')_! IC(X_{L,S}, \tilde{\mathcal{E}})|_{Y_{L,S}}\big) \\ &= \pi^* \big(IC\big(Y'_{L,S}, \pi_! \tilde{\mathcal{E}}\big)\big|_{Y_{L,S}}\big) = \pi^* \pi_! \tilde{\mathcal{E}}. \end{aligned}$$

Consider the following diagram

$$\begin{array}{ccc} G \times_L (N \times_L S^*) & \xrightarrow{b} & \tilde{Y}_{L,S} \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \pi} \\ \tilde{Y}_{L,S} & \xrightarrow{\pi} & Y_{L,S}, \end{array}$$

where $N = \{n \in N_G L;\ nSn^{-1} = S\}$ and $G \times_L (N \times_L S^*)$ is the quotient of $G \times (N \times S^*)$ modulo the $L \times L$-action, $(l, l') \cdot (g, n, s) = (gl^{-1}, ln(l')^{-1}, l's(l')^{-1})$ and the maps a, b are defined by $a(g, n, s) = (g, nsn^{-1})$ and $b(g, n, s) = (gn, s)$. It is easy to see that this is a Cartesian square. Therefore $\pi^* \pi_! \tilde{\mathcal{E}} = b_! a^* \tilde{\mathcal{E}} = \tilde{\mathcal{E}}^{\oplus |N/L|}$.

Since A is a direct summand of $(\pi')_! IC(X_{L,S}, \tilde{\mathcal{E}})[\dim(Y'_{L,S})]$, each direct summand of $((\pi')^* A)|_{\tilde{Y}_{L,S}}$ is a shift of $\tilde{\mathcal{E}}$. In particular, $IC(X_{L,S}, \tilde{\mathcal{E}})[\dim(X_{L,S})]$ is an irreducible constitute of ${}^p H^i((\pi')^* A)$ for some $i \in \mathbb{Z}$.

Notice that A is equivariant for the right Z-action and π' is Z-equivariant, where the Z-action on $X_{L,S} = G \times_P S' U_P$ is defined by $z \cdot (g, s) = (g, sz^{-1})$. Hence ${}^p H^i((\pi')^* A)$ is also Z-equivariant. Therefore $IC(X_{L,S}, \tilde{\mathcal{E}})$ and $IC(X_{L,S}, \tilde{\mathcal{E}})|_{G \times_P SU_P}$ are both Z-equivariant. By definition, the pull back of $IC(X_{L,S}, \tilde{\mathcal{E}})|_{G \times_P SU_P}$ to $G \times SU_P$ is a shift of $\bar{\mathbf{Q}}_{l,G} \boxtimes \mathcal{E} \boxtimes \bar{\mathbf{Q}}_{l,U_P}$. Therefore $\mathcal{E}$ is Z-equivariant. □

As in Section 3.4, we identify $Z_{J,1;\delta}$ with $G \times_{L_{J_\delta}} L_{J_\delta} g_0 / Z(L_J)$. Now we can prove our main theorem.

Theorem 4.3. *Let $J, K \subset I$ with $\delta(K) = K$. Let S be an isolated stratum of $L_K g_0$, S' its closure in $L_K g_0$ and $\mathcal{E} \in \mathcal{S}(S)$. Let $\mathcal{W}$ be the set of $w \in {}^K W^{J_\delta} \cap W^\delta$ with $w^{-1}(K) \subset J_\delta$ and such that $\mathcal{E}$*

is equivariant for the right $\dot{w}Z(L_J)\dot{w}^{-1}$-action on S. Then $IC(\overline{G^1}^{ss}, \mathrm{ind}_{G^1}^{L_K g_0} IC(S', \mathcal{E}))|_{Z_{J,1;\delta}}$ is canonically isomorphic to

$$\bigoplus_{w \in \mathcal{W}} i_{L_{J_\delta}}^G \,\mathrm{ind}_{L_{J_\delta} g_0/Z(L_J)}^{L_{w^{-1}(K)} g_0/Z(L_J)} IC\big({}^{\dot{w}^{-1}} S'/Z(L_J), \mathcal{E}_{J,w}\big),$$

where $\mathcal{E}_{J,w}$ is the local system on ${}^{\dot{w}^{-1}} S/Z(L_J)$ such that $\mathcal{E} = \mathrm{Ad}(\dot{w}^{-1})^* i^* \mathcal{E}_{J,w}$. Here $i : {}^{\dot{w}^{-1}} S \to {}^{\dot{w}^{-1}} S/Z(L_J)$ is the projection map and $\mathrm{Ad}(\dot{w}^{-1}) : S \to {}^{\dot{w}^{-1}} S$, $s \mapsto \dot{w}^{-1} s \dot{w}$.

Proof. Consider the following commuting diagram

$$\begin{array}{ccccc}
\tilde{Y}_{L_K,S} & \hookrightarrow & X_{L_K,S} & \hookrightarrow & \tilde{X}_{L_K,S} \\
\downarrow \pi & & \downarrow \pi' & & \downarrow \pi'' \\
Y_{L_K,S} & \hookrightarrow & Y'_{L_K,S} & \hookrightarrow & \overline{Y'_{L_K,S}} \cap \overline{G^1}^{ss},
\end{array}$$

where $\tilde{X}_{L_K,S} = G \times_{P_K} (\overline{SU_{P_K}} \cap \overline{G^1}^{ss})$, $\pi''(g,z) = (g,g) \cdot z$ and π, π' are the restrictions of π''. Both squares are Cartesian squares. By Theorem 3.5, π'' is a small map. Therefore

$$IC\big(\overline{G^1}^{ss}, \mathrm{ind}_{G^1}^{L_K g_0} IC\big(S', \mathcal{E}\big)\big) = IC\big(\overline{G^1}^{ss}, \pi_! \tilde{\mathcal{E}}\big)\big[\dim(G) - \dim(L_J)\big]$$

is canonically isomorphic to $\pi''_! IC(\tilde{X}_{L_K,S}, \tilde{\mathcal{E}})[\dim(G) - \dim(L_J)]$. This is similar to the argument in Section 4.5.

Therefore $IC(\overline{G^1}^{ss}, \mathrm{ind}_{G^1}^{L_K g_0} IC(S', \mathcal{E}))[-\dim(G) + \dim(L_J)]|_{Z_{J,1;\delta}}$ is canonically isomorphic to

$$\pi''_! IC(\tilde{X}_{L_K,S}, \tilde{\mathcal{E}})|_{G_\Delta \cdot (\overline{SU_{P_K}} \cap Z_{J,1;\delta})} = \big(\pi''|_{Z_{J,1;\delta}}\big)_! \big(IC(\tilde{X}_{L_K,S}, \tilde{\mathcal{E}})|_{G \times_{P_K} (\overline{SU_{P_K}} \cap Z_{J,1;\delta})}\big).$$

We have shown in Theorem 3.3 that

$$\overline{U_{P_K} S} \cap Z_{J,1;\delta} = \bigsqcup_{w \in {}^K W^{J_\delta} \cap W^\delta,\, w^{-1}(K) \subset J_\delta} \big(U_{P_K} S' \dot{w} g_0^{-1}, U_{P_K} \dot{w}\big) \cdot h_{J,\delta}.$$

Similar to the proof of the isomorphism (b) in the proof of Theorem 3.3, we have that

$$\begin{aligned}
&\big(U_{P_K} \cap {}^{\dot{w}} U\big) \times \big(U_{P_K} \cap {}^{\dot{w}} U^-\big) \times \bigsqcup_{J \subset D \subset I} \big(S' \dot{w} g_0^{-1}, U_{P_K} \dot{w}\big) \cdot h_{D,\delta} \\
&\quad \cong \bigsqcup_{J \subset D \subset I} \big(U_{P_K} S' \dot{w} g_0^{-1}, U_{P_K} \dot{w}\big) \cdot h_{D,\delta}.
\end{aligned}$$

By Lemma 4.1, for $w \in {}^K W^{J_\delta} \cap W^\delta$ with $w^{-1}(K) \subset J_\delta$, the restriction of $IC(\tilde{X}_{L_K,S}, \tilde{\mathcal{E}})$ to $G \times_{P_K} (U_{P_K} S' \dot{w} g_0^{-1}, U_{P_K} \dot{w}) \cdot h_{J,\delta}$ is 0 if $w \notin \mathcal{W}$ and is isomorphic to $\bar{\mathbf{Q}}_{(U_{P_K} \cap {}^{\dot{w}} U) \times (U_{P_K} \cap {}^{\dot{w}} U^-)} \boxtimes IC({}^{\dot{w}^{-1}} S'/Z(L_J), \mathcal{E}_{J,w})[|I - J|]$ if $w \in \mathcal{W}$. Now the theorem follows from the isomorphism

$G \times_{P_K} (U_{P_K} S' \dot{w} g_0^{-1}, U_{P_K} \dot{w}) \cdot h_{J,\delta} \cong G \times_{L_{J_\delta}} (L_{J_\delta} \times_{P_{w^{-1}(K)} \cap L_{J_\delta}} {}^{\dot{w}^{-1}} S'/Z(L_J))$ in Sections 3.6 and 3.7 and Section 3.7 (∗). □

4.6. For any $K \subset J \subset I$ with $\delta(K) = K$ and a character sheaf A on $L_K g_0$, we set

$$c_J(A) = \begin{cases} A, & \text{if } A \text{ is equivariant for the right } Z(L_J)\text{-action on } L_K g_0; \\ 0, & \text{otherwise.} \end{cases}$$

If B is a semisimple perverse sheaf on L_K and is a direct sum of some character sheaves $B = \oplus A_i$, then we set $c_J(B) = \oplus c_J(A_i)$. By Lemma 4.2, for any $K' \subset K$ with $\delta(K') = K'$ and a character sheaf A on $L_{K'} g_0$, if $c_J(A) = 0$, then $c_J(\mathrm{ind}_{L_K g_0}^{L_{K'} g_0}(A)) = 0$. By [14, Proposition 4.8] and [15, 27.2 and 38.3], $\mathrm{ind}_{L_{J_\delta} g_0/Z(L_J)}^{L_{w^{-1}(K)} g_0/Z(L_J)} IC({}^{\dot{w}^{-1}} S'/Z(L_J), \mathcal{E}_{J,w})$ is semisimple and perverse. Using Macay type formula [14, Proposition 15.2] and [15, Proposition 38.8], we can reformulate our main theorem in the following way.

Theorem 4.4. *Let $K \subset I$ with $\delta(K) = K$, S be an isolated stratum of $L_K g_0$ and S' be its closure in $L_K g_0$. Let $\mathcal{E}$ be a cuspidal local system on S and $A = \mathrm{ind}_{G^1}^{L_K g_0} IC(S', \mathcal{E}[\dim(S)])$. Then for any $J \supset K$, $IC(\overline{G^1}, A)|_{Z_{J,1;\delta}} = i_{L_{J_\delta}}^G(C)[|I - J|]$, where C is a semisimple perverse sheaf on $L_{J_\delta} g_0/Z(L_J)$ whose pull back to $L_{J_\delta} g_0$ is $c_J \operatorname{res}_{G^1}^{L_{J_\delta} g_0} A[-|I - J|]$.*

By [14, Section 4] and [15, Theorem 30.6], any character sheaf on G^1 is a direct summand of $\mathrm{ind}_{G^1}^{L_K g_0} IC(S', \mathcal{E}[\dim(S)])$ for some pair $(S, \mathcal{E})$ as above. We have that

Corollary 4.5. *Let A' be a character sheaf on G^1 and $J \subset I$. Then $IC(\overline{G^1}, A')|_{Z_{J,1;\delta}}$ is of the form $i_{L_\delta}^G(C)[|I - J|]$ for some semisimple perverse sheaf C on $L_{J_\delta} g_0/Z(L_J)$.*

Furthermore, we conjecture that the semisimple perverse sheaf C is given by the following explicit formula.

Conjecture 4.6. *Let A' be a character sheaf on G^1. Then for any $J \subset I$,*

$$IC(\overline{G^1}, A')\big|_{Z_{J,1;\delta}} = i_{L_{J_\delta}}^G(C)\big[|I - J|\big],$$

where C is the semisimple perverse sheaf on $L_{J_\delta} g_0/Z(L_J)$ whose pull back to $L_{J_\delta} g_0$ is $c_J \operatorname{res}_{G^1}^{L_{J_\delta} g_0}(A')[-|I - J|]$.

Let $\mathcal{E} \in \mathcal{S}(T g_0)$ such that if $w \in W^\delta$ with $\mathrm{Ad}(w)^* \mathcal{E} \cong \mathcal{E}$, then $w = 1$. Then $\mathrm{ind}_{G^1}^{T g_0} \mathcal{E}[\dim(T)]$ is a single perverse sheaf. The character sheaves obtained in this way are called *generic*. See [17, 1.1].

By the above theorem, the conjecture holds for generic character sheaves on G^1. We will show in Proposition 4.7 that this conjecture also holds for Steinberg character sheaf.

4.7. In this subsection, we assume that $G^1 = G$. For any $J \subset I$, we have that $\overline{Z_J} \cap \bar{G}^{ss} \cong G \times_{L_J} \overline{G_J}^{ss}$ (see Section 3.5). Now keep the notation in Theorem 4.4, we can show in the same way as we did for the proof of the main theorem that

$$IC(\bar{G}, A)|_{\overline{Z_J} \cap \bar{G}^{ss}} = i_{L_J}^G IC\big(\overline{G_J}^{ss}, C\big)\big[|I - J|\big].$$

Notice that $i_{L_J}^G IC(\overline{G_J}^{ss}, C)$ is canonically isomorphic to $IC(G \times_{L_J} \overline{G_J}^{ss}, i_{L_J}^G(C))$. Thus $IC(\bar{G}, A)|_{\overline{Z_J} \cap \bar{G}^{ss}}$ is the intermediate extension of its restriction to $Z_J \cap \bar{G}^{ss}$. Since any character sheaf A' on G is a direct summand of some A considered above, we have that

(a) For any $J \subset I$, $IC(\bar{G}, A')|_{\overline{Z_J} \cap \bar{G}^{ss}}$ is canonically isomorphic to $IC(\overline{Z_J} \cap \bar{G}^{ss}, IC(\bar{G}, A')|_{Z_J \cap \bar{G}^{ss}}) = i_{L_J}^G IC(\overline{G_J}^{ss}, IC(\bar{G}, A')|_{G_J})$.

In particular, for any $K \subset J \subset I$, $IC(\bar{G}, A')|_{Z_K \cap \bar{G}^{ss}}$ is canonically isomorphic to $i_{L_K}^G(IC(\overline{G_J}^{ss}, IC(\bar{G}, A')|_{G_J})|_{G_K})$. Hence to verify the above conjecture for $G^1 = G$, it suffices to verify the cases where J is a maximal proper subset of I. However, we still don't know how to do it.

Another thing worth mentioning is that the open embedding $G \to \bar{G}^{ss}$ is an affine map. Hence by [1, Corollary 4.1.12], for any perverse sheaf A on G, $IC(\bar{G}^{ss}, A)|_{\bar{G}^{ss}-G}[-1]$ is perverse. In other words, $IC(\bar{G}, A)|_{\overline{Z_J} \cap \bar{G}^{ss}}[-1]$ is a perverse sheaf for any maximal proper subset J of I. We showed above that for any character sheaf A, $IC(\bar{G}, A)|_{\overline{Z_J} \cap \bar{G}^{ss}}[-|I - J|]$ is perverse for any subset J of I. It would be interesting to see if the result holds for arbitrary perverse sheaves on G.

4.8. In this and next subsections, we assume that k is an algebraic closure of a finite field $\mathbb{F}_q$ and that we are given an $\mathbb{F}_q$-structure on $\tilde{G}$ with a Frobenius morphism $F : \tilde{G} \to \tilde{G}$ such that G^1 is defined over $\mathbb{F}_q$. Then F extends to a Frobenius morphism F on $\overline{G^1}$.

Let A be a character sheaf on G^1 and $\phi : F^*A \to A$ be an isomorphism. Then ϕ extends to an isomorphism $F^*IC(\overline{G^1}, A) \to IC(\overline{G^1}, A)$ which we still denote by ϕ. Then we can define functions $\chi_\phi^A : (G^1)^F \to \bar{\mathbf{Q}}_l$ and $\hat{\chi}_\phi^A : (\overline{G^1})^F \to \bar{\mathbf{Q}}_l$ by

$$\chi_\phi^A(x) = \sum_i (-1)^i Tr\big(\phi_x^i, H^i(A)_x\big),$$

$$\hat{\chi}_\phi^A(x) = \sum_i (-1)^i Tr\big(\phi_x^i, H^i\big(IC\big(\overline{G^1}, A\big)_x\big)\big).$$

The function χ_ϕ^A is called the characteristic function of A and is constant on G^F-conjugacy classes of $(G^1)^F$ and $\hat{\chi}_\phi^A$ is a natural extension of χ_ϕ^A.

Now for any function $f : (G^1)^F \to \bar{\mathbf{Q}}_l$ that is constant on G^F-conjugacy classes, we can naturally extend it to a function $\hat{f} : (\overline{G^1})^F \to \bar{\mathbf{Q}}_l$ as follows.

By [14, Theorem 25.2] and [15, Theorem 21.21], the characteristic functions (for various A) form a basis of the vector space of functions from $(G^1)^F$ to $\bar{\mathbf{Q}}_l$ that are constant on the G^F-conjugacy classes. Hence $f = \sum_A c_A \chi_\phi^A$, where $c_A \in \bar{\mathbf{Q}}_l$ is uniquely determined by f, A and ϕ. Now define

$$\hat{f} = \sum_A c_A \hat{\chi}_\phi^A.$$

We may view the restriction of $\hat{f}$ to $(\overline{G^1})^F - (G^1)^F$ as the boundary values of f. The most interesting case is when $G^1 = G$ and f is an irreducible character of the finite group G^F. The study of the boundary values of irreducible characters of G^F is one of the open problems in Springer's talk [21, Problem 10] at ICM 2006.

4.9. By [16, 12.3], the map $(g_1, g_2) \cdot h_{J,\delta} \mapsto ({}^{g_2}P_J, {}^{g_1}P^-_{\delta(J)}, g_1 H_{P^-_{\delta(J)}} g_0 H_{P_J} g_2^{-1})$ gives a natural isomorphism of $Z_{J,\delta}$ with $\{(P, Q, H_Q g H_P);\ P \in \mathcal{P}_J,\ Q \in \mathcal{P}_{-w_0(\delta(J))},\ g \in G^1,\ {}^gP \cap Q$ is a common Levi of ${}^gP$ and $Q\}$. Here $\mathcal{P}_J \cong G/P_J$ is the variety of parabolic subgroups conjugate to P_J. Now let $x = (P, Q, \gamma) \in (\overline{G^1}^{ss})^F$ and $A = IC(\overline{G^1}, \operatorname{ind}_{G^1}^{L_K g_0} IC(S', \mathcal{E}[\dim(S)]))$, where L_K and S' are defined over $\mathbb{F}_q$ and $\phi: F^*\mathcal{E} \to \mathcal{E}$ is an isomorphism. Then ϕ induces a natural isomorphism $F^*A \to A$, which we also denote by ϕ. By Theorem 4.4, we have that

$$\begin{aligned}\hat{\chi}^A_\phi(x) &= \frac{1}{|(N_{\tilde{G}} P \cap \gamma)^F|} \sum_{g \in (N_{\tilde{G}} P \cap \gamma)^F} \chi^A_\phi(g) \\ &= \frac{1}{|(N_{\tilde{G}} Q \cap \gamma)^F|} \sum_{g \in (N_{\tilde{G}} Q \cap \gamma)^F} \chi^A_\phi(g). \qquad (*)\end{aligned}$$

If Conjecture 4.6 is true, then the formula $(*)$ is true for any character sheaf A on G with ϕ: $F^*A \cong A$ and

$$\hat{f}(x) = \frac{1}{|(N_{\tilde{G}} P \cap \gamma)^F|} \sum_{g \in (N_{\tilde{G}} P \cap \gamma)^F} f(g) = \frac{1}{|(N_{\tilde{G}} Q \cap \gamma)^F|} \sum_{g \in (N_{\tilde{G}} Q \cap \gamma)^F} f(g),$$

for any function $f : (G^1)^F \to \bar{\mathbf{Q}}_l$ that is constant on G^F-conjugacy classes and $x = (P, Q, H_Q g H_P) \in (\overline{G^1}^{ss})^F$.

4.10. In the case where A is a character sheaf on G, the characteristic function of $\operatorname{res}^L_G(A)$ is the truncation of the characteristic function of A as follows

$$\chi_\phi^{\operatorname{res}^L_G(A)}(l) = \frac{1}{|U_P^F|} \sum_{u \in U_P^F} \chi^A_\phi(ul) \quad \text{for } l \in L^F.$$

The truncation, which sends generalized characters of G^F to generalized characters of L^F, is well-known in the representation theory of finite group of Lie type (see [2, Section 8]).

Notice that the formula $(*)$ in the previous subsection for character sheaves on G is

$$\hat{\chi}^A_\phi(x) = \frac{1}{|H_P^F|} \sum_{g \in H_P^F} \chi^A_\phi(gl) = \frac{1}{|H_Q^F|} \sum_{g \in H_Q^F} \chi^A_\phi(gl), \qquad (*)$$

here $x \in \bar{G}^{ss}$ corresponds to the triple $(P, Q, H_Q l H_P)$, where $P \cap Q$ is a common Levi of P and Q and $l \in P \cap Q$. This formula can be regarded as a "truncation" from generalized characters of G^F to generalized characters of $(L/Z(L))^F$.

Now we consider a special character sheaf on G^1 and its intermediate extension to $\overline{G^1}^{ss}$.

4.11. For any $K \subset J \subset I$ with $\delta(K) = K$. The map $L_{J_\delta} \times (P_K \cap L_{J_\delta})g_0/Z(L_J) \to L_{J_\delta}g_0/Z(L_J)$ defined by $(l, z) \mapsto (l, l) \cdot z$ induces a proper map

$$\pi_{J,K,\delta} : L_{J_\delta} \times_{P_K \cap L_{J_\delta}} (P_K \cap L_{J_\delta})g_0/Z(L_J) \to L_{J_\delta}g_0/Z(L_J).$$

The map $\pi_{J,K,\delta}$ is the Springer resolution for not necessarily connected reductive group. It is known that $\pi_{J,K,\delta}$ is a small map. (See for example, [16, 12.6 (a)].) Set

$$C_{J,K,\delta} = (\pi_{J,K,\delta})_! \big(\overline{\mathbb{Q}}_{l, L_{J_\delta} \times_{P_K \cap L_{J_\delta}} (P_K \cap L_{J_\delta})g_0/Z(L_J)} \big[\dim(G_{J_\delta})\big]\big).$$

Moreover, we may identify $(L_{J_\delta}, 1) \cdot h_{J,\delta}$ with $L_{J_\delta}g_0/Z(L_J)$ and $(P_K \cap L_{J_\delta}, 1) \cdot h_{J,\delta}$ with $(P_K \cap L_{J_\delta})g_0/Z(L_J)$ in the natural way. Under this identification, $C_{J,K,\delta}$ is a perverse sheaf on $(L_{J_\delta}, 1) \cdot h_{J,\delta}$.

Define $\pi'_{J,K,\delta} : G \times_{P_K \cap L_{J_\delta}} (P_K \cap L_{J_\delta}, 1) \cdot h_{J,\delta} \to Z_{J,1;\delta}$ by $(g, z) \mapsto (g, g) \cdot z$. Notice that

$$G \times_{L_{J_\delta}} \big(L_{J_\delta} \times_{P_K \cap L_{J_\delta}} (P_K \cap L_{J_\delta}, 1) \cdot h_{J,\delta}\big) \cong G \times_{P_K \cap L_{J_\delta}} (P_K \cap L_{J_\delta}, 1) \cdot h_{J,\delta}.$$

Then

$$i^G_{L_{J_\delta}}(C_{J,K,\delta}) = \big(\pi'_{J,K,\delta}\big)_! \big(\overline{\mathbb{Q}}_{l, G \times_{P_K \cap L_{J_\delta}} (P_K \cap L_{J_\delta}, 1) \cdot h_{J,\delta}} \big[\dim(Z_{J,1;\delta})\big]\big)$$

is a perverse sheaf on $G \times_{L_{J_\delta}} (L_{J_\delta}, 1) \cdot h_{J,\delta} \cong Z_{J,1;\delta}$.

4.12. Let $J, K \subset I$ with $\delta(K) = K$ and $w \in {}^K W^{J_\delta} \cap W^\delta$. Let $\epsilon(w) = \min(wW_J)$. Set $K_1 = \max\{K' \subset K;\ \delta(K') = K',\ \epsilon(w)^{-1}(K') \subset J\}$. By Lemma 2.2, $K_1 = K \cap wJ_\delta$. By Section 1.6 (6),

$$\begin{aligned}
(P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta} &= \big[J, \delta\big(\epsilon(w)\big), \epsilon(w)\big]_{K,\delta} \\
&= (P_K)_\Delta\big(L_{K_1}\delta\big(\dot{\epsilon(w)}\big), \dot{\epsilon(w)}\big) \cdot h_{J,\delta} \\
&= (P_K)_\Delta(L_{K_1}\dot{w}, \dot{w}) \cdot h_{J,\delta} \\
&= (P_K)_\Delta(\dot{w}L_{w^{-1}(K) \cap J_\delta}, \dot{w}) \cdot h_{J,\delta}.
\end{aligned}$$

The map

$$f : G \times_{P_K} (P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta} \to G \times_{{}^{\dot{w}^{-1}}P_K} \big({}^{\dot{w}^{-1}}P_K\big)_\Delta(L_{w^{-1}(K) \cap J_\delta}, 1) \cdot h_J$$

defined by $(g, z) \mapsto (g\dot{w}, (\dot{w}^{-1}, \dot{w}^{-1})z)$ is an isomorphism. Moreover, $\pi_{J,K,w,\delta} = \pi'_{J,K,w,\delta} \circ f$, where

$$\begin{aligned}
\pi_{J,K,w,\delta} &: G \times_{P_K} (P_K)_\Delta(B\dot{w}, B\dot{w}) \cdot h_{J,\delta} \to Z_{J,1;\delta}, \\
\pi'_{J,K,w,\delta} &: G \times_{{}^{\dot{w}^{-1}}P_K} \big({}^{\dot{w}^{-1}}P_K\big)_\Delta(L_{w^{-1}(K) \cap J_\delta}, 1) \cdot h_J \to Z_{J,1;\delta},
\end{aligned}$$

are induced from the map $G \times Z_{J,1;\delta} \to Z_{J,1;\delta}$ defined by $(g, z) \mapsto (g, g) \cdot z$.

Notice that

$$(P_{w^{-1}(K)\cap J_\delta}\cap L_{J_\delta})_\Delta(L_{w^{-1}(K)\cap J_\delta},1)\cdot h_{J,\delta}=(P_{w^{-1}(K)\cap J_\delta}\cap L_{J_\delta},1)\cdot h_{J,\delta}.$$

By Section 3.6 (a), $\pi_{J,K,w,\delta}$ is a small map and

$$\begin{aligned}
&(\pi_{J,K,w,\delta})_!\big(\overline{\mathbb{Q}}_{l,G\times_{P_K}(P_K)_\Delta(B\dot w,B\dot w)\cdot h_{J,\delta}}\big[\dim(Z_{J,1;\delta})\big]\big)\\
&\quad=\big(\pi'_{J,K,w,\delta}\big)_!\big(\overline{\mathbb{Q}}_{l,G\times_{\dot w^{-1}P_K}(\dot w^{-1}P_K)_\Delta(L_{w^{-1}(K)\cap J_\delta},1)\cdot h_{J,\delta}}\big[\dim(Z_{J,1;\delta})\big]\big)\\
&\quad=\big(\pi'_{J,w^{-1}(K)\cap J_\delta,\delta}\big)_!\big(\overline{\mathbb{Q}}_{l,G\times_{P_{w^{-1}(K)\cap J_\delta}\cap L_{J_\delta}}(P_{w^{-1}(K)\cap J_\delta}\cap L_{J_\delta},1)\cdot h_{J,\delta}}\big[\dim(Z_{J,1;\delta})\big]\big)\\
&\quad=i^G_{L_{J_\delta}}(C_{J,w^{-1}(K)\cap J_\delta,\delta}).
\end{aligned}$$

4.13. By [15, 38.11], for any $J\subset I$ with $\delta(J)=J$, there is a unique simple perverse sheaf $St_{J,\delta}$ on $(L_{J_\delta},1)\cdot h_{J,\delta}\cong L_{J_\delta}g_0/Z(L_J)$ such that $St_{J,\delta}$ is a direct summand of $C_{J,\emptyset,\delta}$ and $St_{J,\delta}$ is not a direct summand of $C_{J,K,\delta}$ for any $\emptyset\neq K\subset J$ with $\delta(K)=K$. In fact,

$$St_{J,\delta}\oplus\bigoplus_{K\subset J,\delta(K)=K,2\nmid|K|}C_{J,K,\delta}=\bigoplus_{K\subset J,\delta(K)=K,2||K|}C_{J,K,\delta}.$$

It is known that for any $g\in G^1_J$, $\mathcal{H}^i_g(St_{J,\delta})\neq 0$ for some $i\in\mathbb{Z}$ if and only if the stabilizer of g in G is reductive (i.e., g is quasi-semisimple). In this case, $\sum_{i\in\mathbb{Z}}\dim(\mathcal{H}^i_g(St_{J,\delta}))=1$. See [15, 12.6].

Let $K\subset I$ with $\delta(K)=K$. By Theorem 2.4, $\overline{P^1_K}\cap\overline{G^{1ss}}$ is smooth. By Theorem 2.1 and the previous subsection, $\pi_K:G\times_{P_K}(\overline{P^1_K}\cap\overline{G^{1\text{ß}}})\to\overline{G^{1ss}}$ defined by $(g,z)\mapsto(g,g)\cdot z$ is a small map. Hence $(\pi_K)_!(\overline{\mathbb{Q}}_{l,G\times_{P_K}(\overline{P^1_K}\cap\overline{G^{1\text{ß}}})}[\dim(G)])$ is a perverse sheaf on $\overline{G^{1ss}}$ whose restriction to G^1 is $C_{K,I,\delta}$.

Let S' be the unique simple perverse sheaf on $\overline{G^{1ss}}$ such that $S'|_{G^1}=St_{I,\delta}$. Then

$$\begin{aligned}
&S'\bigoplus\bigoplus_{K\subset I,\ \delta(K)=K,\ 2\nmid|K|}(\pi_K)_!\big(\overline{\mathbb{Q}}_{l,G\times_{P_K}(\overline{P^1_K}\cap\overline{G^{1\text{ß}}})}\big[\dim(G)\big]\big)\\
&\quad=\bigoplus_{K\subset I,\ \delta(K)=K,\ 2||K|}(\pi_K)_!\big(\overline{\mathbb{Q}}_{l,G\times_{P_K}(\overline{P^1_K}\cap\overline{G^{1\text{ß}}})}\big[\dim(G)\big]\big).
\end{aligned}$$

Now we calculate the restriction of S' to $Z_{J,1;\delta}$.

Proposition 4.7. *For $J\subset I$, $S'|_{Z_{J,1;\delta}}=i^G_{L_{J_\delta}}(St_{J,\delta}[|I-J|])$.*

Proof. For $J,K\subset I$ and $w\in{}^KW^{J_\delta}\cap W^\delta$, set $I(J,K,w,\delta)=w^{-1}K\cap J_\delta$.

We have the following Cartesian square

$$\begin{array}{ccc} G \times_{P_K} (\overline{P_K^1} \cap Z_{J,1;\delta}) & \hookrightarrow & G \times_{P_K} (\overline{P_K^1} \cap \overline{G^{1ß}}) \\ \downarrow \pi_K & & \downarrow \pi_K \\ Z_{J,1;\delta} & \hookrightarrow & \overline{G^{1ss}}. \end{array}$$

Hence by Theorem 2.3,

$$\begin{aligned} &\big((\pi_K)_!\big(\overline{\mathbb{Q}}_{l, G \times_{P_K} (\overline{P_K^1} \cap \overline{G^{1ß}})}\big[\dim(G)\big]\big)\big)\big|_{Z_{J,1;\delta}} \\ &\quad = (\pi_K|_{G \times_{P_K} (\overline{P_K^1} \cap Z_{J,1;\delta})})_!\big(\overline{\mathbb{Q}}_{l, G \times_{P_K} (\overline{P_K^1} \cap Z_{J,1;\delta})}\big[\dim(G)\big]\big) \\ &\quad = (\pi_K|_{G \times_{P_K} (\overline{P_K^1} \cap Z_{J,1;\delta})})_! \overline{\mathbb{Q}}_{l, \bigsqcup_{w \in {}^K W^{J_\delta} \cap W^\delta} G \times_{P_K} (P_K)_\Delta (B\dot{w}, B\dot{w}) \cdot h_{J,\delta}}\big[\dim(G)\big] \\ &\quad = \bigoplus_{w \in {}^K W^{J_\delta} \cap W^\delta} (\pi_{J,K,w,\delta})_!\big(\overline{\mathbb{Q}}_{l, G \times_{P_K} (P_K)_\Delta (B\dot{w}, B\dot{w}) \cdot h_{J,\delta}}\big[\dim(G)\big]\big) \\ &\quad = \bigoplus_{w \in {}^K W^{J_\delta} \cap W^\delta} i_{L_{J_\delta}}^G (C_{J, I(J,K,w,\delta),\delta})\big[|I - J|\big]. \end{aligned}$$

Moreover,

$$\begin{aligned} &\bigoplus_{K \subset I,\ \delta(K) = K,\ 2 \mid |K|} \ \bigoplus_{w \in {}^K W^{J_\delta} \cap W^\delta} i_{L_{J_\delta}}^G (C_{J, I(J,K,w,\delta),\delta})\big[|I - J|\big] \\ &\quad = \bigoplus_{w \in W^{J_\delta} \cap W^\delta} \ \bigoplus_{K \subset I,\ \delta(K) = K,\ 2 \mid |K|,\ w \in {}^K W} i_{L_{J_\delta}}^G (C_{J, I(J,K,w,\delta),\delta})\big[|I - J|\big] \\ &\quad = \bigoplus_{w \in W^{J_\delta} \cap W^\delta} \ \bigoplus_{K \subset I(J,I,w,\delta)} i_{L_{J_\delta}}^G (C_{J,K,\delta})\big[|I - J|\big] \bigoplus_{K' \subset I, \delta(K') = K',\ w \in {}^{K'} W,\ I(J,K',w,\delta) = K,\ 2 \mid |K'|} 1. \end{aligned}$$

Similarly,

$$\begin{aligned} &\bigoplus_{K \subset I,\ \delta(K) = K,\ 2 \nmid |K|} \ \bigoplus_{w \in {}^K W^{J_\delta} \cap W^\delta} i_{L_{J_\delta}}^G (C_{J, I(J,K,w,\delta),\delta})\big[|I - J|\big] \\ &\quad = \bigoplus_{w \in W^{J_\delta} \cap W^\delta} \ \bigoplus_{K \subset I(J,I,w,\delta)} i_{L_{J_\delta}}^G (C_{J,K,\delta})\big[|I - J|\big] \bigoplus_{K' \subset I,\ \delta(K') = K',\ w \in {}^{K'} W,\ I(J,K',w,\delta) = K,\ 2 \nmid |K'|} 1. \end{aligned}$$

Fix $w \in W^{J_\delta} \cap W^\delta$. Let $J' = \max\{K \subset I;\ w \in {}^K W\}$. Then $\delta(J') = J'$ and $wI(J, I, w, \delta) \subset J'$. It is easy to see that for any $K \subset I(J, I, w, \delta)$ and $K' \subset I$ with $\delta(K') = K'$, the following conditions are equivalent:

(1) $w \in {}^{K'} W$ and $I(J, K', w, \delta) = K$;
(2) $K = \delta(K)$ and $wK \subset K' \subset wK \bigsqcup (J' - wI(J, I, w, \delta))$.

Notice that $J' - wI(J, I, w, \delta)$ is δ-stable. Therefore, for any $K \subset I(J, I, w, \delta)$,

$$\sum_{K' \subset I,\ w \in {}^{K'}W,\ I(J,K',w,\delta)=K} (-1)^{|K'|} = \begin{cases} (-1)^{|K|}, & \text{if } \delta(K) = K \text{ and } J' - wI(J, I, w, \delta) = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

If $J' = wI(J, I, w, \delta)$, then there is no $i \in I$ such that $ww_0^{J_\delta} \in {}^{\{i\}}W$. Hence $ww_0^{J_\delta} = w_0$ and $w = w_0 w_0^{J_\delta}$. In this case, $I(J, I, w, \delta) = J_\delta$. Therefore for any $w \in W^{J_\delta} \cap W^\delta$ with $w \neq w_0 w_0^{J_\delta}$,

$$\begin{aligned} &\bigoplus_{K \subset I(J,I,w,\delta)} i_{L_{J_\delta}}^G (C_{J,K,\delta})\big[|I - J|\big] \bigoplus_{K' \subset I,\ \delta(K')=K',\ w \in {}^{K'}W,\ I(J,K',w,\delta)=K,\ 2 \mid |K'|} 1 \\ &= \bigoplus_{K \subset I(J,I,w,\delta)} i_{L_{J_\delta}}^G (C_{J,K,\delta})\big[|I - J|\big] \bigoplus_{K' \subset I,\ \delta(K')=K',\ w \in {}^{K'}W,\ I(J,K',w,\delta)=K,\ 2 \nmid |K'|} 1. \end{aligned}$$

Now

$$\begin{aligned} &S'|_{Z_{J,1;\delta}} \bigoplus \bigoplus_{K \subset J_\delta, \delta(K)=K, 2 \nmid |K|} i_{L_{J_\delta}}^G (C_{J,K,\delta})\big[|I - J|\big] \\ &= \bigoplus_{K \subset J_\delta,\ \delta(K)=K,\ 2 \mid |K|} i_{L_{J_\delta}}^G (C_{J,K,\delta})\big[|I - J|\big]. \end{aligned}$$

Hence $S'|_{Z_{J,1;\delta}} = i_{L_{J_\delta}}^G (St_{J,\delta}[|I - J|])$. □

4.14. Let $\tilde{S}$ be the simple perverse sheaf on $\overline{G^1}$ such that $\tilde{S}|_{G^1} = St_{I,\delta}$. Then $\tilde{S}|_{\overline{G^1}_{ss}} = S'$. By the previous Proposition and [11], the following conditions on $z \in \overline{G^1}$ are equivalent:

(1) The stabilizer of z in G is reductive;

(2) $z \in \overline{G^1}^{ss}$ and $\mathcal{H}_z^i(\tilde{S}) \neq 0$ for some $i \in \mathbb{Z}$;

(3) $z \in \overline{G^1}^{ss}$ and $\sum_{i \in \mathbb{Z}} \dim(\mathcal{H}_z^i(\tilde{S})) = 1$.

This verifies Lusztig's conjecture in [16, 12.6] inside $\overline{G^1}^{ss}$. More precisely, by what we have shown above, Lusztig's conjecture is now reduced to the following one:

Conjecture 4.8. *The intermediate extension of S' to $\overline{G^1}$ is the extension by* 0 *outside $\overline{G^1}^{ss}$.*

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