Journal of Geometry and Physics 58 (2008) 1465–1476

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Non-displaceable Lagrangian submanifolds and Floer cohomology with non-unitary line bundle

Cheol-Hyun Cho *

Department of Mathematics, Seoul National University, Kwanakgu Shinrim, San56-1 Seoul, South Korea

ARTICLE INFO

Article history:
Received 23 February 2008
Received in revised form 18 June 2008
Accepted 28 June 2008
Available online 8 July 2008

MSC:
53D12
53D20
53D40

Keywords:
Floer cohomology
Toric manifolds
Lagrangian submanifolds
Non-displaceability

ABSTRACT

We show that in many examples the non-displaceability of Lagrangian submanifolds by Hamiltonian isotopy can be proved via Lagrangian Floer cohomology with non-unitary line bundle. The examples include all monotone Lagrangian torus fibers in a toric Fano manifold (which was also proven by Entov and Polterovich via the theory of symplectic quasi-states) and some non-monotone Lagrangian torus fibers.

We also extend the results by Oh and the author about the computations of Floer cohomology of Lagrangian torus fibers to the case of all toric Fano manifolds, removing the convexity assumption in the previous work.

1. Introduction

This paper provides two improvements from the joint work with Yong-Geun Oh [7]. In [7], we have provided computations of Floer cohomology of Lagrangian torus fibers in toric Fano manifolds, which are important examples of homological mirror symmetry, and symplectic topology of Fano manifolds [14,1,13,8]. But in [7], we have restricted ourselves to the case of some convex toric Fano manifolds because of the technical difficulty involving the transversality of moduli space. One of the result of this paper is to show how to remove this restriction, and prove that the results of [7] hold for all toric Fano manifolds. The other result of this paper is to show that slight modification of Floer cohomology using non-unitary line bundle can be successfully used to prove many Lagrangian intersection properties, which was not possible in the standard definition.

Lagrangian Floer cohomology was first defined by Floer [10], and generalized to the monotone case by Yong-Geun Oh [16]. The definition in full generality, including obstruction and deformation theory has been established by the ingenious work of Fukaya, Oh, Ohta and Ono [12]. The main feature of Floer cohomology of a Lagrangian submanifold is that Floer cohomology ring $HF(L, \psi(L))$ is independent of the choice of the Hamiltonian isotopy ψ. This provides a tool to study an intersection theory of Lagrangian submanifolds, which has played an important role to study symplectic topology in the last two decades.

We show examples of Lagrangian submanifolds which cannot be displaced from itself using Hamiltonian isotopy (non-displaceable, for short). The main tool is the Lagrangian Floer cohomology, with complex-valued two form on the symplectic manifold, which we call, the Floer cohomology with non-unitary line bundle. This turns out to be more efficient than the standard definition for the intersection theoretic applications.

* Tel.: +82 28806534; fax: +82 28874694.
E-mail address: chocheol@snu.ac.kr.

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2008.06.003

The new non-displaceable examples include all monotone Lagrangian torus fibers, which was also proved by Entov and Polterovich via a different method, namely the theory of symplectic quasi-states and quasi-measures [9]. We also find some new non-monotone non-displaceable Lagrangian fibers and general criteria to find one in Fano manifolds. Recall that in the cases of the non-displaceable examples known before via Floer theoretic methods, like the Clifford torus [4,3], or $\mathbb{R}P^n$ [17], the proof relies on the presence of strong symmetry (balanced) for the cancellation of quantum contributions so that Floer cohomology is isomorphic to the singular cohomology. In these new examples, they are not balanced but the new effect enables us more freedom so that we can still cancel quantum contributions out.

To explain this effect, recall that Lagrangian Floer cohomology has a generalization, which came from another motivation, the homological mirror symmetry conjecture [15,11]. On this generalization, one allows, a flat unitary line bundle on a Lagrangian submanifold, or more generally, one allows a unitary line bundle on L whose curvature equals a closed two form $2\pi\sqrt{-1}B$ (defined on M). This is usually called a B-field. The modification we consider is a non-unitary version of them for short. As far as we know, non-unitary case has not been considered before in these cases. This maybe is due to the fact that the Floer homology with unitary bundle has been used mainly for homological mirror symmetry and in such cases one usually identifies the formal parameter in the Novikov ring with a complex number. For example in [7], we have identified $T^{2\pi}$ with a complex number e^{-1} to show the equivalence of the obstruction m_0 with the Landau–Ginzburg superpotential.

But with such identification, the non-unitary version we consider in this paper is equivalent to the standard version (with a different symplectic form) in good cases. Also, there is a delicate point we also address in this paper that when formal parameter is identified with a complex number, the related Floer cohomology (which is called convergent version) does not provide informations which are symplectically invariant. (In the last section, we explain the example of Hirzbruch surface, where some fibers have non-vanishing convergent Bott–Morse Floer cohomology, but still is displaceable by some Hamiltonian isotopy. Hence, we show by this example that $HF(L, L) \neq HF(L, \psi(L))$ for the convergent version of Floer cohomology.)

The computations in this paper, and that of [7,4] was carried out using the Bott–Morse version of Lagrangian Floer homology, or an m_1-homology of an A_∞-algebra of Lagrangian submanifolds defined in [12]. It was shown in [12] that transversal and Bott–Morse versions of Floer cohomologies are isomorphic. To compute the Bott–Morse version of Floer cohomology of a Lagrangian submanifold, one has to classify (pseudo)-holomorphic discs. In [7], we classify all the holomorphic discs with boundary on any Lagrangian torus fiber, and prove the Fredholm regularity of the standard complex structure for all holomorphic discs. This enabled us to compute the Floer cohomology, except one technical problem, which involves the non-regularity of the standard complex structure for holomorphic spheres. Hence, in [7], we were only able to validate our results for convex toric Fano manifolds.

In this paper, we describe how to resolve this problem, using a simple argument and the machinery of [12] by combining the use of holomorphic discs and abstract perturbations of Kuranishi structures together. Hence, we are able to extend the results of [7] to all toric Fano manifolds.

We would like to thank L. Polterovich for asking for the Floer theoretic proof of his results with M. Entov. We also thank for M. Entov, Kenji Fukaya and Yong-Geun Oh for helpful comments on the paper. We remark that the idea in this paper [6], has been generalized by Fukaya, Oh, Ohta and Ono [13].

2. Floer cohomology with a closed two form $\widetilde{B}$

2.1. Floer cohomology with B-fields

We first recall the definition of Floer cohomology with B-fields following the article of Fukaya [11] (See also, [12]). We will consider a slight generalization of it in the next subsection.

Let (M, ω) be a symplectic manifold, and B a closed real-valued two form on M, which is called B-field (in fact B is from $H^2(M, \mathbb{Z})$). Let L_i be a Lagrangian submanifold in M, and $\mathcal{L}_i$ be a complex line bundle over L_i for $i = 1, 2$. Let ∇_i be a unitary connection on $\mathcal{L}_i$, and it is required that the curvature F_{∇_i} of ∇_i satisfies for $i = 1, 2$,

$$F_{\nabla_i} = 2\pi\sqrt{-1}B. \tag{2.1}$$

The definitions and properties of the standard Lagrangian Floer cohomology extends to the case with B-fields without much difficulty. We first recall the definition in the transversal case.

The Λ_{nov}-module $D_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)}$ is defined as

$$D_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)} = \oplus_{p\in L_1\cap L_2} Hom(\mathcal{L}_{1,p}, \mathcal{L}_{2,p}) \otimes \Lambda_{nov,\mathbb{C}}.$$

The differential δ ($=n_{0,0}$ in [12]) is defined using the moduli space of holomorphic strips. For $p, q \in L_1 \cap L_2$, consider the following moduli space

$$\widetilde{M}(L_1, L_2; p, q) = \{\phi : [0, 1] \times \mathbb{R} \to M | \text{Condition 2.1.}\}$$

Condition 2.1. (1) *ϕ is pseudo-holomorphic.*
(2) $\phi(0, \tau) \in L_1, \phi(1, \tau) \in L_2$.
(3) $\lim_{\tau\to-\infty} \phi(t, \tau) = p, \lim_{\tau\to\infty} \phi(t, \tau) = q$

After moding out by $\mathbb{R}$-action, we get a quotient space $\mathcal{M}(L_1, L_2; p, q)$. The moduli space can be divided according to the homotopy classes, say β of the maps ϕ's, and denoted by $\mathcal{M}(L_1, L_2; p, q; \beta)$. The holonomy contribution $Hol(\phi) : Hom(\mathcal{L}_{1,p}, \mathcal{L}_{2,p}) \to Hom(\mathcal{L}_{1,q}, \mathcal{L}_{2,q})$ is defined by

$$Hol(\phi)(\alpha) = h_{\phi(\{1\}\times\mathbb{R})}(\mathcal{L}_2) \circ \alpha \circ h_{\phi(\{0\}\times\mathbb{R})}(\mathcal{L}_1)^{-1}.$$

Here $h_{\phi(\{1\}\times\mathbb{R})}(\mathcal{L}_2) : \mathcal{L}_{2,p} \to \mathcal{L}_{2,q}$ is a parallel transport of $(\mathcal{L}_2, \nabla_2)$ along the path $\phi(\{1\} \times \mathbb{R})$. The map $h_{\phi(\{0\}\times\mathbb{R})}(\mathcal{L}_1)$ is defined similarly.

The symplectic area is given by

$$\omega([\phi]) = \int_{D^2} \phi^*\omega \in \mathbb{R}.$$

Lemma 2.2 ([11] Lemma 2.7). *The following expression for ϕ only depends on the homotopy class of ϕ.*

$$\exp\left(2\pi\sqrt{-1}\int_{D^2}\phi^*B\right) Hol(\phi(\alpha)) \otimes T^{\omega(\beta)}$$

Then the boundary map $\delta : D_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)} \to D_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)}$ is defined for $\alpha \in Hom(\mathcal{L}_{1,p}, \mathcal{L}_{2,p})$ as

$$\delta(\alpha) = \sum_{q,\beta,\phi \text{ with vir.dim.}(\mathcal{M}(L_1,L_2;p,q;\beta)=0)} \pm \exp\left(2\pi\sqrt{-1}\int_{D^2}\phi^*B\right) \cdot Hol(\phi(\alpha)) \otimes T^{\omega(\beta)}.$$

Here signs can be determined explicitly if L_1, L_2 are (relatively) spin and are equipped with choices of spin structures. Additional operations $n_{k,l}$ may be defined as in [12] to give D the structure of A_∞-bimodule over the A_∞-algebras of L_1 and L_2. Consideration of A_∞-bimodule structure is essential when one actually needs to deform the A_∞-algebra to define Floer cohomology, but in our case, it is not necessary, hence we refer readers to [12] for more details on them. For the case of monotone Lagrangian submanifolds, it is not necessary to use the full machinery, and we may use the approach of Oh [17], Cho-Oh [7], or Biran-Cornea [2] with the corresponding modifications. In the case considered in this paper, $\delta^2 = 0$ holds when $(L_2, \mathcal{L}_2, \nabla_2)$ is Hamiltonian equivalent to $(L_1, \mathcal{L}_1, \nabla_1)$ in the following sense. i.e. there exist a function $f : M \times [0, 1] \to \mathbb{R}$, a complex line bundle $\mathcal{L} \to L \times [0, 1]$ and its connection ∇ with the following properties. Let ψ_t be the Hamiltonian isotopy generated by f.

(1) $\psi_1(L_1) = L_2$.
(2) $(\mathcal{L}, \nabla)|_{L\times\{0\}} = (\mathcal{L}_1, \nabla_1)$, $(\mathcal{L}, \nabla)|_{L\times\{1\}} = \psi_1^*(\mathcal{L}_2, \nabla_2)$.
(3) $F_\nabla = 2\pi\sqrt{-1}\,\Psi^*B$. Here $\Psi : L_1 \times [0, 1] \to M$ defined by $\Psi(x, t) = \psi_t(x)$.

We remark that given $(L_1, \mathcal{L}_1, \nabla_1)$ and a Hamiltonian isotopy ψ_2, one can find $(\mathcal{L}_3, \nabla_3)$ such that $(L_1, \mathcal{L}_1, \nabla_1)$ and $(\psi_2(L_1), \mathcal{L}_3, \nabla_3)$ are Hamiltonian equivalent to each other in the above sense. (For this, choose $\mathcal{L} = \pi_1^*\mathcal{L}_1$, $\nabla' = \pi_1^*\nabla_1$ using the projection $\pi_1 : L_1 \times [0, 1] \to L_1$ and notice that $F_{\nabla'} - 2\pi\sqrt{-1}\,\Psi^*B$ is exact. Choose one form α on $L_1 \times [0, 1]$ with $F_{\nabla'} - 2\pi\sqrt{-1}\,\Psi^*B = d\alpha$. Take $\nabla = \nabla' - \alpha$, and define $(\mathcal{L}_2, \nabla_2)$ using the above relation.)

The homology of δ is the Floer cohomology $HF((L_1, \mathcal{L}_1, \nabla_1), (L_2, \mathcal{L}_2, \nabla_2))$ for Hamiltonian equivalent pairs, and it is independent of a Hamiltonian isotopy ψ, and it is isomorphic to the Bott–Morse Floer cohomology, which is a homology of an A_∞-algebra of Lagrangian submanifold with modification as follows. (We refer readers to [12] for details and notations) The A_∞-algebra operations $m_k : C(L)^{\otimes k} \to C(L)$ for $k \in \mathbb{N} \cup \{0\}$ on geometric chains on L are defined as

$$m_{k,\beta}(P_1, \ldots, P_k) = [\mathcal{M}_1(L, J, \beta, P_1, P_2, \ldots, P_k), ev_0] \cdot T^{\omega(\beta)}, \tag{2.2}$$

where P_i's are chains on L, and $\mathcal{M}(L, J, \beta, \ldots, P_*)$ is the moduli space of J-holomorphic discs with homotopy class β intersecting $P_1, \ldots, P_k$ with an additional marked point for evaluation. And this may be modified with the presence of B-field as

$$m^B_{k,\beta}(P_1, \ldots, P_k) = m_{k,\beta}(P_1, \ldots, P_k) \cdot (Hol_{\partial\beta}\mathcal{L}) \exp\left(2\pi\sqrt{-1}\int_\beta B\right) \tag{2.3}$$

The factor $(Hol_{\partial\beta}\mathcal{L})$ records the holonomy around the boundary of J-holomorphic discs. In the case that $B \neq 0$, the holonomy factor alone is not enough to make it well-defined for each homotopy class.

It is well-known that the theorems in [12] can be carried over to the situation with B-fields, since the only modification is adding an additional factor to the boundary operations, which vanishes if the homotopy class β is that of constant discs.

In particular, the following theorem also holds true with the presence of B-fields. In the case of Lagrangian submanifolds that we consider in this paper, it is weakly obstructed with the bounding cochain $B = 0$, hence we omit the bounding cochains from the statement. Also, for simplicity, instead of $HF((L_1, \mathcal{L}_1, \nabla_1), \psi_*(L_1, \mathcal{L}_1, \nabla_1))$, we simply write $HF(L_1, \psi(L_1))$.

Theorem 2.3 (*Theorem G [12]*). *$HF(L, \psi(L))$ is independent of ψ and it coincides with the homology of A_∞-algebra of L.*

As in [4,7], we will show the non-vanishing of the homology of an A_∞-algebra (in a non-unitary setting) to prove the non-displaceability of Lagrangian submanifolds.

2.2. Non-unitary case with a complex valued two form $\widetilde{B}$

In this subsection, we consider slight modification using a complex valued two form $\widetilde{B}$. We consider a closed complex valued two form $\widetilde{B} = B_{re} + \sqrt{-1}B_{im}$ such that the curvature F_{∇_i} of the connection ∇_i on $\mathcal{L}$ satisfies for $i = 1, 2$,

$$F_{\nabla_i} = \widetilde{B},$$

instead of (2.1). Moreover, we do not require $\widetilde{B}/(2\pi i)$ to lie in the image of $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{C})$ as long as there exist $(\mathcal{L}_i, \nabla_i)$ satisfying the above curvature condition for $i = 1, 2$. (i.e. we only require an integrality condition when restricted to L_i, not necessarily on M.)

In the case that $B_{re} = 0$, $B_{re}/(2\pi \mathrm{i}) \in H^2(M,\mathbb{Z})$, then $\widetilde{B}/(2\pi \mathrm{i})$ becomes a B-field discussed in the previous subsection.

The contribution from $\widetilde{B}$ is given as

$$\exp\left(2\pi\sqrt{-1}\int_\beta B\right)(Hol_{\partial\beta}\mathcal{L})\cdot T^{\omega(\beta)} \Rightarrow \exp\left(\int_\beta \widetilde{B}\right)(Hol_{\partial\beta}\mathcal{L})\cdot T^{\omega(\beta)} \tag{2.4}$$

Lemma 2.4. *The new expression in (2.4) also depends only on the homotopy class of β.*

Proof. The part of (2.4), $T^{\omega(\beta)}$ depends only on the homotopy class of β, since ω are closed two forms which vanish on TL. The part, $\exp(\int_\beta \widetilde{B})(Hol_{\partial\beta}\mathcal{L})$ also depends only on the homotopy class of β, due to the relation $F_{\nabla_i} = \widetilde{B}$. □

We remark that in many cases, $\widetilde{B}$ maybe set to be zero, since considering flat *non-unitary* line bundle is good enough to detect non-displaceability of some Lagrangian submanifolds. Also, in this paper, we will choose $\widetilde{B}$ so that its support is away from the given Lagrangian submanifolds. In such a case, an integrality condition is automatically satisfied. It is easy to check that if $\widetilde{B}$ satisfies an integrality condition when restricted on L, then, it also satisfies such a condition on $\psi(L)$ for a Hamiltonian isotopy ψ.

The rest of the story on the definition of Floer homology and their properties are almost identical to the case of B-fields. The benefit of the introducing $\widetilde{B}$ and non-unitary line bundle is that, as it can be seen in (2.4), we now allow the additional contribution to be in $\mathbb{C}$, whereas the traditional approach only allows it to be in $S^1 \subset \mathbb{C}$. (We remark that in [13], one of their idea is to generalize this further to an element of the Novikov ring based on our observation)

Extension of Floer cohomology to the non-unitary line bundle does not seem to be related to the mirror symmetry directly. The reason is that in mirror symmetry, one usually identifies the formal parameter $T^{2\pi}$ with a specific value, say e^{-1}, and after the identification, the above may be interpreted as introducing a new symplectic form $\omega + 2\pi B_{re}$ (if it is symplectic):

$$\mathrm{e}^{\int_\beta \widetilde{B}}(Hol_{\partial\beta}\mathcal{L})\cdot \mathrm{e}^{\omega(\beta)/2\pi} = \mathrm{e}^{\int_\beta (\widetilde{B}+\omega/2\pi)}(Hol_{\partial\beta}\mathcal{L}) \tag{2.5}$$

But, in the realm of symplectic geometry, the difference of formal parameter version and convergent version can be clearly observed (see the last section), hence Floer cohomology with $\widetilde{B}$ in the Novikov ring coefficient, is different from the Floer cohomology with a new symplectic form.

3. Non-displaceable Lagrangian submanifolds

3.1. Lagrangian fibers in toric Fano manifolds

In [4,7], Floer cohomology of a Lagrangian fiber has been computed in a general toric Fano manifold (with convexity assumption, which we remove later in this paper). The Lagrangian fibers with non-vanishing Floer cohomology in these cases, like the Clifford torus, has strong symmetry conditions on the position of the fiber and the shape of the moment polytope that we start with, hence they were called balanced fibers. For example, in [7], the Hirzbruch surface was shown to have no Lagrangian fiber with non-vanishing Floer cohomology.

With the introduction of $\widetilde{B}$, we show that such symmetry condition for the non-vanishing Floer cohomology, can be weakened considerably, hence providing many more examples of non-displaceable Lagrangian fibers. For example, we prove that the monotone fiber in the Hirzbruch surface is non-displaceable. In fact, we prove that every monotone Lagrangian fiber of toric Fano manifold is non-displaceable.

Let us first recall the notations of toric Fano manifolds from [7], to which we refer readers for details. For a given polytope P, and we consider the corresponding symplectic toric manifold (M,ω). Let v_i for $i = 1,\ldots,N$ be the inward normal vectors of each facet of P which define one dimensional generators of the dual fan Σ. Denote by $V(v_i)$ the corresponding toric divisor. Also, recall that the homotopy classes of holomorphic discs with Maslov index two may be enumerated as $\beta_1,\ldots,\beta_N$ where β_i corresponds to each facet v_i.

Lemma 3.1. *There exist closed real-valued two forms B_i on M which vanish on TL such that for $i, j = 1,\ldots,N$, we have*

$$\int_{\beta_j} B_i = \delta_{ij}.$$

Proof. We have shown that the Maslov index is given by twice the intersection number with toric divisors (Theorem 5.1 [7]), and from the classification theorem (Theorem 5.2 [7]), each holomorphic discs of class β_i (with Maslov index two) intersects non-trivially only with $V(v_i)$ and it does not intersect any other toric divisors. Hence we may choose Poincare dual closed two form B_i for each $V(v_i)$, whose support lies in a sufficiently close neighborhood of $V(v_i)$. These satisfy the equations above since each Maslov index two holomorphic disc is a part of a holomorphic sphere which does not intersect the corresponding toric divisor elsewhere. This proves the lemma. □

We consider the case of monotone Lagrangian fibers first, and we will discuss general case in the next subsection. Recall that a Lagragrangian submanifold is called monotone if for any $\beta \in \pi_2(M, L)$, we have $\mu(\beta) = c \cdot \omega(\beta)$ for some fixed constant $c \in \mathbb{R}_{>0}$. This implies that the symplectic manifold is also monotone, in the sense that for any spherical homology class $\alpha \in \pi_2(M)$, $\omega(\alpha) = 2c \cdot c_1(TM)(\alpha)$.

Theorem 3.2. *Let L be any monotone Lagrangian torus fiber in any toric Fano manifold M. There exists $\widetilde{B}$ such that the Bott–Morse Floer cohomology of L with $\widetilde{B}$ is non-trivial. Hence L is not displaceable from itself by any Hamiltonian isotopy of M.*

Remark 3.3. Entov and Polterovich proved the same result using the the theory of quasi-state and quasi-measures [8].

Remark 3.4. There are many examples of monotone Lagrangian torus fibers satisfying this fact. More explicitly, note that toric Fano manifold may be given a symplectic form so that its moment polytope is reflexive. (i.e. its facets are defined by

$$< x, v_i > \geq -1,$$

and 0 being the only integral point in the interior of the polytope). Then, Lagrangian fiber corresponding to the origin, satisfies this property since the symplectic areas of the holomorphic discs are all equal to 2π from the area formula of [7]. Hence, such fibers are always non-displaceable by Hamiltonian isotopy.

Proof. As the Maslov index of any non-trivial holomorphic disc is positive, we may write the Bott–Morse differential m_1 as

$$\begin{aligned} m_1 &= m_{1,0} + \sum_{\beta \in \pi_2(M,L)} m_{1,\beta} \\ &= m_{1,0} + m_{1,2} + m_{1,4} + \cdots. \end{aligned}$$

where $m_{1,0}$ is the classical boundary operator of taking the boundary of a chain and $m_{1,\beta}$ records quantum contributions from J-holomorphic discs of homotopy class β intersecting a chain. And by $m_{1,\mu}$, we denote the sum of $m_{1,\beta}$ over all β with the Maslov index $\mu(\beta) = \mu$. To distinguish with the standard Floer cohomology, we write $m^{\widetilde{B}}_{1,\mu}$ in the case of Floer cohomology with $\widetilde{B}$.

First we choose the following generators C_i of $H^1(L)$ for $i = 1, \ldots, n$.

Definition 3.5. Let l_i be a circle $1 \times \cdots S^1 \cdots \times 1$ where S^1 is the i-th circle of $(S^1)^n \subset (\mathbb{C}^*)^n$. Then torus action of $(\mathbb{C}^*)^n$ on L gives a corresponding cycle in L, which we also denote as l_i by abuse of notation. For $i = 1, \ldots n$, denote by $C_i \in H^1(L)$ the Poincare dual of the cycle

$$(-1)^{i-1}(l_1 \times \cdots \times \hat{l}_i \times \cdots l_n).$$

Now, we show the existence of good $\widetilde{B}$'s for monotone Lagrangian fibers.

Lemma 3.6. *Let L be a monotone Lagrangian fiber in toric Fano manifold M. We may choose a complex valued two form $\widetilde{B}$ on M which vanishes on TL such that for all $i = 1, \ldots, n$, we have*

$$m^{\widetilde{B}}_{1,2}(C_i) = 0.$$

Here we set $\mathcal{L}$ to be the flat complex line bundle over L with trivial holonomy.

Assuming the lemma, we can apply the following proposition to prove non-displaceability of monotone Lagrangian fibers (we state the proposition in a more general setup).

Proposition 3.7. *Let L be a positive Lagrangian torus in a general compact symplectic manifold (See Definition 3.8). Assume that for each i,*

$$m^{\widetilde{B}}_{1,2}(C_i) = 0,$$

Then, the Floer cohomology with $\widetilde{B}$ is isomorphic to the singular cohomology as a Λ_{nov}-module.

$$HF^{\widetilde{B},*}(L, L) \cong HF^*(L; \Lambda_{\mathrm{nov}}).$$

Conversely, if $m^{\widetilde{B}}_{1,2}(C_i) \neq 0$ for some i, then, Floer cohomology vanishes.

The proof of this proposition will be given in the next section. The Lemma 3.6 and the Proposition 3.7 together prove the Theorem 3.2. □

Now we begin proof of the lemma.

Proof. The expression $m_{1,2}^{\widetilde{B}}(C_i)$ is defined as

$$m_{1,2}^{\widetilde{B}}(C_i) = \sum_{j=1}^{N} [\mathcal{M}_1(L, J_0, \beta_j, C_i), ev_0] \cdot \exp\left(\int_{\beta_j} \widetilde{B}\right) \cdot (Hol_{\partial\beta_j}\mathcal{L}) \cdot T^{\omega(\beta_j)}. \tag{3.1}$$

Here, $[\mathcal{M}_1(L, J, \beta_j, C_i), ev_0]$ is a chain obtained as the evaluation image of the moduli space of all J_0-holomorphic discs of class β_j which intersects C_i. This chain has (expected) dimension n, and it is in fact a cycle since the moduli space of Maslov index two holomorphic discs are closed, which is due to the minimality of the index. Hence it is a multiple of fundamental class $[L]$. In fact, it can be exactly computed as (see for [4] for exact signs and details)

$$(-1)^n(+v_{j1}l_1 + \cdots + v_{jn}l_n) \times (-1)^{i-1}(l_1 \times \cdots \times \hat{l_i} \times \cdots l_n) = (-1)^n v_{ji}(l_1 \times \cdots \times l_n) = (-1)^n v_{ji}[L]$$

As $\mathcal{L}$ is a flat bundle with trivial holonomy, hence we may set

$$Hol_{\partial\beta_j}\mathcal{L} \equiv 1.$$

Monotonicity implies that the terms from symplectic area of discs $T^{\omega(\beta_j)}$ are independent of j, and we write $\omega(\beta_j) = A_0 \in \mathbb{R}$ for any j. Now, we will determine $\widetilde{B}$ as a linear combination of B_i's of the Lemma 3.1. Let

$$\widetilde{B} = \sum_{j=1}^{N} c_j B_j, \quad c_j \in \mathbb{C}. \tag{3.2}$$

Then, we define $d_j \in \mathbb{C}^*$

$$d_j := \exp\left(\int_{\beta_j} \widetilde{B}\right) = e^{c_j} \in \mathbb{C}^*$$

Then, the expression (3.1) is given as

$$m_{1,2}^{\widetilde{B}}(C_i) = \sum_{j=1}^{N} (-1)^n v_{ji}[L] d_j T^{A_0} = (-1)^n \left(\sum_{j=1}^{N} v_{ji} d_j\right) T^{A_0}.$$

Hence, to prove the lemma, it suffices to choose non-zero $d_j \in \mathbb{C}^*$ for $j = 1, \ldots N$, satisfying

$$\sum_{j=1}^{N} v_{ji} d_j = 0, \quad \text{forall } i = 1, \ldots, n. \tag{3.3}$$

Recall that v_i's are the one dimensional generators of the dual fan Σ of the polytope P, hence we have $N > n$ and $\{v_j | j = 1, \ldots, N\}$ span the vector space $\mathbb{R}^n$. Therefore, the Eq. (3.3) clearly has a non-trivial solution by linear algebra.

Now we show that we may obtain a solution of (3.3) such that $d_j \neq 0$ for all j. To prove this claim, we show that the solution space of (3.3) is not contained in any coordinate plane of $\mathbb{R}^N$. To prove the latter, for a fixed index $i \in \{1, \ldots, n\}$, we find a solution $(d_1, \ldots, d_N)$ of (3.3) such that $d_i \neq 0$.

We choose $v_{i_2}, \ldots, v_{i_n}$ such that if we set $v_i = v_{i_1}$, the collection $\{v_{i_1}, \ldots, v_{i_n}\}$ defines an n-dimensional cone in the fan Σ. This implies that $\{v_{i_1}, \ldots, v_{i_n}\}$ generate $\mathbb{R}^n$. We may take another vector $v_{i_{n+1}} \neq v_1$ for some $1 \leq i_{n+1} \leq N$ such that $v_{i_{n+1}}$ does not lie on the subspace generated by $\{v_{i_2}, \ldots, v_{i_n}\}$. This is possible since Σ is a complete fan. Then, it is clear that since $\{v_{i_1}, \ldots, v_{i_{n+1}}\}$ is not linearly independent, and we may write

$$v_{i_{n+1}} = \sum_{j=1}^{n} d_j v_{i_j}. \tag{3.4}$$

From the construction, we have $d_1 \neq 0$. This proves the claim that the solution space is not contained in any coordinate plane. This finishes the proof of the lemma. □

3.2. General positive Lagrangian submanifolds

The proof in the monotone case suggests the following generalization in the general symplectic manifold. In short, we prove that if there exist an identity similar to (3.3) at each symplectic energy level (of holomorphic discs), then it can be shown to be non-displaceable.

First, we define

Definition 3.8. Let (M, ω) be a compact symplectic manifold. A Lagrangian torus L is called positive if there exist a compatible almost complex structure J, such that

(1) Maslov index of any non-trivial J-holomorphic disc, and $c_1(TM)(\alpha)$ for any non-trival J-holomorphic sphere α are always positive.
(2) J is Fredholm regular for any J-holomorphic discs of Maslov index two, and the evaluation map $ev_0 : \mathcal{M}_1(L, \beta) \to L$ is submersive for homotopy classes β of J-holomorphic discs of Maslov index two.

Let L be a positive Lagrangian torus in a symplectic manifold (M, ω) with a compatible almost complex structure J as in the definition. If we denote the homotopy classes of J-holomorphic discs of Maslov index two as $\beta_1, \ldots, \beta_N$ and consider

$$\{\omega(\beta_1), \ldots, \omega(\beta_N)\} = \{\lambda_1, \ldots, \lambda_k\}$$

where energy levels, $\lambda_i \in \mathbb{R}$'s are set to be distinct from each other. Using the map

$$\partial : \pi_2(M, L) \to \pi_1(L) \to H_1(L) \cong \mathbb{Z}^n,$$

one can define $v_j = \partial\beta_j \in \mathbb{Z}^n$ for $j = 1, \ldots, N$.

We define the set of indices with energy λ_i as

$$I_{\lambda_i} = \{l | \omega(\beta_l) = \lambda_i\} \subset \{1, 2, \ldots, n\}.$$

We list the elements of I_{λ_i} as

$$I_{\lambda_i} = \{l_{i1}, \ldots, l_{ia_i}\}$$

Theorem 3.9. *Let (M, ω), L, J, β_*, λ_* as above. We assume*

(1) *L be a positive Lagrangian torus.*
(2) *For each energy level λ_i, there exists $c_{l_{i1}}, \ldots, c_{l_{ia_i}} \in \mathbb{C}^*$ such that*

$$\sum_{j=1}^{a_i} c_{l_{ij}} v_{l_{ij}} = 0.$$

(3) *There exist a set of complex-valued two forms on M, $\{B_i\}$ for $i = 1, \ldots, N$, which vanishes on TL and satisfies*

$$\exp\left(\int_{\beta_j} B_i\right) = \begin{cases} 1 & \text{if } i \neq j \\ c_i & \text{if } i = j \end{cases} \tag{3.5}$$

Then, there exist a closed complex valued two form $\widetilde{B}$ such that Floer cohomology with $\widetilde{B}$ is non-vanishing, and in fact, isomorphic to the singular cohomology $H^*(L, \Lambda_{\text{nov}})$ as a module.

Remark 3.10. In case $c_i = 1$, then B_i can be chosen as zero.

Proof. The proof is exactly the same as monotone case, but the difference is that we apply the same argument for each energy level to achieve the vanishing of $m_{1,2}^{\widetilde{B}}(C_*)$. □

In the last section, we provide some examples of non-monotone Lagrangian torus fibers which is shown to be non-displaceable as an application.

4. Computation of Floer cohomology

This section consists of two parts. In the first part, we give the proof of Proposition 3.7, which claims that in certain cases, we can determine the Floer cohomology of Lagrangian tori from the Maslov index two boundary map computation on the codimension one generators. In the second part, we explain the technical problem of transversality on the computation of Floer cohomology of Lagrangian tori on the general toric Fano manifolds from [7], and explain how to overcome the problem with the idea of the first part. We learned that similar idea to that of the first part has been used by Biran-Cornea in a different setting of Lagrangian Floer cohomology in [2].

4.1. Computation from Maslov index two contribution.

We begin the proof of the Proposition 3.7. In the case of $\widetilde{B} = 0$, this will be used to to remove the convexity assumption in [7].

Recall that we have an A_∞-algebra of Lagrangian submanifolds by the construction of [12]. In our case, we assume the positivity of Maslov index for non-constant holomorphic discs and this implies that m_0 is a multiple of the fundamental cycle $[L]$. Hence, in the language of [12], it is weakly obstructed, and we may set the bounding cochain to be $B = 0$. We also assume that the A_∞-algebra of [12] is unital. (We may need to take the canonical model if necessary [12] for this. But in this positive case, the $m^{\widetilde{B}}_{1,2}(C_i)$ part can be made to be the same after such process due to dimensional restriction)

In Proposition 3.7, we have assumed that $m^{\widetilde{B}}_{1,2}(C_i) = 0$. (From now on we omit $\widetilde{B}$ from the notation for simplicity). The above condition actually implies that

$$m_1(C_i) = 0.$$

From the dimension calculation, $m_{1,\mu}(C_i)$ is expected to be $n - 2 + \mu$ dimensional chain ($n = \dim(L)$), hence for $\mu \geq 4$, $m_{1,\mu}(C_i)$ maybe regarded as zero.

Now, it remains to show that C_i's are *not* m_1-coboundaries. (It may become coboundary because the dimension of $m_{1,\mu}(P)$ equals $n - 1 = \dim(C_i)$ if $\mu = \deg(P) - 1$.) But we can show that C_i's are not m_1 coboundaries by the following simple argument.

By the Leibniz rule from the A_∞-algebra equations, we have

$$m_1(m_2(x, y)) + m_2(m_1(x), y) + (-1)^{|x|+1} m_2(x, m_1(y)) = 0. \tag{4.1}$$

If x, y are m_1-cycles, then $m_2(x, y)$ is also a m_1-cycle. Recall that its energy zero part is $m_{2,\beta_0}(x, y)$ which is given by cup product of (or intersection product of chains) x and y (here β_0 is the homotopy class of constant discs). And if x or y is a m_1-coboundary, then $m_2(x, y)$ is also a m_1-coboundary.

For the torus, we have chosen C_i's so that the following intersection is transversal and equal to

$$(\cdots (PD(C_1) \cap PD(C_2)) \cap \cdots) \cap PD(C_n)) = \pm pt \in T^n$$

Hence, we may define $m_{2,0}(C_i, C_j)$ so that

$$\pm PD(pt) = m_{2,0}(m_{2,0}(\cdots (m_{2,0}(C_1, C_2), C_3), \ldots), C_n) \tag{4.2}$$

Therefore, we may write

$$m_2(m_2(\cdots (m_2(C_1, C_2), C_3), \ldots), C_n) = \pm PD(pt) \pm \sum_k a_k P_k T^{\lambda_k}, \tag{4.3}$$

where $a_k \in \mathbb{Q}$, and $\lambda_k > 0$ and P_k's are chains of positive dimensions in L. These P_k's are the result of successive operations $m_{2,\beta}$'s and easy computation shows that they are of positive dimensions except (4.2). We remark that in [5] we observed that $PD(pt)$ is a singular cohomology cycle, but not a m_1-cycle in general. And one needs additional chains such as P_k to get m_1-cycle in the chain level.

Notice that the expression (4.3) is an m_1-cycle since we assumed that each C_i's are m_1-cycles. We claim that the expression (4.3) cannot be a m_1-coboundary since its energy zero part is $PD(pt)$.

Suppose the expression (4.3) equals $m_1(\sum_j Q_j T^{\sigma_j})$ for some Q_j, σ_j's. First, note that $m_{1,\beta}(Q_j)$ is of positive dimension if $\mu(\beta) \geq 2$ since

$$\dim(Q_j) + \mu(\beta) - 1 > 0.$$

Hence, to have $PD(pt)$ as a boundary image, it should be of the form $m_{1,0}(\sum_{j'} Q_{j'})$ for some $Q_{j'}$'s, but clearly this cannot become a (pt), since the (signed) sum of the coefficients of the zero dimensional chain in $m_{1,0}(\sum_{j'} Q_{j'})$ vanishes, since $m_{1,0}$ is the standard boundary operation ∂.

Hence, this proves that (4.3) is a non-trivial m_1-homology cycle. Therefore, we conclude that all the intermediate products $m_2(\cdots, m_2(C_1, C_2), \ldots, C_i)$ as well as all C_i's are in fact also non-trivial in Floer cohomology. This proves the main statement of the Proposition.

To prove the converse statement, suppose that $m_{1,2}(C_i) = c_i[L]T^{Area}$ for some $c_i \neq 0$. This in fact implies that $m_{1,2}(pt) \neq 0$: If we write

$$m_{1,2}(pt) = \sum_{j=1}^{n} a_j [l_j],$$

then it is not hard to check that $a_i = c_i$ from above. This implies that $m_{1,2}(pt) \neq 0$. Now, recall that in Theorem 10.1 of [7], we showed the equivalence of $m_{1,2}(pt) \neq 0$ and the vanishing of Floer cohomology.

The computation here is very similar to that of [7] and the only difference is the fact that in [7], we have crucially used the fact that $m_{1,\mu} \equiv 0$ for $\mu \geq 4$, which may not be valid anymore.

But we can use spectral sequence coming from the Maslov index, so that at the stage of Maslov index two boundary maps, exactly same argument as in [7] Theorem 10.1 can be used. As the entries of the spectral sequence vanishes after taking the Maslov index two boundary maps, one can easily see that the Floer homology also should vanish.

Such a spectral sequence in the monotone case has been used by Oh [18]. We remark that in the general case, one should use spectral sequence using the energy filtration [12] which is equivalent to the filtration coming from Maslov index in the monotone case.

As we also deal with non-monotone case, we explain in more detail about the spectral sequence using Maslov index. Recall that the boundary map m_1 can be written as a sum

$$m_1 = \sum_{\mu, even} m_{1,\mu}.$$

We remark that in the positive case each $m_{1,\mu}$ consists of contributions from finitely many homotopy classes due to index restrictions (without the positivity assumption, this is not true).

Note that $m_{1,\mu}$ is a map of degree $1 - \mu$. Hence all the maps have negative degree except the map $m_{1,0}$ which have positive degree. Assuming $m_{1,0}$ vanishes, we can use the decomposition of the boundary map to define spectral sequence just as in [18]. To make $m_{1,0}$ vanish, we consider the canonical model of A_∞-algebra.

Recall that filtered A_∞-algebra is called canonical if $m_{1,0}$ vanishes. In [12] Chap 23.4, they construct the canonical model of A_∞-algebra using the modified ribbon trees similar to the construction of Kontsevich and Soibelman. In their construction, in the positive case, it is easy to see that for elements $x \in \mathcal{H}^*(L)$ of (Hodge type decomposition of the chain complex),

$$m^{can}_{1,\mu}(x) = pr \circ m_{1,\mu}(x)$$

where the pr is the projection map to $\mathcal{H}^*(L)$. In the case of $m^{can}_{1,2}$ on $\mathcal{H}$, the operation is equivalent to the one before taking the canonical model. Hence, by first taking the canonical model, and then using the spectral sequence coming from the Maslov index, one can show that Floer homology vanishes if $m_{1,2}(pt) \neq 0$.

We refer readers to [13] for another alternative argument for the results in this subsection.

4.2. The case of torus fibers in the general toric Fano manifolds

We first describe the problem we have encountered in [7], and we explain how to resolve it. Recall that for toric Fano manifolds, we have classified all holomorphic discs with boundary on Lagrangian torus fibers generalizing the results of [4]. By proving the Fredholm regularity of the standard complex structure J_0 for all the holomorphic discs, and computing symplectic areas of holomorphic discs, we have explicitly computed Floer differential $m_{1,\beta}$. The only problem was that although J_0 is Fredholm regular for all holomorphic discs, it is *not* Fredholm regular for holomorphic spheres for general toric Fano manifolds. It means that for the holomorphic spheres, the actual dimensions of the moduli spaces may be bigger than the expected (virtual) dimensions of them due to this non-regularity.

Hence, the moduli space of holomorphic discs of Maslov index ≥ 4 may not have a good compactification since it may bubble off a holomorphic sphere, and in this case the boundary strata may have bigger dimension than the main stratum. So we have restrict ourselves to the case of, so called, convex symplectic manifolds in [7], which guarantees the regularity of J_0-holomorphic spheres.

This non-regularity problem can be resolved if we make (abstract Kuranishi) perturbations of the moduli spaces, then the perturbed moduli space is almost impossible to use for computation directly. Recall that without perturbation, the image of holomorphic discs of Maslov index ≥ 4 did not have any non-trivial contribution. More precisely, in Proposition 7.2 [7], we have proved that $m_\mu \equiv 0$ for $\mu \geq 4$. This is because although the moduli spaces of holomorphic discs have correct dimensions, their evaluation images with one marked point is of smaller dimension than expected, and it was regarded as zero. In this way, the Floer cohomology depended only on $m_{1,2}$. But as soon as we introduce abstract perturbations to resolve the sphere bubble issues, this argument is no longer true since the images of the virtual chains after abstract perturbations will have images of expected dimension in general. Hence, their contribution may not vanish.

Now, the argument in the last subsection tells us how to overcome this problem, since Floer cohomology can be determined by $m_{1,2}(C_i)$ only. In our case, since the moduli spaces of holomorphic discs of Maslov index two are always well-defined without perturbation, we will use unperturbed moduli spaces for the index two case and we can make explicit computations on $m_{1,2}(C_i)$. Now, we can introduce an abstract perturbation for $\mu \geq 4$ if necessary. Even though $m_{1,\mu}$ for $\mu \geq 4$ is not computable, the arguments in the previous subsection can be used to determine Floer cohomology completely. Now it is clear that the results of [7] holds for all toric Fano manifolds from the arguments in the first part, together with the machinery of [12].

Proposition 4.1. *The theorems in* [7] *also holds true for all toric Fano manifolds*

Now the interesting open question is what happens in the non-Fano case. The argument itself works fine even in non-Fano case, but the problem is that the moduli space of holomorphic discs of Maslov index two also needs to be perturbed, since there might be a sphere bubble with non-positive Chern number from the Maslov index two disc. For detailed discussions on non-Fano case, we refer readers to [13].

5. More examples

In the first subsection, we consider the Hirzebruch surface F_1 which illustrates very well the subtle differences between various versions of Lagrangian Floer cohomology considered in this paper. Later, we discuss non-monotone examples.

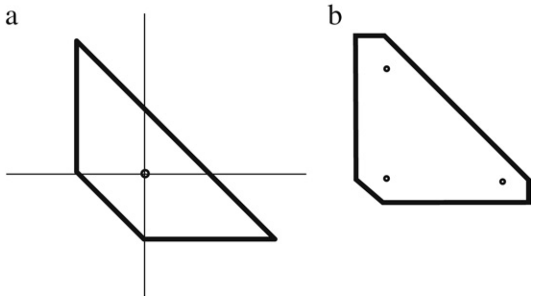

Fig. 1. (a) Hirzbruch surface, (b) Three point blow up of $\mathbb{C}P^2$.

5.1. Hirzebruch surface

Recall that F_1 is obtained from $\mathbb{C}P^2$ by blowing up a fixed point of the torus action. Symplectic form can be given by specifying its moment polytope, which we define as the region in $\mathbb{R}^2$ (See Fig. 1 (a)) bounded by

$$x = -1, \; y = -1, \quad x + y = -1, \qquad x + y = 1. \tag{5.1}$$

First, recall that there are four homotopy classes of holomorphic discs of Maslov index two, β_i corresponding to each normal vector v_i for $i = 1, 2, 3, 4$. Also recall the following area formula of holomorphic discs from [7].

Theorem 5.1. *(Theorem 8.1 [7]) Let P be a polytope defining a toric Fano manifold M, which is defined as*

$$\{x \in \mathbb{R}^n | \langle x, v_j \rangle \geq \lambda_j\}.$$

Let $A = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$. Then the area of holomorphic disc $D(v_j)$ of homotopy class β_j whose boundary lies in the Lagrangian submanifold $\mu_P^{-1}(A)$ is

$$2\pi(\langle A, v_j \rangle - \lambda_j).$$

Hence, one can easily notice that for the $\mu_P^{-1}((0,0))$, the areas of $\beta_1, \beta_2, \beta_3, \beta_4$ all equals 2π from the formula. The fiber $\mu_P^{-1}((0,0))$ is in fact a monotone Lagrangian submanifold.

Now, one can explicitly compute the Floer cohomology of $\mu_P^{-1}((\theta_1, \theta_2))$ for any $(\theta_1, \theta_2) \in P$ as follows. Let l_1, l_2 be two generators of $H_1(T^2)$, and without considering $\widetilde{B}$, we have (See [7] for details)

$$m_1(l_1) = -T^{2\pi(1-\theta_1-\theta_2)} \mathrm{e}^{-h_1 \mathrm{i} - h_2 i} + T^{2\pi(1+\theta_1+\theta_2)} \mathrm{e}^{h_1 \mathrm{i} + h_2 \mathrm{i}} + T^{2\pi(1+\theta_2)} \mathrm{e}^{h_2 \mathrm{i}} \tag{5.2}$$

$$m_1(l_2) = -T^{2\pi(1-\theta_1-\theta_2)} \mathrm{e}^{-h_1 \mathrm{i} - h_2 \mathrm{i}} + T^{2\pi(1+\theta_1+\theta_2)} \mathrm{e}^{h_1 \mathrm{i} + h_2 \mathrm{i}} + T^{2\pi(1+\theta_1)} \mathrm{e}^{h_1 \mathrm{i}} \tag{5.3}$$

To have non-vanishing Floer cohomology, we should have $m_1(l_1) = m_1(l_2) = 0$ from the Proposition 3.7. Since two expressions (5.2) and (5.3) differ by the last term, one can easily see that

$$\theta_1 = \theta_2 = \theta, \qquad h_1 = h_2 = h. \tag{5.4}$$

Now, we will discuss three versions of Lagrangian Floer homology separately.

5.2. The standard Floer cohomology

As there are three terms to be canceled, it is necessary that all the terms T^{area} should have equal area, which implies $\theta_1 = \theta_2 = 0$. Hence we need to solve the equation.

$$\mathrm{e}^{4hi} + \mathrm{e}^{3hi} + 1 = 0.$$

but this equation does not have a solution for $h \in \mathbb{R}$ (See (5.6)). Hence, the standard Floer cohomologies of all Lagrangian torus fibers vanish in this case.

5.3. Convergent version of Floer cohomology

We first explain the computation in the convergent case and explain its implications afterward. We recall that in the convergent version of Floer cohomology which appears in Mirror symmetry, one substitute a formal parameter T in the Novikov ring with a specific complex value. In general, the operations m_* is an infinite sum over all quantum contributions where infinite sum was justified by using the Novikov ring coefficients. After the substitution, there is a convergence issue of the infinite sum. But with the positivity assumptions, it is only a finite sum from the dimension argument. Hence in this case there is no convergence problem.

In [7], the substitution $T^{2\pi} = e^{-1}$ was used to identify m_0 and the Landau–Ginzburg superpotential. Hence, we will consider the case of the substitution $T^{2\pi} = e^{-1}$. Note that in the convergent version, T is of explicit value, hence the terms with different exponents could add up. Then, we set

$$z = e^{-\theta + ih}.$$

Then, vanishing of (5.2) and (5.3), with (5.4) is equivalent to

$$z^4 + z^3 - 1 = 0. \tag{5.5}$$

This equation has four solutions which are approximately (obtained from Matlab).

$$e^{-0.0614\pm 1.8063i},\ e^{0.1995},\ e^{0.3223+\pi i}. \tag{5.6}$$

Note that none of them corresponds to the origin where $|z| = 1$. But, in [8], it was observed that all the fibers other than $\mu_P^{-1}((0,0))$ are displaceable by some Hamiltonian isotopy. Hence, the fibers corresponding to (5.6) have non-vanishing Bott–Morse Floer cohomology $HF(L, L)$, but is displaceable by some Hamiltonian isotopy. This implies that for the convergent version of Floer cohomology, we have

$$HF(L, L) \neq HF(L, \psi(L)).$$

This shows that the convergent version does not have the same property as the standard version at least in the Morse–Bott case. This is in part because the presence of formal parameter T was crucially used in proving various isomorphisms and the symplectic invariance property of Floer cohomology. But mirror symmetry correspondence seems to be related to the convergent version of Floer cohomology.

5.4. With a complex valued closed two from $\widetilde{B}$

Now, we show that we may choose a good $\widetilde{B}$ which gives rise to a non-vanishing Floer cohomology for the monotone Lagrangian fiber $\mu_P^{-1}((0,0))$. Let

$$v_1 = (-1, -1), \qquad v_2 = (1, 0), \qquad v_3 = (1, 1), \qquad v_4 = (0, 1).$$

Then, we can find a solution (c_1, c_2, c_3, c_4) with $c_i \in \mathbb{C}^*$ satisfying

$$\sum_{i=1}^{4} c_i v_i = 0.$$

In fact, (5.2) and (5.3) becomes (with trivial holonomy for simplicity)

$$\begin{aligned} m_1(l_1) &= \left(((-l_1 - l_2) \times l_1) e^{\int_{\beta_1} \widetilde{B}} + ((l_1 + l_2) \times l_1) e^{\int_{\beta_3} \widetilde{B}} + (l_2 \times l_1) e^{\int_{\beta_4} \widetilde{B}}\right) T^{2\pi} \\ &= \left(e^{\int_{\beta_1} \widetilde{B}} - e^{\int_{\beta_3} \widetilde{B}} - e^{\int_{\beta_4} \widetilde{B}}\right) T^{2\pi} = (c_1 - c_3 - c_4) T^{2\pi} \end{aligned} \tag{5.7}$$

$$\begin{aligned} m_1(l_2) &= \left(((-l_1 - l_2) \times l_2) e^{\int_{\beta_1} \widetilde{B}} + (l_1 \times l_2) e^{\int_{\beta_2} \widetilde{B}} + ((l_1 + l_2) \times l_2) e^{\int_{\beta_3} \widetilde{B}}\right) T^{2\pi} \\ &= \left(-e^{\int_{\beta_1} \widetilde{B}} + e^{\int_{\beta_2} \widetilde{B}} + e^{\int_{\beta_4} \widetilde{B}}\right) T^{2\pi} = (-c_1 + c_2 + c_3) T^{2\pi}. \end{aligned} \tag{5.8}$$

Such c_i's are easy to find, for example, we may take

$$c_1 = 2, c_2 = c_3 = c_4 = 1.$$

Hence $\widetilde{B}$ may be taken as a suitable constant multiple (so that $c_1 = 2$) of the Poincare dual of the divisor $D(v_1)$ whose support is sufficiently close to $D(v_1)$. Then, the resulting Floer cohomology with $\widetilde{B}$ is non-vanishing, hence proves the non-displaceability of this monotone fiber. In this particular example, another way to prove this is to set $\widetilde{B} = 0$ and use non-unitary flat connections to obtain the same results.

5.5. Non-monotone examples

We provide some non-monotone Lagrangian submanifolds which are non-displaceable by Hamiltonian isotopy in this subsection, which were not known previously.

The first example is the case of $\mathbb{C}P^2$ blown up at three fixed points of the torus action (in the same way at each three points) (See Fig. 1(b)). In fact, there are three non-displaceable fibers as in the figure.

We may define the polytope as a region bounded by the lines

$$x + y = 1, \quad x = 0, \quad y = 0, \quad x = 1 - \epsilon, \quad y = 1 - \epsilon, \quad x + y = \epsilon.$$

Then, we can check that the fibers corresponding to

$$(\epsilon, \epsilon), \quad (\epsilon, 1 - 2\epsilon), \quad (1 - 2\epsilon, 1 - 2\epsilon)$$

are non-displaceable by the Theorem 3.9. Let us denote the normal vectors as $v_1, \ldots, v_6$ in this case counting counter clockwise starting from $v_1 = (-1, -1)$.

One may notice that there are two energy levels, $\lambda_1 = 2\pi\epsilon$, $\lambda_2 = 2\pi(1 - 2\epsilon)$ of holomorphic discs of Maslov index two, and at the energy level $2\pi\epsilon$, there are three normal vectors v_3, v_4, v_5 with

$$v_3 + \mathrm{e}^{\pi \mathrm{i}} v_4 + v_5 = 0,$$

and at the energy level $2\pi(1 - 2\epsilon)$, there are three normal vectors v_1, v_2, v_6 with

$$\mathrm{e}^{\pi \mathrm{i}} v_1 + v_2 + v_6 = 0.$$

Hence the non-displaceability follows from the theorem.

There is a higher dimensional analogue of this example. Namely for $\mathbb{C}P^n$, one can consider the blow up at $(n + 1)$ fixed points of the torus action. Then, there will be $(n + 1)$ non-displaceable fibers as in the above example near each blow up, whose non-displaceability can be proved analogously.

As dimension goes up, it gets easier to find more examples, since there are more normal vectors to play with. Another example may be obtained by blowing up $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ at the two fixed points where one is located at the other end of the diagonal from the other. More precisely, one can set the moment polytope to be the cube with verticies $(\pm 1, \pm 1, \pm 1)$, and the normal vectors corresponding to the blow ups to be $(1, 1, 1)$, $(-1, -1, -1)$. Then, if the corresponding facets are given by $x + y + z = 1$ and $x + y + z = -1$, then one gets a reflexive polytope. Now, to get non-monotone examples, one moves four facets (out of eight) together. Namely

(1) Consider $x + y + z = 1 - \epsilon$, together with $x = y = z = -1 + \epsilon$ for some small $\epsilon > 0$ and four other facets remain to be the same. Then, the fiber corresponding to $(0, 0, 0)$ is not monotone, we can see that there are four normal vectors at the energy level 2π and $2\pi(1 - \epsilon)$ each. One can show that the fiber corresponding to $(0, 0, 0)$ is non-displaceable, in fact by the standard Floer cohomology, from the cancellation argument.
(2) Consider the case of $x + y + z = 1 - \epsilon$, $x = y = z = 1 - \epsilon$. In this case the fiber corresponding to $(0, 0, 0)$ is non-displaceable, using the Floer cohomology with $\widetilde{B}$ since cancellation arguments work with the help of $\mathrm{e}^{\pi \mathrm{i}}$ as in the previous example.

Acknowledgement

This work is supported by Startup Research Fund of College of Natural Siences, Seoul National University.

References

[1] D. Auroux, Mirror symmetry and T-duality in the complement of an anticanonical divisor, preprint. arXiv:0706.3207.
[2] P. Biran, O. Cornea, Quantum structures for Lagrangian submanifolds, preprint. arXiv:0708.4221.
[3] P. Biran, M. Entov, L. Polterovich, Calabi quasimorphisms for the symplectic ball., Commun. Contemp. Math. 6 (5) (2004) 793–802.
[4] C.-H. Cho, Holomorphic discs, spin structures and the Floer cohomology of the Clifford torus, Int. Math. Res. Not. (35) (2004) 1803–1843.
[5] C.-H. Cho, Product structures of the Floer cohomology of Lagrangian torus fibers in toric Fano manifolds, Comm. Math. Phys. 260 (2006) 613–640.
[6] C.-H. Cho, Non-displaceable Lagrangian submanifolds and Floer cohomology with non-unitary line bundle, Preprint. arXiv:0710.5454.
[7] C.-H. Cho, Y.-G. Oh, Floer cohomology and disc instantons of Lagrangian torus fibers in toric Fano manifolds, Asian J. Math. 10 (2006) 773–814.
[8] M. Entov, L. Polterovich, Quasi-states and symplectic intersections, Comment. Math. Helv. 81 (1) (2006) 75–99.
[9] M. Entov, L. Polterovich, Rigid subsets of symplectic manifolds preprint. arXiv:0704.0105.
[10] A. Floer, Morse theory for Lagrangian intersections, J. Differential Geom. 28 (1988) 513–547.
[11] K. Fukaya, Floer homology for families—a progress report, in: Integrable Systems, Topology, and Physics (Tokyo, 2000), in: Contemp. Math., vol. 309, Amer. Math. Soc, 2002, pp. 33–68.
[12] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono, Lagrangian intersection Floer theory-anomaly and obstruction, Kyoto University preprint. 2000, revised version, (2007).
[13] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono, Lagrangian Floer theory of toric Fano manifolds I, Preprint. arXiv:0802.1703.
[14] K. Hori, C. Vafa, Mirror symmetry, preprint. 2000, hep-th/0002222.
[15] M. Kontsevich, Homological algebra of mirror symmetry, in: ICM-1994 Proceedings, Birkhäuser, Zürich, 1995.
[16] Y.-G. Oh, Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs I, Comm. Pure Appl. Math. 46 (1993) 949–994; Y.-G. Oh, Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs I, Comm. Pure Appl. Math. 48 (1995) 1299–1302 (Addenda).
[17] Y.-G. Oh, Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs II & III, Comm. Pure Appl. Math. 46 (1993) 995–1012; H. Hofer, et al. (Eds.), The Floer Memorial Volume, Birkhäuser, Basel, 1995, pp. 555–573.
[18] Y.-G. Oh, Floer cohomology, spectral sequence, and the Maslov class of Lagrangian embeddings, IMRN (7) (1996) 305–346.