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On the obstructed Lagrangian Floer theory ☆

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Abstract

Lagrangian Floer homology in a general case has been constructed by Fukaya, Oh, Ohta and Ono, where they construct an A_∞-algebra or an A_∞-bimodule from Lagrangian submanifolds. They developed obstruction and deformation theories of the Lagrangian Floer homology theory. But for obstructed Lagrangian submanifolds, the standard Lagrangian Floer homology cannot be defined.

We explore several well-known homology theories on these A_∞-objects, which are Hochschild and cyclic homology for an A_∞-objects and Chevalley–Eilenberg or cyclic Chevalley–Eilenberg homology for their underlying L_∞-objects. We show that these homology theories are well-defined and invariant even in the obstructed cases. Due to the existence of m_0, the standard homological algebra does not work and we develop analogous homological algebra over Novikov fields.

We provide computations of these homology theories in some cases: We show that for an obstructed A_∞-algebra with a non-trivial primary obstruction, Chevalley–Eilenberg Floer homology vanishes, whose proof is inspired by the comparison with cluster homology theory of Lagrangian submanifolds by Cornea and Lalonde.

In contrast, we also provide an example of an obstructed case whose cyclic Floer homology is non-vanishing.

Keywords: Symplectic geometry; Lagrangian submanifold; Floer homology; Hochshild homology; Cyclic homology

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1. Introduction

Floer homology invented by Floer [10], has proven to be a very powerful tool in the symplectic geometry and related areas. For Floer cohomology of a Lagrangian submanifold in a symplectic manifold, Fukaya, Oh, Ohta and Ono [14] have constructed the A_∞-algebra of a Lagrangian submanifold and the A_∞-bimodule of a pair of Lagrangian submanifolds in full generality. An A_∞-algebra is given by sequence of operations m_k for $k = 0, 1, 2, \ldots,$ satisfying quadratic relations (see Definition 2.2). If $m_0 = 0$, the m_1 is a differential and the homology of m_1 becomes Lagrangian Floer cohomology for A_∞-algebra of a Lagrangian submanifold. But in general, $m_0 \neq 0$ and in such a case, one cannot consider m_1-homology. In [14], they have developed obstruction and deformation theory and showed that if the obstructions vanish, then one can deform the given A_∞-structure to m_k^b for $k = 0, 1, 2, \ldots,$ so that the deformed A_∞-algebra has vanishing m_0^b. In such unobstructed (or weakly obstructed) cases, the Floer cohomology theories can be defined, and can be applied to the study of symplectic topology or homological mirror symmetry (see [21,3,6,4,14,15] for example).

In this paper, we will investigate the obstructed cases, for which the standard Floer cohomology cannot be defined. We explore alternative ways of defining homology, by considering Hochschild and cyclic homology of A_∞-objects and Chevalley–Eilenberg or cyclic Chevalley–Eilenberg homology of their underlying L_∞-objects.

Such homology theories are well known for associative (or Lie) algebras (see for example [23]). Also for A_∞, or L_∞-algebras with $m_0 = 0$ such homology theories have been known (we refer readers to [19] for the definitions using non-commutative geometry). The definition easily extends to the case with non-vanishing m_0, but it turns out that the usual homological algebra does not immediately extend as the usual contracting homotopy of the bar complex does not work with $m_0 \neq 0$. We show that by working with Novikov fields, modified contraction homotopy exists and that we still have the reduced Hochschild homology, and (b, B)-cyclic complex where Connes–Tsygan B-operator actually has an additional term compared to the standard case.

The main motivation to study these homology theories is to have a well-defined Floer homology theory even for obstructed Lagrangian submanifolds. We show that even for obstructed A_∞-algebras, these homology theories are well-defined and invariant under various choices involved and hence define invariants of a homotopy class of A_∞-objects.

One can obtain cyclic homology complex of an A_∞-algebra or a cyclic Chevalley–Eilenberg homology complex from the bar complex. Recall that A_∞-algebra $(C, \{m_k\})$ is algebraically a tensor-coalgebra $T(C[1])$ with a codifferential $\widehat{d} = \sum_k \widehat{m}_k$. The complex $(T(C[1]), \widehat{d})$ is called a bar complex, whose homology is trivial (see Lemma 3.1). One can consider cyclic or symmetric bar complex, which is a subcomplex of the bar complex, by considering the fixed elements of the natural cyclic or symmetric group action. The homology of these subcomplexes is in fact the cyclic homology of A_∞-algebra or cyclic Chevalley–Eilenberg homology of the induced L_∞-algebra. Here, as any associative algebra can be regarded as a Lie algebra whose bracket is given by the commutator, an A_∞-algebra (A_∞-module) gives rise to an underlying L_∞-algebra (L_∞-module) by symmetrizing all A_∞-operations and Chevalley–Eilenberg homology is their Lie algebra homology. We remark that the existence and invariance of homology of these subcomplexes has been known to authors of [14]. (Note that what we call cyclic bar complex is different from the cyclic bar complex of Getzler and Jones [18].)

We also remark that there have been different approaches to consider obstructed cases, by Cornea and Lalonde [7] using Morse functions and by Fukaya [11] using the relationship with loop space homology and Floer homology.

We observe that the topological dual theory of the Chevalley–Eilenberg homology theory of L_∞-objects is related to the cluster homology theory announced by Cornea and Lalonde in [7]. Their cluster homology corresponds to the extended cyclic Chevalley–Eilenberg homology of L_∞-algebras and their symmetric fine Floer homology corresponds to Chevalley–Eilenberg homology of L_∞-modules. Unfortunately, analytic details of the construction of cluster homology theory in [7] have not been rigorously established yet, but the homology theories in this paper may provide an alternative way to consider such theories in the obstructed cases.

Such a study of the dual geometry of an A_∞ or L_∞-algebra was initiated by Kontsevich [20]. The dual of an A_∞-algebra (as a coalgebra) is a differential graded algebra (DGA). Actually, in contact geometry, the dual language has been mostly used (see Chekanov [2], Eliashberg, Givental, and Hofer [9] for example) and the analogue of unobstructed condition is the notion of augmentation (see Lemma 7.4).

In Section 9, we explain carefully such a dualization process over Novikov field coefficients. To take appropriate duals of the filtered A_∞ or L_∞-objects, we consider topological duals induced by energy filtrations. In fact, the completion used in this paper is somewhat different from that of Cornea and Lalonde, resulting different behavior of the homology theories in obstructed cases. (Cornea has informed us that the filtration used here also works in the cluster homology setting.)

We consider Hochschild homology of A_∞-bimodule of a pair of Lagrangian submanifolds when one Lagrangian submanifold is obtained as a Hamiltonian isotopy of the other. Such Hochschild homology contains information about their intersections. Similarly, one can consider the induced L_∞-module of such pair over the L_∞-algebra of such a Lagrangian submanifold and consider its Chevalley–Eilenberg homology.

Theorem 1.1. *If A is an obstructed A_∞-algebra of Lagrangian submanifold with non-trivial primary obstruction, then the Chevalley–Eilenberg Floer (co)homology, and the extended cyclic Chevalley–Eilenberg Floer cohomology vanish.*

Recall that in [7], cluster complex with free terms has vanishing cluster homology. This may be interpreted as the vanishing of Chevalley–Eilenberg Floer homology when $m_0 \neq 0$ if we use the filtration of [7]. Hence one can notice the subtlety in choosing filtrations.

In contrast, we find a very different phenomenon of cyclic homology of A_∞-algebra in some cases.

Theorem 1.2. *Let L be a relatively spin compact Lagrangian submanifold in a closed (or with convex boundary) symplectic manifold, which admits only non-positive Maslov index pseudo-holomorphic discs with boundary on L. Then, its cyclic homology of the A_∞-algebra of L is non-trivial even when L is obstructed.*

The proof of the above theorem relies on the construction of explicit non-vanishing element of the cyclic homology in such a case.

Theorem 1.3. *Let L be a relatively spin compact Lagrangian submanifold in a closed (or with convex boundary) symplectic manifold, which is displaceable by a Hamiltonian isotopy. Here A_∞-algebra of L may be obstructed. Then, the Hochschild homology of the A_∞-algebra of L vanishes and also the Chevalley–Eilenberg homology vanishes.*

Now, the following question is related to the Maslov class conjecture in the obstructed case, but we do not know whether it is true or not.

Question 1.1. Let L be a relatively spin compact Lagrangian submanifold in a closed (or with convex boundary) symplectic manifold, which is displaceable by a Hamiltonian isotopy. Does the cyclic homology of the A_∞-algebra of L vanish? Also does the cyclic Chevalley–Eilenberg homology vanish?

If the answer to the first question is yes, then together with Theorem 1.2, we would conclude that the Maslov index of a relatively spin compact Lagrangian submanifold in C^n vanishes even if it is obstructed; this is the Maslov class conjecture (see [25,26,29,12]) and we leave this for future research.

This paper is organized in the following way. In Section 2, we recall basic notions of A_∞-algebra, A_∞-bimodule, and their cyclic and symmetric versions. We also recall various notions from [14] which are needed to prove the results. In Section 3, we show the isomorphism property of related homology theories for homotopy equivalent objects. In Section 4, we explain the Hochschild and Chevalley–Eilenberg homology and isomorphisms under weakly filtered homotopy equivalences. In Section 5, we consider cyclic version of the theories in Section 4, and show the modification of homological algebra with the presence of m_0. In Section 6, we recall results of [14], and apply the discussed homology theories. In Section 7, we explain the relation between Maurer–Cartan element and Hochschild homology and augmentation. In Section 8, we find a non-trivial element in the cyclic Floer homology. In Section 9, we consider topological dual theories of the above and in Section 10, we consider dualizations of Chevalley–Eilenberg homology and show its comparison to cluster homology theory of [7].

2. Algebraic setup

We briefly recall the notions about A_∞-algebras and A_∞-bimodules, and their cyclic and symmetric versions. We use the same sign convention as in [14], and in [5]. We also recall gapped condition and A_∞-homotopy from [14] to which we refer readers for more details.

2.1. Novikov fields

Let R be the field $\mathbb{R}$. We can also consider $\mathbb{C}$ or $\mathbb{Q}$, but $\mathbb{Q}$ cannot be used in the last section due to Theorem 10.4. Novikov rings are defined as follows (T and e are formal parameters)

$$\Lambda_{nov} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{q_i} \;\middle|\; a_i \in R,\ \lambda_i \in \mathbb{R},\ q_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty \right\},$$

$$\Lambda_{0,nov} = \left\{ \sum_i a_i T^{\lambda_i} e^{q_i} \in \Lambda_{nov} \;\middle|\; \lambda_i \geqslant 0 \right\}, \qquad \Lambda^+_{0,nov} = \left\{ \sum_i a_i T^{\lambda_i} e^{q_i} \in \Lambda_{0,nov} \;\middle|\; \lambda_i > 0 \right\}.$$

We define a valuation $\tau : \Lambda_{nov} \to \mathbb{R}$ which is the minimum energy of the expression:

$$\tau\left(\sum_i a_i T^{\lambda_i} e^{q_i} \right) = Min\bigl(\{\lambda_i \mid \forall i\}\bigr). \tag{2.1}$$

We also define the energy filtration as

$$F^{\lambda}\Lambda_{0,nov} = \{x \in \Lambda_{0,nov} \mid \tau(x) \geqslant \lambda\}.$$

We remark that in [14], they work with $\Lambda_{0,nov}$ coefficient to define these A_∞-objects and as Λ_{nov} is flat over $\Lambda_{0,nov}$, it does not cause any trouble. But one should be careful since the Floer cohomology of a pair of Lagrangian submanifolds over Λ_{nov} is invariant under the Hamiltonian isotopy, but not over $\Lambda_{0,nov}$. This is because the related maps are weakly filtered A_∞-bimodule maps. We remark that A_∞-algebras over R (unfiltered case) are denoted as $\overline{C}$ and their A_∞-maps as $\overline{m}_i$'s.

We work with Novikov field coefficients in the following cases: first, to find the reduced Hochschild homology. Second, when we try to construct the Connes–Tsygan's B-operator and finally when we make dualizations to obtain *DGA*'s. But note that Λ_{nov} is not a field due to the formal parameter e. For example, $(1+e)$ is not an invertible element. To overcome this, we may take one of the following two approaches. First, one defines universal Novikov ring without e:

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in R,\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i = \infty \right\}. \tag{2.2}$$

In this case, Λ is a field, but as one loses the track of the index, one should work with $\mathbb{Z}/2$-graded complexes instead of $\mathbb{Z}$-graded ones except for the cases of vanishing Maslov index. Here there exists at least $\mathbb{Z}/2$-grading as the Maslov index of a holomorphic disc with Lagrangian boundary conditions is always even for an orientable Lagrangian submanifold. (We learned this approach from Fukaya and this will be used in their upcoming work on toric manifolds.)

On the other hand, to keep e alive, we can also take the following approach. We consider a field of rational functions $R(e)$ of the variable e, and consider the tensor product

$$\Lambda_{nov}^{(e)} = \Lambda_{nov} \otimes_{R[e,e^{-1}]} R(e). \tag{2.3}$$

Then, we obtain a field $\Lambda_{nov}^{(e)}$ and as it is obtained via tensoring the field $R(e)$, it does not affect the homology theories very much. We remark that in most of the construction of [14], they work with $\Lambda_{0,nov}$ and only when one needs to work with Λ_{nov}, they take tensor product $\bigoplus \Lambda_{nov}$ to work with Λ_{nov} coefficients. We will take a similar approach when using the field $\Lambda_{nov}^{(e)}$.

2.2. A_∞-algebras

Let $\overline{C} = \bigoplus_{j\in\mathbb{Z}} \overline{C}^j$ be a graded vector space over R. We denote the parity change (or suspension) as $(\overline{C}[1])^m = \overline{C}^{m+1}$, and denote by $|x_i|$ (resp. $|x_i|'$) is the degree of (resp. the shifted degree of) the element x_i. Hence $|x_i| = |x_i|' + 1$. We define

$$T_k(\overline{C}[1]) = \underbrace{\overline{C}[1] \otimes \cdots \otimes \overline{C}[1]}_{k}, \qquad T_{1,\dots,k}(\overline{C}[1]) = \bigoplus_{j=1}^{k} T_j(\overline{C}[1]). \tag{2.4}$$

To simplify the notation, we set

$$B_k(C) := T_k(\overline{C}[1]), \qquad B_{1,\dots,k}(C) = T_{1,\dots,k}(\overline{C}[1]).$$

Definition 2.1. The **tensor-coalgebra** of $\overline{C}[1]$ over R is given by $B\overline{C} := \bigoplus_{k\geqslant 1} T_k(\overline{C}[1])$, with the comultiplication defined by

$$\Delta : B\overline{C} \to B\overline{C} \otimes B\overline{C}, \qquad \Delta(v_1 \otimes \cdots \otimes v_n) := \sum_{i=1}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n).$$

Now, consider a family of maps

$$\overline{m}_k : T_k\big(\overline{C}[1]\big) \to \overline{C}[1], \quad \text{for } k = 1, 2, \ldots.$$

We can extend $\overline{m}_k$ uniquely to a coderivation

$$\begin{aligned} &\widehat{\overline{m}}_k(x_1 \otimes \cdots \otimes x_n) \\ &\quad = \sum_{i=1}^{n-k+1} (-1)^{|x_1|' + \cdots + |x_{i-1}|'} x_1 \otimes \cdots \otimes m_k(x_i, \ldots, x_{i+k-1}) \otimes \cdots \otimes x_n \end{aligned} \tag{2.5}$$

for $k \leqslant n$ and $\widehat{\overline{m}}_k(x_1 \otimes \cdots \otimes x_n) = 0$ for $k > n$.

The coderivation $\widehat{\overline{d}} = \sum_{k=1}^{\infty} \widehat{\overline{m}}_k$ is well defined as a map from $B\overline{C}$ to $B\overline{C}$. The A_∞-equations are equivalent to the equality $\widehat{\overline{d}} \circ \widehat{\overline{d}} = 0$, or equivalently,

Definition 2.2. An A_∞-algebra $(\overline{C}, \{m_*\})$ consists of a $\mathbb{Z}$-graded vector space $\overline{C}$ over R with a collection of multi-linear maps $m := \{m_n : \overline{C}[1]^{\otimes n} \to \overline{C}[1]\}_{n\geqslant 1}$ of degree one satisfying the following equation for each $k = 1, 2, \ldots$:

$$0 = \sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1-1} (-1)^{\epsilon_1} m_{k_1}\big(x_1, \ldots, x_{i-1}, m_{k_2}(x_i, \ldots, x_{i+k_2-1}), \ldots, x_k\big) \tag{2.6}$$

where $\epsilon_1 = |x_1|' + \cdots + |x_{i-1}|'$.

Now, we define filtered A_∞-algebra. Let C be a free graded $\Lambda_{0,nov}$-module. We define similarly $T_k(C[1])$, $B_k(C)$ for $k \geqslant 1$ as in (2.4), and set $T_0(C[1]) = B_0(C) = \Lambda_{0,nov}$.

Definition 2.3. A filtered A_∞-algebra $(C, \{m_*\})$ consists of a $\Lambda_{0,nov}$-module $\overline{C}$ with a collection of multi-linear maps $m := \{m_n : C[1]^{\otimes n} \to C[1]\}_{n\geqslant 0}$ of degree one satisfying the A_∞-equations (2.6) for $k = 0, 1, 2, \ldots$.

The module C also has a filtration from the filtration of $\Lambda_{0,nov}$. The filtration on $B_k(C)$ is defined as

$$F^\lambda B_k(C) = \bigcup_{\lambda_1+\cdots+\lambda_k \geqslant \lambda} \big(F^{\lambda_1} C \otimes \cdots \otimes F^{\lambda_k} C\big).$$

Definition 2.4. Define

$$BC = \bigoplus_{k=0}^{\infty} B_k(C),$$

and $\widehat{B}C$ be its completion with respect to energy filtration.

Then, one can define $\widehat{m}_k$ as in (2.5), and note that when $k=0$, $\widehat{m}_0$ is defined as

$$\widehat{m}_0(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n-k+1} (-1)^{|x_1|'+\cdots+|x_{i-1}|'} x_1 \otimes \cdots \otimes x_{i-1} \otimes m_0(1) \otimes \cdots \otimes x_n.$$

Then by setting $\widehat{d} = \sum_{k=0}^{\infty} \widehat{m}_k$, the A_∞-equations are equivalent to the equality $\widehat{d} \circ \widehat{d} = 0$. The complex $(\widehat{B}C, \widehat{d})$ is called the *bar complex* of an A_∞-algebra A.

The above equality gives rise to countably many relations among $\{m_k\}$ where the first two are given as

$$\begin{cases} m_1\bigl(m_0(1)\bigr) = 0, \\ m_2\bigl(m_0(1), x\bigr) + (-1)^{\deg x+1} m_2\bigl(x, m_0(1)\bigr) + m_1\bigl(m_1(x)\bigr) = 0. \end{cases} \tag{2.7}$$

If $m_0 = 0$, we have $m_1^2 = 0$, hence it defines the homology of the A_∞-algebra. In general m_0 does not vanish, hence m_1 is not necessarily a differential. The obstruction and deformation theory when $m_0 \neq 0$ was developed in [14], and in an unobstructed case, one can define (deformed) Floer cohomology. See Section 7 for more discussion on unobstructedness.

An element $I \in C^0 = C^{-1}[1]$ is called a unit if

$$\begin{cases} m_{k+1}(x_1, \ldots, I, \ldots, x_k) = 0 \quad \text{for } k \geqslant 2 \text{ or } k = 0, \\ m_2(I, x) = (-1)^{\deg x} m_2(x, I) = x. \end{cases} \tag{2.8}$$

For filtered A_∞-algebras, we assume the maps $\{m_k\}$ satisfy

$$\begin{cases} m_k\bigl(F^{\lambda_1} C^{m_1} \oplus \cdots \oplus F^{\lambda_k} C^{m_k}\bigr) \subset F^{\lambda_1+\cdots+\lambda_k} C^{m_1+\cdots+m_k-k+2}, \\ m_0(1) \in F^{\lambda'} C[1] \quad \text{for some } \lambda' > 0. \end{cases} \tag{2.9}$$

We remark that

$$\Lambda_{0,nov}/\Lambda_{0,nov}^+ \cong R\bigl[e, e^-\bigr].$$

For a given filtered A_∞-algebra, $(C, \{m_k\})$, by considering modulo $\Lambda_{0,nov}^+$, we obtain

$$\overline{m}_k : B_k(\overline{C}) \otimes_R R\bigl[e, e^-\bigr] \to C[1] \otimes_R R\bigl[e, e^-\bigr].$$

We assume that all the $\overline{m}_k$-maps in fact are induced from

$$\overline{m}_k : B_k(\overline{C}) \to \overline{C}[1].$$

We make similar assumptions for all unfiltered A_∞-homomorphisms, unfiltered A_∞-bimodules and their homomorphisms in this paper as in [14]. In geometric situations as in [14], the classical informations (such as singular cohomology) are recorded in $\overline{m}_k$ whereas quantum informations using pseudo-holomorphic curves are recorded in m_k. Symplectic energy λ is recorded as T^λ and Maslov indices μ as e^μ. Hence, the above assumption amounts to the fact that for classical contributions, we have $\mu \equiv 0$.

Remark 2.5. We clarify our notation of A_∞-algebra. What we call A_∞-algebra here is called in some literature *weak* A_∞-algebra which may have a non-trivial m_0 term. A case without m_0 term is called a strict A_∞-algebra.

2.3. L_∞-algebras

Now, we explain the definition of an L_∞-algebra. First, consider an element σ of the group S_k of all permutations of the set $\{1, 2, \dots, k\}$. The group S_k acts on $T_k(\overline{C}[1])$ or $T_k(C[1])$ by

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_k) = (-1)^{\epsilon(\sigma,\vec{x})} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}, \tag{2.10}$$

where

$$\epsilon(\sigma, \vec{x}) = \sum_{i,j \text{ with } i<j,\ \sigma(i)>\sigma(j)} \big(|x_i|' \cdot |x_j|'\big). \tag{2.11}$$

For example, if we denote the cyclic element $\sigma_0 = (1, 2, \dots, k) \in S_k$, note that

$$\sigma_0 \cdot (x_1 \otimes \cdots \otimes x_k) = (-1)^{(|x_1|'(\sum_{i=2}^k |x_i|'))} x_2 \otimes \cdots \otimes x_k \otimes x_1.$$

Definition 2.6. Let $B_k^{cyc}\overline{C}$ be the set of fixed elements of the above σ_0-action on $B_k\overline{C}$, and denote

$$B^{cyc}\overline{C} = \bigoplus_{k=1}^{\infty} B_k^{cyc}\overline{C}.$$

Let $B_k^{cyc}C$ be the set of fixed elements of the cyclic group action on B_kC, and denote

$$\widehat{B}^{cyc}C = \widehat{\bigoplus}_{k=0}^{\infty} B_k^{cyc}C, \qquad \widehat{B}_{\geqslant 1}^{cyc}C = \widehat{\bigoplus}_{k=1}^{\infty} B_k^{cyc}C,$$

where we take a completion with respect to energy filtration.

Hence, B^{cyc} consists of cyclically invariant words.

Definition 2.7. We define $E_k(\overline{C})$ to be the submodule of $B_k(\overline{C})$ consisting of *fixed* elements of S_k action on $B_k(\overline{C})$ and let

$$E\overline{C} = \bigoplus_{k=1}^{\infty} E_k(\overline{C}).$$

We define $E_k(C)$ to be the submodule of $B_k(C)$ consisting of *fixed* elements of symmetric group action on $B_k(C)$ and let

$$\widehat{E}C = \widehat{\bigoplus}_{k=0}^{\infty} E_k(C), \qquad \widehat{E}_{\geqslant 1}C = \widehat{\bigoplus}_{k=1}^{\infty} E_k(C).$$

Remark 2.8. Equivalently, one can instead use the quotient complex by defining the equivalence relation by the above cyclic or symmetric group action.

For convenience, we use the following notation for the generators of $E\overline{C}$ and $\widehat{E}C$:

$$[x_1, \ldots, x_k] = \sum_{\tau \in S_k} (-1)^{\epsilon(\tau,\vec{x})} x_{\tau(1)} \otimes \cdots \otimes x_{\tau(k)}.$$

Lemma 2.1. *There is a coalgebra structure on* $E\overline{C}$

$$\Delta : E\overline{C} \to E\overline{C} \otimes E\overline{C},$$

restricted from $(B\overline{C}, \Delta)$. *This is graded commutative and coassociative.*

Proof. One can see that

$$\begin{aligned}
&\Delta\left(\sum_{\sigma \in S_k} (-1)^{\epsilon(\sigma,\vec{x})} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}\right) \\
&\quad = \sum_{\sigma \in S_k, i} (-1)^{\epsilon(\sigma,\vec{x})} (x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(k)}) \\
&\quad = \sum_{i} \sum_{\sigma \in (i,k-i)\ \text{shuffle}} (-1)^{\epsilon(\sigma,\vec{x})} \big([x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i-1)}]\big) \otimes \big([x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(k)}]\big).
\end{aligned}$$

Define $T : E\overline{C} \otimes E\overline{C} \to E\overline{C} \otimes E\overline{C}$ by

$$T(\alpha \otimes \beta) = (-1)^{|\alpha|'|\beta'|} \beta \otimes \alpha,$$

where α, β are homogeneous elements of degree $|\alpha|'$, $|\beta'|$ respectively. Then, it is not hard to see that $T \circ \Delta = \Delta$ which proves the cocommutativity. □

Similarly, one can prove the following lemma (see [14, Remark 3.2.21]).

Lemma 2.2. *There is a formal coalgebra structure on* $\widehat{E}C$

$$\Delta : \widehat{E}C \to \widehat{E}C \mathbin{\widehat{\otimes}} \widehat{E}C.$$

This is graded commutative and coassociative.

Remark 2.9. The above coalgebra structure will be used to define commutative algebra structure to the dual space of $\widehat{E}C$. Note also that $B^{cyc}\overline{C}$ does *not* have an induced coalgebra structure from $B\overline{C}$.

Lemma 2.3. *(See [14,11].) The codifferential $\widehat{d}$ (resp. $\widehat{\overline{d}}$) restricts to the map $\widehat{B}^{cyc}C \to \widehat{B}^{cyc}C$ (resp. $B^{cyc}\overline{C} \to B^{cyc}\overline{C}$). Also $\widehat{d}$ (resp. $\widehat{\overline{d}}$) restricts to the codifferential $\widehat{d}: \widehat{E}C \to \widehat{E}C$ (resp. $\widehat{\overline{d}}: E\overline{C} \to E\overline{C}$).*

Proof. It is easy to check the claim for the generators $\sum_\sigma \sigma(x_1 \otimes \cdots \otimes x_k)$ where the summand is over $\sigma \in \mathbb{Z}/k\mathbb{Z}$ for the cyclic case or $\sigma \in S_k$ for the symmetric case. □

L_∞-algebra structure on $\overline{C}$ is the codifferential $\widehat{\overline{d}}$ on $E\overline{C}$ or equivalently,

Definition 2.10. An L_∞-algebra $(\overline{C}, \{l_*\})$ consists of a $\mathbb{Z}$-graded vector space $\overline{C}$ over R with a collection of multi-linear maps $l := \{l_n : E_n\overline{C} \to \overline{C}[1]\}_{n \geqslant 1}$ of degree one satisfying the following equation for each $k = 1, 2, \ldots$:

$$0 = \sum_{k_1+k_2=k+1} \sum_{\sigma \in (k_1,k_2)\ \text{shuffle}} (-1)^{\epsilon(\sigma,\vec{x})} l_{k_1}\big(\big[l_{k_2}\big([x_{\sigma(1)}, \ldots, x_{\sigma(k_2)}]\big), x_{\sigma(k_2+1)}, \ldots, x_{\sigma(k)}\big]\big). \tag{2.12}$$

Definition 2.11. A filtered L_∞-algebra $(C, \{l_*\})$ consists of a $\Lambda_{0,nov}$-module C with a collection of multi-linear maps $l := \{l_n : E_nC \to C[1]\}_{n \geqslant 0}$ of degree one satisfying (2.12) for each $k = 0, 1, 2, \ldots$.

Note that for any A_∞-algebra, there exists the underlying L_∞-algebra obtained by the restriction to fixed elements of symmetric group action.

2.4. Filtered A_∞-homomorphisms

We recall the notion of filtered A_∞-homomorphism between two filtered A_∞-algebras (unfiltered case is similar and omitted). The family of maps of degree 0

$$f_k : B_k(C_1) \to C_2[1] \quad \text{for } k = 0, 1, \ldots$$

induce the coalgebra map $\widehat{f}: \widehat{B}C_1 \to \widehat{B}C_2$, which for $x_1 \otimes \cdots \otimes x_k \in B_kC_1$ is defined by the formula

$$\widehat{f}(x_1 \otimes \cdots \otimes x_k) = \sum_{0 \leqslant k_1 \leqslant \cdots \leqslant k_n \leqslant k} f_{k_1}(x_1, \ldots, x_{k_1}) \otimes \cdots \otimes f_{k-k_n}(x_{k_n+1}, \ldots, x_k).$$

We remark that the above can be an infinite sum due to the possible existence of $f_0(1)$. In particular, $\widehat{f}(1) = e^{f_0(1)}$. It is assumed that

$$\begin{cases} f_k\big(F^\lambda B_k(C_1)\big) \subset F^\lambda C_2[1], & \text{and} \\ f_0(1) \in F^{\lambda'} C_2[1] & \text{for some } \lambda' > 0. \end{cases} \tag{2.13}$$

The map $\widehat{f}$ is called a filtered A_∞-homomorphism if

$$\widehat{d} \circ \widehat{f} = \widehat{f} \circ \widehat{d}.$$

It is easy to check the following, whose proof is left as an exercise.

Lemma 2.4. *For any filtered A_∞-homomorphism $f : C_1 \to C_2$, the map $\widehat{f}$ restricts to the chain maps*

$$\widehat{f} : \widehat{B}^{cyc} C_1 \to \widehat{B}^{cyc} C_2, \qquad \widehat{f} : \widehat{E} C_1 \to \widehat{E} C_2.$$

In particular the latter provides the notion of a filtered L_∞-homomorphism between filtered L_∞-algebras.

2.5. A_∞ and L_∞-modules

Definition 2.12. For graded vector spaces $\overline{C}_1, \overline{C}_0$ and $\overline{M}$ over R, one writes

$$T^{\overline{M}}(\overline{C}_1, \overline{C}_0) := \bigoplus_{k \geqslant 0,\, l \geqslant 0} \overline{C}_1^{\otimes k} \otimes \overline{M} \otimes \overline{C}_0^{\otimes l}. \tag{2.14}$$

Furthermore, let

$$\Delta^{\overline{M}} : T^{\overline{M}}(\overline{C}_1, \overline{C}_0) \to \big(T\overline{C}_1 \otimes T^{\overline{M}}(\overline{C}_1, \overline{C}_0)\big) \oplus \big(T^{\overline{M}}(\overline{C}_1, \overline{C}_0) \otimes T\overline{C}_0\big)$$

be given by

$$\begin{aligned}
&\Delta^{\overline{M}}(v_1 \otimes \cdots \otimes v_k \otimes \underline{w} \otimes v_{k+1} \otimes \cdots \otimes v_{k+l}) \\
&\quad := \sum_{i=1}^{k} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes \underline{w} \otimes \cdots \otimes v_n) \\
&\qquad + \sum_{i=k}^{k+l-1} (v_1 \otimes \cdots \otimes \underline{w} \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_{k+l}).
\end{aligned}$$

Here we underlined elements of the module $\overline{M}$ for convenience.

Let $(\overline{C}_1, \widehat{\overline{d}}_1)$ and $(\overline{C}_0, \widehat{\overline{d}}_0)$ be A_∞-algebras. An A_∞-bimodule $\overline{M}$ over $(\overline{C}_1, \overline{C}_0)$ is a map D^M defined as follows: for simplicity, we first denote

$$\overline{B}^M(C_1, C_0) = T^{\overline{M}[1]}\big(\overline{C}_1[1], \overline{C}_0[1]\big).$$

We consider a map $D^M : \overline{B}^M(C_1, C_0) \to \overline{B}^M(C_1, C_0)$ of degree one satisfying the following commutative diagram:

$$\begin{array}{ccc}
\overline{B}^M(C_1, C_0) & \xrightarrow{\Delta^M} & (B\overline{C}_1 \otimes \overline{B}^M(C_1, C_0)) \oplus (\overline{B}^M(C_1, C_0) \otimes B\overline{C}_0) \\
\Big\downarrow {\scriptstyle D^M} & & \Big\downarrow {\scriptstyle (id \otimes D^M + \widehat{\overline{d}}_1 \otimes id) \oplus (D^M \otimes id + id \otimes \widehat{\overline{d}}_0)} \\
\overline{B}^M(C_1, C_0) & \xrightarrow[\Delta^M]{} & (B\overline{C}_1 \otimes \overline{B}^M(C_1, C_0)) \oplus (\overline{B}^M(C_1, C_0) \otimes B\overline{C}_0)
\end{array}$$

One can show (see [28]) that such D^M is determined by the family of maps

$$\eta_{k_1,k_0} : T_{k_1}\overline{C}_1[1] \otimes \overline{M}[1] \otimes T_{k_0}\overline{C}_0[1] \to \overline{M}[1].$$

We say $\overline{M}$ has the structure of an A_∞-bimodule over $(\overline{C}_1, \overline{C}_0)$ if

$$D^M \circ D^M = 0.$$

Now, the definitions of L_∞-bimodule can be obtained by symmetrizing the above construction. Namely, let $(\overline{C}, \widehat{\overline{d}})$ be an A_∞-algebra and consider an A_∞-bimodule $\overline{M}$ over $(\overline{C}, \overline{C})$. We can consider a symmetric group action on $\overline{B}^M(C, C)$ defined by

$$\begin{aligned}&\sigma \cdot (x_1 \otimes \cdots \otimes x_k \otimes \underline{x_{k+1}} \otimes x_{k+2} \otimes \cdots \otimes x_{k+l+1})\\ &\quad = (-1)^{\epsilon(\sigma,\vec{x})}(x_{\sigma(1)} \otimes \cdots \otimes \underline{x_{\sigma(j)}} \otimes \cdots \otimes v_{\sigma(k+l+1)}),\end{aligned}$$

where $\sigma(j) = k+1$. For example, $\sigma_0 = (1, 2, 3) \in S_3$,

$$\sigma_0 \cdot (x_1 \otimes \underline{x_2} \otimes x_3) = (-1)^{|x_1|'(|x_2|'+|x_3|')}\underline{x_2} \otimes x_3 \otimes x_1.$$

Let $\overline{E}^M(C)$ be the fixed elements of the symmetric group action on $\overline{B}^M(C, C)$. We denote

$$[\underline{x}_0, x_1, \ldots, x_k] := \sum_{\tau \in S_{k+1},\ j=\sigma^{-1}(0)} (-1)^{\epsilon(\tau,\vec{x})} x_{\tau(1)} \otimes \cdots \otimes \underline{x}_{\sigma(j)} \otimes \cdots \otimes x_{\tau(k)}.$$

Note that there exists one-to-one correspondence between $\overline{E}^M(C)$ and $\overline{M}[1] \otimes E_k\overline{C}$ by taking a term of $\overline{E}^M(C)$ in $\overline{M}[1] \otimes E_k\overline{C}$. We will identify them (with the sign rules as above) for convenience.

Let $\overline{\widetilde{C}}$ be an induced L_∞-algebra from the A_∞-algebra $\overline{C}$. We define L_∞-bimodule $\overline{M}$ over an L_∞-algebra $\overline{\widetilde{C}}$ as a map

$$D^M : \overline{E}^M(C) \to \overline{E}^M(C)$$

obtained by symmetrizing the above construction such that $D^M \circ D^M = 0$. Or equivalently,

Definition 2.13. Let $(\overline{C}, \{l_*\})$ be an L_∞-algebra. Then, an L_∞-bimodule structure on a graded vector space $\overline{M}$ is given by a collection of maps $\eta := \{\eta_k : \overline{M}[1] \otimes E_k\overline{C} \to \overline{M}[1]\}_{k \geqslant 0}$ of degree one satisfying the following equation for each $k = 0, 1, 2, \ldots$

$$\begin{aligned}\sum_{k_1+k_2=k}\ \sum_{\sigma \in (k_1,k_2)\ \text{shuffle}} &(-1)^{\epsilon(\sigma,\vec{x}_{\geqslant 1})}\bigl(\eta_{k_1}\bigl(\bigl[\eta_{k_2}[\underline{x}_0, x_{\sigma(1)}, \ldots, x_{\sigma(k_2)}], x_{\sigma(k_2+1)}, \ldots, x_{\sigma(k)}\bigr]\bigr)\\ &+ \eta_{k_1}\bigl(\bigl[\underline{x}_0, l_{k_2}[x_{\sigma(1)}, \ldots, x_{\sigma(k_2)}], x_{\sigma(k_2+1)}, \ldots, x_{\sigma(k)}\bigr]\bigr)\bigr) = 0. \qquad (2.15)\end{aligned}$$

In fact, the notions such as A_∞-homomorphism, A_∞-bimodule map, A_∞-homotopy induce those of L_∞-homomorphism, L_∞-bimodule map, L_∞-homotopy without much difficulty via the process of symmetrization. These notions will be explained more in the setting of filtered A_∞-algebras now.

2.6. Filtered bimodules

The notion of a filtered A_∞ or L_∞-bimodule can be easily defined as in unfiltered case. Let M be a graded free filtered $\Lambda_{0,nov}$-module and denote by $F^\lambda M$ its filtration. We complete M with respect to this filtration. Let $(C_1, \{m_k^1\})$, $(C_0, \{m_k^0\})$ be filtered A_∞-algebras over $\Lambda_{0,nov}$.

A family of operations for $k_1, k_0 \in \mathbb{Z}_{\geqslant 0}$

$$n_{k_1,k_0} : B_{k_1}(C_1) \otimes M[1] \otimes B_{k_0}(C_0) \to M[1]$$

of degree one define an A_∞-bimodule structure on M if they satisfy Eq. (2.18). We also assume that $n_{*,*}$ preserves the filtration in an obvious way.

These operations can be extended to

$$\widehat{D}^M : \widehat{B}(C_1)\widehat{\otimes} M[1]\widehat{\otimes}\widehat{B}(C_0) \to \widehat{B}(C_1)\widehat{\otimes} M[1]\widehat{\otimes}\widehat{B}(C_0), \tag{2.16}$$

which is defined as follows (here, codifferentials of $\widehat{B}C_i$ are denoted as $\widehat{d_i}$):

$$\begin{aligned}
&\widehat{D}^M(x_1 \otimes \cdots \otimes x_k \otimes y \otimes z_1 \otimes \cdots \otimes z_l) \\
&\quad = \widehat{d}_1(x_1 \otimes \cdots \otimes x_k) \otimes y \otimes z_1 \otimes \cdots \otimes z_l \\
&\qquad + \sum_{p\leqslant k, q\leqslant l} (-1)^{|x_1|'+\cdots+|x_{k-p}|'} x_1 \otimes \cdots \otimes x_{k-p} \otimes n_{p,q}(x_{k-p+1} \otimes \cdots \otimes y \otimes \cdots \otimes z_q) \\
&\qquad \otimes \cdots \otimes z_l + (-1)^{\sum(|x_i|'+|y|')} x_1 \otimes \cdots \otimes x_k \otimes y \otimes \widehat{d}_0(z_1 \otimes \cdots \otimes z_l).
\end{aligned} \tag{2.17}$$

The family of maps $\{n_{k_1,k_0}\}_{k_1,k_0 \in \mathbb{Z}_{\geqslant 0}}$ defines an A_∞-bimodule if

$$\widehat{D}^M \circ \widehat{D}^M = 0. \tag{2.18}$$

One can rewrite the above equation into countably many equations involving n_{k_1,k_0}'s. The first equation is

$$n_{0,0} \circ n_{0,0}(a) + n_{1,0}\big(m_0^1(1), a\big) + (-1)^{|a|'} n_{0,1}\big(a, m_0^0(1)\big) = 0. \tag{2.19}$$

In an unfiltered A_∞-bimodule case, Eq. (2.19) becomes $\bar{n}_{0,0} \circ \bar{n}_{0,0} = 0$.

Now we recall the notion of an A_∞-bimodule homomorphism. Let C_i, C_i' be filtered A_∞-algebras ($i = 0, 1$). Let M and M' be (C_1, C_0) and (C_1', C_0') filtered A_∞-bimodules respectively. Let $f^i : C_i \to C_i'$ be filtered A_∞-algebra homomorphisms. Then, a filtered A_∞-bimodule homomorphism $\phi : M \to M'$ over (f^1, f^0) is a family of $\Lambda_{0,nov}$-module homomorphisms $\{\phi_{k_1,k_0}\}$

$$\phi_{k_1,k_0} : B_{k_1}(C_1)\widehat{\otimes}_{\Lambda_{0,nov}} M[1]\widehat{\otimes}_{\Lambda_{0,nov}} B_{k_0}(C_0) \to M'[1]$$

which respects the filtration in an obvious way, and satisfies

$$\widehat{\phi} \circ \widehat{D}^M = \widehat{D}^{M'} \circ \widehat{\phi}. \tag{2.20}$$

Here $\widehat{\phi} : B(C_1)\widehat{\otimes} M[1]\widehat{\otimes} B(C_0) \to B(C_1')\widehat{\otimes} M'[1]\widehat{\otimes} B(C_0')$ is defined by

$$\widehat{\phi}(x_1 \otimes \cdots \otimes x_k \otimes y \otimes z_1 \otimes \cdots \otimes z_l)$$
$$= \sum_{p \leqslant k, q \leqslant l} \widehat{f}^1(x_1 \otimes \cdots \otimes x_{1,k-p}) \otimes n_{p,q}(x_{k-p+1} \otimes \cdots \otimes y \otimes \cdots \otimes z_q)$$
$$\otimes \widehat{f}^0(z_{q+1} \otimes \cdots \otimes \cdots z_l).$$

In the case that C_1 and C_0 (resp. C_1' and C_0') are the same A_∞-algebra, we denote $\widetilde{C}$ (resp. $\widetilde{C}'$) the induced L_∞-algebras from C_i (resp. C_i') for $i = 1$ or 2. By taking the fixed elements of symmetric group action in the above construction, one can define a notion of filtered L_∞-morphism $\phi : M \to M'$ over f.

Now, we recall the notion of a pull-back of an A_∞-bimodule. (See [14, Lemma 26.7-9].) Let (M, n) be a filtered (C_1', C_0') A_∞-bimodule, and let $f^i : C_i \to C_i'$ $(i = 0, 1)$ be filtered A_∞-homomorphisms. Then (M, n) give rise to a (C_1, C_0) A_∞-bimodule $((f^1, f^0)^* M, (f^1, f^0)^* n)$ with

$$\left(f^1, f^0\right)^* n(\vec{x}, y, \vec{z}) = n\left(\widehat{f}^1(\vec{x}), y, \widehat{f}^0(\vec{z})\right).$$

The pull-back operation is also functorial. Namely, let $g^i : C_i' \to C_i''$ be filtered A_∞-homomorphisms and M' a filtered (C_1'', C_0'') A_∞-bimodule. Then, a filtered A_∞-bimodule homomorphism $\phi : M \to M'$ over (g^1, g^0) induces a filtered A_∞-bimodule homomorphism over the identity

$$\left(f^1, f^0\right)^* \phi : \left(f^1, f^0\right)^* M \to \left(g^1 \circ f^1, g^0 \circ f^0\right)^* M',$$

where

$$\left(f^1, f^0\right)^* \phi(\vec{x}, y, \vec{z}) = \phi\left(\widehat{f}^1(\vec{x}), y, \widehat{f}^0(\vec{z})\right).$$

For the case $(f^1, f^0) = (id, id)$, it states that an A_∞-bimodule homomorphism $\phi : M \to M'$ over (g^1, g^0) can be considered as an A_∞-bimodule homomorphism $\widetilde{\phi} : M \to (g^1, g^0)^* M'$ over (id, id).

2.7. Gapped condition and spectral sequences

We recall the gapped condition and the spectral sequence arising from the related energy filtration. Let G be a submonoid of $\mathbb{R}_{\geqslant 0} \times 2\mathbb{Z}$ satisfying the following conditions:

(1) Let $\pi : \mathbb{R}_{\geqslant 0} \times 2\mathbb{Z} \to \mathbb{R}_{\geqslant 0}$ be the projection to the first component. Then $\pi(G) \subset \mathbb{R}_{\geqslant 0}$ is discrete.
(2) $G \cap (\{0\} \times 2\mathbb{Z}) = \{(0, 0)\}$.
(3) $G \cap (\{\lambda\} \times 2\mathbb{Z})$ is finite set for any λ.

We may denote its components as λ, μ: For $\beta \in G$,

$$\beta = \left(\lambda(\beta), \mu(\beta)\right) \in \mathbb{R}_{\geqslant 0} \times 2\mathbb{Z}.$$

A filtered A_∞-algebra is called to be *G-gapped*, if there exist R-module homomorphisms $m_{k,\beta}: B_k\overline{C}[1] \to \overline{C}[1]$ for $k = 0, 1, 2, \ldots$ and $\beta \in G$ such that

$$m_k = \sum_{\beta \in G} T^{\lambda(\beta)} e^{\mu(\beta)} m_{k,\beta}.$$

A filtered A_∞-algebra (C, m) is said to be gapped if it is G-gapped for some G. Similarly one can define gapped A_∞-homomorphisms. We remark that the A_∞-algebra of Lagrangian submanifolds constructed in [14] is gapped due to Gromov Compactness theorem. Here G is defined to be the submonoid of $\mathbb{R}_{\geqslant 0} \times 2\mathbb{Z}$ generated by

$$G(L)_0 = \bigl\{ \bigl(\omega(\beta), \mu_L(\beta)\bigr) \bigm| \beta \in \pi_2(M, L),\ \mathcal{M}(L, \beta, J) \neq 0 \bigr\}.$$

If an A_∞-algebra is gapped, then for any $k \geqslant 0$,

$$m_k(\mathbf{x}) - \overline{m}_k(\mathbf{x}) \in F^{\lambda_0} C \quad \text{for some } \lambda_0 > 0.$$

This is because 0 is discrete in $\pi(G)$, and hence there exists some $\lambda_0 > 0$ such that $\pi(\beta) \geqslant \lambda_0$ for any non-zero $\beta \in G$.

One can define in a similar way, gapped filtered A_∞-bimodules, gapped filtered A_∞-bimodule homomorphisms. For weakly filtered A_∞-bimodule homomorphism, [14] introduces a notion of a G-set, G' and G'-gapped weakly filtered A_∞-bimodule homomorphism. This is analogous to the above definition but to allow energy loss up to a fixed amount. We refer readers to [14, Definition 21.3] for details.

Let (C, δ) be a chain complex over $\Lambda_{0,nov}$, which is gapped. A new energy filtration is introduced by setting $\mathcal{F}^n C = F^{n\lambda_0} C$ for each $n \in \mathbb{Z}_{\geqslant 0}$. This filtration gives rise to the spectral sequence. This is a spectral sequence of a filtration over a filtered ring, and we recall it here from [14]. We put

$$\begin{cases} Z_r^{p,q}(C) = \bigl\{ x \in \mathcal{F}^q C^p \bigm| \delta(x) \in \mathcal{F}^{q+r-1} C^{p+1} \bigr\} + \mathcal{F}^{q+1} C^p, \\ B_r^{p,q}(C) = \bigl(\delta\bigl(\mathcal{F}^{q-r+2} C^{p-1}\bigr) \cap \mathcal{F}^q C^p \bigr) + \mathcal{F}^{q+1} C^p, \\ \mathcal{E}_r^{p,q}(C) = \dfrac{Z_r^{p,q}(C)}{B_r^{p,q}(C)}. \end{cases} \tag{2.21}$$

We denote $\Lambda_{0,nov}^{(0)}$ to be the degree zero part of $\Lambda_{0,nov}$. We define a filtration on $\Lambda_{0,nov}^{(0)}$ by $\mathcal{F}^n \Lambda_{0,nov}^{(0)} = F^{n\lambda_0} \Lambda_{0,nov}^{(0)}$. We denote $\Lambda^{(0)}(\lambda) = \Lambda_{0,nov}^{(0)} / F^\lambda \Lambda_{0,nov}^{(0)}$. Then the associated graded module is given by $gr_*(\mathcal{F}\Lambda_{0,nov}^{(0)}) = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} gr_n(\mathcal{F}\Lambda_{0,nov}^{(0)})$, where each $gr_n(\mathcal{F}\Lambda_{0,nov}^{(0)})$ is naturally isomorphic to $\Lambda^{(0)}(\lambda)$. Each $\mathcal{E}_r^{p,q}$ has a structure of $\Lambda^{(0)}(\lambda)$-module.

Lemma 2.5. *(See [14, Lemma 26.20].) There exists $d_r^{p,q}: \mathcal{E}_r^{p,q} \to \mathcal{E}_r^{p+1,q+r-1}$, which is a $\Lambda^{(0)}(\lambda)$-module homomorphism such that*

(1) $\delta_r^{p+1,q+r-1} \circ \delta_r^{p,q} = 0$.
(2) $Ker(\delta_r^{p,q}) / Im(\delta_r^{p-1,q-r+1}) \cong \mathcal{E}_{r+1}^{p,q}(C)$.
(3) $e^{\pm 1} \circ \delta_r^{p,q} = \delta_r^{p\pm 2,q} \circ e^{\pm 1}$.

The convergence of this spectral sequence is a non-trivial question since the filtration is not bounded and δ_r does not vanish for large r in general. In the case of the Floer cohomology (with respect to m_1), convergence of the spectral sequence was proved in [14].

2.8. A_∞ (and L_∞)-homotopies

We recall the notions of A_∞-homotopies between two A_∞-homomorphisms and between two A_∞-bimodule homomorphisms. In [14], it is defined using the notion of a model of $[0,1] \times C$.

A filtered A_∞-algebra $\mathfrak{C}$ together with filtered A_∞-homomorphisms

$$Incl : C \to \mathfrak{C}, \qquad Eval_{s=0} : \mathfrak{C} \to C, \qquad Eval_{s=1} : \mathfrak{C} \to C$$

is said to be a *model* of $[0,1] \times C$ if the following holds.

(1) $Incl_k : B_k C \to \mathfrak{C}$ is zero unless $k = 1$. The same holds for $Eval_{s=0}$ and $Eval_{s=1}$.
(2) $Eval_{s=0} \circ Incl = Eval_{s=1} \circ Incl = identity$.
(3) $Incl_1$ induces a cochain homotopy equivalence of the complex $(\overline{C}, \overline{m}) \to (\overline{\mathfrak{C}}, \overline{m})$, and $(Eval_{s=0})_1$, $(Eval_{s=1})_1$ induce cochain homotopy equivalences of the complex $(\overline{\mathfrak{C}}, \overline{m}) \to (\overline{C}, \overline{m})$.
(4) The homomorphism $(Eval_{s=0})_1 \oplus (Eval_{s=1})_1 : \mathfrak{C} \to C \oplus C$ is surjective.

Let C_1, C_2 be filtered A_∞-algebras and $f, g : C_1 \to C_2$ filtered A_∞-homomorphisms between them. Then f is said to be homotopic to g if there exists a filtered A_∞-homomorphism $\mathfrak{F} : C_1 \to \mathfrak{C}_2$ such that $Eval_{s=0} \circ \mathfrak{F} = f$, $Eval_{s=1} \circ \mathfrak{F} = g$:

$$\begin{array}{ccc} & & C_2 \\ & \overset{f}{\nearrow} & \uparrow{\scriptstyle Eval_0} \\ C_1 & \xrightarrow{\mathcal{F}} & \mathfrak{C}_2 \\ & \underset{g}{\searrow} & \downarrow{\scriptstyle Eval_1} \\ & & C_2 \end{array} \tag{2.22}$$

Here $\mathfrak{C}_2$ is a model of $[0,1] \times C_2$, and the above definition is independent of the choice of a model.

Now, we recall the model for A_∞-bimodules. Let M be a filtered (C_1, C_0) A_∞-bimodule and $\mathfrak{C}_i$ a model of $[0,1] \times C_i$. A model of $[0,1] \times M$ is a filtered $(\mathfrak{C}_1, \mathfrak{C}_0)$ A_∞-bimodule $\mathfrak{M}$ equipped with A_∞-bimodule homomorphisms $Eval_{s=s_0} : \mathfrak{M} \to M$ over $Eval_{s=s_0} : \mathfrak{C}_i \to C_i$ (for $s_0 = 0, 1$), and $Incl : M \to \mathfrak{M}$ over $Incl : C_i \to \mathfrak{C}_i$ with the following properties.

(1) $Eval_{s=s_0} \circ Incl$ is equal to the identity.
(2) $(Eval_{s=s_0})_{k_1,k_0} = (Incl)_{k_1,k_0} = 0$ for $(k_1, k_0) \neq (0,0)$.
(3) $(Eval_{s=0})_{0,0} \oplus (Eval_{s=1})_{0,0} : \mathfrak{M} \to M \oplus M$ is split surjective.
(4) $(Incl)_{0,0} : M \to \mathfrak{M}$ induces a cochain homotopy equivalence between $\overline{n}_{0,0}$ complexes.

There also exists a notion of L_∞-homotopy which is defined in an analogous way, and we refer readers to [13] for explicit statements on them.

2.9. Weakly filtered bimodule homomorphisms

For a filtered A_∞-bimodule (M, n) over $\Lambda_{0,nov}$, we get a filtered A_∞-bimodule $(\widetilde{M}, n)$ over Λ_{nov} by

$$\widetilde{M} = M \otimes_{\Lambda_{0,nov}} \Lambda_{nov}.$$

Note that $\widetilde{M}$ has a filtration $F^\lambda \widetilde{M}$ over $\lambda \in \mathbb{R}$.

Let $\widetilde{M}$ be a filtered (C_1, C_0) A_∞-bimodule over Λ_{nov}, and $\widetilde{M}'$ be a filtered (C_1', C_0') A_∞-bimodule over Λ_{nov}. Let $f^{(i)} : C_i \to C_i'$ be filtered A_∞-homomorphisms. A *weakly filtered A_∞-bimodule homomorphism* $\widetilde{M} \to \widetilde{M}'$ over $(f^{(0)}, f^{(1)})$ is a family of Λ_{nov}-module homomorphisms

$$\phi_{k_1,k_0} : B_{k_1}(C_1) \widehat{\otimes} \widetilde{M} \widehat{\otimes} B_{k_0}(C_0) \to \widetilde{M}'$$

with the following properties:

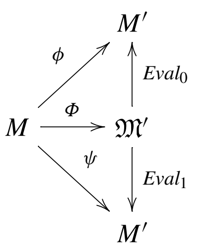
(1) There exists $c \geqslant 0$ independent of k_0, k_1 such that

$$\phi_{k_1,k_0}\big(F^{\lambda_1} B_{k_1}(C_1) \widehat{\otimes} F^\lambda \widetilde{M} \widehat{\otimes} F^{\lambda_0} B_{k_0}(C_0)\big) \subset F^{\lambda_1+\lambda+\lambda_0-c} \widetilde{M}'.$$

(2) $\widehat{\phi} \circ \widehat{d} = \widehat{d'} \circ \widehat{\phi}$.

Weakly filtered homomorphisms arise when we study the invariance property of the Floer cohomology $HF(L_0, L_1) \cong HF(L_0, \phi(L_1))$ where the constant c is related to the Hofer norm of the Hamiltonian isotopy ϕ.

Let $\phi, \psi : M \to M'$ be (weakly) filtered A_∞-bimodule homomorphisms over $(f^{(1)}, f^{(0)})$ and $(g^{(1)}, g^{(0)})$ respectively. Here $f^{(i)} : C_i \to C_i'$, $g^{(i)} : C_i \to C_i'$ are filtered A_∞-homomorphisms. Then, ϕ is said to be homotopic to ψ if there exist models $\mathfrak{M}', \mathfrak{C}_i'$ of $[0, 1] \times M', [0, 1] \times C_i'$ respectively, homotopies $\mathfrak{F}^{(i)} : C_i \to \mathfrak{C}_i'$ between $f^{(i)}$ and $g^{(i)}$, and a (weakly) filtered A_∞-bimodule homomorphism $\Phi : M \to \mathfrak{M}'$ over $(\mathfrak{F}^{(1)}, \mathfrak{F}^{(0)})$ such that $Eval_{s=0} \circ \Phi = \phi$, $Eval_{s=1} \circ \Phi = \psi$:



(2.23)

Here we also recall the definitions of homotopy equivalences. A filtered A_∞-homomorphism $f : C \to C'$ is called a *homotopy equivalence* if there exists a filtered A_∞-homomorphism $g : C' \to C$ such that $f \circ g$ and $g \circ f$ are homotopic to identity.

A (weakly) filtered A_∞-bimodule homomorphism $\phi : M \to M'$ over (f^0, f^1) is said to be a *homotopy equivalence* if there exists a (weakly) filtered A_∞-bimodule homomorphism

$\psi : M' \to M$ over (g^0, g^1) where $\phi \circ \psi$ and $\psi \circ \phi$ are homotopic to identity. Here g^1 and g^0 are homotopy inverses of f^1 and f^0 respectively.

The notions discussed so far can be defined in the L_∞-setting also.

3. Bar cohomology and isomorphisms

Consider the bar complexes $(B\overline{C}, \widehat{\overline{d}})$ and $(\widehat{B}C, \widehat{d})$ and their subcomplexes $(B^{cyc}\overline{C}, \widehat{\overline{d}})$, $(E\overline{C}, \widehat{\overline{d}})$, $(\widehat{B}^{cyc}C, \widehat{d})$ and $(\widehat{E}C, \widehat{d})$. In this section, we first show that cohomology of the bar complex is trivial for weakly unital A_∞-algebras. But the cohomology of subcomplexes are not trivial in general. In fact, cohomology of $(\widehat{B}^{cyc}C, \widehat{d})$ is isomorphic to cyclic homology of A_∞-algebra C and the cohomology of $(\widehat{E}C, \widehat{d})$ is the cyclic Chevalley–Eilenberg homology of the induced L_∞-algebra $\widetilde{C}$ of the given A_∞-algebra C. We prove that quasi-isomorphisms induce isomorphisms of these homology theories.

3.1. Bar complex

Let $(\overline{C}, \overline{m})$ be a unital strict A_∞-algebra. The following theorem is well known.

Lemma 3.1. *The cohomology of a bar complex $(B\overline{C}, \widehat{\overline{d}})$ is trivial.*

Proof. Let I be the unit of the A_∞-algebra. One can define the contracting homotopy s of the bar complex as follows: $s : B\overline{C} \to B\overline{C}$ is defined as

$$s(x_1 \otimes \cdots \otimes x_n) = I \otimes x_1 \otimes \cdots \otimes x_n. \tag{3.1}$$

One can check without much difficulty that on $B\overline{C}$

$$\widehat{\overline{d}} \circ s + s \circ \widehat{\overline{d}} = id - 0,$$

which provides the contracting homotopy of the bar complex. □

In the filtered case, we have

Lemma 3.2. *The cohomology of a bar complex $(\widehat{B}C, \widehat{d})$ of a filtered homotopy unital A_∞-algebra C is isomorphic to $\Lambda_{0,nov}$.*

Proof. In the next subsection, we will show that quasi-isomorphic A_∞-algebras have isomorphic cohomologies of the bar complexes. Hence, we may assume that the filtered A_∞-algebra is unital, by taking the canonical model of the given filtered A_∞-algebra which is homotopy unital.

In the case with $m_0 = 0$, the same homotopy s defined as in the unfiltered case provides the contracting homotopy. The only difference in this case is that $\widehat{B}C$ has in addition $B_0C = \Lambda_{0,nov}$. As for $1 \in \Lambda_{0,nov}$, we have $\widehat{d}(1) = m_0(1) = 0$, and 1 is a $\widehat{d}$-cycle. But clearly, the image of $\widehat{d}$ never contains 1 as one of its components. Hence 1 generates $\widehat{d}$-cohomology in this case and this proves the lemma for $m_0 = 0$.

Let us assume that $m_0 \neq 0$. Unfortunately in this case, s does not define a contracting homotopy (see Lemma 5.5), hence this case is a bit more complicated. As $\widehat{d}(1) = m_0(1) \neq 0$, it is not clear whether the cohomology of the bar complex is trivial or isomorphic to $\Lambda_{0,nov}$. But we claim that it is always isomorphic to $\Lambda_{0,nov}$. Namely, we can always find a $\widehat{d}$-cycle γ which includes 1 as one of its components and as before this gives rise to a non-trivial $\widehat{d}$-cohomology element. We define γ as

$$\gamma := 1 - I \otimes m_0 + I \otimes m_0 \otimes I \otimes m_0 + \cdots = \sum_{k=0}^{\infty} (-1)^k (I \otimes m_0)^{\otimes k}.$$

Note that the sum is well-defined as the energy of the summand goes to infinity as $k \to \infty$. We claim that $\widehat{d}(\gamma) = 0$. This follows from the following, which uses unitality of the A_∞-algebra,

$$\widehat{d}\big((I \otimes m_0)^{\otimes k}\big) = m_0 \otimes (I \otimes m_0)^{\otimes k} - m_0 \otimes (I \otimes m_0)^{\otimes k-1}.$$

Now, to prove that $\widehat{d}$-cohomology is isomorphic to $\Lambda_{0,nov}$, we can proceed as we do in the next subsection that we consider energy filtration and use the vanishing of bar cohomology in the unfiltered case and spectral sequence arguments to prove the vanishing of $\widehat{d}$-cohomology for tensors of positive length.

Also note that in the special case that A_∞-algebra is unobstructed and has a bounding cochain $b \in C$ satisfying $\widehat{d}(e^b) = 0$, e^b can be used instead of γ. One can show that any two such $\widehat{d}$-cycles which equal 1 mod $\Lambda_{0,nov}^{+}$ are cohomologous to each other from the vanishing results. □

3.2. *Isomorphisms*

Let us call the cohomologies of the complex $(B\overline{C}, \widehat{\overline{d}})$, $(B^{cyc}\overline{C}, \widehat{\overline{d}})$, $(E\overline{C}, \widehat{\overline{d}})$, $(\widehat{B}C, \widehat{d})$, $(\widehat{B}^{cyc}C, \widehat{d})$ and $(\widehat{E}C, \widehat{d})$ as bar cohomology for short. In this section, we prove that two quasi-isomorphic (filtered) A_∞-algebras have isomorphic bar cohomology.

For the cyclic case, it can be also proved by showing its equivalence to the cyclic homology of A_∞-algebra, but we show the proof here as the similar arguments are used at several instances of this paper. Exactly the same argument works for all cases, so we present the proof in the symmetric case only.

We first consider the unfiltered case (in particular, we have $m_0 = 0$).

Proposition 3.3. *Let $\overline{C}_1, \overline{C}_2$ be unfiltered A_∞-algebras over a ring R, and let $\overline{f} : \overline{C}_1 \to \overline{C}_2$ be an unfiltered A_∞-homomorphism which induces an isomorphism on $\overline{m}_1$-cohomologies. Then, f induces an isomorphism on bar cohomology.*

Proof. Let $\widehat{\overline{f}} : B\overline{C}_1 \to B\overline{C}_2$ be the associated cohomomorphism between two coalgebras. By Lemma 2.4, we can regard it as a chain map

$$\widehat{\overline{f}} : (E\overline{C}_1, \widehat{\overline{d}}_1) \to (E\overline{C}_2, \widehat{\overline{d}}_2).$$

To prove that $\widehat{\overline{f}}$ induces an isomorphism on bar cohomology, we will use the spectral sequences induced by the following number filtrations on $E\overline{C}_i$'s. Namely, we set

$$N^k(E\overline{C}_i) = E\overline{C}_i \cap B_{0,\ldots,k}\overline{C}_1. \tag{3.2}$$

Note that

$$0 = N^0(E\overline{C}_i) \subset N^1(E\overline{C}_i) \subset \cdots \subset E\overline{C}_i.$$

And the filtration is exhaustive, and Hausdorff:

$$\bigcup_k N^k(E\overline{C}_i) = E\overline{C}_i, \qquad \bigcap_k N^k(E\overline{C}_i) = \emptyset.$$

(We remark that for a filtered A_∞-algebra, the number filtration is not exhaustive because of the completion with respect to the energy, even in the case that $m_0 = 0$.)

It is also easy to check that $\widehat{\overline{d}}$ preserves the filtration. Therefore for each i, there exists a spectral sequence with

$$\mathcal{E}_1^{p,q}(E\overline{C}_i) = N^q\big(E\overline{C}_i^p\big)/N^{q-1}\big(E\overline{C}_i^p\big),$$

which converges to the homology of $\widehat{\overline{d}}$ on $E\overline{C}_i$ (see [24,30] for example). Convergence can be easily seen as the filtration is bounded below, exhaustive and Hausdorff [30, Theorem 5.5.1]. Here the differential δ_1 on $\mathcal{E}_1$ is induced by $\widehat{\overline{m}}_1$. Hence,

$$\mathcal{E}_2^{p,q}(E\overline{C}_i) = E\big(H^p(\overline{C}_i)\big) \cap B_q H^p(\overline{C}_i).$$

Note that $\widehat{\overline{f}}$ induces a map of spectral sequences since $\widehat{\overline{f}}$ preserves the number filtration:

$$\widehat{\overline{f}}\big(N^k(E\overline{C}_1)\big) \subset N^k(E\overline{C}_2).$$

Since f_1 induces an isomorphism on $\overline{m}_1$-cohomologies, it is easy to see that $\widehat{\overline{f}}$ induces an isomorphism on the $\mathcal{E}_2$-levels of the spectral sequences. Hence by the standard arguments of the spectral sequences, $\widehat{\overline{f}}$ induces an isomorphism between $\widehat{\overline{d}}$-cohomologies of $E\overline{C}_i$. The other cases follow from the same argument. □

Now, we consider a filtered case.

Proposition 3.4. *Let C_1, C_2 be gapped filtered A_∞-algebras over $\Lambda_{0,nov}$, and let $f : C_1 \to C_2$ be a gapped filtered A_∞-homomorphism, where $\overline{f}_1$ induces an isomorphism on $\overline{m}_1$-homologies. Then, f induces an isomorphism on bar cohomology of C_1 and C_2.*

Proof. First, note that $\widehat{d}$ and $\widehat{f}$ do not preserve the number filtration. Namely, $f_0(1), m_0(1) \in \Lambda_{0,nov}^+$ increases the number of tensor products in a given term. We will consider the energy filtration and consider the associated spectral sequences. Then, we prove that the induced map between spectral sequences on the $\mathcal{E}_2$-level is an isomorphism by considering a number filtration on $\mathcal{E}_1$-level of the spectral sequences.

Recall that $\widehat{E}C_i$ has an energy filtration, which we denoted as $F^\lambda(\widehat{E}C)$. By the gapped condition on C_1, C_2 and f, we can take $\lambda_0 > 0$ which works for both of them. We consider the filtration $\mathcal{F}^n(\widehat{E}C_i) = F^{n\lambda_0}(\widehat{E}C_i)$. It is easy to check that this filtration is complete:

$$\widehat{E}C_i = \lim_{\leftarrow} \widehat{E}C_i/\mathcal{F}^n(\widehat{E}C_i).$$

Note that by (2.9) and (2.13)

$$\widehat{d}\big(\mathcal{F}^{\lambda}(\widehat{E}C_i)\big) \subset \mathcal{F}^{\lambda}(\widehat{E}C_i), \qquad \widehat{f}\big(\mathcal{F}^{\lambda}(\widehat{E}C_1)\big) \subset \mathcal{F}^{\lambda}(\widehat{E}C_2).$$

Hence for each i, we have a spectral sequence with

$$\mathcal{E}_1^{p,q}(\widehat{E}C_i) = \mathcal{F}^q\big(\widehat{E}C_i^p\big)/\mathcal{F}^{q+1}\big(\widehat{E}C_i^p\big),$$

and a morphism of spectral sequences induced from $\widehat{f}$. But the convergence of these spectral sequences is not clear as $d_r \neq 0$ is even for large r in general.

But we need the spectral sequences for comparison purposes only and for such a purpose, convergence of the spectral sequences is not required by the following general theorem on spectral sequences.

Theorem 3.5. *(See [30, Eilenberg–Moore Comparison Theorem 5.5.11].) Let $f : V \to W$ be a map of filtered complexes of modules, where both V and W are complete and exhaustive. Fix $r \geqslant 0$. Suppose f_r: $\mathcal{E}_r^{p,q}(V) \cong \mathcal{E}_r^{p,q}(W)$ is an isomorphism for all p and q. Then $f : H^*(V) \to H^*(W)$ is an isomorphism.*

The idea of the proof of the above theorem is to use the mapping cone complex, which is also filtered by $F^q cone(f) = F^{q+r}V[1] \oplus F^q W$. And the fact that f^r is an isomorphism of $\mathcal{E}_r$, implies that $\mathcal{E}^r_{p,q}(cone(f)) = 0$ for all p, q by the related long exact sequence. In this case, spectral sequence obviously collapses and one can apply the complete convergence theorem (see [30]) to conclude that $H_* cone(f)$ is trivial. Since $cone(f)$ is an exact complex, this implies the above theorem.

In our case, note that we have

$$\mathcal{E}_1^{p,q} \cong E\overline{C}^p \otimes_R gr_*(\mathcal{F}\Lambda_{0,nov}).$$

And the differential δ_1 on $\mathcal{E}_1$-level is induced from $\widehat{\overline{d}}$ which is the energy zero part of $\widehat{d}$.

Now, we show that the induced map f_* between $\mathcal{E}_2$-levels of the spectral sequences is an isomorphism. Note that the induced map from f between $\mathcal{E}_1$-levels of the spectral sequences is induced by $\widehat{\overline{f}}$, which is the energy zero part of $\widehat{f}$. Note also that $\overline{f}_0(1) = \overline{m}_0 = 0$ and elements of $E\overline{C}$ are of finite sum since there cannot be an infinite sum without having the energy going to infinity.

Hence, for each fixed p and q, we consider the number filtration $N^k(E_1^{p,q}(E\overline{C}_i))$ as in (3.2) and there exists another spectral sequence arising from this number filtration converging to the homology of $(\mathcal{E}_1^{p,q}, \widehat{\overline{d}})$. Note that $\widehat{\overline{f}}$ induces an isomorphism between $\overline{m}_1$-cohomologies. Hence, the $\widehat{\overline{f}}$ induces an isomorphism of the spectral sequences from the number filtration, and induces an isomorphism between homologies of $(\mathcal{E}_1^{p,q}, \widehat{\overline{d}})$.

This shows that the induced map f_* on the $\mathcal{E}_2$-levels of the spectral sequences (with respect to the energy filtration) is indeed an isomorphism. Hence by the Eilenberg–Moore comparison theorem, $\widehat{f}$ induces an isomorphism on bar cohomology. The other cases follow exactly from the same argument. □

Remark 3.1. If $\widetilde{C}_1$ and $\widetilde{C}_2$ are filtered A_∞-algebras over Λ_{nov} obtained from the above C_1 and C_2 by taking tensor products with Λ_{nov}, then $f : \widetilde{C}_1 \to \widetilde{C}_2$ also induces an isomorphism on bar cohomology.

Remark 3.2. In [14], the spectral sequence of (C, m_1) (not $\widehat{B}C$ nor $\widehat{E}C$) with respect to the energy filtration was shown to converge, by using the fact that A_∞-algebras of Lagrangian submanifolds are weakly finite.

4. Hochschild, Chevalley–Eilenberg homology and isomorphisms

In this section, we recall a definition of Hochschild (resp. Chevalley–Eilenberg) homology of an A_∞ (resp. L_∞)-bimodule and consider their isomorphism properties under weakly filtered gapped homotopy equivalences. Weakly filtered case where the related map is not filtration preserving, is essential to discuss the invariance of Lagrangian Floer homology.

4.1. Definition of Hochschild homology

We recall the definition of Hochschild homology of an A_∞-bimodule $(M, \{n_{*,*}\})$ of an A_∞-algebra $A = (C, \{m_*\})$. (See [18,22] or [27] for more details on this subsection.)

We begin with a remark that the Hochschild homology can be regarded as a bar cohomology in the following way. Consider a complex $(B^M(C,C), D^M)$ from the definition (2.14). In fact, one can consider a cyclic group action and an induced subcomplex $(B^M(C,C)^{cyc}, D^M)$ considering the fixed elements of such a cyclic action. The homology of this subcomplex is called a Hochschild homology of a bimodule M over A.

Now, we give more detailed and conventional description of Hochschild homology. We denote

$$C^k(A, M) = M[1] \otimes C[1]^{\otimes k}.$$

We will denote its degree $\bullet$ part as $C^k_\bullet(A, M)$.

We define the Hochschild chain complex

$$C_\bullet(A, M) = \widehat{\bigoplus_{k \geqslant 0}} C^k_\bullet(A, M), \tag{4.1}$$

after completion with respect to energy filtration and with the boundary operation

$$d^{Hoch} : C_\bullet(A, M) \to C_{\bullet+1}(A, M)$$

defined as follows: we will underline the module element for reader's convenience. For $v \in M$ and $x_i \in A$,

$$\begin{aligned} & d^{Hoch}(\underline{v} \otimes x_1 \otimes \cdots \otimes x_k) \\ & \quad = \sum_{\substack{1 \leqslant j \leqslant k+1-i \\ 1 \leqslant i}} (-1)^{\epsilon_1} \underline{v} \otimes \cdots \otimes x_{i-1} \otimes m_j(x_i, \ldots, x_{i+j-1}) \otimes \cdots \otimes x_k \end{aligned}$$

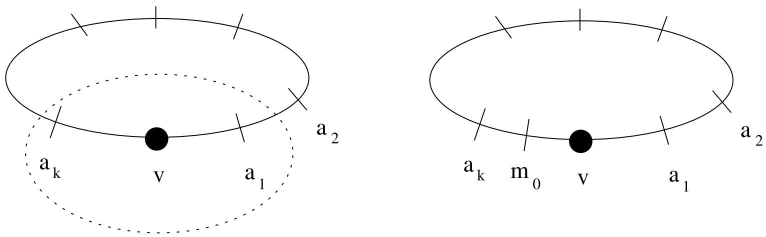


Fig. 1. Hochschild boundary.

$$+\sum_{i=1}^{k+1}(-1)^{\epsilon_2}\underline{v}\otimes x_1\otimes\cdots\otimes x_{i-1}\otimes m_0(1)\otimes x_i\otimes\cdots\otimes x_k$$
$$+\sum_{\substack{0\leqslant i,j\leqslant k\\ i+j\leqslant k}}(-1)^{\epsilon_3}\underline{n_{i,j}(x_{k-i+1},\dots,x_k,v,x_1,\dots,x_j)}\otimes x_{j+1}\otimes\cdots\otimes x_{k-i}. \tag{4.2}$$

Here the signs ϵ_1, ϵ_2, ϵ_3 are obtained from Koszul sign convention as usual. More explicitly, we have

$$\epsilon_1=\epsilon_2=|v|'+|x_1|'+\cdots+|x_{i-1}|',$$
$$\epsilon_3=\left(\sum_{s=1}^{i}|x_{k-i+s}|'\right)\left(|v|'+\sum_{t=1}^{j}|x_t|'\right).$$

The second and third type expressions in (4.2) arise as in Fig. 1. One considers an element $v\otimes x_1\otimes\cdots\otimes x_k$ as placed in a circle with special marking on the module element $v\in M$. And the boundary operation d^{Hoch} may be understood as taking an appropriate operation on elements placed on a connected arc of the circle or the insertion of m_0, and reading off the resulting element starting from the special marking. In particular, in the second terms of (4.2), we do not insert m_0 ahead of v, because in the operation corresponding to the right-hand side figure, m_0 will be inserted in the last position, after x_k.

The following is standard and can be easily understood from the figure.

Lemma 4.1.

$$d^{Hoch}\circ d^{Hoch}=0.$$

The homology of d^{Hoch} is called the *Hochschild homology* of M over A and is denoted as $H_\bullet(A,M)$. In the case of $M=A$, where the A_∞-bimodule structure on A is given by

$$n_{i,j}=m_{i+j+1},$$

we have the Hochschild homology $H_\bullet(A,A)=HH_\bullet(A)$ of an A_∞-algebra A.

4.2. Definition of Chevalley–Eilenberg homology

Here, we recall the definition of Chevalley–Eilenberg (L_∞-algebra) homology with coefficient in an L_∞-bimodule.

Let M be an L_∞-bimodule over an L_∞-algebra $\widetilde{A} = (C, \{l_*\})$ (see Definition 2.13). We denote

$$CE^k(\widetilde{A}, M) = M[1] \otimes E_k C$$

and denote its degree $\bullet$ part as $CE^k_\bullet(\widetilde{A}, M)$.

We define the chain complex

$$CE_\bullet(\widetilde{A}, M) = \widehat{\bigoplus_{k \geqslant 0}} CE^k_\bullet(\widetilde{A}, M),$$

after completion with respect to energy filtration.

The boundary operation

$$d^{CE} : CE_\bullet(\widetilde{A}, M) \to CE_{\bullet+1}(\widetilde{A}, M)$$

defined as follows, using the L_∞-module structure maps η: for $v \in M$ and $x_i \in C$ for $i = 1, \dots, k$, we define

$$\begin{aligned} & d^{CE}\big(\underline{v} \otimes [x_1, \dots, x_k]\big) \\ &\quad = \sum_{k_1 \geqslant 0} \sum_{(k_1, k-k_1) \text{ shuffle}} (-1)^{\epsilon(\sigma, \vec{x}_{\geqslant 1})} \underline{\eta_{k_1}\big(\underline{v} \otimes [x_{\sigma(1)}, \dots, x_{\sigma(k_1)}]\big)} \otimes [x_{\sigma(k_1+1)}, \dots, x_{\sigma(k)}] \\ &\qquad + \sum_{k_1 \geqslant 0} \sum_{(k_1, k-k_1) \text{ shuffle}} (-1)^{|v|' + \epsilon(\sigma, \vec{x}_{\geqslant 1})} \underline{v} \otimes \big[l_{k_1}\big([x_{\sigma(1)}, \dots, x_{\sigma(k_1)}]\big), x_{\sigma(k_1+1)}, \dots, x_{\sigma(k)}\big]. \end{aligned}$$

By the definition of L_∞-bimodule, we have

$$d^{CE} \circ d^{CE} = 0,$$

and we denote its homology as $H^{CE}_\bullet(\widetilde{A}, M)$.

4.3. Weakly filtered homotopy equivalences and isomorphisms

We show isomorphism properties of the Hochschild or Chevalley–Eilenberg homology under weakly filtered homotopy equivalences. To do so, we first show that there is a canonical chain map between the corresponding chain complexes.

Suppose we have an A_∞-bimodule homomorphism $\phi : M \to N$ between two A_∞-bimodules over A. Namely, we have a family of maps $\phi_{i,j} : A^{\otimes i} \otimes M \otimes A^{\otimes j} \to N$ satisfying A_∞-bimodule equations.

We define a chain map $\phi_* : C_\bullet(A, M) \mapsto C_\bullet(A, N)$ as

$$\phi_*(v \otimes x_1 \otimes \cdots \otimes x_k) = \sum_{\substack{0 \leqslant i,j \leqslant k \\ i+j \leqslant k}} (-1)^{\epsilon_2} \phi_{i,j}(x_{k-i+1}, \ldots, x_k, v, x_1, \ldots, x_j) \otimes x_{j+1} \otimes \cdots \otimes x_{k-i},$$

where ϵ_2 is as given above. The following is easy to check.

Lemma 4.2. *We have*

$$\phi_* \circ d_M^{Hoch} = d_N^{Hoch} \circ \phi_*,$$

and hence it induces a map $\phi_* : H_\bullet(A, M) \to H_\bullet(A, N)$.

More generally, let $A = (C, m)$, $A' = (C, m')$ be two filtered A_∞-algebras, and let $\alpha : A \to A'$ be a filtered A_∞-homomorphism. Let M (resp. M') be an A_∞-bimodule over A (resp. over A'). For an A_∞-bimodule homomorphism $\phi : M \to M'$ over (α, α), one can define a chain map $\phi_* : C_\bullet(A, M) \to C_\bullet(A', M')$:

$$\phi_*(v \otimes x_1 \otimes \cdots \otimes x_k) = \sum_{\substack{0 \leqslant i,j \leqslant k \\ i+j \leqslant k}} (-1)^{\epsilon_2} \phi_{i,j}(x_{k-i+1}, \ldots, x_k, v, x_1, \ldots, x_j) \otimes \widehat{\alpha}(x_{j+1} \otimes \cdots \otimes x_{k-i}).$$

One can also obtain similar maps for L_∞-case. Namely, let $\widetilde{A}$ (resp. $\widetilde{A'}$) be an induced L_∞-algebra from A (resp. A'), and $\widetilde{f}$ an induced filtered L_∞-homomorphism between A and A'. And consider M (resp. N) as an induced L_∞-module over $\widetilde{A}$ (resp. $\widetilde{A'}$). The A_∞-bimodule map ϕ induces an L_∞-bimodule homomorphism between M and N over $\widetilde{f}$. One can also check that such a map is a chain map.

Proposition 4.3. *Let M and N be gapped filtered A_∞-bimodules over a gapped filtered A_∞-algebra $A = (C, m)$. Let $\phi : M \to N$ be filtered or weakly filtered gapped A_∞-bimodule homomorphism, which is a homotopy equivalence. Then the map ϕ induces an isomorphism between Hochschild homology of M and N over A and also ϕ induces an isomorphism between Chevalley–Eilenberg homology of M and N over $\widetilde{A}$:*

$$H_\bullet(A, M) \cong H_\bullet(A, N), \qquad H_\bullet^{CE}(\widetilde{A}, M) \cong H_\bullet^{CE}(\widetilde{A}, N).$$

Proof. The proof follows by considering the definition of homotopy via the models of A_∞ or L_∞-homotopy. As the proof of L_∞-case is exactly the same as that of A_∞-case, we only consider A_∞-case.

From the definition of homotopy equivalence and the A_∞-homotopy, it is enough to prove that if $f, g : M \to N$ are filtered or weakly filtered A_∞-bimodule homomorphisms over A, and if they are A_∞-homotopic to each other, then they induce the same map on Hochschild homology (i.e. $f_* = g_*$).

Denote by $\mathfrak{N}$ the model of $[0, 1] \times N$, and by $\mathfrak{A}$ the model of $[0, 1] \times A$. We will use the notation *Incl*, *Eval* without distinction between two models $\mathfrak{N}$ and $\mathfrak{A}$, which should be clear from the context.

We denote by H the A_∞-homotopy $H: M \to \mathfrak{N}$ which is a (weakly) filtered A_∞-bimodule homomorphism over $(Incl, Incl)$ ($\mathfrak{N}$ is an A_∞-bimodule over $\mathfrak{A}$). It satisfies the following commutative diagram:

$$\begin{array}{ccc} & & N \\ & \overset{f}{\nearrow} & \uparrow {\scriptstyle Eval_0} \\ M & \xrightarrow{H} & \mathfrak{N} \\ & \underset{g}{\searrow} & \downarrow {\scriptstyle Eval_1} \\ & & N \end{array} \tag{4.3}$$

Note that $Eval \circ Incl = id$ by the definition of the model, hence, the composition $Eval_s \circ H : M \to N$ is an A_∞-bimodule map over (id, id) for $s = 0, 1$.

Since the induced maps are $f_* = (Eval_0)_* \circ H_*$ and $g_* = (Eval_1)_* \circ H_*$, it is enough to show the composition $Eval_s \circ Incl$ induces an isomorphism on Hochschild homology as it implies that

$$g_* = (Eval_1)_* \circ H_* = (Eval_0 \circ Incl)_* \circ (Eval_1)_* \circ H_* = (Eval_0)_* \circ (Incl \circ Eval_1)_* \circ H_* = f_*.$$

Even though H is only weakly filtered, the maps $Eval$, $Incl$ are filtered, and are of very simple forms that $(Eval_s)_{i,j} = 0$ and $(Incl_s)_{i,j} = 0$ for $(i, j) \neq (0, 0)$ and $s = 0$ or 1. We can use these good properties of $Eval$ and $Incl$-maps to prove the desired isomorphism property. Now the rest of the proof is very similar to that of the last section and we leave the details to the reader. □

Remark 4.1. If an A_∞-bimodule map is a quasi-isomorphism in an unfiltered case, then it is a homotopy equivalence. In the filtered case, note that $n_{0,0}$ does not necessarily define a complex (namely $n_{0,0}^2 \neq 0$), but $\bar{n}_{0,0}$ (modulo $\Lambda^+_{0,nov}$) does define a chain complex. For an A_∞-bimodule map $\phi : M \to N$, if $\bar{\phi} : \overline{M} \to \overline{N}$ is a quasi-isomorphism, then ϕ is a homotopy equivalence as proved in [14, Theorem 5.2.35]. Hence, the above proposition holds in such a case. But for weakly filtered case, an appropriate notion of "modulo $\Lambda^+_{0,nov}$" cannot be defined and analogous statements are not known. Namely, what should be the definition of quasi-isomorphisms for weakly filtered case is not clear as explained in [14, Remark 5.2.36].

4.4. Reduced Hochschild homology with m_0 terms

It is well known that for a unital A_∞-algebra A with $m_0 = 0$, the Hochschild homology of an A_∞-bimodule M over A can be computed using the reduced Hochschild chain complex. Similarly one considers in the filtered case,

$$C^{red}_\bullet(A, M) = \widehat{\bigoplus_k} M[1] \otimes \big(C/(k \cdot I)\big)[1]^{\otimes k},$$

and it is easy to check that

$$\widehat{d} : C^{red}_\bullet(A, M) \to C^{red}_{\bullet+1}(A, M)$$

is well-defined and defines a complex whose homology is called the reduced Hochschild homology $H_\bullet^{red}(A, M)$.

We prove that with Novikov field coefficients, it is quasi-isomorphic to the standard Hochschild chain complex. Instead of Λ_{nov}, we use the coefficients Λ or $\Lambda_{nov}^{(e)}$ which are Novikov fields defined in the definition (2.2) or (2.3). We consider $\Lambda_{nov}^{(e)}$ only for simplicity. We give a proof since the standard proof does not generalize immediately due to the presence of m_0.

Proposition 4.4. *Reduced Hochschild homology is isomorphic to the Hochschild homology $H_\bullet(A, M)$ for filtered A_∞-algebra A with a strict unit I (even with $m_0 \neq 0$).*

Proof. We modify the proof given in the book of Loday [23, Section 1.6]. Define s_i: $C_\bullet(A, M) \to C_{\bullet-1}(A, M)$ by

$$s_i(v \otimes a_1 \otimes \cdots \otimes a_k) = (-1)^{|v|'+\cdots+|a_i|'} v \otimes \cdots \otimes a_i \otimes I \otimes a_{i+1} \otimes \cdots \otimes a_k,$$

and define $t_i : C_\bullet(A, M) \to C_\bullet(A, M)$ by

$$t_i(v \otimes a_1 \otimes \cdots \otimes a_k) = (-1)^{|v|'} v \otimes \widehat{m}_0(a_1 \otimes \cdots \otimes a_i \otimes I \otimes a_{i+1} \otimes \cdots \otimes a_k).$$

Here, we set $s_i = t_i = 0$ if $i < 0$ or $i > k$.

The maps t_i's are introduced to make the filtration below compatible with the Hochschild differential. Note that for short we may write

$$t_i = d_0^{Hoch} \circ s_i, \tag{4.4}$$

where d_0^{Hoch} denotes the 2nd term of the definition of d^{Hoch} in (4.2). Let $D_\bullet$ be a submodule of $C_\bullet(A, M)$ which is the completion of the submodule generated by the images of the maps $\{s_i\}_{i\in\mathbb{N}\cup\{0\}}$, $\{t_i\}_{i\in\mathbb{N}\cup\{0\}}$ or equivalently by the images of $\{s_i\}_{i\in\mathbb{N}\cup\{0\}}$.

One can check easily that $(D_\bullet, d^{Hoch})$ is a subcomplex of $(C_\bullet(A, M), d^{Hoch})$.

Lemma 4.5. *$(D_\bullet, d^{Hoch})$ is acyclic subcomplex.*

Proof. We prove this by introducing the following filtration: Consider a filtration $F_p D_\bullet$ which is generated by the images of $s_0, \ldots, s_p, t_0, \ldots, t_p$. One can check that the filtration is compatible with d^{Hoch}.

Then, by the spectral sequence argument, it is enough to show that $Gr_p D_\bullet$ is acyclic for any p, which will be shown in the next lemma. Here, $F_p D_\bullet$ is not exhaustive filtration as we have used completion, but the filtration is complete. And in this case, acyclicity for each p implies that the spectral sequence is weakly convergent and hence proves the acyclic property of the subcomplex $(D_\bullet, d^{Hoch})$ (see [30]). □

Lemma 4.6. *$Gr_p D_\bullet$ is acyclic for any p. More precisely, we have a chain homotopy α_p between identity and zero map: i.e. they satisfy the identities*

$$\left(d^{Hoch} \circ \alpha_p + \alpha_p \circ d^{Hoch}\right) \circ s_p = s_p \mod F_{p-1},$$
$$\left(d^{Hoch} \circ \alpha_p + \alpha_p \circ d^{Hoch}\right) \circ t_p = t_p \mod F_{p-1}.$$

Proof. We define a chain homotopy $\alpha_p : Gr_p D_\bullet \to Gr_p D_{\bullet-1}$ as follows:

$$\alpha_p\big(s_p(v \otimes a_1 \otimes \cdots \otimes a_k)\big) = s_p \circ s_p(v \otimes a_1 \otimes \cdots \otimes a_k),$$
$$\alpha_p\big(t_p(v \otimes a_1 \otimes \cdots \otimes a_k)\big) = -t_p\big(s_p(v \otimes a_1 \otimes \cdots \otimes a_k)\big).$$

From (4.4), we may also write

$$\alpha_p \circ t_p = -d_0^{Hoch} \circ s_p \circ s_p.$$

We write

$$d^{Hoch} = d_0^{Hoch} + d_+^{Hoch}.$$

One can check as in the standard case (although complicated)

$$\big(d_+^{Hoch} \circ \alpha_p + \alpha_p \circ d_+^{Hoch}\big) \circ s_p = s_p \mod F_{p-1},$$
$$\big(d_+^{Hoch} \circ \alpha_p + \alpha_p \circ d_+^{Hoch}\big) \circ t_p = t_p \mod F_{p-1}.$$

Now, we check the same identity for d_0^{Hoch}. Note that

$$\big(d_0^{Hoch} \circ \alpha_p + \alpha_p \circ d_0^{Hoch}\big) \circ s_p = d_0^{Hoch} \circ s_p \circ s_p - d_0^{Hoch} \circ s_p \circ s_p = 0,$$
$$\big(d_0^{Hoch} \circ \alpha_p + \alpha_p \circ d_0^{Hoch}\big) \circ t_p = -d_0^{Hoch} \circ d_0^{Hoch} \circ s_p \circ s_p + \alpha_p\big(d_0^{Hoch} \circ d_0^{Hoch} \circ s_p\big) = 0,$$

as $d_0^{Hoch} \circ d_0^{Hoch} = 0$. This proves the lemma. □

Hence $D_\bullet$ is acyclic, and the quotient complex is $C_\bullet^{red}(A, M)$. □

We remark that this reduced version should be helpful for computations of Hochschild homology for weakly obstructed Lagrangian submanifolds as one can then ignore Hochschild boundary operation coming from m_0 by using the reduced version.

5. Cyclic and cyclic Chevalley–Eilenberg homology

We recall the standard definitions of cyclic homology of A_∞-algebra and cyclic Chevalley–Eilenberg homology of an induced L_∞-algebra, and explain its homological algebra with $m_0 \neq 0$. It turns out that there are some modifications to be made due to the presence of m_0. We refer readers to the book by Loday [23] in the standard case of associative and Lie algebras and in the homotopy algebra (with $m_0 = 0$) case to the paper by Hamilton and Lazarev [19] for an approach using non-commutative de Rham theory.

Let $A = (C, \{m_*\})$ be a filtered A_∞-algebra, and let $\widetilde{A}$ be an induced L_∞-algebra. First, consider the subcomplexes of the bar complex $(\widehat{B}^{cyc}C, \widehat{d})$ and $(\widehat{E}C, \widehat{d})$ introduced in Section 2.

Definition 5.1. We define the *cyclic homology* of an A_∞-algebra to be the homology of the complex $(\widehat{B}^{cyc}_{\geqslant 1}C, \widehat{d})$, and denote it as $HC_\bullet(A)$. We define the *cyclic Chevalley–Eilenberg homology* of an L_∞-algebra $\widetilde{A}$ to be the homology of the complex $(\widehat{E}_{\geqslant 1}C, \widehat{d})$ and denote it as $HC_\bullet^{CE}(\widetilde{A})$.

Remark 5.2. Instead of invariants, the standard definition is given by considering the quotient as in (5.3) of Connes' complex. The resulting homologies are isomorphic, and we use the above for the simplicity of presentation.

In fact cyclic Chevalley–Eilenberg homology for L_∞-algebra is just a Chevalley–Eilenberg homology with trivial coefficient. The reason that it is called cyclic Chevalley–Eilenberg homology is that there is a uniform approach for A_∞, L_∞ and C_∞-algebras to define Hochschild and cyclic homology theories via considering formal manifolds and their non-commutative de Rham theory. We refer readers to [19] for more detailed explanations and references.

5.1. Cyclic bicomplex and Connes complex

We show that the two standard approaches to define cyclic homology can be used when $m_0 \neq 0$ and also are equivalent to each other. The bicomplex for cyclic homology was introduced by B. Tsygan, and we can consider an analogous bicomplex for filtered A_∞-algebras. (See [23] for the classical case and we assume that the reader is familiar with the construction in [23].) Consider the Hochschild chain complex $C_\bullet(A, A)$ defined in (4.1). For the cyclic generator $t_{n+1} \in \mathbb{Z}/(n+1)\mathbb{Z}$, we define its action on $A^{\otimes(n+1)}$ as in (2.10):

$$t_{n+1} \cdot (x_0, x_1, \dots, x_n) = (-1)^{|x_n|'(|x_0|'+\cdots+|x_{n-1}|')}(x_n, x_0, \dots, x_{n-1}).$$

Here, we set t_1 to be identity on A and write the identity map as 1. Consider $N_{n+1} := 1 + t_{n+1} + t_{n+1}^2 + \cdots + t_{n+1}^n$.

As in the classical case, we have the natural augmented exact sequence:

$$A^{\otimes(n+1)} \xleftarrow{1-t_{n+1}} A^{\otimes(n+1)} \xleftarrow{N_{n+1}} A^{\otimes(n+1)} \xleftarrow{1-t_{n+1}} A^{\otimes(n+1)} \xleftarrow{N_{n+1}} \cdots.$$

We consider $\bigoplus_{n=1}^\infty N_n$-action on $\bigoplus_{n=1}^\infty A^{\otimes n}$ and denote it as

$$N : C_\bullet(A, A) \mapsto C_\bullet(A, A). \tag{5.1}$$

We can also similarly define $(1-t) : C_\bullet(A, A) \mapsto C_\bullet(A, A)$.

Recall that in the classical case, cyclic bicomplex has even columns which are the copies of the Hochschild complex, and odd columns which are the copies of the bar complex. We will construct the bicomplex in the similar way: even columns will be given by $(C_\bullet(A, A), d^{Hoch})$. For odd columns, note that $\widehat{B}C = \Lambda_{0,nov} \oplus C_\bullet(A, A)$ as $B_0C = \Lambda_{0,nov}$ is not present in the Hochschild chains. Consider $\widehat{d}$-operation on $C_\bullet(A, A)$ considered as a subspace of $\widehat{B}C$. Due to Lemma 3.2, the homology of the chain complex $(C_\bullet(A, A), \widehat{d})$ vanishes, and this will be the odd columns.

These two differentials are certainly different. For example, given any $x \in C$, we have

$$d^{Hoch}(x) = (-1)^{|x|'} x \otimes m_0 + m_1(x)$$

whereas

$$\widehat{d}(x) = m_0 \otimes x + (-1)^{|x|'} x \otimes m_0 + m_1(x).$$

To follow the standard notation, we set $b = d^{Hoch}$ and $b' = \widehat{d}$.

Lemma 5.1. *We have on $C_\bullet(A, A)$ the following identities:*

$$b(1-t) = (1-t)b', \qquad b'N = Nb. \tag{5.2}$$

We thus obtain the cyclic bicomplex (analogous to Tsygan's) defined as follows.

Definition 5.3. Define

$$CC_{pq}(A) = C_q(A, A) \quad \text{for all } p \geqslant 0,\ q \in \mathbb{Z}.$$

We define differentials as

$$\begin{aligned} b : CC_{pq}(A) \mapsto CC_{p(q+1)}(A) &\quad \text{for } p \text{ even},\\ -b' : CC_{pq}(A) \mapsto CC_{p(q+1)}(A) &\quad \text{for } p \text{ odd},\\ 1-t : CC_{pq}(A) \mapsto CC_{(p-1)q}(A) &\quad \text{for } p \text{ odd},\\ N : CC_{pq}(A) \mapsto CC_{(p-1)q}(A) &\quad \text{for } p \text{ even} \end{aligned}$$

$$\begin{array}{ccccccccc}
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C_1(A,A) & \xleftarrow{1-t} & C_1(A,A) & \xleftarrow{N} & C_1(A,A) & \xleftarrow{1-t} & C_1(A,A) & \xleftarrow{N} & \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C_0(A,A) & \xleftarrow{1-t} & C_0(A,A) & \xleftarrow{N} & C_0(A,A) & \xleftarrow{1-t} & C_0(A,A) & \xleftarrow{N} & \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C_{-1}(A,A) & \xleftarrow{1-t} & C_{-1}(A,A) & \xleftarrow{N} & C_{-1}(A,A) & \xleftarrow{1-t} & C_{-1}(A,A) & \xleftarrow{N} & \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & &
\end{array}$$

For example, for associative algebras (whose degree is concentrated at zero), the standard bicomplex of cyclic homology can be seen in the 4th quadrant. All elements have degree -1 after degree shifting, hence the negative of the length gives the degree of an expression.

Proposition 5.2. *The homology of the above (completed) total complex $\widehat{\mathrm{Tot}}(CC(A))$ is isomorphic to cyclic homology $HC_\bullet(A)$.*

Proof. In the standard case, there exists an isomorphism of the homology of the total complex of the bicomplex, and the homology of Connes' complex. Recall that Connes complex is defined as

$$C^\lambda_\bullet(A) := coker(1-t) = C_\bullet(A, A)/im(1-t). \tag{5.3}$$

It is easy to check that this complex has the same homology as cyclic homology defined in Definition 5.1 (which is the standard invariant and coinvariant relation).

Consider a natural surjection $p : \widehat{\mathrm{Tot}}(CC_\bullet)(A) \to C^\lambda_\bullet(A)$, where the quotient map is given from the first column. Recall that the rows of the bicomplex are acyclic augmented complexes with $H_0 = C^\lambda_\bullet(A)$. Consider the standard horizontal increasing filtrations on $CC_\bullet(A)$ and $C^\lambda_\bullet(A)$, and use the spectral sequence arguments as in the classical case or as in the last section to prove the proposition. □

As usual, there exists the Connes exact sequence, relating Hochschild homology and cyclic homology.

Lemma 5.3. *We have the following exact sequence:*

$$\to H_\bullet(A, A) \to HC_\bullet(A) \to HC_{\bullet+2}(A) \to H_{\bullet+1}(A, A) \to .$$

5.2. (b, B)-complex

There is also the (b, B)-complex, which is obtained from the bicomplex using the acyclicity of the even columns. We will see that the contraction homotopy of the bar complex in the standard case does not work for filtered A_∞-algebras with $m_0 \neq 0$. We first find a modified contraction homotopy, and we will also discuss normalized (b, B)-complex. For this, we need to work on Novikov ring which is a field. Instead of Λ_{nov}, we can use the coefficients Λ or $\Lambda^{(e)}_{nov}$ which are Novikov fields defined in the definition (2.2) or (2.3). For simplicity, we only consider $\Lambda^{(e)}_{nov}$.

In the case that $m_0 = 0$, we have a contracting homotopy as in the standard case. Namely, $s : C_\bullet(A, A) \to C_\bullet(A, A)$ is defined by

$$s(x_1 \otimes \cdots \otimes x_n) = I \otimes x_1 \otimes \cdots \otimes x_n.$$

It is easy to check that we have

$$s \circ \widehat{d} + \widehat{d} \circ s = id - 0.$$

But in the filtered case with non-trivial m_0, this is no longer true: for any x_1, we have

$$\begin{aligned}(s \circ \widehat{d} + \widehat{d} \circ s)(x_1) &= I \otimes \big(m_1(x_1) + m_0 \otimes x_1 + (-1)^{|x_1|'} x_1 \otimes m_0\big) \\ &\quad + m_2(I, x_1) + m_1(I) \otimes x_1 - I \otimes m_1(x_1) + m_0 \otimes I \otimes x_1 \\ &\quad - I \otimes m_0 \otimes x_1 - (-1)^{|x_1|'} I \otimes x_1 \otimes m_0 \\ &= x_1 + m_0 \otimes I \otimes x_1 \neq x_1.\end{aligned}$$

We find a modified contracting homotopy $\tilde{s}$. First, recall from Lemma 3.2 that the homology of the bar complex $(\widehat{B}C, \widehat{d})$ is isomorphic to $\Lambda_{0,nov}$. It implies that the homology of $(C_\bullet(A, A), \widehat{d})$ vanishes (with $\Lambda^{(e)}_{nov}$ or $\Lambda_{0,nov}$ coefficient) as we do not consider the part B_0C.

Lemma 5.4. *There exists a $\Lambda^{(e)}_{nov}$-module V such that*

$$C_\bullet(A, A) = Ker(\widehat{d}) \oplus V.$$

Proof. We first note that $Ker(\widehat{d}) = Im(\widehat{d})$ is $\Lambda_{nov}^{(e)}$-module as $\widehat{d}$ is $\Lambda_{nov}^{(e)}$-linear. As $\Lambda_{nov}^{(e)}$ is a field, it is then easy to find such subspace V. □

Lemma 5.5. *Consider $C_\bullet(A, A)$ with $\Lambda_{nov}^{(e)}$ coefficient. There exists a contracting homotopy $\tilde{s}$ of the complex $(C_\bullet(A, A), \widehat{d})$ for a filtered A_∞-algebra A.*

Proof. We define $\tilde{s}$ using the decomposition in Lemma 5.4. Define for $p \in V$, $\tilde{s}(p) = s(p)$ as before. As $Ker(\widehat{d}) = Im(\widehat{d})$, one can find a linear map $h : Ker(\widehat{d}) \to V$ such that $\widehat{d} \circ h = id|_{Ker(\widehat{d})}$. For $p \in Ker(\widehat{d}) = Im(\widehat{d})$, define

$$\tilde{s}(p) = I \otimes p - m_0 \otimes I \otimes h(p).$$

Then, $\tilde{s}$ now satisfies

$$\tilde{s} \circ \widehat{d} + \widehat{d} \circ \tilde{s} = id - 0. \tag{5.4}$$

To see this, for $p \in V$,

$$\tilde{s} \circ \widehat{d}(p) + \widehat{d} \circ \tilde{s}(p) = I \otimes \widehat{d}(p) - m_0 \otimes I \otimes p + \widehat{d}(I \otimes p).$$

For convenience, write $\widehat{d} = \widehat{d}_0 + \widehat{d}_+$ where $\widehat{d}_0 = \widehat{m}_0$. From the unital property of $\widehat{d}$, we have

$$\widehat{d}(I \otimes p) = \widehat{d}_0(I \otimes p) + p - I \otimes \widehat{d}_+(p).$$

Here, the second term is result of $\widehat{d}_+$-operation containing I in its input and the third term is that of $\widehat{d}$ which does not contain the unit I. Hence the (5.4) follows by adding up and computing $\widehat{d}_0$.

For $p \in Ker(\widehat{d})$, we have

$$\begin{aligned}\tilde{s} \circ \widehat{d}(p) + \widehat{d} \circ \tilde{s}(p) &= 0 + \widehat{d} \circ \tilde{s}\big(\widehat{d}\big(h(p)\big)\big) \\ &= 0 + \widehat{d}\big(I \otimes \widehat{d}\big(h(p)\big) - m_0 \otimes I \otimes h(p)\big).\end{aligned} \tag{5.5}$$

To simplify notations, we write $q = h(p)$ and hence $\widehat{d}q = p$. Note that $\widehat{d}(I \otimes \widehat{d}(h(p)))$ equals

$$\begin{aligned}&\widehat{d}_0(I \otimes \widehat{d}_0 q) + \widehat{d}_+(I \otimes \widehat{d}_0 q) + \widehat{d}_0(I \otimes \widehat{d}_+ q) + \widehat{d}_+(I \otimes \widehat{d}_+ q) \qquad (5.6)\\ &\quad = \big(m_0 \otimes I \otimes \widehat{d}_0 q + I \otimes \widehat{d}_0\big(\widehat{d}_0(q)\big)\big) + \big(\widehat{d}_0(q) - I \otimes \widehat{d}_+\big(\widehat{d}_0(q)\big)\big) \\ &\qquad + \big(m_0 \otimes I \otimes \widehat{d}_+ q - I \otimes \widehat{d}_0\big(\widehat{d}_+(q)\big)\big) + \big(\widehat{d}_+ q - I \otimes \widehat{d}_+\big(\widehat{d}_+(q)\big)\big) \\ &\quad = m_0 \otimes I \otimes \widehat{d}q + \widehat{d}(q) - I \otimes \big(\widehat{d} \circ \widehat{d}(q)\big) \\ &\quad = m_0 \otimes I \otimes \widehat{d}q + \widehat{d}(q). \qquad (5.7)\end{aligned}$$

Here, we used the fact that $\widehat{d}_0 \circ \widehat{d}_0 = 0$.

Now, the third term of (5.5) equals

$$\begin{aligned} -\widehat{d}(m_0 \otimes I \otimes q) &= -\widehat{d_0}(m_0 \otimes I \otimes q) - \widehat{d_+}(m_0 \otimes I \otimes q) \\ &= -m_0 \otimes I \otimes \widehat{d_0}(q) - m_0 \otimes I \otimes \widehat{d_+}(q) \\ &= -m_0 \otimes I \otimes \widehat{d}q. \end{aligned} \tag{5.8}$$

The claim follows by adding (5.7) with (5.8). □

This lemma is now used to define a new Connes operator $B = (1-t)\tilde{s}N$. As we have used $\tilde{s}$ instead of s, the resulting B is slightly different from the standard B which may contain additional terms of the form $(1-t)(m_0 \otimes I \otimes \alpha)$.

And we obtain the following bicomplex whose homology is isomorphic to cyclic homology:

$$\begin{array}{ccccccc} \uparrow b & & \uparrow b & & \uparrow b & & \\ C_1(A,A) & \xleftarrow{B} & C_2(A,A) & \xleftarrow{B} & C_3(A,A) & \xleftarrow{B} & \\ \uparrow b & & \uparrow b & & \uparrow b & & \\ C_0(A,A) & \xleftarrow{B} & C_1(A,A) & \xleftarrow{B} & C_2(A,A) & \xleftarrow{B} & \\ \uparrow b & & \uparrow b & & \uparrow b & & \\ C_{-1}(A,A) & \xleftarrow{B} & C_0(A,A) & \xleftarrow{B} & C_1(A,A) & \xleftarrow{B} & \\ \uparrow b & & \uparrow b & & \uparrow b & & \end{array} \tag{5.9}$$

Now, as we have (b, B)-complex, we can also consider normalized (b, B)-complex by considering $C_i^{red}(A, A)$ instead of $C_i(A, A)$ in the (b, B)-complex above. There is an obvious surjection from (b, B)-complex to normalized (b, B)-complex which can be shown to be quasi-isomorphism.

We remark that even though we have normalized (b, B)-complex, we do not have normalized Tsygan's bicomplex. Also, the additional terms of Connes operator B in the filtered case will disappear in the normalized (b, B)-complex, hence giving rise to the standard B-operator. One can also define variants of cyclic homologies as in the standard case.

6. Lagrangian Floer theory

We recall some of the main results of [14], and apply to them the homology theories discussed so far. Their invariance properties can be proved as a corollary.

Theorem 6.1. *(See [14, Theorem A].) To each relatively spin Lagrangian submanifold L, we can associate a structure of gapped filtered A_∞-algebra structure $\{m_k\}$ on $H^*(L, \Lambda_{0,nov})$, which is well-defined up to isomorphism. If $\psi : (M, L) \to (M', L')$ is a symplectic diffeomorphism, then we can associate to it an isomorphism $\psi_* := (\psi^{-1})^* : H^*(L, \Lambda_{0,nov}) \to H^*(L', \Lambda_{0,nov})$*

of filtered A_∞-algebras whose homotopy class depends only on the isotopy class of symplectic diffeomorphism ψ.

The Poincaré dual $PD[L] \in H^0(L, \Lambda_{0,nov})$ of the fundamental class $[L]$ is the unit of our filtered A_∞-algebra.

They first construct a filtered A_∞-algebra $(C(L, \Lambda_{0,nov}), m)$ which is homotopy unital, and use the following theorem of the canonical model construction, to obtain a filtered A_∞-algebra structure on homology $H^*(L, \Lambda_{0,nov})$, which is unital.

Theorem 6.2. *(See [14, Theorem 23.2].) Any gapped filtered A_∞-algebra (C, m) is homotopy equivalent to a gapped filtered A_∞-algebra (C', m') with $\overline{m}_1 = 0$. The homotopy equivalence can be taken as a gapped filtered A_∞-homomorphism. If (C, m) is homotopy unital, then its canonical model is unital.*

Recall that a filtered A_∞-algebra is called canonical if $\overline{m}_1 = 0$. Note that the canonical model still may have non-trivial $m_0 \in \Lambda_{0,nov}^+$ and $m_{1,\beta}$ for $\beta \neq 0$.

Now, we apply the homology theories discussed so far.

Definition 6.1. Let $(C(L), m)$ be a gapped filtered A_∞-algebra of a Lagrangian submanifold L. The Hochschild (resp. cyclic) homology of $(C(L), m)$ is called *Hochschild* (resp. *cyclic*) *Floer homology* of L and denoted as

$$HH_\bullet\big(C(L), m\big) =: HH_\bullet(L) \quad \big(\text{resp. } HC_\bullet\big(C(L), m\big) =: HC_\bullet(L)\big).$$

Let $(C(L), l)$ be the induced L_∞-algebra from $(C(L), m)$. The Chevalley–Eilenberg homology of $(C(L), l)$ with coefficient in $(C(L), l)$ (resp. Λ_{nov}) is called *Chevalley–Eilenberg* (resp. *cyclic Chevalley–Eilenberg*) *Floer homology* of L and denoted as

$$H_\bullet^{CE}\big(\big(C(L), l\big), \big(C(L), l\big)\big) =: H_\bullet^{CE}(L, L) \quad \big(\text{resp. } H_\bullet^{CE}\big(\big(C(L), l\big), \Lambda_{nov}\big) =: HC_\bullet^{CE}(L)\big).$$

As mentioned in the introduction, the main motivation to study these homology theories is that they provide well-defined homology theories even when the original A_∞-structure is obstructed, and they are invariant under various choices involved.

Corollary 6.3. *Hochschild, cyclic and (cyclic) Chevalley–Eilenberg Floer homologies are well-defined up to isomorphism depending only on the homotopy class of the A_∞-algebra of Lagrangian submanifold.*

Proof. Theorem A of [14], with Propositions 3.4, 4.3 proves the corollary. □

To study the Lagrangian intersection theory, the case of a pair of Lagrangian submanifolds (L_1, L_0) is considered.

Theorem 6.4. *(See [14, Theorem 12.72].) Let L_1, L_0 be a relatively spin pair of Lagrangian submanifolds, which are of clean intersection. Then we have*

$$\big(C(L_1, L_0), n\big)$$

which has the structure of filtered A_∞-bimodule over the pair

$$\big(\big(C(L_1, \Lambda_{0,nov}), m_*\big), \big(C(L_0, \Lambda_{0,nov}), m_*\big)\big).$$

Now, we restrict to the case when the Lagrangian submanifold L_1 is obtained as a Hamiltonian isotopy of $L_0 = L$ (namely, $L_1 = \phi_1(L)$ where ϕ_s ($s \in [0,1]$) is a Hamiltonian isotopy with $\phi_0 = id$). By [14, Theorem 19.1], we have a homotopy equivalence

$$f : \big(C(L, \Lambda_{0,nov}), m\big) \to \big(C\big(\phi_1(L), \Lambda_{0,nov}\big), m\big).$$

By using f, we can pull back the A_∞-bimodule $(C(\phi_1(L), L), n)$ to be an A_∞-bimodule $(C(\phi_1(L), L), (f, id)^* n)$ over a pair $((C(L, \Lambda_{0,nov}), m), (C(L, \Lambda_{0,nov}), m))$ as explained in Section 2.4.

Theorem 6.5. *(See [14, Theorem 12.75].) Let us assume $L = L_1 = L_0$. We also assume J_t is independent of t. Then the A_∞-bimodule structure on $C(L_1, L_0)$ can be taken as the same as the A_∞-algebra structure on $C(L, \Lambda_{0,nov})$.*

The following theorem of [14] proves the invariance of Floer cohomology.

Theorem 6.6. *(See [14, Theorem 22.14].) There exists an ϵ-weakly filtered A_∞-bimodule homomorphism $\Phi : (C(\phi_1(L), L; \Lambda_{nov}), n) \to (C(L, L; \Lambda_{nov}), n)$ over (f, id), which is a homotopy equivalence. Here ϵ is any number greater than the Hofer length of the Hamiltonian isotopy $\{\phi_s\}_s$.*

Consider the A_∞-algebra $A = (C(L; \Lambda_{nov}), m)$, which also can be regarded as A_∞-bimodule $(C(L, \Lambda_{nov}), n)$ over (A, A). Denote by M the A_∞-bimodule

$$\big(C\big(\phi_1(L), L; \Lambda_{nov}\big), (f, id)^* n\big) \quad \text{over } (A, A).$$

Then, from the above theorem, together with the pull-back construction, we obtain the weakly filtered homotopy equivalence $\Phi : M \to (C(L, \Lambda_{nov}), n)$ over (id, id) between the two A_∞-bimodules over (A, A).

Let us denote also by $\widetilde{A}$ the induced L_∞-algebra from the A_∞-algebra A, and let $\widetilde{M}$ be the induced L_∞-bimodule over $\widetilde{A}$ obtained from the A_∞-bimodule $(M, (f, id)^* n)$ over A.

We emphasize that the following two theorems hold even for obstructed Lagrangian submanifolds.

Corollary 6.7. *We have isomorphisms of Hochschild and Chevalley–Eilenberg homology:*

$$H_\bullet(A, M) \cong H_\bullet(A, A) = HH_\bullet(L),$$
$$H_\bullet^{CE}(\widetilde{A}, \widetilde{M}) \cong H_\bullet^{CE}(\widetilde{A}, \widetilde{A}) = H_\bullet^{CE}(L, L).$$

Proof. This follows from Theorem 22.14 of [14] and Proposition 4.3. □

Corollary 6.8. *If a Lagrangian submanifold L is displaceable via Hamiltonian isotopy ϕ^1 (i.e. $L \cap \phi^1(L) = \emptyset$), then, its Hochschild Floer homology and Chevalley–Eilenberg Floer homology of L vanish.*

Proof. This directly follows from the above corollary as the module M or $\widetilde{M}$ would be void in such a case. □

This proves Theorem 1.3 stated in the introduction.

7. Unobstructedness and Hochschild homology

In this section, we discuss bounding cochains and its relation to Hochschild homology of an A_∞-algebra. We briefly recall the definition of unobstructedness. Consider a filtered A_∞-algebra $A = (C, m)$ with $\widehat{d}(1) = m_0(1) \neq 0$. Then, we have $m_1^2 \neq 0$ in general. Suppose there exists an element $b \in C^1$ with $b \in F^{\lambda_0}C$ for some $\lambda_0 > 0$ which satisfies the following equation:

$$\widehat{d}e^b = \widehat{d}(1 + b + b \otimes b + b \otimes b \otimes b + \cdots) = 0.$$

If such an element exists, the A_∞-algebra A is called *unobstructed*, and b is called a bounding cochain or Maurer–Cartan elements.

With any such b, one can deform the A_∞-algebra (C, m) into another A_∞-algebra $A^b = (C, m^b)$ by defining the new A_∞-structure as

$$m_k^b(x_1 \otimes x_2 \otimes \cdots \otimes x_k) := m\big(e^b \otimes x_1 \otimes e^b \otimes \cdots \otimes e^b \otimes x_k \otimes e^b\big),$$

for $x_1 \otimes \cdots \otimes x_k \in B_k C$. Here

$$m\big(e^b \otimes x_1 \otimes e^b \otimes \cdots \otimes e^b \otimes x_k \otimes e^b\big) = \sum_* m_{k+*}(b, \ldots, b, x_1, b, \ldots, b, x_k, b, \ldots, b).$$

Note that if b is a Maurer–Cartan element, we have

$$\begin{aligned}\widehat{d}\big(e^b \otimes x_1 \otimes e^b\big) &= \widehat{d}\big(e^b\big) \otimes x_1 \otimes e^b + e^b \otimes m\big(e^b \otimes x_1 \otimes e^b\big) \otimes e^b + e^b \otimes x_1 \otimes \widehat{d}\big(e^b\big) \\ &= e^b m_1^b(x_1) e^b.\end{aligned}$$

This implies that

$$0 = \widehat{d} \circ \widehat{d}\big(e^b \otimes x_1 \otimes e^b\big) = \widehat{d}\big(e^b \otimes m_1^b(x_1) \otimes e^b\big) = e^b \otimes \big(\big(m_1^b\big)^2 x_1\big) \otimes e^b.$$

Hence, m_1^b defines a deformed chain complex whose homology in the Lagrangian case is called Lagrangian Floer homology. See [14] for more details.

In many known examples, the A_∞-algebra of Lagrangian submanifolds are in fact weakly obstructed. If there exists an element $b \in C^1$ with $b \in F^{\lambda_0}C$ for some $\lambda_0 > 0$ satisfying

$$\widehat{d}e^b = CI$$

for some $C \in \Lambda_{0,nov}^+$, then b is called weak bounding cochains, or weak Maurer–Cartan elements.

Proposition 7.1. *Let $A = (C, m)$ be unobstructed or weakly obstructed A_∞-algebra. For any (weak) bounding cochain b of A, consider a deformed A_∞-algebra $A^b = (C, m^b)$. Then, Hochschild homology of A^b is independent of b and for each b we have*

$$HH_\bullet(A) \cong HH_\bullet\big(A^b\big).$$

Proof. To prove this, we only need to show that (A, m^b) and (A, m) are homotopy equivalent as in [14, Lemma 5.2.12]. Such homotopy equivalence $i^b : (C, m^b) \to (C, m)$ can be given by defining

$$i_0^b(1) = b, \qquad i_1^b = id, \qquad i_{\geqslant 2}^b = 0. \qquad \square$$

Remark 7.1. This proposition does *not* assert that any deformation of an A_∞-algebra has isomorphic Hochschild homology. The deformations that are realized by bounding cochains are special and not every deformation is realized by that of bounding cochain (see for example the form of the Hochschild cycle in the following proposition). For example, a Lagrangian isotopy would provide a deformation of A_∞-algebra of Lagrangian submanifolds, which may not be realized by bounding cochains. For example, the Clifford torus in $\mathbb{C}P^n$ can be Lagrangian isotoped to a nearby torus fiber, but all of them except the Clifford torus are displaceable, and hence has vanishing Hochschild homology. The Hochschild homology of the Clifford torus is non-trivial (see [17]).

Proposition 7.2. *Let b be a (resp. weak) Maurer–Cartan element of a unital A_∞-algebra A. Then, the following element γ_b gives a (resp. reduced) Hochschild homology cycle of A:*

$$\gamma_b = I \otimes e^b.$$

Proof. As b is a Maurer–Cartan element, we have $m(e^b) = 0$. Then, consider γ_b defined as above, and it is easy to check that $d^{Hoch}(\gamma_b) = 0$. We have

$$\begin{aligned} d^{Hoch}(\gamma_b) &= m\big(I \otimes e^b\big) \otimes e^b - I \otimes e^b \otimes m\big(e^b\big) \otimes e^b + m\big(e^b \otimes I\big) \otimes e^b \\ &= m_2(I, b) + m_2(b, I) = b + (-1)^{|b|} b = 0. \qquad \square \end{aligned}$$

We remark that the correspondence does not guarantee a non-vanishing Hochschild homology class. The reason is that when the Lagrangian submanifold is unobstructed and displaceable, then its Hochschild homology should vanish, due to Corollary 6.8.

Lemma 7.3. *If two bounding cochains in Proposition* 7.2 *are gauge equivalent, then the induced Hochschild homology cycles are homologous.*

Proof. Let $\mathfrak{A}$ be the unital model of $[0, 1] \times A$. By definition, two bounding cochains b_0 and b_1 are gauge equivalent, if there exists a bounding cochain $\mathbf{b}$ of $\mathfrak{A}$ such that $Eval_s(\mathbf{b}) = b_s$ for $s = 0$ and 1. As $\mathfrak{A}$ is unital A_∞-algebra, let $\mathbf{I}$ be the unit of $\mathfrak{A}$. Then, $\mathbf{I} \otimes e^{\mathbf{b}}$ defines a Hochschild cycle of $HH_\bullet(\mathfrak{A})$. Also, note that $Eval_s$ induces a map between Hochschild cycles and in fact as $(Eval_s)_k = 0$ for $k \neq 0$, we have

$$I \otimes e^{b_s} = Eval_s\big(\mathbf{I} \otimes e^{\mathbf{b}}\big).$$

But as $Eval_s$ and $Incl$ induce an isomorphism of Hochschild homology, we may proceed as in the proof of Proposition 4.3 to prove that $I \otimes e^{b_s}$ has the same Hochschild homology for $s = 0$ and 1. □

We also remark that with a suitable Hochschild homology class (or in general a negative cyclic homology class), we can find explicit homotopy cyclic inner product structure on the A_∞-algebra which will be explained in an upcoming joint work with Sangwook Lee (see also [5]).

Now, we show that after dualization (see Section 9), unobstructedness corresponds to the notion of an augmentation (see for example [2,9] for more details on augmentation). Here an augmentation of a differential graded algebra (B, d) is an algebra homomorphism $\epsilon : B \to \boldsymbol{k}$ to its coefficient ring $\boldsymbol{k}$ such that $\epsilon \circ d = 0$. The correspondence follows easily from the formalism of [14].

Lemma 7.4. *Let (A, m) be a filtered A_∞-algebra over the Novikov field Λ. Suppose (A, m) is unobstructed. Then, the differential graded algebra $((\widehat{B}A)^*, \widehat{d}^*)$ has an augmentation.*

Proof. Consider the A_∞-automorphsim i^b defined above, and also an induced map $\widehat{i}^b : \widehat{B}A \to \widehat{B}A$. Then, the corresponding augmentation $\epsilon : (\widehat{B}A)^* \to \Lambda$ is defined as a composition of the algebra map $(\widehat{i}^b)^* : (\widehat{B}A)^* \to (\widehat{B}A)^*$ with the projection $\pi_0 : (\widehat{B}A)^* \to \Lambda$ to its component of length zero:

$$\epsilon = \pi_0 \circ \widehat{i}^b.$$

Hence it remains to show that $\epsilon \circ \widehat{d}^* = 0$. Given $f \in (\widehat{B}A)^*$, we have

$$\epsilon \circ \widehat{d}^*(f) = \pi_0 \circ \widehat{i}^b \circ \widehat{d}^*(f) = f\big(\widehat{d}\big(\widehat{i}^b(1)\big)\big) = f\big(\widehat{d}\big(e^b\big)\big) = 0. \qquad \square$$

8. Non-trivial element in cyclic Floer homology

In this section, we find a condition of an obstructed case which has non-trivial cyclic Floer homology. Let L be a Lagrangian submanifold which only admits non-positive Maslov index pseudo-holomorphic discs. Namely, we assume that $\mu(\beta) \leqslant 0$ for any homotopy class β which is realized by J-holomorphic discs. Consider the unital A_∞-algebra A on $(H_*(L, \Lambda_{nov}), m)$, which is given by Theorem A of [14].

We assume that A is obstructed. In an unobstructed case, the same result holds true with much easier proof using the last part of the proof given here, and in this case $PD[L]$ gives a non-trivial element of cyclic Floer homology. Hence we assume that A is obstructed. Now we find a non-trivial element in $HC_\bullet(L)$. Denote by $m_0 = m_0(1) \neq 0$ and also recall that $PD[L]$ defines a unit on this gapped filtered A_∞-algebra. To simplify expression we will write L instead of $PD[L]$.

Note that L is not a cycle in the bar complex as we have

$$\widehat{d}(L) = m_0 \otimes L - L \otimes m_0 \neq 0.$$

Our idea is to consider the following additional terms to cancel these m_0 terms successively. Recall the cyclic symmetrization operation N from (5.1) and define

$$\alpha_{2k+1} = N_{2k+1}(\underbrace{L \otimes m_0 \otimes L \otimes m_0 \otimes \cdots \otimes m_0 \otimes L}_{2k+1}).$$

We let $\alpha_1 = L$ and consider the sum

$$\alpha = \sum_{k=0}^{\infty} (-1)^k \alpha_{2k+1} \in \widehat{B}^{cyc} H(L, \Lambda_{0,nov}).$$

Proposition 8.1. *With the above assumptions, the element α defines a non-trivial homology class in $HC_{(-1)}(L)$.*

Proof. Note that in the expression of cyclic permutation of α, any two of m_0 are always separated by L. Because L is a unit of the A_∞-algebra, the only non-trivial operations of $\widehat{d}$ on α are $\widehat{m}_0$, $\widehat{m}_1$ and $\widehat{m}_2$. Since $m_1(L) = m_1(m_0) = 0$, we have $\widehat{m}_1 = 0$. Therefore, it suffices to prove the following lemma to prove the proposition.

Lemma 8.2. *We have*

$$\widehat{m}_0(\alpha_{2k-1}) = \widehat{m}_2(\alpha_{2k+1}).$$

Proof. We will compute both sides and show that they are indeed equal. We first point out that both m_0 and L have shifted degree one, hence when they pass across each other the negative sign will appear. We also have from (2.8) that

$$m_2(m_0, L) = m_2(L, m_0) = m_0. \tag{8.1}$$

The left-hand side can be computed by the following elementary lemma, whose proof is left for the reader.

Lemma 8.3. *Suppose a_i for $i = 1, \ldots, 2k+1$ are elements of degree one. Then we have*

$$\widehat{m}_0\big(N_{2k+1}(a_1 \otimes a_2 \otimes \cdots \otimes a_{2k+1})\big) = N_{2k+2}\big(\widehat{m}_0(a_1 \otimes a_2 \otimes \cdots \otimes a_{2k-1}) \otimes a_{2k+1}\big).$$

Now, by using the lemma, we can compute

$$\begin{aligned}
\widehat{m}_0(\alpha_{2k-1}) &= \widehat{m}_0\big(N_{2k-1}(\underbrace{L \otimes m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L}_{2k-1})\big) \\
&= N_{2k}\big(\widehat{m}_0(L \otimes m_0 \otimes L \otimes \cdots \otimes m_0) \otimes L\big) \\
&= N_{2k}(\underbrace{m_0 \otimes L \otimes m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L}_{2k}) \\
&= k(m_0 \otimes L \otimes m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L) - k(L \otimes m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L \otimes m_0).
\end{aligned}$$

The second line follows from the previous lemma, and the third line follows from the cancellation (the terms with $\cdots m_0 \otimes m_0 \cdots$ occur twice with the opposite signs).

Now we compute $\widehat{m}_2(\alpha_{2k+1})$. Note that α_{2k+1} may be divided into the following 5 types from the cyclic permutations:

$$\begin{aligned}\alpha_{2k+1} &= N_{2k+1}(\underbrace{L \otimes m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L}_{2k+1}) \\ &= L \otimes m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L \\ &\quad + L \otimes L \otimes m_0 \otimes \cdots \otimes L \otimes m_0 \\ &\quad + m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L \otimes L \\ &\quad + m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L \otimes L \otimes m_0 \otimes \cdots \otimes m_0 \otimes L + \cdots \\ &\quad + L \otimes m_0 \otimes \cdots \otimes m_0 \otimes L \otimes L \otimes m_0 \otimes \cdots \otimes L \otimes m_0 + \cdots.\end{aligned}$$

Note that the last two types have $(k-1)$ such elements each. For each type, one can easily compute using (8.1)

$$\widehat{m}_2(L \otimes m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L) = m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L - L \otimes m_0 \otimes L \otimes \cdots \otimes m_0,$$

$$\widehat{m}_2(L \otimes L \otimes m_0 \otimes \cdots \otimes L \otimes m_0) = 0,$$

$$\widehat{m}_2(m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L \otimes L) = 0,$$

$$\widehat{m}_2(m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L \otimes L \otimes m_0 \otimes \cdots \otimes m_0 \otimes L) = m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L,$$

$$\widehat{m}_2(L \otimes m_0 \otimes \cdots \otimes m_0 \otimes L \otimes L \otimes m_0 \otimes \cdots \otimes L \otimes m_0) = -L \otimes m_0 \otimes L \otimes \cdots \otimes m_0.$$

Hence we have

$$\widehat{m}_2(\alpha_{2k+1}) = k(m_0 \otimes L \otimes m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L) - k(L \otimes m_0 \otimes L \otimes \cdots \otimes m_0 \otimes L \otimes m_0).$$

Hence this proves Lemma 8.2. □

So far we have proved that $\widehat{d}(\alpha) = 0$. To prove Proposition 8.1, we need to prove that α is a non-trivial element of the cyclic Floer homology. We will need the assumption that Maslov index is non-positive for J-holomorphic discs for this purpose.

Recall that we have

$$m_k = \sum_{\beta \in G} T^{\lambda(\beta)} e^{\mu(\beta)/2} m_{k,\beta}.$$

Here $m_{k,\beta} : (H^*(L)[1])^{\otimes k} \to H^*(L)[1]$ has (after degree shift) degree $1 - \mu(\beta)$. And before degree shift, $m_{k,\beta}$ has degree $(2 - \mu(\beta) - k)$.

Note that the available degrees of elements in $H^*(L)[1]$ are from (-1) to $n-1$, and the only degree (-1) element is L. Also the shifted degree of $m_{k,\beta}(x_1, \ldots, x_k)$ is

$$|x_1|' + |x_2|' + \cdots + |x_k|' + 1 - \mu(\beta).$$

Hence, if $x_i \neq L$ for all i, then $|x_i|' \geqslant 0$ hence the $m_{k,\beta}(x_1, \ldots, x_k)$ has degree (before shift) $\geqslant 2 - \mu(\beta) > 0$. Hence they cannot produce L as its image.

But as it is unital, if one of $x_i = L$, then most of the m_k operations vanish (see (2.8)) and the only non-trivial operation which can have L as its image is $m_2(L, L)$. But $L \otimes L$ is not an element of $\widehat{B}^{cyc}(H^*(L, \Lambda_{nov}))$, since

$$N_2(L \otimes L) = L \otimes L - L \otimes L = 0.$$

Hence, this proves that α is a non-trivial homology element in cyclic Floer homology, as the leading term of α is not in the image of $\widehat{d}$. This proves the proposition. □

We remark that similar approach in the cyclic Chevalley–Eilenberg complex does not work. For example, one may check that the symmetric sum of the expression $L \otimes m \otimes L$ vanishes due to the cancellation of pairs occurring in the permutation of two L's.

The element α can be also seen as a cycle of the bicomplex given in Definition 5.3. To see this, note that α has degree (-1) and satisfies $(1-t)\alpha = 0$. One should put α in the augmented bicomplex of the one given in Definition 5.3. Namely, consider α as an element in $C_{-1,-1}(A, A)$. As $(1-t)\alpha = 0$, we can find α'_0 with $N(\alpha') = \alpha$. Also as $b'(\alpha) = 0$, from the commutative diagram of the bicomplex, we have $N(b(\alpha'_0)) = 0$, hence one can find α'_1 with $(1-t)\alpha'_1 = b(\alpha'_0)$. One can continue in a similar way to obtain a cycle in cyclic bicomplex.

The original motivation for our interest in this non-trivial element was to prove the non-displaceability of Lagrangian submanifolds with Maslov class zero. To prove such a result, one may prove the non-vanishing of the Hochschild homology of the A_∞-algebra of such a Lagrangian submanifold. Unfortunately, we do not know how to prove such a non-vanishing property of Hochschild homology of an A_∞-algebra using α. We remark that the Connes exact sequence in Lemma 5.3 does not imply the desired non-vanishing property.

9. Dualization

As A_∞-algebras (resp. L_∞-algebras) are given by coalgebras with codifferentials, the suitable dualization provides non-commutative differential graded algebras (resp. commutative DGA) or a formal manifold in the language of Kontsevich and Soibelman [22]. This point of view is particularly interesting to study homological algebras of these infinity algebras (see [19]) or homotopy cyclic infinity structures (see [22,5]).

As mentioned in the introduction, in contact geometry, the dual language has been mostly used ([2,9] for example) and it also has a certain advantage as algebras can be easier to deal with than coalgebras. But as we deal with Novikov fields, the dualization process is more complicated.

We explain an appropriate procedure to take a dual of a completed infinite-dimensional space over $\Lambda_{nov}^{(e)}$. We will work with $\Lambda_{nov}^{(e)}$ in this section, as we would like to work with field coefficients (see (2.3)) for dualization.

Let V be a vector space over the field $\Lambda_{nov}^{(e)}$. Here, we assume V have at most countably many generators $\{v_i\}_{i\in\mathbb{N}}$ and

$$V = \bigoplus_i \bigl(\Lambda_{nov}^{(e)}\langle v_i\rangle\bigr).$$

We will consider V as a topological vector space by defining a fundamental system of neighborhoods of V at 0: first define the filtrations $F^{>\lambda}V$ as

$$F^{>\lambda}V = \left\{ \sum_{j=1}^{k} a_j v_{i_j} \;\middle|\; a_i \in \Lambda_{nov}^{(e)},\ \tau(a_i) > \lambda,\ \forall i \right\}.$$

Here τ is the valuation of $\Lambda_{nov}^{(e)}$ which gives the minimal exponent of q defined in (2.1). We regard $F^{>\lambda}V$ for $\lambda = 0, 1, 2, \ldots$ as fundamental system of neighborhoods at 0, and neighborhoods at $v \in V$ then are given by $v + F^{>\lambda}V$.

The completion of V with respect to energy, $\widehat{V}$, has been considered throughout the paper, and it can be also considered as a completion using the Cauchy sequences in V in this topological vector space (see [1] for example). Let $\widehat{F}^{>\lambda}V$ be the induced open set of $\widehat{V}$ from $F^{>\lambda}V$ for each λ.

This topology has been introduced to consider the topological dual space $\widehat{V}^*$ of $\widehat{V}$. We define $\widehat{V}^*$ to be the set of all continuous $\Lambda_{nov}^{(e)}$-linear maps from $\widehat{V}$ to $\Lambda_{nov}^{(e)}$:

$$\widehat{V}^* = Hom_{cont}\big(\widehat{V}, \Lambda_{nov}^{(e)}\big).$$

More explicitly, we can characterize the continuity of $\Lambda_{nov}^{(e)}$-linear maps in the following way. Denote by $v_i^* \in \widehat{V}^*$ a map which is defined as $\Lambda_{nov}^{(e)}$-linear extension of

$$v_i^*(v_i) = 1, \qquad v_i^*(v_j) = 0 \quad \text{for } j \neq i.$$

The map v_i^* is continuous and so is any finite sum of such v_i^*'s.

Lemma 9.1. *For any $\lambda_0 \in \mathbb{R}$, any map given by an infinite sum*

$$v^* = \sum_{j=1}^{\infty} a_j v_{i_j}^*, \quad \text{with } a_i \in \Lambda_{nov}^{(e)},\ \tau(a_i) > \lambda_0,\ \forall i,$$

is always continuous.

Moreover, a map given by an infinite sum whose $\tau(a_i)$'s are not bounded below, is not continuous.

Proof. Note that for any open set $F^{>\lambda}\Lambda_{nov}^{(e)}$ of $\Lambda_{nov}^{(e)}$, we have $v^*(\widehat{F}^{>\lambda-\lambda_0}V) \subset F^{>\lambda}\Lambda_{nov}^{(e)}$. Hence v^* is continuous. For the second assertion, consider $w^* = \sum_{j=1}^{\infty} b_j v_{i_j}^*$ with $\tau(b_j) \to -\infty$ as $j \to \infty$. Then, for a given open set $F^{>\lambda}\Lambda_{nov}^{(e)}$ and for any $v \in \widehat{V}$ and any $\lambda_1 \in \mathbb{R}$, we can find $y \in v + \widehat{F}^{>\lambda_1}(V)$ such that $w^*(y) \notin F^{>\lambda}\Lambda_{nov}^{(e)}$. Such y can be chosen for example as

$$v + \sum_{j=s}^{\infty} v_{i_j} q^{\lambda - \tau(b_j) - 1},$$

where s is any number with $(\lambda - \tau(b_j) - 1 > \lambda_1)$ for all $j > s$. One can find such s as $(\lambda - \tau(b_j) - 1)$ converges to infinity as $j \to \infty$. $\square$

The above lemma explains what are the elements of $\widehat{V}^*$.

10. Chevalley–Eilenberg cohomology

In this section, we consider the dual of (cyclic) Chevalley–Eilenberg homologies, which we call (cyclic) Chevalley–Eilenberg cohomology. Then, we express the cochain complex in a more explicit form and compare with the work of Cornea and Lalonde in [7]. We make computations of (cyclic) Chevalley–Eilenberg cohomology when the A_∞-algebra has non-vanishing primary obstruction cycle, and show the vanishing of cohomology using the natural algebra structure on them.

10.1. Cyclic Chevalley–Eilenberg cohomology

We apply the construction in the previous section to define the dual of the cyclic Chevalley–Eilenberg chain complexes introduced in Definition 5.1. Recall that we have a bar subcomplex $(\widehat{E}C, \widehat{d})$ over $\Lambda_{0,nov}$ coefficient from Section 2. We may change the coefficient of $(\widehat{E}C, \widehat{d})$ to be $\Lambda_{nov}^{(e)}$ and denote it again with the same notation. We assume that C has at most countable generators. We regard $\widehat{E}C$ as a topological vector space as in the previous subsection, and take the topological dual

$$\widehat{E}C^* := Hom_{cont}\big(\widehat{E}C, \Lambda_{nov}^{(e)}\big).$$

One can see that $\widehat{d}^*$ also naturally defines a differential and $(\widehat{E}C^*, \widehat{d}^*)$ forms a chain complex.

Recall that $\Lambda_{nov}^{(e)}$ is a field. By following a standard proof of the universal coefficient theorem (see for example [8]), we have

Lemma 10.1. *There exists a natural map from the homology of $(\widehat{E}C^*, \widehat{d}^*)$ to the topological dual of the homology of $(\widehat{E}C, \widehat{d})$ which is an isomorphism*

$$H_\bullet\big(\widehat{E}C^*, \widehat{d}^*\big) \xrightarrow{\cong} \big(H_\bullet(\widehat{E}C, \widehat{d})\big)^*.$$

The same statement holds for $(\widehat{E}_{\geqslant 1}C, \widehat{d})$ also.

In fact, the difference between $H_\bullet(\widehat{E}C^*, \widehat{d}^*)$ and $H_\bullet(\widehat{E}_{\geqslant 1}C^*, \widehat{d}^*)$ can be easily seen as follows. In $\widehat{E}C^*$, there exists the linear functional $\widehat{E}C \to \Lambda_{nov}^{(e)}$ given by the projection to the length zero component and hence identity on $E_0C = \Lambda_{nov}^{(e)}$. Note that we have a short exact sequence

$$0 \to (\widehat{B}_{\geqslant 1}C, \widehat{d}) \to (\widehat{B}C, \widehat{d}) \to \big(\Lambda_{nov}^{(e)}, 0\big) \to 0.$$

By considering its dual exact sequence and its associated long exact sequence, we have

$$\begin{aligned} 0 &\to H_1\big(\widehat{E}C^*, \widehat{d}^*\big) \to H_1\big(\widehat{E}_{\geqslant 1}C^*, \widehat{d}^*\big) \to \Lambda_{nov}^{(e)} \\ &\to H_0\big(\widehat{E}C^*, \widehat{d}^*\big) \to H_0\big(\widehat{E}_{\geqslant 1}C^*, \widehat{d}^*\big) \to 0. \end{aligned}$$

The generator of $\Lambda_{nov}^{(e)}$ in the middle of the above will correspond to $H_1(\widehat{E}_{\geqslant 1}C^*, \widehat{d}^*)$ if $m_0(1) \neq \widehat{d}(\alpha)$ for any $\alpha \in \widehat{E}_{\geqslant 1}C$. If $m_0(1) = \widehat{d}(\alpha)$, we have a non-trivial element $(1-\alpha) \in H_0(\widehat{E}C^*, \widehat{d}^*)$.

Despite Lemma 10.1, we remark that there is an advantage to consider the cohomology theory in this case as algebras are generally easier to work with than coalgebras and this will be essentially used to prove the vanishing results later. More precisely, from Lemma 2.2, we have

$$\widehat{E}C^* \widehat{\otimes} \widehat{E}C^* \to \big(\widehat{E}C \widehat{\otimes} \widehat{E}C\big)^* \xrightarrow{\Delta^*} \widehat{E}C^*.$$

This provides an algebra structure on $\widehat{E}C^*$ with a unit 1, where the unit is a map $\widehat{E}C \to \Lambda_{nov}^{(e)}$ which is identity on $E_0C = \Lambda_{nov}^{(e)}$ and vanishes elsewhere. For later arguments, it is essential to have a unit of commutative DGA. Hence we will consider $(\widehat{E}C^*, \widehat{d}^*)$ mostly, and call it the extended cyclic Chevalley–Eilenberg cohomology.

Now, we express more explicitly the dual space $\widehat{E}C^*$ with generators. Suppose that the $\Lambda_{nov}^{(e)}$-module C has generators $\{e_i\}_{i\in I}$ where I is at most a countable set. We may also assume that the valuation $\tau(e_i)=0$ and e_i is homogeneous of degree $|e_i|'$. We write the dual $e_i^*=x_i$ and define the degree of x_i as $|x_i|'=-|e_i|'$. We may write

$$[e_{i_1},\dots,e_{i_k}]^*=x_{i_1}x_{i_2}\cdots x_{i_k}$$

where we define the variables x_i's to be graded commutative:

$$x_i\cdot x_j=(-1)^{|x_i|'|x_j|'}x_j\cdot x_i.$$

We call the number of variables x_i's in the monomial to be its length.

Consider the vector space $\mathbb{R}\langle x_i\rangle_{i\in I}$ generated by these variables and consider also the free graded commutative algebra over the vector space $\mathbb{R}\langle x_i\rangle_{i\in I}$ and denote them by $S(\mathbb{R}\langle x_i\rangle)$, in which elements are given by finite sum of monomials of finite length. By Lemma 9.1, we can give the following definition.

Definition 10.1. We define the extended cyclic Chevalley–Eilenberg cochain $\widehat{E}C^*$ alternatively as

$$CE^\bullet(C)=\big(S\big(\mathbb{R}\langle x_i\rangle_{i\in I}\big)\otimes\Lambda_{nov}^{(e)}\big)^\wedge$$

where in the completion $(\ )^\wedge$, we allow infinite sums with the valuations of its coefficients bounded from below. Coboundary operation is given by $\widehat{d}^*$, to define extended Chevalley–Eilenberg cohomology.

We remark that Cornea and Lalonde have announced a cluster homology theory of Lagrangian submanifolds in [7]. They have used the Morse function and gradient flows and allowed several disc components connected by Morse flows. The construction of [14] is based on singular chains rather than Morse functions and gradient flows. Here, the analogy is that one may think of singular chains as unstable manifolds of the given Morse function.

To obtain the actual cluster complex of [7], one should take the topological dual of cyclic Chevalley–Eilenberg complex of the following A_∞-algebra recently constructed by Fukaya, Oh, Ohta and Ono.

Theorem 10.2. *(See [16, Theorem 5.1].) Let L be a relatively spin Lagrangian submanifold in a closed symplectic manifold (M,ω). Then there exists a Morse function f such that the Morse complex $CM^*(f)\otimes\Lambda_{0,nov}$ carries a structure of a filtered A_∞-algebra, which is homotopy equivalent to the filtered A_∞-algebra constructed in* [14].

Recall that the construction of the A_∞-algebra in the above theorem is given by first constructing $A_{n,K}$-algebra for each (n,K) and for $(n,K)\prec(n',K')$, $A_{n,K}$-equivalence between such $A_{n,K}$ and $A_{n',K'}$-algebras. From this, they construct A_∞-algebra in a purely algebraic way, by pulling back higher A_∞-structures.

Hence, the following comparison will only hold up to large (n,K). Now, to construct $A_{n,K}$-algebra, the chains χ_g are constructed inductively for $g\in\mathbb{N}$ so that for any (g_0), there exists $g_1>g_0$ such that they construct $A_{n,K}$-algebra structure on χ_g with the following properties.

Namely, $A_{n,K}$-structure is defined on χ_{g_0} in a geometric way (using Kuranishi perturbation and fiber products), and then they are extended to χ_g algebraically using the sum over tree formula or homological perturbation lemma.

In [16], they consider a specific choice of f constructed in a way compatible with triangulation of L, so that the sum over tree formula may be interpreted as counting gradient flow trees whose vertices represent pseudo-holomorphic discs and whose gradient flows represent gradient flow lines (see their Fig. 6 of [16]). This is exactly as in the cluster complex case (where the only difference is the direction of flows). Hence, by taking the dualization as in the previous subsection, the construction of [16] becomes in fact quite similar to that proposed by Cornea and Lalonde (for large (n, K)). We refer readers to [16] for more details on their construction.

In any case, after taking the dual of [16], we obtain the completed symmetric algebra on generators and obtain differential graded commutative algebra (comm. DGA) as in [7]. Cornea and Lalonde also introduced symmetric fine Floer homology which is defined for a pair, Lagrangian submanifold and its Hamiltonian isotopy image. This corresponds to the Chevalley–Eilenberg cohomology for L_∞-modules which will be explained in the next subsection.

But there is a subtlety regarding the filtrations. Namely, the filtration we use here is different from that of Cornea and Lalonde. Here we recall their filtration of the cluster complex of [7, Eq. (1)]:

$$L^k\big(S\mathbb{Q}\langle Crit(f)[1]\rangle \otimes \Lambda_{nov}\big) = Q\langle x_1 x_2 \cdots x_s e^{\lambda}\colon\ s \geqslant k \text{ or } \omega(\lambda) \geqslant k\rangle. \tag{10.1}$$

Hence infinite sums either have length of each term converging to infinity or energy converging to infinity with the above filtration (10.1). In particular, infinite sum of monomials whose length goes to infinity while energy converging to negative infinity is allowed.

But in our case, due to Lemma 9.1, we do not allow such infinite sums of unbounded negative energy. And as we will see, this will cause different behaviors of resulting homology theories.

Remark 10.2. We have been informed by Cornea that the filtration used here also can be used in the cluster homology theory, and we thank him for his comments. But we do not know whether the filtration used in [7] can be used here to provide an invariant homology theory as we prove the invariance before we take the dualization and then use Lemma 10.1.

We define $\widetilde{\tau}: CE^\bullet(C) \to \mathbb{R}$ as in (2.1), which gives the minimal exponent of q used in the coefficients of an element in $CE^\bullet(C)$. The product structure of $\widehat{E}C^*$ corresponds to the natural product structure on $CE^\bullet(C)$ which may be considered as a usual product of formal series of commuting variables.

As we work on DGA, we can use the clever argument from the work of Cornea and Lalonde:

Proposition 10.3. *(Cf. [7, Proposition 1.3].) Suppose that for some $x \in CE^\bullet(C)$, we have*

$$\widehat{d}^*(x) = 1 + h,$$

for $h \in CE^\bullet(C)$ with $\widetilde{\tau}(h) \geqslant 0$ and h has only terms with positive length. Then the homology of $(CE^\bullet(C), \widehat{d}^*)$ vanishes.

Remark 10.3. The condition $\widetilde{\tau}(h) \geqslant 0$, which is rather restrictive, is not required in [7] due to the different choice of filtration.

Proof of Proposition 10.3. The condition on h guarantees that the following is an element of $CE^\bullet(C)$:

$$h' = \sum_{j=0}^{\infty} (-1)^j h^j.$$

As $\widehat{d}^* \circ \widehat{d}^* = 0$, and $\widehat{d}^*(1) = 0$, we have $\widehat{d}^* h = 0$. As $\widehat{d}^*$ is a derivation of the DGA $CE^\bullet(C)$, we also have $\widehat{d}^* h' = 0$. Hence,

$$\widehat{d}^*\big(x \cdot h'\big) = \big(\widehat{d}^*(x) \cdot h'\big) = (1+h)\left(\sum_{j=0}^{\infty} (-1)^j h^j\right) = 1. \tag{10.2}$$

As 1 is a coboundary, this implies that any $\widehat{d}^*$-cocycle $y \in CE^\bullet(C)$ is a coboundary:

$$y = 1 \cdot y = \widehat{d}^*\big(x \cdot h'\big) \cdot y = \widehat{d}^*\big(x \cdot h' \cdot y\big). \qquad \square$$

In the case that there is no quantum contribution from pseudo-holomorphic discs, one can compute the extended cyclic Chevalley–Eilenberg cohomology easily. First, recall the following theorem:

Theorem 10.4. *(See [14, Theorem X].) In the case that there is no quantum contribution, the A_∞-algebra of Lagrangian submanifold $(H^*(L,\mathbb{R}), m)$ is homotopy equivalent to the de Rham complex of L as an A_∞-algebra.*

Corollary 10.5. *In the case that there is no quantum contribution, the extended cyclic Chevalley–Eilenberg cohomology is isomorphic to*

$$\big(S\big(H_*(L,\mathbb{R})[1]\big) \otimes \Lambda_{nov}^{(e)}\big)^\wedge.$$

Proof. Note that de Rham complex is a differential graded algebra, hence $\overline{m}_k \equiv 0$ for $k \geqslant 3$. And the product $\overline{m}_2$ is graded commutative. Hence $\bar{l}_k \equiv 0$ for $k \geqslant 2$. Using the canonical (minimal) model theorem, one can find a finite-dimensional minimal A_∞-algebra B with an A_∞-homotopy equivalence f from B to the de Rham complex. It is easy to see that the induced L_∞-structure of B is again trivial. Hence the A_∞-algebra of Lagrangian submanifold in this case is homotopy equivalent to another A_∞-algebra structure on the singular homology $H_*(L,\mathbb{R})$ whose induced L_∞-structure is trivial. As the (extended) cyclic Chevalley–Eilenberg cohomology is an invariant of the homotopy class, and all the differentials vanish in the latter case, hence the claim follows. $\square$

Now, we can prove the theorem stated in the introduction.

Theorem 10.6. *Let L be a relatively spin Lagrangian submanifold in a symplectic manifold (M,ω) with non-vanishing primary obstruction cycle. Let A be the A_∞-algebra of L. Then its extended cyclic Chevalley–Eilenberg cohomology vanishes.*

Proof. We will construct an element x which satisfies the assumption of Proposition 10.3.

We briefly recall the definition of a primary obstruction cycle. We label

$$0 = \beta_0, \beta_1, \dots, \beta_k, \dots,$$

the equivalence classes of homotopy classes of pseudo-holomorphic discs with boundary on L, where two homotopy classes are equivalent if they have the same Maslov indices and symplectic energies. Here enumeration is made so that $\omega(\beta_i) \leqslant \omega(\beta_{i+1})$ for a symplectic form ω.

Suppose that $\lambda := \omega(\beta_1) = \cdots = \omega(\beta_j) < \omega(\beta_{j+1})$ for some $j \geqslant 1$. As we consider equivalence classes, we have

$$\mu(\beta_s) \neq \mu(\beta_t) \quad \text{for any } 1 \leqslant s \neq t \leqslant j.$$

As the classes $\beta_1, \dots, \beta_j$ are minimal classes, the boundary image of holomorphic discs in the class β_s, which is $m_{0,\beta_s}(1)$, defines a cycle of $\overline{m}_1$ for each $s = 1, \dots, j$. Primary obstruction cycles are defined as $\mathcal{O}_s = m_{0,\beta_s}(1)$ for each s.

Now we assume that we work on the canonical model A_{can} of A, and the induced L_∞-algebra structure $\widetilde{A}_{can}$ is trivial as in the above corollary. This means that we have $l_{k,\beta_0} = \bar{l}_k \equiv 0$.

The $\overline{m}_1$-cycle $\mathcal{O}_s$ is non-trivial and in the canonical model, we still have $m_{0,\beta_s}(1) = \mathcal{O}_s$. Here we may work on the canonical model as we have proved that extended cyclic Chevalley–Eilenberg cohomology is an invariant of homotopy class of A.

We set x_s to be a dual variable to $\mathcal{O}_s$ in $CE^\bullet(C)$. Then,

$$\begin{aligned}\widehat{d}^* x_s(1) = x_s\big(\widehat{d}(1)\big) &= x_s\big(m_{0,\beta_1} T^\lambda e^{\mu(\beta_1)} + \cdots + m_{0,\beta_1} T^\lambda e^{\mu(\beta_s)}\big) + \text{higher energy terms} \\ &= 1 \cdot T^\lambda \big(e^{\mu(\beta_s)} + \xi\big) + T^\lambda \eta =: a_0 \cdot T^\lambda.\end{aligned}$$

Here as m_{0,β_t} may have non-trivial x_s value for $t \neq s$, hence we write such contribution as ξ, where there cannot be any cancellation as each $\mu(\beta_t)$ is distinct. And by $\eta \in F^{>0}\Lambda_{nov}^{(e)}$, we denote the rest with higher energy. Clearly,

$$a_0 T^\lambda = \widehat{d}^* x_s(1) \neq 0.$$

Note that a_0 is invertible and consider its inverse $1/a_0$. Consider

$$y = \frac{1}{a_0} T^{-\lambda} x_s.$$

Then, we have $\widehat{d}^* y(1) = 1$ by definition. Hence we have

$$\widehat{d}^* y = 1 + h,$$

where h has terms of positive length. Also note that we have $\widetilde{\tau}(h) \geqslant 0$ because λ is the minimal energy with non-trivial L_∞-algebra operation. Hence, y satisfies the assumption of Proposition 10.3 and implies the desired vanishing property. $\square$

We remark that the related Proposition 1.3 in [7] is somewhat different due to a different choice of filtration (10.1). It seems that in such a case it does not recognize the unobstructedness as in the above theorem. Assume in the above proof that we work in the chain level (not in the canonical model) and suppose all the primary obstructions vanish, i.e. there exists a chain b_s with $\overline{m}_1(b_s) = -\mathcal{O}_s$. Let us assume that m_{0,β_s} is a chain which is not zero (i.e. b_s is not zero). Then, consider a dual variable x_s of m_{0,β_s}. Note that m_{0,β_s} is homologically trivial, but as we take dual on the chain level we have a corresponding dual variable. Then we have as before

$$\widehat{d}^* x_s(1) = a_0 T^{\lambda},$$

for a non-trivial a_0 with $\tau(a_0) = 1$. But also

$$\widehat{d}^* x_s(b_s) = x_s\big(m_1(b_s)\big) = x_s\big(\overline{m}_1(b_s)\big) + \text{higher energy terms.}$$

Here we have

$$x_s\big(\overline{m}_1(b_s)\big) = x_s(-\mathcal{O}_s) = -1.$$

If we denote the dual variable of b_s to be x'_s, then we have

$$\widehat{d}^* x_s = a_0 T^{\lambda} - x'_s + h,$$

for some h. Hence, $\widehat{d}^*(T^{-\lambda}x_s)$ will have a component $-T^{-\lambda}x'_s$ which has negative energy.

Recall that in the proof of Proposition 10.3, one takes $\sum_{j=0}^{\infty}(-1)^j(T^{-\lambda}x'_s)^j$ which would have unbounded negative energy. With the filtration (10.1) of [7], such an expression is allowed and it will prove the vanishing of the homology.

But in the case of our paper, such an expression with unbounded energy is not allowed and hence such an argument cannot be used to prove the vanishing of Chevalley–Eilenberg cohomology.

10.2. Chevalley–Eilenberg cohomology

Similarly, we take the topological dual of the Chevalley–Eilenberg chain complexes defined in Section 4.2 for L_∞-modules M over L_∞-algebra $\widetilde{A} = (C, l)$, and call its homology a Chevalley–Eilenberg cohomology $CE^\bullet(\widetilde{A}, M)$. In fact we will only consider the case $M = \widetilde{A}$. By proceeding as in the previous subsection, we obtain

Definition 10.4. We define the Chevalley–Eilenberg cochain $(CE_\bullet(\widetilde{A}, \widetilde{A}))^*$ alternatively as

$$CE^\bullet(C, C) = C \otimes CE^\bullet(C) = C \otimes \big(S\mathbb{R}\langle x_i\rangle_{i\in I} \otimes \Lambda_{nov}^{(e)}\big)^\wedge.$$

Coboundary operation is given by $(d^{CE})^*$, to define the Chevalley–Eilenberg cohomology.

By proceeding as in the standard universal coefficient theorem, one can prove that

Lemma 10.7. *There exists a natural map from the homology of $(CE^\bullet(\widetilde{A}, \widetilde{A}), (d^{CE})^*)$ to the topological dual of the homology of $(CE_\bullet(\widetilde{A}, \widetilde{A}), d^{CE})$ which is an isomorphism.*

Corollary 10.8. *If L is displaceable from itself via Hamiltonian isotopy, its Chevalley–Eilenberg cohomology vanishes.*

Proof. This follows from Corollary 6.8 and the above lemma. □

Now, we can prove the remaining part of Theorem 1.1.

Theorem 10.9. *If a Lagrangian submanifold L has a non-trivial primary obstruction class, then its Chevalley–Eilenberg cohomology vanishes.*

Proof. This proceeds as in [7, Remark 1.11]. Namely, one can see that $CE^{\bullet}(C, C)$ has a differential graded right module structure over a differential graded algebra $CE^{\bullet}(C)$ (which is obtained as a dual of a comodule). Hence when the homology of $CE^{\bullet}(C)$ is trivial, it is easy to show that the homology of $(CE^{\bullet}(C, C), (d^{CE})^{*})$ is also trivial. □

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