# On the Optimal Rates of Convergence for Nonparametric Deconvolution Problem

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# ON THE OPTIMAL RATES OF CONVERGENCE FOR NONPARAMETRIC DECONVOLUTION PROBLEM

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## Abstract

Suppose we have n observations from $Y = X + \varepsilon$, where $\varepsilon$ is measurement error with known distribution, and the density f of X is unknown with the non-parametric constraint $f \in \{f(x): |f^{(m)}(x) - f^{(m)}(x+\delta)| \le B\delta^{\alpha},\ |f| \le C\}$. Suppose the functional of interest is $T(f) = f^{(l)}(x_0)$ ; for $l = 0$, T(f) is the density function at a point. Then the optimal rate of estimating $T(f)$ is $O\left((\log n)^{-\frac{m+\alpha-l}{\beta}}\right)$ if the tail of the characteristic function of $\varepsilon$ is of order $|t|^{\beta_0}\exp(-|t|^{\beta}/\gamma)$ *as* $t \to \infty$ , and is $O\left(n^{-\frac{m+\alpha-l}{2(m+\alpha)+2\beta+1}}\right)$ if the tail of the characteristic function is of order $O(t^{-\beta})$. Moreover, the optimal rate of convergence of a distribution function is also found, which is no longer "root-n consistency" as in the ordinary case. In addition, the optimal rate of estimating the functional $T(f) = \sum_1^l a_j f^{(j)}(x_0)$ is also addressed.

*KEY WORDS:* Deconvolution; nonparametric density estimation; Estimation of Distribution; Optimal rates of convergence; Minimax risk; Kernel estimation; Fourier Transformation; Smoothness of error.

**Abbreviated title**: Optimal rates of Deconvolution

## 1. Introduction

Suppose we have observations $Y_1, \cdots, Y_n$ having the same distribution as that of $Y$ available to estimate the unknown density f(x) of a random variable X, where

$$Y = X + \varepsilon \tag{1.1}$$

with measurement error $\varepsilon$ of known distribution. Assume furthermore that the random variables $X$ and $\varepsilon$ are independent. We will discuss herein how well the unknown density and its distribution function can be estimated nonparametrically under certain smoothness conditions.

The usual smoothness condition imposed on f is the set with kth bounded derivatives. More generally, we shall assume $f^{(m)}$ satisfies Lipschitz condition of order $\alpha$, i.e. f is in the set

$$C_{m,\alpha,B} = \{ f(x) \colon |f^{(m)}(x) - f^{(m)}(x+\delta)| \le B\delta^{\alpha},\ |f| \le C \} \tag{1.2}$$

where $B$, $C$, and $0 \le \alpha < 1$ are constants. Then the optimal rate of convergence will depends on $m$, $\alpha$ through $m + \alpha$.

Such a model of measurements being contaminated with error exists in many different fields and has been widely studied. For example, the observation Y is the survival time of an animal, $X$ is the time that a tumor occurs, and $\varepsilon$ is the time from tumor occurring to death. Some other examples, described in Liu and Taylor (1987), Carroll and Hall (1987), are Crump and Seinfeld (1982), Mendelsohn and Rice (1982), and Medgyessy (1977).

The applications for such a model in theoretical settings are mentioned in Carroll and Hall (1987). For example, our results can be applied to the estimation of the prior of nonparametric empirical Bayes problem (Berger, 1980). Also, the theory can be applied to nonparametric regression estimation for estimation of regression function and to the generalized linear model ( Stefanski and Carroll (1987) ), and other models such as $Y = X\varepsilon$.

Carroll and Hall (1987) give the optimal rates of density estimation at a point when the error is normal, and give the result for gamma distributions omitting the proof. As far the author can determine from their proof, we don't not know exactly why the lower rates should depend on the tail of $\phi_\varepsilon$. In our heuristic argument in section 3, the reason for this dependence are clearly stated. We will generalize the result to more general setting by assuming that the tail of characteristic function is either of form

$$|\phi_\varepsilon(t)| = O(|t|^{\beta_0}\exp(-|t|^{\beta}/\gamma)) \quad (as\ t \to \infty) \tag{1.3}$$

or

$$\phi_\varepsilon(t) = O(t^{-\beta}) \quad (as\ t \to \infty) \tag{1.4}$$

Moreover, we will give the rate of convergence of estimating distribution , which is not $n^{1/2}$ convergence any more.

We will address how the difficulty of deconvolution depends heavily on both imposed smoothness condition on density f and on the smoothness condition of distribution of error. By smoothness of the distribution of error, we mean the order of characteristic function $\phi_\varepsilon(t)$ of $\varepsilon$ as $t \to \infty$. The difficulty can be explained intuitively: on the basis of finite observations, the tail of $\phi_\varepsilon(t)$ makes it difficult to identify the tail of characteristic function of X, and hence the distribution of X. It can also be explained by the fact that the empirical characteristic function of Y does not go to 0 ( as $t \to \infty$ ), and hence to use inversion formula heuristically, we have to truncate the integration somewhere. The smoother of the distribution of $\varepsilon$, the more we have to truncate, which increases the variance and bias of the estimator. Thus, we have to pay some extra cost for estimating the density of f in deconvolution form. Therefore, the optimal rate depends on how much we pay for the extra cost.

To find the optimal rate of convergence, we usually have to find a lower bound and an upper bound. Suppose we want to estimate a functional T of f. In principle, we can apply the modulus lower bound $b(\frac{c}{\sqrt{n}})$ invented by Donoho and Liu (1987), and defined by

$$b(\varepsilon) = sup\{\ |T(f_1) - T(f_2)|: f_1, f_2 \in C_{m,\alpha,B}, H(f_{Y1}, f_{Y2}) \le \varepsilon\} \tag{1.5}$$

in current setting, where $H(f_{Y1}, f_{Y2})$ is the Hellinger metric between $f_{Y1}$ and $f_{Y2}$ (see Lecam (1973), (1985)) and $f_{Y1}$ is the density of Y under the convolution of model (1.1). But constructing a lower bound in this way may obscure the essential difficulty inside. Instead of using Hellinger metric, we will use $\chi^2$ metric. To formulate the idea, let

$$\chi^2(f_1, f_2) = \int (f_1 - f_2)^2 f_1^{-1} dx \tag{1.6}$$

be the $\chi^2$ metric between two densities, and

$$b_T(\varepsilon) = sup\{ |T(f_1) - T(f_2)| : f_1, f_2 \in C_{m,\alpha,B}, \chi(f_{Y1}, f_{Y2}) \le \varepsilon \} \tag{1.7}$$

Then, the lower bound $b_T(\frac{1}{\sqrt{n}})$ will be the attainable lower bound, and the rate can be computed explicitly. In other words, when we cannot distinguish between two densities based on n observations of Y, then the change of functional is a lower bound.

To find an achievable upper bound, we will use Parzen's (1962) type of density estimator. A similar construction is used by Stefanski and Carroll (1987) and Liu and Taylor (1987). For a nice kernel function K(x), let $\phi_K(t)$ be its Fourier transform with $\phi_K(0) = 1$. Then the kernel density estimator is defined by

$$\hat{f}_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx)\, \phi_K(th) \frac{\hat{\phi}_n(t)}{\phi_\varepsilon(t)}\, dt \tag{1.8}$$

for suitable choice of bandwidth h and kernel function. Note (1.8) can also be written in kernel type of density (see (2.3)). Moreover, we will use (1.8) to construct the estimators of the derivatives of the unknown density and its distribution.

In section 2, we will exhibit the rates of kernel type of density estimators , which are the optimal in terms of rates of convergence. In section 3, we will construct the lower bounds and give heuristic argument of the results which allow us to say the optimal rate of convergence. In section 4, we will give some brief discussion and comments. In section 5, we will give the proofs of the results.

## 2. Kernel density estimators

Let's start with the kernel density estimator (1.8) with empirical characteristic function defined by

$$\hat{\phi}_n(t) = \frac{1}{n}\sum_{1}^{n} \exp(itY_n) \tag{2.1}$$

and a given known function $\phi_K(t)$. Let K(x) be the Fourier inversion of $\phi_K(t)$ defined by

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} exp(-itx)\phi_K(t)dt, \tag{2.2}$$

a smooth kernel function. Then (1.8) can be rewritten as a kernel type of estimator:

$$\hat{f}_n(x_0) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{h}\, g_h\left(\frac{x_0 - Y_j}{h}\right) \tag{2.3}$$

where

$$g_h(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-itx)\frac{\phi_K(t)}{\phi_\varepsilon(t/h)}\, dt \tag{2.4}$$

Define the maximum mean square error (MMSE) of an estimator $\hat{f}$ to be

$$MMSE(\hat{f}) = \sup_{f \in C_{m,\alpha,B}} E(\hat{f}(x_0) - f(x_0))^2 \tag{2.5}$$

To compute (2.5), first compute its bias and see what kind of kernel or equivalently its Fourier transformation $\phi_K(t)$, we should use.

$$E\hat{f}_n(x_0) - f(x_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-itx_0)\,\phi_K(th)\phi_X(t)\, dt - f(x_0)$$

$$= f(x) * \frac{1}{h}K\left(\frac{x}{h}\right)\Big|_{x_0} - f(x_0) \tag{2.6}$$

The last expression does not depend on the error distribution. Thus, the minimum conditions we have to impose are that the Kernel function K satisfies the conditions of those without convolution. We will state them on its Fourier domain.

The conditions we are going to impose on K and on $\phi_\varepsilon$ are

A1) $\phi_\varepsilon(t) \neq 0$, for *any* $t$.

A2) $\phi_K(t)$ is a symmetric function, having m + 2 bounded integrable derivatives on $(-\infty, +\infty)$.

A3) $\phi_K(t) = 1 + O(|t|^{m+\alpha})$, as $t \to 0$.

Note that A2) and A3) is imposed simply to make kernel $K(\cdot)$ satisfy the condition of "classical" (without convolution) kernel function. Additional conditions will be specified below.

More generally, we can use $\hat{f}_n^{(l)}(x_0)$ to estimate $f^{(l)}(x_0)$, the $l^{th}$ derivative of the unknown density at $x_0$. For the exponential decay of $\phi_\varepsilon$ case (we will call this the supersmooth case), we have the following rates of convergence.

**Theorem 1**: Under the assumptions A1) ~ A3) and

E1) $\phi_K(t) = 0$ for $|t| \geq 1$.

E2) $|\phi_\varepsilon(t)|\ |t|^{-\beta_0}\exp(|t|^\beta/\gamma) > c$ $(as\ t \to \infty)$ with $\beta, \gamma, c > 0$.

Then by choosing the bandwidth $h = (4/\gamma)^{\frac{1}{\beta}}(log\ n)^{-\frac{1}{\beta}}$,

$$\sup_{f \in C_{m,\alpha,B}} E(\hat{f}_n^{(l)}(x_0) - f^{(l)}(x_0))^2 = O((\log n)^{-2(m+\alpha-l)/\beta}) \tag{2.7}$$

**Remark 1**: When $l = 0$, $\hat{f}_n(x_0)$ is the estimator of density function itself, which has rate of convergence of $O((\log n)^{-(m+\alpha)/\beta})$. The constant 1 in condition E1) is not essential. It can be replaced by any positive constant. The reason we impose such a condition is simply to make (1.8) converge, and for easy calculation in the proof.

For the case of geometric decay of $\phi_\varepsilon$ (called the smoothness case), we have the following result.

**Theorem 2**: Under the assumptions A1) ~ A3) and

G1) $\phi_\varepsilon'(t)t^{\beta+1} \to -\beta c$, $\phi_\varepsilon(t)t^\beta \to c$, $(t \to +\infty)$ with $c > 0$.

G2) $\int_{-\infty}^{+\infty} |\phi_K(t)| t^{\beta+l-1} dt < \infty$, and $\int_{-\infty}^{+\infty} |\phi_K'(t)| t^{\beta+l} dt < \infty$

Then by choosing the bandwidth $h = O(n^{-1/[2(m+\alpha)+2\beta+1]})$,

$$\sup_{f \in C_{m,\alpha,B}} E(\hat{f}_n^{(l)}(x_0) - f^{(l)}(x_0))^2 = O(n^{-\frac{2(m+\alpha-l)}{2(m+\alpha)+2\beta+1}}) \tag{2.8}$$

**Remark 2**: If we want to estimate $T(f) = \sum_1^l a_j f^{(j)}(x_0)$ in $C_{m,\alpha,B}$, then the kernel density estimator $T(\hat{f}_n) = \sum_1^l a_j \hat{f}_n^{(j)}(x_0)$ $(a_l \neq 0)$ has the optimal rate of $O((log\ n)^{-(m+\alpha-l)/\beta})$ or $O(n^{-\frac{m+\alpha-l}{2(m+\alpha+\beta)+1}})$, depending on the rate of the tail of $\phi_\varepsilon$. The proof of such a result follows easily from the proofs given in this and the next section under the same assumptions.

Note that the rate of convergence given by (2.7) and (2.8) is the optimal one, which will be shown in the next section.

Now, we consider an estimator of distribution function. Define the estimator of distribution $F(x_0)$ by

$$\hat{F}_n(x_0) = \int_{-M_n}^{x_0} \hat{f}_n(t)\, dt \tag{2.9}$$

where $\hat{f}_n(t)$ is the kernel density estimator given by (2.3).

**Theorem 3**: Under assumptions E1), E2), A1) of Theorem 1, suppose that $\phi_K(t)$ is a symmetric function, having m+3 bounded integrable derivatives on $(-\infty, +\infty)$, and $\phi_K(t) = 1 + O(|t|^{m+1+\alpha})$, as $t \to 0$. Then by choosing the same bandwidth as for Theorem 1 and $M_n = n^{1/3}$, we have:

$$\sup_{f \in C'_{m,\alpha,B}} E_f(\hat{F}_n(x_0) - F(x_0))^2 = O((log\ n)^{-2(m+\alpha+1)/\beta})$$

where $C'_{m,\alpha,B} = \{f \in C_{m,\alpha,B} : F(-n) \leq D(log\ n)^{-(m+2)/\beta}\}$.

**Remark 3:** In the proofs of Theorem 5 & 7, we will see that the rate given in Theorem 3 is the optimal one, because the least favorable pair we choose is in $C'_{m,\alpha,B}$.

## 3. Lower bounds

In this section, we will find lower bounds for estimating densities and distributions. More generally, suppose we want to estimate T(f) from observation (1.1). Then we have the following lower bound of $b_T(\frac{c}{\sqrt{n}})$.

**Theorem 4:** If for some sequence of positive constants $\{a_n : n \geq 1\}$, we have

$$\liminf_{n \to \infty} \inf_{f \in C_{m,\alpha,B}} P_f\{|\hat{T}_n - T(f)| \leq a_n\} = 1$$

then

$$\liminf_{n \to \infty} a_n / b_T(\frac{c}{\sqrt{n}}) \geq 1/2 \tag{3.1}$$

Moreover for any estimator $\hat{T}_n$ of T(f), we have for $\forall$ c,

$$\sup_{f \in C_{m,\alpha,B}} E_f\ (\hat{T}_n - T(f))^2 > Cb_T^2(\frac{c}{\sqrt{n}}) \tag{3.2}$$

for some constants C. In other words, no estimator can estimate better that $b_T(\frac{c}{\sqrt{n}})$.

**Remark 4:** (3.2) is implied by the result of Donoho and Liu (1987) in the current setting. In fact, such a theorem may be familiar to some authors. For the purpose of later use we state here and give a proof.

Now, let's study the lower bound of $b_T(\frac{1}{\sqrt{n}})$. To begin with, suppose the functional of interest is $T(f) = f(x_0)$, density at a point. Let's give a heuristic argument to see why the result should depend on the tail of $\phi_\varepsilon$. Rigorous proof will be given in section 5, which involves mathematical details and more careful constructions.

Let's assume without loss of generality that $x_0 = 0$ by relocating $x_0$ to origin. To calculate the abstract bound $b_T(f)$, take a pair $f_0(x) \in C_{m,\alpha,B}$, $f_1(x) \in C_{m,\alpha,B}$, for which

$$f_1(x) = f_0(x) + c\,\delta^k H(x/\delta) \tag{3.3}$$

where k $= m + \alpha$, $H(0) \neq 0$, $\int_{-\infty}^{+\infty} H(x)\,dx = 0$ and the $m^{th}$ derivative of H(x) satisfies Lipschitz condition of order $\alpha$. Then by suitable choice of the tail of $H(x)$, $f_0(x)$ and constant c, $f_1$ will be a density in $C_{m,\alpha,B}$ for small $\delta$. $\delta$ is chosen such that $\chi^2$-distance

$$\int_{-\infty}^{+\infty} (f_{Y1} - f_{Y2})^2 f_{Y1}^{-1} dx \le \frac{c}{n} \tag{3.4}$$

and the lower bound of density estimation at a point will be half of the change of functional

$$|T(f_0) - T(f_1)|/2 = \frac{c}{2}\ |H(0)|\delta^k = O(\delta^k) \tag{3.5}$$

Thus, we have to find $\delta$ as larger as possible so that (3.4) holds, or equivalently such that

$$\delta^{2k+1} \int_{-\infty}^{+\infty} ( \int_{-\infty}^{+\infty} H(x-y)dF_\varepsilon(\delta y))^2\, g_0^{-1}(\delta x)\, dx \le O(\frac{1}{n}) \tag{3.6}$$

where $F_\varepsilon$ is the distribution function of the random variable $\varepsilon$, $g_0 = f_0 {*} F_\varepsilon$.

Suppose we can prove that as $\delta \to 0$,

$$\int_{-\infty}^{+\infty} [ \int_{-\infty}^{+\infty} H(x-y)dF_\varepsilon(\delta y)]^2\, g_0^{-1}(\varepsilon x)\, dx \tag{3.7}$$

$$\le C \int_{-\infty}^{+\infty} [ \int_{-\infty}^{+\infty} H(x-y)dF_\varepsilon(\delta y)\,]^2\, dx$$

where C is a constant independent of n. Then by Parseval's identity, to make (3.5) hold, we have to choose $\delta$ from

$$\delta^{2k+1} \int_{-\infty}^{+\infty} ( \int_{-\infty}^{+\infty} H(x-y)dF_\varepsilon(\delta y)^2\, dx = O(\frac{1}{n}) \tag{3.8}$$

or equivalently from

$$\delta^{2k+1} \int_{-\infty}^{+\infty} |\phi_H(t)\, \phi_\varepsilon(t/\delta)|^2\, dt \le O(\frac{1}{n}) \tag{3.9}$$

where $\phi_H$ is the Fourier transformation of H. Thus, the result will depend on the tail of $\phi_\varepsilon$ only. It is not hard to choose $\delta$ from (3.9) and consequently to get a desired lower bound.

**Theorem 5**: Suppose that the tail of $\phi_\varepsilon$ satisfies

$$|\phi_\varepsilon(t)|\ |t|^{-\beta_0} \exp(|t|^\beta/\gamma) \le c \quad (as\ t \to \infty)$$

and $P\ \{x + |x|^{\alpha_0} \ge \varepsilon \ge x - |x|^{\alpha_0}\} = O(|x|^{-(a+1-\alpha_0)})$for $0 \le \alpha_0 < 1$ (as $x \to \pm\infty$) for $0 \le \alpha_0 < 1$, $a$>0.5, then no estimator can estimate $T(f) = \hat{f}_n^{(l)}(x_0)$ knowing $f \in C_{m,\alpha,B}$ faster than $O((log\ n)^{-(m+\alpha-l)/\beta})$ in the sense that if

$$\liminf_{n\to\infty} \inf_{f \in C_{m,\alpha,B}} P_f\{|\hat{f}^{(l)}(x_0) - f^{(l)}(x_0)| \le a_n\} = 1 \tag{3.10}$$

then

$$(log\ n)^{(m+\alpha-l)/\beta}\, a_n \to \infty \tag{3.11}$$

Moreover, for any estimator $\hat{T}_n$,

$$\sup_{f \in C_{m,\alpha,B}} E_f\ (\hat{T}_n - T(f))^2 > O((log\ n)^{-2(m+\alpha-l)/\beta}) \tag{3.12}$$

From the result given in Theorem 1, we know that the optimal rate of estimating a density in the supersmooth noise case is only of order $O((log n)^{-(m+\alpha)/\beta})$. Specifically, when the error is distributed as Cauchy, then the optimal rate is $O((\log n)^{-(m+\alpha)})$, and when the error is normal, then the optimal rate is $O(\log n)^{-(m+\alpha)/2})$.

**Theorem 6**: Suppose that the tail of $\phi_\varepsilon$ satisfies the condition G1) of Theorem 2, and $\phi_\varepsilon''(t)t^{(\alpha+2)} \to \alpha(\alpha+1)c\ \ (t \to \infty)$, then no estimator can estimate $T(f) = f^{(l)}(x_0)$, under the constraint that $f \in C_{m,\alpha,B}$, faster than $O(n^{-\frac{m+\alpha-l}{2m+2\alpha+2\beta+1}})$ in the sense of (3.1) and

(3.2).

**Remark 5:** In some cases, $\phi_\varepsilon(t) = \exp(it\varepsilon_0)\phi(t)$, where $\phi(t)$ satisfies the condition of Theorem 6. Then $\phi_\varepsilon$ itself doesn't satisfy the conditions of Theorem 6, but by translation the result still holds. Note that the constant c can be 0 in Theorem 6.

**Remark 6:** For estimating the functional $T(f)$ in Remark 2, the lower bounds are exactly the same as those given in Theorem 5 and 6 using the same constructions.

Thus, we get the optimal rates for the smooth cases and the supersmooth cases. In practice, those conditions are easy to check. The cases of error distributions satisfying Theorem 2 & 6 include gamma distribution, double exponential distribution , etc. And the cases of error distributions satisfying Theorem 1 & 5 are normal, cauchy, mixture normal, and many other distributions. Now, we state some lower bounds for estimating the distribution function.

**Theorem 7:** Under the condition of Theorem 5, then no estimator of estimation the distribution function of X at a point under constraint (1.2) can be faster than $O((\log n)^{-\frac{m+\alpha+1}{\beta}})$. in the sense of (3.10) ~ (3.12).

**Theorem 8:** Under the condition of Theorem 6, then no estimator of estimation the distribution function of X at a point under constraint (1.2) can be faster than $O(n^{-\frac{m+\alpha+1}{2m+2\alpha+2\beta+1}})$ when $\beta \geq 0.5$ and $O(n^{-1/2})$ when $\beta < 0.5$.

**Remark 7:** For estimating $P_F\{ X \in (a,b] \}$, the lower bounds are the same as those in Theorem 7 & 8. However, the lower bound given by Theorem 8 may not be attainable. The attainable one might be $O(n^{-\frac{m+\alpha+1}{2(m+\alpha+\beta+1)}})$.

## 4. Discussion

We hope to decompose the difficulty of deconvolution into two parts: the difficulty of deconvolving a functional T, and modulus function without convolution (does not depend on the tail of characteristic function). The first part tells us how difficult of deconvolution is for a functional, and the second part tells us the difficulty of estimating a functional even though no convolution exists. To formula the idea, let modulus function

$$b(\delta) = \sup_{f_1, f_2 \in C_{m,\alpha,B}} \{ |T(f_1) - T(f_2)| : \int_{-\infty}^{+\infty} (f_1 - f_2)^2 f_1^{-1} \le \delta \}$$

be the difficulty function of estimating T(f) without convolution ( i.e. the lower bound if error $\varepsilon = 0$ ). And one way to define the difficulty of deconvolving a functional is

$$D_T(\delta) = \sup_{(f_1, f_2) \in C_\delta} \{ \int_{-\infty}^{+\infty} (f_{Y1} - f_{Y2})(f_{Y1})^{-1} dx : \int_{-\infty}^{+\infty} (f_1 - f_2)^2 f_1^{-1} dx \le \delta \}$$

where $C = \{(f_1, f_2): f_1, f_2 \in C_{m,\alpha,B},\ |T(f_1) - T(f_2)| \ge b(\delta)/2\}$ and then the lower bound for estimating $T(f)$ is $b(D_T^{-1}(\frac{1}{n}))$. The lower bound suggests that we find a pair of density functions which is the least favorable in the situation without convolution and such that the $\chi^2$ distance is as small as possible. Of course, we hope to find a difficulty function D, which does not depend on T. But, it is impossible simply looking at estimating the mean and density of the normal error case $\varepsilon \sim N(0, 1)$. In this case, $D(\delta) = O(\delta)$ for estimating the mean, while $D(\delta) = O(\delta^{\frac{2(m+\alpha+\beta)+1}{2(m+\alpha)+1}})$ for estimating the density.

When error $\varepsilon$ is uniformly distributed on [0,1], say, the model (1.1) itself is identifiable. Theorem 6 tells us that no estimator can estimate the density of X at a point faster than $O(n^{-\frac{m+\alpha}{2(m+\alpha+b)+1}})$ for any b < 1. However, we cannot use kernel density estimator (1.8) to estimate the density, because (1.8) is not integrable almost surely.

## 5. Proofs

### Proof of Theorem 1

According to our remark in section 2, the function K(t) satisfies the conditions of a kernel function in density estimation in the situation of no convolution. Thus, we can apply the result of classical kernel density estimation (2.6) (see Rao (1983), P 46 ~ 47), and it follows that

$$\sup_{f \in C_{m,\alpha,B}} |E\hat{f}_n^{(l)}(x_0) - f^{(l)}(x_0)|$$

$$= \sup_{f \in C_{m,\alpha,B}} | \int_{-\infty}^{+\infty} f^{(l)}(x_0 - y)\frac{1}{h}K(\frac{y}{h})\, dy - f^{(l)}(x_0)|$$

$$\leq Ch^{k-l}$$

for some constant C, where $k = m + \alpha$. Now the variance of $\hat{f}_n^{(l)}(x_0)$ is

$$var(\hat{f}_n^{(l)}(x_0)) = \frac{1}{(2\pi)^2 n}\, var\Big[\int_{-\infty}^{+\infty} (-it)^l \exp(-it(x_0 - Y_1))\,\frac{\phi_K(th)}{\phi_\varepsilon(t)}\, dt\Big]$$

$$\leq \frac{1}{(2\pi)^2 n} E\,|\int_{-\infty}^{+\infty} (-it)^l \exp(-it(x_0 - Y_1))\,\frac{\phi_K(th)}{\phi_\varepsilon(t)}\, dt\,|^2$$

$$\leq \frac{1}{(2\pi)^2 nh^2}\Big[\int_{-1}^{+1} \frac{|\phi_K(t)|}{|\phi_\varepsilon(t/h)|}dt\Big]^2 \qquad (5.1)$$

By assumption E2, when $Mh \leq |t| \leq 1$ (for large but fixed M ),

$$|\phi_\varepsilon(t/h)| \geq \frac{c}{2}(t/h)^\beta \exp(-h^{-\beta}/\gamma)$$

Moreover, by (A1)

$$|\phi_\varepsilon(t/h)| \geq \min_{|t| \leq M} \phi_\varepsilon(t) > 0, \qquad when\ |t| \leq Mh$$

Thus, by (5.1)

$$var(\hat{f}_n^{(l)}(x_0)) \leq \frac{1}{(2\pi)^2 nh^2}\, O(\exp(2h^{-\beta}/\gamma))$$

$$= o(n^{-\frac{1}{3}})$$

by choosing the bandwidth $h = O((4/\gamma)^{\frac{1}{\beta}}(log\ n)^{-\frac{1}{\beta}})$. Hence, the conclusion follows.

**Lemma 5.1**: Under the assumption of Theorem 2,

$$h^{2\beta}[g_h^{(l)}(x)]^2 \le \frac{D}{x^2}, \quad \text{(uniformly in small h).}$$

for some constant D.

**Proof of Lemma 5.1:**

By integration by parts,

$$g_h^{(l)}(x) = \frac{1}{ix}\int_{-\infty}^{+\infty} \exp(-itx)\Big[(-it)^l \frac{\phi_K(t)}{\phi_\varepsilon(t/h)}\Big]' dt$$

Thus, by the usual argument,

$$h^{2\beta}[g_h^{(l)}(x)]^2 \le \frac{1}{x^2}\Big\{\int_{-\infty}^{+\infty} h^\beta \, \Big|\Big[\frac{t^l\ \phi_K(t)}{\phi_\varepsilon(t/h)}\Big]'\Big| dt\Big\}^2$$

$$\le \frac{C}{x^2}\Big\{\int_{-\infty}^{+\infty} (|\phi_K(t)t^{\beta+l-1}| + |\phi'_K(t)|\ t^{\beta+l})dt\Big\}^2$$

(uniformly in small h ) for some constant C. Hence, the assertion follows.

**Proof of Theorem 2:**

By choosing the bandwidth as given by Theorem 2 and by the calculation of Theorem 1, we have

$$\sup_{f \in C_{m,\alpha,B}} |E\hat{f}_n^{(l)}(x_0) - f^{(l)}(x_0)| \le O(h^{k-l}) = O(n^{-\frac{k-l}{2k+2\alpha+1}})$$

where $k = m + \alpha$. Now, we need only to compute the variance of the estimator. Let $g_{hl} = g_h^{(l)}$. Then by (2.3),

$$var(\hat{f}_n^{(l)}(x_0)) \le \frac{1}{nh^{2-2l}} E g_{hl}^2(\frac{x_0 - Y_1}{h}) \tag{5.2}$$

Let $f_Y(y)$ be the density of $Y = X + \varepsilon$, then

$$E g_{hl}^2(\frac{x_0 - Y_1}{h}) = \int_{-\infty}^{+\infty} g_{hl}^2(\frac{y}{h}) f_Y(x_0 - y)\, dy \tag{5.3}$$

Note that $|f_Y(x)| \le C$ for all $f \in C_{m,\alpha,B}$. Hence, the following result follows unformly in $f \in C_{m,\alpha,B}$. For any small $\eta$,

$$| \int_{-\infty}^{+\infty} f_Y(x_0 - y) \frac{1}{h} g_{hl}^2(\frac{y}{h})\, dy - f_Y(x_0) \int_{-\infty}^{+\infty} g_{hl}^2(y)\, dy |$$

$$= | \int_{-\infty}^{+\infty} (f_Y(x_0 - y) - f_Y(x_0)) \frac{1}{h} g_{hl}^2(\frac{y}{h})\, dy |$$

$$\le \max_{|y| \le \eta} |f_Y(x_0 - y) - f_Y(x_0)| \int_{|y| \le \eta} \frac{1}{h} g_{hl}^2(\frac{y}{h})\, dy$$

$$+ \int_{|y| \ge \eta} \frac{f_Y(x_0 - y)}{y} \frac{y}{h} g_{hl}^2(\frac{y}{h})\, dy + f_Y(x_0) \int_{|y| \ge \eta} \frac{1}{h} g_{hl}^2(\frac{y}{h})\, dy$$

$$\triangleq \max_{|y| \le \eta} |f_Y(x_0 - y) - f_Y(x_0)|\, I_1 + I_2 + I_3$$

It is easy to check by definition that $|g_h^{(l)}(y)| \le Ch^{-2\beta}$ unformly for small $h$ and some constant C. By Lemma 5.1,

$$I_1 = \int_{|y| \le \eta} \frac{1}{h} g_{hl}^2(\frac{y}{h})\, dy$$

$$\le \int_{|y| \ge 1} g_{hl}^2(y)\, dy + \int_{|y| \le 1} g_{hl}^2(y)\, dy \tag{5.4}$$

$$\le Dh^{-2\beta} \int_{|y| \ge 1} \frac{1}{y^2}\, dy + 2Ch^{-2\beta}$$

$$= O(h^{-2\beta}) \tag{5.5}$$

Applying Lemma 5.1 again, we have

$$I_2 = \int_{|y| \geq \eta} \frac{f(x_0 - y)}{y} \frac{y}{h} g_{hl}^2(\frac{y}{h})\, dy$$

$$\leq \frac{1}{\eta} \sup_{|y| \geq \eta/h} |y\, g_{hl}^2(y)|$$

$$\leq \frac{1}{\eta} \sup_{|y| \geq \eta/h} \frac{D}{y} h^{-2\beta}$$

$$= o(h^{-2\beta}) \qquad (5.6)$$

Similar reason shows

$$I_3 = |f_Y(x_0)| \int_{|y| \geq \eta/h} \frac{1}{h} g_{hl}^2(\frac{y}{h})\, dy = o(h^{-2\beta}) \qquad (5.7)$$

Note that (5.4) and (5.5) implies that

$$\int_{-\infty}^{+\infty} g_{hl}^2(y)\, dy = O(h^{-2\beta}) \qquad (5.8)$$

Combining (5.5) ~ (5.8), we conclude from (5.3) that

$$\frac{1}{nh^{2-2l}} E\, g_{hl}^2(\frac{x_0 - Y_1}{h})$$

$$\leq \frac{1}{n} O(h^{-2\beta - 2l - 1}) = O(n^{-\frac{2(k-l)}{2k+2\beta+1}})$$

Hence, we get the desired conclusion.

**Proof of Theorem 3**

$$E\hat{F}_n(x_0) = \int_{-n^{1/3}}^{x_0} \int_{-\infty}^{+\infty} f(u - y) \frac{1}{h} K(\frac{y}{h})\, dy\, du$$

$$= \int_{-\infty}^{+\infty} \frac{1}{h} (F(x_0 - y) - F(-n^{1/3} - y))\, K(\frac{y}{h})\, dy$$

Now by the standard proof, we can show that

$$\sup_{f \in C_{m,\alpha,B}} |E\hat{F}_n(x_0) - F(x_0)|$$

$$\leq | \int_{-\infty}^{+\infty} F(x_0 - y)\frac{1}{h}K(\frac{y}{h})\, dy - F(x_0)| + \int_{-\infty}^{+\infty} F(-n^{1/3} - hy)|K(y)|dy$$

$$\leq Ch^{m+\alpha+1} + O(F(-n^{-1/3}/2)) + \int_{-\infty}^{-n^{1/3}} |K(y)|\, dy$$

$$= O\left((\log n)^{-\frac{m+\alpha+1}{\beta}}\right)$$

On the other hand, the variance of $\hat{F}_n(x_0)$ is

$$var(\hat{F}_n(x_0)) \leq (n^{1/3} + x_0)\frac{1}{(2\pi)^2 nh^2}\left[\int_{-\infty}^{+\infty} |\phi_K(t)|/|\phi_\varepsilon(\frac{t}{h})|\, dt\right]^2$$

$$\leq O(n^{1/3}\frac{1}{nh^2}\exp(2h^{-\beta}/\gamma))$$

The conclusion follows.

**Proof of Theorem 4**

Take a pair $f_1, f_2 \in C_{m,\alpha,B}$ such that it satisfies (3.4) and

$$b_T(\frac{c}{n}) \leq |T(f_1) - T(f_2)| + o(a_n) \tag{5.9}$$

Then

$$E_{f_{Y1}^n}\left[\frac{f_{Y2}(y_1)\cdots f_{Y2}(y_n)}{f_{Y1}(y_1)\cdots f_{Y1}(y_n)}\right]^2$$

$$= (1 + \int_{-\infty}^{+\infty} (f_{Y1} - f_{Y2})^2/f_{Y1}\, dx)^n \leq e^c \tag{5.10}$$

where $f_{Y1}$ is the convolution of $f_1$ with the distribution of $\varepsilon$. Hence by the Cauchy-Schwartz inequality,

$$\left[P_{f_2}\{|\hat{T}_n - T(f_2)| \le a_n\}\right]^2 \le e^c\, P_{f_1}\{|\hat{T}_n - T(f_2)| \le a_n\} \tag{5.11}$$

On the other hand, by (5.11) we have

$$P_{f_1}\{|\hat{T}_n - T(f_2)| \le 2a_n\}$$

$$\ge P_{f_1}\{|T(f_1) - \hat{T}_n| \le a_n,\ |T(f_2) - \hat{T}_n| \le a_n\}$$

$$= P_{f_1}\{|T(f_2) - \hat{T}_n| \le a_n\} + o(1)$$

$$\ge e^{-c} \text{ (as } n \to \infty)$$

Hence, we conclude that

$$|T(f_1) - T(f_2)| \le 2a_n$$

and the conclusion follows.

We need the following Lemma in order to prove theorem 5 ~ 8.

**Lemma 5.2:** Suppose that F is a distribution function, then the convolution density

$$g_0(x) = \int_{-\infty}^{+\infty} \frac{C_r}{(1 + (x - y)^2)^r} dF(y)$$

satisfies

$$g_0(x) \ge D\, |x|^{-2r}$$

as $x \to \infty$ with D > 0.

**Proof:** Choose M large enough such that

$$F(M) - F(-M) > 0$$

Then when |x| is large,

$$g_0(x) > \int_{-M}^{M} \frac{C_r}{(1 + (x - y)^2)^r} dF(y) > D \frac{1}{|x|^{2r}}$$

**Lemma 5.3:** Suppose $P\ \{x + |x|^{\alpha_0} \geq \varepsilon \geq x - |x|^{\alpha_0}\} = O(|x|^{-(a+1-\alpha_0)})$ for $0 \leq \alpha_0 < 1$ and H(x) is bounded with $H(x) = o(|x|^{-m_0})$ (as $x \to \pm\infty$). Then there exists a large M and a constant C such that when $|\delta x| \geq M$,

$$\int_{-\infty}^{+\infty} H(x-y)\, dF_\varepsilon(\delta y) \leq C(\delta |x|)^{-1.5-(a-0.5)/2}$$

if $m_0(a - 0.5) > 1.5 + (a - 0.5)/2$.

**Proof:** Divide the real line into two parts:

$$I_1 = \{y: |x - y/\delta| \leq |x|^\alpha\}, \quad I_2 = \{y: |x - y/\delta| > |x|^\alpha\}$$

Then, by simple algebra,

$$\int_{-\infty}^{+\infty} H(x-y)\, dF_\varepsilon(\delta y)$$

$$\leq \int_{I_1} + \int_{I_2} H(x - y/\delta)\, dF_\varepsilon(y)$$

$$\leq O((\delta|x|)^{-1-a+\alpha}) + O(|x|^{-m_0\alpha})$$

Now choosing $\alpha = (a - 0.5)/2$, the conclusion follows.

**Proof of Theorem 5:**

By relocating $x_0$ to the origin, without loss of generality assume that $x_0 = 0$. Denote $k = m + \alpha$. Take a real function function $H(.)$ satisfying the following conditions:

1. $H^{(l)}(0) \neq 0$.

2. $H^{(k)}(x)$ is bounded continuous for each k.

3. $H(x) = O(x^{-m_0})$, as $x \to \infty$, for some given $m_0$.

4. $\int_{-\infty}^{+\infty} H(x)\, dx = 0.$

5. $\int_{-\infty}^{0} H(x)dx \neq 0.$

6. $\phi_H(t) = 0$, when $|t|$ is outside [1, 2], where $\phi_H$ is the Fourier transformation of $H$.

To see why such a function $H(\cdot)$ exists, let's take a nonnegative symmetric function $\phi(t)$ which vanishes outside [1, 2] when $t \geq 0$ and has continuous first $m_0$ bounded derivatives ( $m_0$ is large enough such that Lemma 5.3 holds). Moreover, $\phi(t)$ satisfies

$$h^{(l)}(0) \neq h^{(l)}(1) \tag{5.12}$$

and

$$\int_{1}^{2} \frac{\sin t}{t} \phi(t)\, dt \neq 0$$

where h(x) is the Fourier inversion of $\phi(t)$ defined by

$$h(x) = \frac{1}{\pi} \int_{1}^{2} \cos(tx)\, \phi(t) dt \tag{5.13}$$

Such a $\phi(\cdot)$ exists because all functions satisfying the above conditions are infinite dimensional. Let $H(x) = h(x) - h(x+1)$, then its Fourier transformation $\phi_H(t) = (1 - e^{-it})\phi(t)$, and $H(x)$ satisfies the conditions 1 ~ 6.

Now take a pair of densities

$$f_0 = \frac{C_r}{(1+x^2)^r}, \quad \text{and } f_1 = f_0 + c\,\delta^k\, H(x/\delta)$$

Then, by Lemma 5.2, $f_1$ is a density when $\delta$ is small, and by choosing $r$ close to 0.5 and c close to 0, $f_0$, and $f_1 \in C_{m,\alpha,B}$.

Denote $g_0 = f_0 * F_\varepsilon$. Now the $\chi^2$ distance between the two densities in convolution space is of order (c.f. (3.6))

$$\delta^{2k+1} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} H(x-y) dF_\varepsilon(\delta y) \right)^2 g_0^{-1}(\delta x)\, dx$$

$$\leq \delta^{2k+1}\sqrt{\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} H(x-y)dF_\varepsilon(\delta y)\right)^2 dx}$$

$$\times\sqrt{\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} H(x-y)dF_\varepsilon(\delta y)/g_0(\delta x)\right)^2 dx} \tag{5.14}$$

Note that by Parseval's identity the first term of (5.14) is

$$\int_{-\infty}^{+\infty} |\phi_H(t)|^2 |\phi_\varepsilon(t/\delta)|^2 \, dt$$

$$= 2\int_1^2 |\phi_H(t)|^2 |\phi_\varepsilon(t/\delta)|^2 \, dt$$

$$\leq O(\delta^{-2\beta_0}\exp(-2\delta^{-\beta}/\gamma))$$

uniformly in small $\delta$. Let the minimum value of $g_0(x)$ over $[-M, M]$ be $m_g$, which is bigger than 0. By Lemma 5.2 and 5.3, the second term is bounded by

$$m_g^{-2}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} H((x-y))dF_\varepsilon(\delta y)\right]^2 dx + \int_{|\delta x| > M}\left[\frac{(\delta x)^{-1.5-(a-0.5)/2}}{D(\delta x)^{-2r}}\right]^2 dx = O(\delta^{-1}).$$

Consequently, when $\delta \to 0$,

$$\int_{-\infty}^{+\infty} (f_{Y1} - f_{Y2})^2 (f_{Y1})^{-1} \, dx \leq C\delta^c \exp(-\delta^{-\beta}/\gamma) \tag{5.15}$$

for some constants c and C. Taking

$$\delta = (\log n + (c+1)\log(\log n))^{-\frac{1}{\beta}}\gamma^{-\frac{1}{\beta}}$$

$$(5.15) \leq \frac{C}{n \log n} = o(\frac{1}{n})$$

and the change of the functional is

$$|f_1^{(l)}(0) - f_2^{(l)}(0)| = O(\delta^{(k-l)})$$

Thus,

$$b_T(\frac{1}{\sqrt{n}}) = O((log\ n)^{-(k-l)/\beta})$$

and the conclusion follows.

**Proof of Theorem 6:**

Use the same notation as in the proof of Theorem 5. Take the same $\phi(t)$ except only the first two continuous derivatives are required in this case. Now take a pair of densities

$$f_0 = \frac{C_r}{(1+x^2)^r}, \quad \text{and } f_1 = f_0 + c\,\delta^k\, H(x/\delta)$$

Let $\phi_H(t) = (1 - e^{-it})\phi(t)$ be the Fourier transformation of $H(x)$, and define

$$\phi_\delta(t) = (\phi_H(t)\ \phi_\varepsilon(t/\delta))''$$

and

$$\phi_0(t) = \lim_{\delta \to 0} \delta^{-\beta}\ \phi_\delta(t) \tag{5.16}$$

which converges uniformly in $|t| \in [1, 2]$. Now, by Fourier inversion formula,

$$\int_{-\infty}^{+\infty} H(x-y)dF_\varepsilon(\delta y)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-itx)\phi_H(t)\phi_\varepsilon(t/\delta)\ dt$$

$$= -\frac{1}{2\pi x^2} \int_{1 \le |t| \le 2} e^{-itx}\ \phi_\delta(t)\ dt \tag{5.17}$$

Let $N_\delta = \int_{1 \le |t| \le 2} |\delta^{-\beta}\ \phi_\delta(t) - \phi_0(t)|\,dt$, which goes to 0 ( as $\delta \to 0$ ). Then by (5.17),

$$|\ \delta^{-\beta} \int_{-\infty}^{+\infty} H(x-y)\ dF_\varepsilon(\delta y)\ | \le \frac{N_\delta}{x^2} + \frac{1}{2\pi x^2} \int_{1 \le |t| \le 2} |\ \phi_0(t)\ |\ dt \tag{5.18}$$

Now, we are ready to compute (3.6). By Parseval's indentity, when $\delta$ is small,

$$I_1 \triangleq \int_{|x| \le 1} \Big( \int_{-\infty}^{+\infty} H(x-y) dF_\varepsilon(\delta y) \Big)^2 g_0^{-1}(\delta x)\, dx$$

$$\le C \int_{-\infty}^{+\infty} \Big( \int_{-\infty}^{+\infty} H(x-y) dF_\varepsilon(\delta y) \Big)^2 dx$$

$$\le C \int_{-\infty}^{+\infty} |\phi_H(t)\, \phi_\varepsilon(t/\delta)|^2\, dt \tag{5.19}$$

$$= O(\delta^{2\beta})$$

where $g_0 = f_0 * F_\varepsilon$ does not vanish, and hence C is a finite constant. By Lemma 5.2, and (5.18)

$$I_2 \triangleq \delta^{-2\beta} \int_{|x| \ge 1} \Big( \int_{-\infty}^{+\infty} H(x-y) dF_\varepsilon(\delta y) \Big)^2 g_0^{-1}(\delta x)\, dx$$

$$\le \int_{|x| \ge 1} \Big[ \frac{N_\delta}{x^2} + \frac{1}{2\pi x^2} \int_{1 \le |t| \le 2} |\phi_0(t)|\, dt \Big] g_0^{-1}(\delta x)\, dx$$

$$= O(1) \tag{5.20}$$

Consequently, the $\chi^2$-distance in the Y variable ( see (3.6) ) is

$$\delta^{2(m+\alpha)+1} (I_1 + \delta^{2\beta} I_2) = O(n^{-1})$$

and the change of the functional is

$$|T(f_1) - T(f_0)| = \delta^{m+\alpha-l} |h^{(l)}(1) - h^{(l)}(0)| = O(n^{-\frac{m+\alpha-l}{2(m+\alpha+\beta)+1}})$$

Hence the conclusion follows.

**Proof of Theorem 7 & 8**: By translation, without loss of generality assume that $x_0 = 0$. Take the same least favorable pairs as used in Theorem 5 and 6. Then the change of functional is

$$|F_1(0) - F_0(0)| = \delta^{m+\alpha} \left| \int_{-\infty}^{0} H(x/\delta)\, dx \right|$$

$$= O(\delta^{m+\alpha+1})$$

Hence the result follows.

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