

Asymptotics of Empirical Eigen-structure for Ultra-high Dimensional Spiked Covariance Model

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Abstract

We derived the asymptotic distributions of the spiked eigenvalues and eigenvectors under a generalized and unified asymptotic regime, which takes into account the spikeness of leading eigenvalues, sample size, and dimensionality. This new regime allows high dimensionality and diverging eigenvalue spikes and provides new insights on the roles the leading eigenvalues, sample size, and dimensionality played in the principal component analysis. The results are proven by a new technical device, which swaps the role of rows and columns and converts the high-dimensional problems into low-dimensional ones. Our results are a natural extension of those in Paul (2007) to more general setting with new insights and solve the rates of convergence problems in Shen et al. (2013). They also reveal the biases of the estimation of leading eigenvalues and eigenvectors by using the principal component analysis, and lead to a new covariance estimator for the approximate factor model, called shrinkage principal orthogonal complement thresholding (S-POET), which corrects the biases. Our results are successfully applied to outstanding problems in estimation of risks of large portfolios and false discovery proportions for dependent test statistics and are illustrated by simulation studies.

Keywords: Asymptotic distributions; Principal component analysis; Spiked covariance model; Ultra-high dimension; Diverging eigenvalues; Approximate factor model; Relative risk management; False discovery proportion.

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1 Introduction

Principal Component Analysis (PCA) has widely been used as a powerful tool for dimensionality reduction and data visualization. Its theoretical properties such as the consistency and asymptotic distributions of empirical eigenvalues and eigenvectors are perceived to be challenging especially in high dimensional regime. For the past half century substantial amount of efforts have been devoted to understanding empirical eigen-structures. An early effort is Anderson (1963) who established the asymptotic normality of eigenvalues and eigenvectors under the classical regime with large sample size n and fixed dimension p . However, as dimensionality diverges at the same rate as the sample size, sample covariance matrix is a notoriously bad estimator with substantial different eigen-structure from the population one. Recent literature made the endeavor to understand the behaviors of eigenvalues and eigenvectors under high dimensional regime where both n and p go to infinity. See for example Baik et al. (2005); Bai (1999); Paul (2007); Johnstone and Lu (2009); Onatski (2012); Shen et al. (2013) and many related papers. For additional developments and references, see Bai and Silverstein (2009).

Most of studies focus on the situations where signals are weak or semi-weak (Onatski, 2012) with leading asymptotic eigenvalues bounded (Paul, 2007; Bai and Silverstein, 2009) or slow growing (Onatski, 2012). However, Fan et al. (2013a) shows that for factor models with pervasive factors, the leading eigenvalues can grow linearly with the dimensionality and hence their corresponding eigenvectors can be consistently estimated as long as sample size diverges. This leads to the question of how the asymptotics of eigen-structure depends on the interplay of spikiness of leading eigenvalues, dimensionality, and sample size. An interesting study on this topic is Shen et al. (2013), which focuses only on the consistency of the problem. The question then arises naturally on the rates of convergence and asymptotic structures of empirical eigenvalues and eigenvectors. This forms the subject of this study.

In this paper, we consider a high dimensional spiked covariance model with the first several eigenvalues significantly larger than the rest. Typically, the spike part is of importance and of interest. We provides new understanding on how the spiked empirical eigenvalues and eigenvectors fluctuate around their theoretical counterparts and what their asymptotic biases are. For the spiked covariance model, three quantities play an essential role in determining the asymptotic behaviors of empirical eigen-structure: the sample size n , the dimension p , and the magnitude of leading eigenvalues $\{\lambda_j\}_{j=1}^m$. Theoretical properties of PCA have been investigated from three different perspectives.

The first angle is through a low-rank plus sparse decomposition, where the covariance matrix is perceived as the sum of a low-rank and a sparse matrix. The low-rank part contributes to the signal to be recovered whereas the sparse part serves as noise. In Fan et al. (2008), the low-rank matrix corresponds to the dependence induced by the

common factors or covariates whereas the sparse matrix corresponds to the idiosyncratic noise. Candès et al. (2011) considered the principal component pursuit and showed that it can recover the decomposition structure under the incoherence condition. Cai et al. (2013b) studied the optimal rates for estimating eigenvalues and eigenvectors of spiked covariance matrices with jointly k -sparse eigenvectors. This type of work assumes bounded eigenvalues, which limit the signals we can get from the data. Correspondingly, they reduce the possibility of noise accumulation by assuming certain eigenvector structure in the sense of either incoherence condition or jointly k -sparse or other conditions.

A different line of efforts is to analyze PCA through random matrix theories, where it is typically assumed $p/n \rightarrow \gamma \in (0, \infty)$ with bounded spike sizes. It is well known that if the true covariance matrix is identity, the empirical spectral distribution converges almost surely to the Marcenko-Pastur distribution (Bai, 1999) and when $\gamma < 1$ the largest and smallest eigenvalues converge almost surely to $(1 + \sqrt{\gamma})^2$ and $(1 - \sqrt{\gamma})^2$ respectively (Bai and Yin, 1993). If the true covariance structure takes the form of a spiked matrix, Baik et al. (2005) showed that the asymptotic distribution of the empirical eigenvalues exhibit an $n^{2/3}$ scaling when the eigenvalue lies below a threshold $1 + \sqrt{\gamma}$, and an $n^{1/2}$ scaling when it is above the threshold. For the case where we have the regular scaling, Paul (2007) investigated the asymptotic behavior of the corresponding empirical eigenvectors and showed that the major part of an eigenvector which corresponds to the spiked eigenvalues is normally distributed with regular scaling $n^{1/2}$. The same random matrix regime has also been considered by Onatski (2012) in studying the principal component estimator for high-dimensional factor models.

Deviating from the previous literature, we will consider the ultra-high dimensional regime allowing $p/n \rightarrow \infty$. If $p/n \rightarrow \infty$, to ensure sufficiently strong signal for PCA, it is natural to also have the spike sizes go to infinity, namely, $\lambda_j \rightarrow \infty$ for the first m leading eigenvalues. This leads to the third perspective of understanding PCA from this ultra high dimensional setting. Shen et al. (2013) adopted this point of view and considered the regime of $\frac{p}{n\lambda_j} \rightarrow c_j$ where $0 \leq c_j < \infty$ for leading eigenvalues. This is more general than the bounded eigenvalue condition. Specifically if eigenvalues are bounded, we require the ratio p/n converges to a bounded constant. On the other hand, if the dimension is much larger than sample size, we offset the dimensionality by assuming increased signals. In particular, the pervasive factor model considered in Fan et al. (2013a) corresponds to $c_j = 0$ since the leading eigenvalues $\lambda_j \asymp p$. The weak factor model considered by Onatski (2012) also implies $c_j = 0$, with p/n bounded and $\lambda_j \asymp p^\theta$ for some $\theta \in (0, 1)$.

Shen et al. (2013) revealed an interesting fact that spiked sample eigenvalues almost surely converges to a biased quantity of the true eigenvalues; furthermore the corresponding sample eigenvectors show an asymptotic conical structure. We will consider the same regime as theirs, but focus more on the asymptotic distributions of the eigen-structure under more relaxed conditions. Our results are a natural extension of Paul (2007) to

more general setting and solve the rates of convergence problems in Shen et al. (2013).

In addition to the different regime we take on, we also introduce a simple and novel technique to proceed our proofs. The idea is to flip the roles of rows and columns and treat p as the sample size and n as the dimension. When p is higher than n , sample covariance is clearly degenerate. Switching the roles of n and p allows us to utilize the existing results on eigen-structures. To be specific, if we have n samples generated from $N(\mathbf{0}, \mathbf{D})$ where $\mathbf{D} = \text{diag}(d_1, \dots, d_p)$ is diagonal, then all the information we have is just an n by p data matrix with independent entries. We can simply treat the data as p independent vectors of dimension n each with distribution $N(\mathbf{0}, d_i \mathbf{I}_n)$. Even when the data are not normally distributed and hence p n -dimensional vectors are then not independent, the idea is still extremely powerful and leads to better understanding of relationship between high and low dimensionality. We illustrate how this simple trick entails the asymptotic behaviors of both empirical eigenvalues and eigenvectors of sample covariance matrix.

The rest of the paper is organized as follows. Section 2 introduces the notations, assumptions, and an important fact which serves as basis of our proofs. The fact will help unravel the relationship between high and low dimensions. Sections 3.1 and 3.2 devote to the theoretical conclusions of the sample eigenvalues and eigenvectors of the spiked covariance matrix under our asymptotic regime. Section 4 applies our results to estimating risks of large portfolios and controlling false discovery proportions. In Section 5, simulations are conducted to illustrate the theoretical results at the finite sample. All the proofs are provided in Sections 6 and 7.

2 Assumptions and a simple fact

Assume that $\{\mathbf{Y}_i\}_{i=1}^n$ is a sequence of i.i.d. random variables with zero mean and covariance matrix $\Sigma_{p \times p}$. Let $\lambda_1, \dots, \lambda_p$ be the eigenvalues of Σ in descending order. We consider the spiked population model as follows.

Assumption 2.1. $\lambda_1 > \lambda_2 > \dots > \lambda_m \gg \lambda_{m+1} \geq \dots \geq \lambda_p > 0$, where the spiked eigenvalues go to infinity while the non-spiked eigenvalues are bounded, i.e. $c_0 \leq \lambda_j \leq C_0, j > m$ for constants $c_0, C_0 > 0$.

The eigenvalues are divided into the diverging ones and bounded ones. For simplicity, we only consider distinguishable eigenvalues (multiplicity 1) for the largest m eigenvalues and a fixed number m , independent of n and p . This assumption is satisfied by the factor model $\mathbf{y} = \mathbf{B}\mathbf{f} + \boldsymbol{\varepsilon}$ considered by Fan et al. (2013a) as follows. Assume without loss of generality that $\text{var}(\mathbf{f}) = \mathbf{I}_m$, the $m \times m$ identity matrix. Then, the model implied covariance matrix $\Sigma = \mathbf{B}\mathbf{B}' + \Sigma_\varepsilon$, where $\Sigma_\varepsilon = \text{var}(\boldsymbol{\varepsilon})$. If the factor loadings $\{\mathbf{b}_i\}$ (the transpose of rows of \mathbf{B}) are an i.i.d. sample from a population with mean zero and covariance Σ_b , then by the law of large numbers, $p^{-1}\mathbf{B}'\mathbf{B} = p^{-1}\sum_{i=1}^p \mathbf{b}_i\mathbf{b}_i' \rightarrow \Sigma_b$. In

other words, the eigenvalues of $\mathbf{B}\mathbf{B}'$ are approximately

$$p\lambda_1(\boldsymbol{\Sigma}_b)(1 + o(1)), \dots, p\lambda_m(\boldsymbol{\Sigma}_b)(1 + o(1)), 0, \dots, 0,$$

where $\lambda_j(\boldsymbol{\Sigma}_b)$ is the j^{th} eigenvalue of $\boldsymbol{\Sigma}_b$. If we assume that $\|\boldsymbol{\Sigma}_\varepsilon\|$ is bounded, then by Weyl's theorem, we conclude that

$$\lambda_j = p\lambda_j(\boldsymbol{\Sigma}_b)(1 + o(1)), \quad \text{for } j = 1, \dots, m, \quad (2.1)$$

and the remaining is bounded.

In the spiked covariance models, three essential factors come into play: the sample size n , dimension p and spikes λ_j 's. The following relationship is assumed as in Shen et al. (2013).

Assumption 2.2. For $1 \leq j \leq m$, $p/(n\lambda_j) = c_j + o(n^{-1/2})$, where $0 = c_1 = \dots = c_{m_0-1} < c_{m_0} \dots < c_m < \infty$ and $1 \leq m_0 \leq m + 1$. In addition, for the non-spiked part, $(p - m)^{-1} \sum_{j=m+1}^p \lambda_j = \bar{c} + o(n^{-1/2})$.

The regime $p > n$ here is more general than what has been used in random matrix theories or sparse PCA literature. We allow $p/n \rightarrow \infty$ in any manner, though λ_j needs also grow fast enough. In particular, $c_j = 0$ is allowed. We do not assume the non-spiked eigenvalues the same, as in most spiked covariance model in literature (Paul, 2007; Johnstone and Lu, 2009). If $m_0 = 1$, all spiked eigenvalues are growing at the speed p/n while $m_0 = m + 1$ means that those eigenvalues are growing at faster rate as in the case of Fan et al. (2013a). This can easily be seen from (2.1).

By the spectral decomposition, $\boldsymbol{\Sigma} = \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}'$, where the orthonormal matrix $\boldsymbol{\Gamma}$ is constructed by the eigenvectors of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_p)$. Let $\mathbf{X}_i = \boldsymbol{\Gamma}'\mathbf{Y}_i$. Since the empirical eigenvalues are invariant and the empirical eigenvectors are equivariant under an orthonormal transformation, we focus the analysis on the transformed domain of \mathbf{X} and the results can be translated into the original data. Note that $\text{var}(\mathbf{X}) = \boldsymbol{\Lambda}$. Let $Z_j = X_j/\sqrt{\lambda_j}$ be the standardized column for $j \leq p$.

Assumption 2.3. $\{\mathbf{X}_i\}_{i=1}^n$ are i.i.d copies of \mathbf{X} . The standardized random variables $\{Z_j\}_{j=1}^p$ are sub-Gaussian with independent entries of mean zero and variance one. The sub-Gaussian norms of all components are uniformly bounded: $\max_j \|Z_j\|_{\psi_2} \leq C_0$, where $\|Z_j\|_{\psi_2} = \sup_{q \geq 1} q^{-1/2}(E|Z_j|^q)^{1/q}$.

Since $\text{Var}(\mathbf{X}_i) = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_p)$, the first m population eigenvectors are simply unit vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m$. Denote the n by p transformed data matrix by $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n)'$. Then the sample covariance matrix is

$$\hat{\boldsymbol{\Sigma}}_{p \times p} = \frac{1}{n} \mathbf{X}'\mathbf{X} = \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i \mathbf{X}_i'$$

whose eigenvalues are denoted by $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_p$ ($\hat{\lambda}_j = 0$ for $j > n$) with corresponding eigenvectors $\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2, \dots, \hat{\boldsymbol{\xi}}_p$. Note that the empirical eigenvectors of data \mathbf{Y}_i 's are $\hat{\boldsymbol{\xi}}_j^{(Y)} = \mathbf{\Gamma} \hat{\boldsymbol{\xi}}_j$.

Let \mathbf{Z}_j be the j^{th} column of the standardized \mathbf{X} . Then each \mathbf{Z}_j has i.i.d sub-Gaussian entries with zero mean and unit variance. Exchanging the role of rows and columns, we get the n by n sample covariance matrix

$$\tilde{\boldsymbol{\Sigma}}_{n \times n} = \frac{1}{n} \mathbf{X} \mathbf{X}' = \frac{1}{n} \sum_{j=1}^p \lambda_j \mathbf{Z}_j \mathbf{Z}_j',$$

with the same nonzero eigenvalues $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_n$ as $\hat{\boldsymbol{\Sigma}}$ and the corresponding eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. It is well known that for $i = 1, 2, \dots, n$

$$\hat{\boldsymbol{\xi}}_i = (n \hat{\lambda}_i)^{-1/2} \mathbf{X}' \mathbf{u}_i \quad \text{and} \quad \mathbf{u}_i = (n \hat{\lambda}_i)^{-1/2} \mathbf{X} \hat{\boldsymbol{\xi}}_i, \quad (2.2)$$

while the other eigenvectors of $\hat{\boldsymbol{\Sigma}}$ corresponding to eigenvalue 0 constitute a $(p - n)$ -dimensional orthogonal complement of $\hat{\boldsymbol{\xi}}_1, \dots, \hat{\boldsymbol{\xi}}_n$.

By using this simple fact and under the same regime assumptions with $c_0 = C_0 = 1$ in Assumption 2.1, $\lambda_j = 1$ for $j > m$ in Assumption 2.2, and Gaussian data in Assumption 2.3, Shen et al. (2013) showed that

$$\frac{\hat{\lambda}_j}{\lambda_j} \xrightarrow{\text{a.s.}} 1 + c_j, \quad 1 \leq j \leq m;$$

and

$$\left| \langle \hat{\boldsymbol{\xi}}_j, \mathbf{e}_j \rangle \right| \xrightarrow{\text{a.s.}} (1 + c_j)^{-\frac{1}{2}},$$

where $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the inner product of two vectors. However, even under the Gaussian assumption, they do not establish any results on convergence rates nor asymptotic distributions of the sample eigenvalues and eigenvectors. This motivates the current paper.

The aim of this paper is to establish the asymptotic normality of the empirical eigenvalues and eigenvectors under the more relaxed conditions. Our results are a natural extension of Paul (2007) to more general setting with new insights, who derived the asymptotic normality of sample eigenvectors using complicated random matrix techniques for Gaussian samples under the regime $p/n \rightarrow \gamma \in [0, 1)$. Compared to them, our proof, based on the relationship (2.2), is much simpler and insightful to understand the behavior of ultra high dimensional PCA.

Here are some notations that we will use in the paper. For a symmetric matrix \mathbf{M} , we define $\lambda_j(\mathbf{M})$ to be the j^{th} largest eigenvalue of \mathbf{M} and $\lambda_{\max}(\mathbf{M})$, $\lambda_{\min}(\mathbf{M})$ to be the maximal and minimal eigenvalue respectively. We denote $\text{tr}(\mathbf{M})$ to be the trace of \mathbf{M} . If \mathbf{M} is a general matrix, we denote its matrix entry-wise maximum value as $\|\mathbf{M}\|_{\max} =$

$\max_{i,j}\{|M_{i,j}|\}$ and define the quantities $\|\mathbf{M}\| = \lambda_{\max}^{1/2}(\mathbf{M}'\mathbf{M})$, $\|\mathbf{M}\|_F = (\sum_{i,j} M_{i,j}^2)^{1/2}$, $\|\mathbf{M}\|_\infty = \max_i \sum_j |M_{i,j}|$ to be its spectral, Frobenius and induced ℓ_∞ norms. For any vector \mathbf{v} , its ℓ_2 norm is represented by $\|\mathbf{v}\|$ while ℓ_1 norm is written as $\|\mathbf{v}\|_1$. We denote $\text{diag}(\mathbf{v})$ to be the diagonal matrix with the same diagonal entries as \mathbf{v} . For two random vectors \mathbf{a}, \mathbf{b} of the same length, we say $\mathbf{a} = \mathbf{b} + O_P(\delta)$ if $\|\mathbf{a} - \mathbf{b}\| = O_P(\delta)$ and $\mathbf{a} = \mathbf{b} + o_P(\delta)$ if $\|\mathbf{a} - \mathbf{b}\| = o_P(\delta)$. We denote $\mathbf{a} \xrightarrow{d} \mathcal{L}$ for some distribution \mathcal{L} if there exists $\mathbf{b} \sim \mathcal{L}$ such that $\mathbf{a} = \mathbf{b} + o_P(1)$. In the following, C is a generic constant that may differ from line to line.

3 Asymptotic behavior of empirical eigen-structure

3.1 Asymptotic normality of empirical eigenvalues

Let us first investigate the behavior of the first m empirical eigenvalues of $\hat{\Sigma}$. Denote by $\lambda_j(\mathbf{A})$ the j th largest eigenvalue of matrix \mathbf{A} and recall that $\hat{\lambda}_j = \lambda_j(\hat{\Sigma})$. We have the following asymptotic normality of $\hat{\lambda}_j$.

Theorem 3.1. *Under Assumptions 2.1 - 2.3, $\{\hat{\lambda}_j\}_{j=1}^m$'s have independent limiting distributions. In addition,*

$$\sqrt{n} \left\{ \frac{\hat{\lambda}_j}{\lambda_j} - \left(1 + \bar{c}c_j + O_P(c_j \sqrt{n/p}) \right) \right\} \xrightarrow{d} N(0, \kappa_j - 1), \quad (3.1)$$

where κ_j is the kurtosis of X_j .

The theorem shows that the bias of $\hat{\lambda}_j/\lambda_j$ is $\bar{c}c_j + O_P(c_j \sqrt{n/p})$. In our regime, the second term is negligible in comparison with the first term. Using the definition of c_j , the second term is of order $\sqrt{p/n}/\lambda_j$, which is of order $o_P(n^{-1/2})$ if $\sqrt{p} = o(\lambda_j)$. The latter assumption is satisfied by the strong factor model in Fan et al. (2013a) and a part of weak factor model in Onatski (2012). To get the asymptotically unbiased estimate, it requires $c_j = 0$ or $p/(n\lambda_j) \rightarrow 0$ for $j \leq m$. This is the same conclusion as that of Shen et al. (2013). In addition, under the typical spiked covariance model as in Baik et al. (2005), Johnstone and Lu (2009) and Paul (2007), where it is assumed $\lambda_j = c_0 = C_0, j \geq m$, we have $\bar{c} = c_0$ equal to the minimum eigenvalue of the population covariance matrix. The theorem reveals the bias is controlled at the rate $p/(n\lambda_j)$. Our result is also consistent with Anderson (1963)'s result that

$$\sqrt{n}(\hat{\lambda}_j - \lambda_j) \xrightarrow{d} N(0, 2\lambda_j^2),$$

for Gaussian distributions and fixed p and λ_j 's.

3.2 Behavior of empirical eigenvectors

Let us consider the asymptotic distribution of the empirical eigenvectors $\hat{\boldsymbol{\xi}}_j$'s corresponding to $\hat{\lambda}_j$, $j = 1, 2, \dots, m$. As in Paul (2007), each $\hat{\boldsymbol{\xi}}_j$ is divided into two parts corresponding to the spike and non-spike components, i.e. $\hat{\boldsymbol{\xi}}_j = (\hat{\boldsymbol{\xi}}'_{jA}, \hat{\boldsymbol{\xi}}'_{jB})'$ where $\hat{\boldsymbol{\xi}}_{jA}$ is of length m .

Theorem 3.2. *Under Assumptions 2.1 - 2.3, we have*

(i) *For the spike part,*

$$\sqrt{n} \left(\hat{\boldsymbol{\xi}}_{jA} - \frac{1}{\sqrt{1 + \bar{c}c_j}} \mathbf{e}_{jA} + O_P \left(c_j \sqrt{\frac{n}{p}} \right) \right) \xrightarrow{d} N_m(\mathbf{0}, \boldsymbol{\Sigma}_j), \quad (3.2)$$

for $j = 1, 2, \dots, m$, where

$$\boldsymbol{\Sigma}_j = (1 + \bar{c}c_j)^{-1} \sum_{k \in [m] \setminus j} \frac{\lambda_j \lambda_k}{(\lambda_j - \lambda_k)^2} \mathbf{e}_{kA} \mathbf{e}'_{kA},$$

where $[m] = \{1, \dots, m\}$, \mathbf{e}_{kA} is the first m elements of unit vector \mathbf{e}_k , and $a_{jk} = \lim_{\lambda_j, \lambda_k \rightarrow \infty} \sqrt{\lambda_j \lambda_k} / (\lambda_j - \lambda_k)$, which is assumed to exist. Furthermore, the normality implies the inner product between empirical eigenvector and the population one converges to $\frac{1}{\sqrt{1 + \bar{c}c_j}}$ in probability and

$$\langle \hat{\boldsymbol{\xi}}_j, \mathbf{e}_j \rangle - \frac{1}{\sqrt{1 + \bar{c}c_j}} = O_P(c_j \sqrt{n/p}) + o_P(n^{-1/2}). \quad (3.3)$$

(ii) *For the noise part, if we further assume the data is Gaussian,*

$$\boldsymbol{\Omega} \hat{\boldsymbol{\xi}}_{jB} + O_P(c_j \sqrt{n/p}) \xrightarrow{d} \text{Unif} \left(B_{p-m} \left(\sqrt{\frac{\bar{c}c_j}{1 + \bar{c}c_j}} \right) \right), \quad (3.4)$$

where $\boldsymbol{\Omega} = \text{diag}(\sqrt{\bar{c}/\lambda_{m+1}}, \dots, \sqrt{\bar{c}/\lambda_p})$ and $\text{Unif}(B_k(r))$ denotes the uniform distribution over the centered sphere of radius r . In addition, the max norm of $\hat{\boldsymbol{\xi}}_{jB}$ satisfies

$$\|\hat{\boldsymbol{\xi}}_{jB}\|_{\max} = O_P(c_j \sqrt{n/p} + \sqrt{c_j \log p/p}). \quad (3.5)$$

In the above theory, we assume that $a_{jk} = \lim_{\lambda_j, \lambda_k \rightarrow \infty} \frac{\sqrt{\lambda_j \lambda_k}}{\lambda_j - \lambda_k}$ exists. When $m_0 = 1$ in Assumption 2.2, all c_j 's are positive and $\lambda_k/\lambda_j \rightarrow c_j/c_k$ and $a_{jk} = \sqrt{c_j c_k} / (c_k - c_j)$. The assumption also holds for the pervasive factor model (Fan et al., 2013a), in which $a_{jk} = \sqrt{\lambda_j(\boldsymbol{\Sigma}_b) \lambda_k(\boldsymbol{\Sigma}_b)} / (\lambda_k(\boldsymbol{\Sigma}_b) - \lambda_j(\boldsymbol{\Sigma}_b))$.

Theorem 3.2 is an extension of random matrix results into ultra high dimensional regime. Its proof sheds light on how to use the smaller $n \times n$ matrix $\tilde{\boldsymbol{\Sigma}}$ as a tool to

understand the behavior of the larger $p \times p$ covariance matrix $\hat{\Sigma}$. Specifically, we start from $\tilde{\Sigma}\mathbf{u}_j = \hat{\lambda}_j\mathbf{u}_j$ or identity (6.3) and then use the simple fact (2.2) to get a relationship (6.4) of eigenvector $\hat{\xi}_j$. This makes the whole proof much simpler in comparison with Paul (2007) who showed a similar type of results on a different spike model through a complicated representation of $\hat{\xi}_j$ and $\hat{\lambda}_j$. From this simple trick, we can understand deeply how some important high and low dimensional quantities link together and differ from each other.

Several remarks are in order. First, since $\hat{\xi}_j^{(\mathbf{Y})} = \mathbf{\Gamma}\hat{\xi}_j$ for empirical eigenvectors based on observed data \mathbf{Y} , we have decomposition

$$\hat{\xi}_j^{(\mathbf{Y})} = \mathbf{\Gamma}_A\hat{\xi}_{jA} + \mathbf{\Gamma}_B\hat{\xi}_{jB},$$

where $\mathbf{\Gamma} = (\mathbf{\Gamma}_A, \mathbf{\Gamma}_B)$. Note that $\mathbf{\Gamma}_A\hat{\xi}_{jA}$ converges to the true eigenvector deflated by a factor of $\sqrt{1 + \bar{c}c_j}$ with the convergence rate $O_P(c_j\sqrt{n/p} + n^{-1/2})$ while $\mathbf{\Gamma}_B\hat{\xi}_{jB}$ creates a random bias, which is distributed uniformly on an ellipse of $(p - m)$ dimension and projected into the p dimensional space spanned by $\mathbf{\Gamma}_B$. The two parts intertwined in such a way that correction for the bias of estimating eigenvectors is almost impossible. More details are discussed in Section 4 for factor models. Secondly, it is clearly as in the eigenvalue case, the term $c_j\sqrt{n/p}$ disappears when $\sqrt{p} = o(\lambda_j)$. In particular, for the stronger factor given by (2.1), $\hat{\xi}_j^{(\mathbf{Y})}$ is a consistent estimator.

Theorem 3.2 obviously implies the result of Shen et al. (2013). It also generalizes the asymptotic distribution of non-spiked part from pure orthogonal invariant case of Paul (2007) to more general setting and regime. In particular, the asymptotic distribution of the non-spiked component is not uniform over a sphere any more, but over an ellipse. In addition, our result can be compared with the low dimensional case, where Anderson (1963) showed that

$$\sqrt{n}(\hat{\xi}_j - \mathbf{e}_j) \xrightarrow{d} N_p\left(\mathbf{0}, \sum_{k \in [m] \setminus j} \frac{\lambda_j \lambda_k}{(\lambda_j - \lambda_k)^2} \mathbf{e}_k \mathbf{e}_k'\right), \quad (3.6)$$

for fixed p and λ_j 's. Under our assumptions, if the spiked eigenvalues go to infinity, the constants in the asymptotic covariance matrix are replaced by the limits a_{jk} 's. Again similar to the behavior of eigenvalues, the spiked part $\hat{\xi}_{jA}$ preserves the normality property except for a bias factor $1/(1 + \bar{c}c_j)$ caused by the high dimensionality. Last but not least, it has been shown by Johnstone and Lu (2009) that PCA generates consistent eigenvector estimation if and only if $p/n \rightarrow 0$ when the spike sizes are fixed. This motivates the work of sparse PCA. We take the spikeness of eigenvalues into account and provide additional insights by showing that PCA consistently estimate eigenvalues and eigenvectors when $p/(n\lambda_j) \rightarrow 0$. This explains why Fan et al. (2013a) can consistently estimate the eigenvalues and eigenvectors while Johnstone and Lu (2009) can not.

4 Applications to factor models

In this section, we propose a method named shrinkage principal orthogonal complement thresholding or S-POET for estimating the large covariance matrix induced by the approximate factor models. The estimator is based on the correction of bias of estimated eigenvalues as specified in (3.1). We derive for the first time the relative estimation errors of covariance matrix under the spectral norm. The results are then applied to management of portfolio risk and estimation of false discovery proportion.

4.1 Approximate factor models

Factor models have been widely used in various disciplines such as finance and genomics. Consider the approximate factor model

$$y_{it} = \mathbf{b}'_i \mathbf{f}_t + u_{it}, \quad (4.1)$$

where y_{it} is the observed data for the i th ($i = 1, \dots, p$) individual (e.g. returns of stocks) or components (e.g. expression of genes) at time $t = 1, \dots, T$; \mathbf{f}_t is a $m \times 1$ vector of latent common factors and \mathbf{b}_i is the factor loadings for the i th individuals or components; u_{it} is the idiosyncratic error, uncorrelated with the common factors. In genomics application, t can also index individuals or repeated experiments. For simplicity we assume there is no time dependency.

The factor model can also be written into a matrix form as follows:

$$\mathbf{Y} = \mathbf{B}\mathbf{F}' + \mathbf{U}, \quad (4.2)$$

where $\mathbf{Y}_{p \times T}$, $\mathbf{B}_{p \times m}$, $\mathbf{F}_{T \times m}$, $\mathbf{U}_{p \times T}$ are respectively the matrix form of observed data, factor loading matrix, factor matrix, and error matrix. For identifiability issue, we impose the condition that $\text{cov}(\mathbf{f}_t) = \mathbf{I}$ and $\mathbf{B}'\mathbf{B}$ is a diagonal matrix. Thus, the covariance matrix is given by

$$\Sigma = \mathbf{B}\mathbf{B}' + \Sigma_u, \quad (4.3)$$

where Σ_u is the covariance matrix of the idiosyncratic error at any time t .

Under the assumption that Σ_u is sparse with its eigenvalues bounded away from zero and infinity, the population covariance exhibit a low-rank plus sparse structure. The sparsity is measured by the following quantity

$$m_p = \max_{i \leq p} \sum_{j \leq p} |\sigma_{u,ij}|^q,$$

for some $q \in [0, 1]$ (Bickel and Levina, 2008). In particular, m_p with $q = 0$ is the maximum number of nonzero elements in each row of Σ_u .

In order to estimate the true covariance matrix with the above factor structure, Fan et al. (2013a) proposed a method called ‘‘POET’’ to recover the unknown factor matrix as well as the factor loadings. The idea is simply to first decompose the sample covariance matrix into the spike part and non-spike part and estimate them separately. Specifically, let $\hat{\Sigma} = T^{-1}\mathbf{Y}\mathbf{Y}'$ and $\{\hat{\lambda}_j\}$ and $\{\hat{\xi}_j\}$ be its corresponding eigenvalues and eigenvectors. They define

$$\hat{\Sigma}^\top = \sum_{j=1}^m \hat{\lambda}_j \hat{\xi}_j \hat{\xi}_j' + \hat{\Sigma}_u^\top, \quad (4.4)$$

where $\hat{\Sigma}_u^\top$ is the matrix after applying thresholding method (Bickel and Levina, 2008) to $\hat{\Sigma}_u = \hat{\Sigma} - \sum_{j=1}^m \hat{\lambda}_j \hat{\xi}_j \hat{\xi}_j'$.

They showed that the above estimation procedure is equivalent to the least square approach that minimizes

$$(\hat{\mathbf{B}}, \hat{\mathbf{F}}) = \arg \min_{\mathbf{B}, \mathbf{F}} \|\mathbf{Y} - \mathbf{B}\mathbf{F}'\|_F^2 \text{ s.t. } \frac{1}{T}\mathbf{F}'\mathbf{F} = \mathbf{I}_m, \mathbf{B}'\mathbf{B} \text{ is diagonal.} \quad (4.5)$$

The columns of $\hat{\mathbf{F}}/\sqrt{T}$ are the eigenvectors corresponding to the m largest eigenvalues of the $T \times T$ matrix $T^{-1}\mathbf{Y}'\mathbf{Y}$ and $\hat{\mathbf{B}} = T^{-1}\mathbf{Y}\hat{\mathbf{F}}$. After \mathbf{B} and \mathbf{F} are estimated, the sample covariance of $\hat{\mathbf{U}} = \mathbf{Y} - \hat{\mathbf{B}}\hat{\mathbf{F}}'$ can be formed: $\hat{\Sigma}_u = T^{-1}\hat{\mathbf{U}}\hat{\mathbf{U}}'$. Finally thresholding is applied to $\hat{\Sigma}_u$ to generate $\hat{\Sigma}_u^\top = (\hat{\sigma}_{u,ij}^\top)_{p \times p}$, where

$$\hat{\sigma}_{u,ij}^\top = \begin{cases} \hat{\sigma}_{u,ij}, & i = j, \\ s_{ij}(\hat{\sigma}_{u,ij})I(|\hat{\sigma}_{u,ij}| \geq \tau_{ij}), & i \neq j, \end{cases} \quad (4.6)$$

where $s_{ij}(\cdot)$ is the generalized shrinkage function (Antoniadis and Fan, 2001; Rothman et al., 2009) and $\tau_{ij} = \tau(\hat{\sigma}_{u,ii}\hat{\sigma}_{u,jj})^{1/2}$ is the entry-dependent threshold. The above adaptive threshold corresponds to applying thresholding with parameter τ to the correlation matrix of $\hat{\Sigma}_u$. The positive parameter τ will be determined later.

Fan et al. (2013a) showed that under Assumptions 7.1 - 7.4 listed in Section 6,

$$\|\hat{\Sigma}^\top - \Sigma\|_{\Sigma, F} = O_P\left(\frac{\sqrt{p} \log p}{T} + m_p \left(\frac{\log p}{T} + \frac{1}{p}\right)^{(1-q)/2}\right), \quad (4.7)$$

where $\|\mathbf{A}\|_{\Sigma, F} = p^{-1/2}\|\Sigma^{-1/2}\mathbf{A}\Sigma^{-1/2}\|_F$ and $\|\cdot\|_F$ is the Frobenius norm. Note that

$$\|\hat{\Sigma}^\top - \Sigma\|_{\Sigma, F} = p^{-1/2}\|\Sigma^{-1/2}\hat{\Sigma}^\top\Sigma^{-1/2} - \mathbf{I}_p\|_F,$$

which measures the relative errors in Frobenius norm. A more natural metric is relative errors under the operator norm $\|\mathbf{A}\|_{\Sigma} = p^{-1/2}\|\Sigma^{-1/2}\mathbf{A}\Sigma^{-1/2}\|$, which can not be obtained by using the technical device of Fan et al. (2013a). Using our new tools, we will establish such a result under somewhat weaker conditions than their pervasiveness assumption.

Note that the relative error convergence is particularly meaningful for spiked covariance matrix, as eigenvalues are in different scales.

4.2 Shrinkage POET under relative spectral norm

The discussion above reveals that POET has several drawbacks. First, the spike size has to be of order $\Theta(p)$ which rules out relatively weaker factors. Second, it is well known that the empirical eigenvalues are inconsistent if the spike eigenvalues do not significantly dominate the non-spike part. Therefore, proper correction or shrinkage is needed. See a recent paper by Donoho et al. (2014) for optimal shrinkage of eigenvalues.

Regarding to the first drawback, we relax the assumption $\|p^{-1}\mathbf{B}'\mathbf{B} - \mathbf{\Omega}\| = o(1)$ in Assumption 7.1 to the following weaker assumption.

Assumption 4.1. $\|\mathbf{\Lambda}_A^{-1/2}\mathbf{B}'\mathbf{B}\mathbf{\Lambda}_A^{-1/2} - \mathbf{\Omega}\| = o(1)$ for some $\mathbf{\Omega}$ with eigenvalues bounded from above and below, where $\mathbf{\Lambda}_A = \text{diag}(\lambda_1, \dots, \lambda_m)$ as the notation before. In addition, we assume $\sqrt{p}(\log T)^{1/r_2} = o(\lambda_m)$ and λ_1/λ_m is bounded from above and below.

This assumption does not require the first m eigenvalues of $\mathbf{\Sigma}$ take on any specific rates. They can still be much smaller than p . As we assume bounded $\|\mathbf{\Sigma}_u\|$, the assumption $\lambda_m \rightarrow \infty$ is also imposed to avoid the issue of identifiability. When λ_m does not diverge, more sophisticated condition is needed to guarantee identifiability (Chandrasekaran et al., 2011).

In order to handle the second drawback, we propose the Shrinkage POET (S-POET) method. Inspired by (3.1), the shrinkage POET modifies the first part in POET estimator (4.4) as follows:

$$\hat{\mathbf{\Sigma}}^S = \sum_{j=1}^m \hat{\lambda}_j^S \hat{\boldsymbol{\xi}}_j \hat{\boldsymbol{\xi}}_j' + \hat{\mathbf{\Sigma}}_u^\top, \quad (4.8)$$

where $\hat{\lambda}_j^S = \hat{\lambda}_j / (1 + \bar{c}c_j)$. When c_j is unknown, we use $\hat{\lambda}_j^S = \max\{\hat{\lambda}_j - \bar{c}p/n, 0\}$, which is simply a soft thresholding correction. Obviously if $\hat{\lambda}_j$ is sufficiently large, $\hat{\lambda}_j^S/\lambda_j \approx \hat{\lambda}_j/\lambda_j - \bar{c}c_j = 1 + o_P(1)$.

We also need the following additional assumption:

Assumption 4.2. (i) $\{\mathbf{u}_t, \mathbf{f}_t\}_{t \geq 1}$ are independently and identically distributed with $\mathbb{E}[u_{it}] = \mathbb{E}[u_{it}f_{jt}] = 0$ for all $i \leq p, j \leq m$ and $t \leq T$.

(ii) There exist positive constants c_1 and c_2 such that $\lambda_{\min}(\mathbf{\Sigma}_u) > c_1$, $\|\mathbf{\Sigma}_u\|_\infty < c_2$, and $\min_{i,j} \text{Var}(u_{it}u_{jt}) > c_1$.

(iii) There exist positive constants r_1, r_2, b_1 and b_2 such that for $s > 0, i \leq p, j \leq m$,

$$\mathbb{P}(|u_{it}| > s) \leq \exp(-(s/b_1)^{r_1}) \quad \text{and} \quad \mathbb{P}(|f_{jt}| > s) \leq \exp(-(s/b_2)^{r_2}).$$

(iv) There exists $M > 0$ such that for all $i \leq p, j \leq m$, $|b_{ij}| \leq M\sqrt{\lambda_j/p}$.

The first three conditions are common in factor model literature. If we write $\mathbf{B} = (\tilde{\mathbf{b}}_1, \dots, \tilde{\mathbf{b}}_m)$, by Weyl's inequality we have $\max_{1 \leq j \leq m} \|\tilde{\mathbf{b}}_j\|^2 / \lambda_j \leq 1 + \|\boldsymbol{\Sigma}_u\| / \lambda_j = 1 + o(1)$. Thus it is also reasonable to assume the magnitude $|b_{ij}|$ of factor loadings is of order $\sqrt{\lambda_j/p}$ in the last condition.

Now we are ready to investigate $\|\hat{\boldsymbol{\Sigma}}^S - \boldsymbol{\Sigma}\|_{\Sigma}$. With a bit abuse of notation, suppose the SVD decomposition of $\boldsymbol{\Sigma}$,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Gamma}_{p \times m} & \boldsymbol{\Omega}_{p \times (p-m)} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Lambda}_{m \times m} & \\ & \boldsymbol{\Theta}_{(p-m) \times (p-m)} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Gamma}' \\ \boldsymbol{\Omega}' \end{pmatrix}.$$

Then obviously

$$\begin{aligned} \|\hat{\boldsymbol{\Sigma}}^S - \boldsymbol{\Sigma}\|_{\Sigma} &\leq \left\| \boldsymbol{\Sigma}^{-\frac{1}{2}} (\hat{\boldsymbol{\Gamma}} \hat{\boldsymbol{\Lambda}}^S \hat{\boldsymbol{\Gamma}}' - \mathbf{B} \mathbf{B}') \boldsymbol{\Sigma}^{-\frac{1}{2}} \right\| + \|\boldsymbol{\Sigma}^{-\frac{1}{2}} (\hat{\boldsymbol{\Sigma}}_u^{\top} - \boldsymbol{\Sigma}_u) \boldsymbol{\Sigma}^{-\frac{1}{2}}\| \\ &=: \Delta_L + \Delta_S, \end{aligned} \quad (4.9)$$

and

$$\Delta_S \leq \|\boldsymbol{\Sigma}^{-1}\| \|\hat{\boldsymbol{\Sigma}}_u^{\top} - \boldsymbol{\Sigma}_u\| \leq C \|\hat{\boldsymbol{\Sigma}}_u^{\top} - \boldsymbol{\Sigma}_u\|. \quad (4.10)$$

It can be shown

$$\begin{aligned} \Delta_L &= \left\| \begin{pmatrix} \boldsymbol{\Lambda}^{-\frac{1}{2}} \boldsymbol{\Gamma}' \\ \boldsymbol{\Theta}^{-\frac{1}{2}} \boldsymbol{\Omega}' \end{pmatrix} (\hat{\boldsymbol{\Gamma}} \hat{\boldsymbol{\Lambda}}^S \hat{\boldsymbol{\Gamma}}' - \mathbf{B} \mathbf{B}') \begin{pmatrix} \boldsymbol{\Gamma} \boldsymbol{\Lambda}^{-\frac{1}{2}} & \boldsymbol{\Omega} \boldsymbol{\Theta}^{-\frac{1}{2}} \end{pmatrix} \right\| \\ &\leq \Delta_{L1} + \Delta_{L2}, \end{aligned} \quad (4.11)$$

where $\Delta_{L1} = \|\boldsymbol{\Lambda}^{-\frac{1}{2}} \boldsymbol{\Gamma}' (\hat{\boldsymbol{\Gamma}} \hat{\boldsymbol{\Lambda}}^S \hat{\boldsymbol{\Gamma}}' - \mathbf{B} \mathbf{B}') \boldsymbol{\Gamma} \boldsymbol{\Lambda}^{-\frac{1}{2}}\|$ and $\Delta_{L2} = \|\boldsymbol{\Theta}^{-\frac{1}{2}} \boldsymbol{\Omega}' (\hat{\boldsymbol{\Gamma}} \hat{\boldsymbol{\Lambda}}^S \hat{\boldsymbol{\Gamma}}' - \mathbf{B} \mathbf{B}') \boldsymbol{\Omega} \boldsymbol{\Theta}^{-\frac{1}{2}}\|$. Thus in order to find the convergence rate of relative spectral norm, we need to consider the terms Δ_{L1} , Δ_{L2} and Δ_S separately. Notice that Δ_{L1} measures the relative error of the estimated spiked eigenvalues, Δ_{L2} reflects the goodness of the estimated eigenvectors, and Δ_S controls the error of estimating the sparse idiosyncratic covariance matrix. The following vital theorem reveals the rate of each term.

Theorem 4.1. *Under Assumptions 2.1, 2.2, 2.3, 4.1 and 4.2, if $p \log p > \max\{T(\log T)^{4/r_2}, T(\log(pT))^{2/r_1}\}$, we have*

$$\Delta_{L1} = O_P\left(c_m \sqrt{T/p} + T^{-1/2}\right), \quad \Delta_{L2} = O_P\left(\frac{p}{T} + \frac{1}{\lambda_m}\right),$$

and by applying adaptive thresholding estimator (4.6) with

$$\tau_{ij} = C \omega_T (\hat{\sigma}_{u,ii} \hat{\sigma}_{u,jj})^{1/2}, \quad \text{and} \quad \omega_T = \sqrt{\log p/T} + \sqrt{1/p},$$

we have

$$\Delta_S = O_P\left(m_p \omega_T^{1-q}\right).$$

Combining the three terms, $\|\hat{\Sigma}^S - \Sigma\|_{\Sigma} = O_P(\Delta_{L1} + \Delta_{L2} + \Delta_S)$.

The relative error convergence in spectral norm characterizes the accuracy of estimation for spiked covariance matrix. In contrast with the previous results on Frobenius or max norm, this is the first time that the relative rate under spectral norm is derived. When $\lambda_m \asymp p$ and $q = 0$, we have

$$\|\hat{\Sigma}^S - \Sigma\|_{\Sigma} = O_P\left(\frac{p}{T} + m_p \sqrt{\frac{\log p}{T}} + \sqrt{\frac{1}{p}}\right).$$

Comparing the rate with (4.7), we see the difference under two different norms. The term $\sqrt{p} \log p/T$ in (4.7) is enlarged to rate p/T , which is due to the incoherence of the low-rank signal matrix and sparse error matrix spaces. Specifically this rate comes from Δ_{L2} . If we care only the relative error of the low-rank and sparse matrix spaces separately, we should only consider Δ_{L1} and Δ_S .

If $c_j = 0$ as in Fan et al. (2013a), the proposed S-POET method does not shrink the spiked empirical eigenvalues. However, when we have semi-weak factors (Fan et al., 2014) whose corresponding eigenvalues are as weak as of order p/T , shrinkage is necessary to guarantee the convergence of Δ_{L1} . On the other hand, if instead POET is applied to estimate covariance matrix, $\Delta_{L1} = O_P(p/(\lambda_m T) + T^{-1/2})$ which is only bounded. However since the empirical eigenvectors are not corrected, POET and S-POET attain the same rate for Δ_{L2} , which actually dominates Δ_{L1} and Δ_S in high dimensional setting. Nevertheless, as to be seen in the simulation studies, shrinkage POET can stabilize the estimator and improve the estimation accuracy. For this reason, we recommend S-POET in practice.

In the next two subsections, we assume error and factors are Gaussian distributed and consider a more specific case where $\lambda_j \propto p^{\alpha}$ for $\alpha > 1/2$, $j = 1, \dots, m$ and sample size $T \geq Cp^{\beta}$ with $0 \leq \beta < 1, \alpha + \beta > 1$. In this case $p/(T\lambda_j) \leq p/(p^{\alpha+\beta}) \rightarrow 0$, thus S-POET and POET are approximately the same. The stated results are achievable for POET too. We will see that POET performs almost as well as S-POET when the eigenvalues are sufficiently spiked.

4.3 Portfolio risk management

Portfolio allocation and risk management have been a fundamental problem in finance since Markowitz (1952)'s groundbreaking work on minimizing the volatility of portfolios with a given expected return. Specifically, the risk of a given portfolio with allocation vector \mathbf{w} is conventionally measured by its variance $\mathbf{w}'\Sigma\mathbf{w}$, where Σ is the volatility (covariance) matrix of the returns of underlying assets. Therefore in order to estimate portfolio risk, it is essential to estimate the covariance matrix Σ , although the task is challenging due to curse of dimensionality. Thus, factor models can be introduced to

reduce the dimensionality. This was the idea of Fan et al. (2013b) in which they used POET estimator to estimate Σ . However, the basic method for bounding the risk error $|\mathbf{w}'\hat{\Sigma}\mathbf{w} - \mathbf{w}'\Sigma\mathbf{w}|$ in their paper as well as another earlier paper of similar topic (Fan et al., 2012) was

$$|\mathbf{w}'\hat{\Sigma}\mathbf{w} - \mathbf{w}'\Sigma\mathbf{w}| \leq \|\mathbf{w}\|_1^2 \|\hat{\Sigma} - \Sigma\|_{\max}.$$

They assumed that the gross exposure of the portfolio is bounded, mathematically $\|\mathbf{w}\|_1 = O(1)$, which made it possible to only focus on the max error norm. Technically, when p is large, $\mathbf{w}'\Sigma\mathbf{w}$ can converge to zero faster than the rate of $\|\hat{\Sigma} - \Sigma\|_{\max}$. What an investor cares mostly is the relative risk error $\text{RE}(\mathbf{w}) = |\mathbf{w}'\hat{\Sigma}\mathbf{w}/\mathbf{w}'\Sigma\mathbf{w} - 1|$. Often \mathbf{w} is a data-driven investment strategy, which is a random variable itself. Regardless of what \mathbf{w} is,

$$\max_{\mathbf{w}} \text{RE}(\mathbf{w}) = \|\hat{\Sigma} - \Sigma\|_{\Sigma},$$

which does not converge by Theorem 4.1. The question is what kind of portfolio \mathbf{w} will make the relative error converge. Decompose \mathbf{w} as a linear combination of the eigenvectors of Σ , namely $\mathbf{w} = (\Gamma, \Omega)\boldsymbol{\eta}$ and $\boldsymbol{\eta} = (\boldsymbol{\eta}'_A, \boldsymbol{\eta}'_B)'$. We have the following useful result for risk management.

Theorem 4.2. *Under Assumptions 2.1, 2.2, 4.1, 4.2 and the factor model (4.1) with Gaussian noises and factors, if there exists $C_1 > 0$ such that $\|\boldsymbol{\eta}_B\|_1 \leq C_1$, and assume $\lambda_j \propto p^\alpha$ and $T \geq Cp^\beta$ for $\alpha > 1/2, 0 \leq \beta < 1, \alpha + \beta > 1$ and $j = 1, \dots, m$, then the relative risk error is of order*

$$\text{RE}(\mathbf{w}) = \left| \frac{\mathbf{w}'\hat{\Sigma}^S\mathbf{w}}{\mathbf{w}'\Sigma\mathbf{w}} - 1 \right| = O_P\left(T^{-\min\{\frac{\alpha+\beta-1}{\beta}, \frac{1}{2}\}} + m_p w_T^{1-q}\right).$$

If we further assume $\|\boldsymbol{\eta}_A\| \geq C_2$, $\text{RE}(\mathbf{w}) = O_P(T^{-1/2} + m_p w_T^{1-q})$.

The condition $\|\boldsymbol{\eta}_B\|_1 \leq C_1$ is obviously much weaker than $\|\mathbf{w}\|_1 = O(1)$. It does not limit the total exposure of investor's position, but only put constraint on investment of the non-spiked section.

4.4 Estimation of false discovery proportion

Another important application of the factor model is the estimation of false discovery proportion. For simplicity, we assume Gaussian data $\mathbf{X}_i \sim N(\boldsymbol{\mu}, \Sigma)$ with an unknown correlation matrix Σ and wish to test separately which coordinates of $\boldsymbol{\mu}$ are nonvanishing. Consider the test statistic $\mathbf{Z} = \sqrt{n}\bar{\mathbf{X}}$ where $\bar{\mathbf{X}}$ is the sample mean of all data. Then $\mathbf{Z} \sim N(\boldsymbol{\mu}^*, \Sigma)$ with $\boldsymbol{\mu}^* = \sqrt{n}\boldsymbol{\mu}$ and the problem is to test

$$H_{0j} : \mu_j^* = 0 \quad \text{vs} \quad H_{1j} : \mu_j^* \neq 0.$$

Define the number of discoveries $R(t) = \#\{j : P_j \leq t\}$ and the number of false discoveries $V(t) = \#\{\text{true null} : P_j \leq t\}$, where P_j is the p-value associated with the j th test. Note that $R(t)$ is observable while $V(t)$ needs to be estimated. The false discovery proportion (FDP) is defined as $\text{FDP}(t) = V(t)/R(t)$.

Recently Fan and Han (2013) proposed to employ the factor structure

$$\boldsymbol{\Sigma} = \mathbf{B}\mathbf{B}' + \mathbf{A}, \quad (4.12)$$

where $\mathbf{B} = (\sqrt{\lambda_1}\boldsymbol{\xi}_1, \dots, \sqrt{\lambda_m}\boldsymbol{\xi}_m)$ and $\lambda_j, \boldsymbol{\xi}_j$ are respectively the j th eigenvalue and eigenvector of $\boldsymbol{\Sigma}$ as before. For simplicity, assume the maximal number of nonzero elements of each row of \mathbf{A} is bounded. Then \mathbf{Z} can be stochastically decomposed as

$$\mathbf{Z} = \boldsymbol{\mu}^* + \mathbf{B}\mathbf{W} + \mathbf{K},$$

where $\mathbf{W} \sim N(\mathbf{0}, \mathbf{I}_m)$ are m common factors and $\mathbf{K} \sim N(\mathbf{0}, \mathbf{A})$ independent of \mathbf{W} are the idiosyncratic errors. In Fan and Han (2013), they demonstrated that a good approximation for $\text{FDP}(t)$ is

$$\text{FDP}_A(t) = \sum_{i=1}^p [\Phi(a_i(z_{t/2} + \eta_i)) + \Phi(a_i(z_{t/2} - \eta_i))]/R(t), \quad (4.13)$$

where $z_{t/2}$ is the $t/2$ -quantile of the standard normal distribution, $a_i = (1 - \|\mathbf{b}_i\|^2)^{-1/2}$, $\eta_i = \mathbf{b}_i'\mathbf{W}$ and \mathbf{b}_i' is the i th row of \mathbf{B} .

Realized factors \mathbf{W} and the loading matrix \mathbf{B} are typically unknown. If a generic estimator $\hat{\boldsymbol{\Sigma}}$ is provided, then we are able to estimate \mathbf{B} and thus \mathbf{b}_i from its empirical eigenvalues and eigenvectors $\hat{\lambda}_j$'s and $\hat{\boldsymbol{\xi}}_j$'s. \mathbf{W} can be estimated by the least-squares estimate $\hat{\mathbf{W}} = (\hat{\mathbf{B}}'\hat{\mathbf{B}})^{-1}\hat{\mathbf{B}}'\mathbf{Z}$. Fan and Han (2013) proposed the following estimator for $\text{FDP}_A(t)$:

$$\widehat{\text{FDP}}_U(t) = \sum_{i=1}^p [\Phi(\hat{a}_i(z_{t/2} + \hat{\eta}_i)) + \Phi(\hat{a}_i(z_{t/2} - \hat{\eta}_i))]/R(t), \quad (4.14)$$

where $\hat{a}_i = (1 - \|\hat{\mathbf{b}}_i\|^2)^{-1/2}$ and $\hat{\eta}_i = \hat{\mathbf{b}}_i'\hat{\mathbf{W}}$. The following assumptions are in their paper.

Assumption 4.3. *There exists a constant $h > 0$ such that (i) $R(t)/p > hp^{-\theta}$ for $h > 0$ and $\theta \geq 0$ as $p \rightarrow \infty$ and (ii) $\hat{a}_i \leq h, a_i \leq h$ for all $i = 1, \dots, p$.*

They showed that if $\hat{\boldsymbol{\Sigma}}$ is based on the POET estimator with a spike size $\lambda_m \asymp p$, under Assumptions 7.1 - 7.4 together with Assumption 4.3,

$$|\widehat{\text{FDP}}_{U,POET}(t) - \text{FDP}_A(t)| = O_P\left(p^\theta \left(\sqrt{\frac{\log p}{T}} + \frac{\|\boldsymbol{\mu}^*\|}{\sqrt{p}}\right)\right). \quad (4.15)$$

Again we can relax the assumption of spikeness from order p to much weaker Assumption

4.1. Since Σ is a correlation matrix, $\lambda_1 \leq \text{tr}(\Sigma) = p$. This, together with Assumption 4.1, leads us to consider that all leading eigenvalues are of order proportional to p^α for $1/2 < \alpha \leq 1$.

Now apply the proposed S-POET method to obtain $\hat{\Sigma}^S$ and use it for FDP estimation. Then we have the following theorem.

Theorem 4.3. *If Assumptions 2.1, 2.2, 4.1, 4.2, 4.3 are applied to Gaussian independent data $\mathbf{X}_i \sim N(\boldsymbol{\mu}, \Sigma)$, and $\lambda_j \propto p^\alpha$, $T \geq Cp^\beta$ for $1/2 < \alpha \leq 1, 0 \leq \beta < 1, \alpha + \beta > 1$, $j = 1, \dots, m$, we have*

$$|\widehat{FDP}_{U,SPOET}(t) - FDP_A(t)| = O_P\left(p^\theta(\|\boldsymbol{\mu}^*\|p^{-\frac{1}{2}} + T^{-\min\{\frac{\alpha+\beta-1}{\beta}, \frac{1}{2}\}})\right).$$

Comparing the result with (4.15), this convergence rate attained by S-POET is more general than the rate achieved before. The only difference is the second term, which is $O(T^{-1/2})$ if $\alpha + \frac{1}{2}\beta \geq 1$ and $T^{-(\alpha+\beta-1)/\beta}$ if $\alpha + \frac{1}{2}\beta < 1$. But we relax the condition from $\alpha = 1$ in Fan and Han (2013) to $\alpha \in (1/2, 1]$, which means a weaker signal than order p is actually allowed to obtain a consistent estimate of false discovery proportion.

5 Simulations

We conducted some simulations to demonstrate the finite sample behaviors of eigenstructure, the performance of S-POET, and validity of applying it to estimating false discovery proportion.

5.1 Eigen-structure

In this simulation, we set $n = 50$, $p = 500$ and $\Sigma = \text{diag}(50, 20, 10, 1, \dots, 1)$, which has three spikes ($m = 3$) $\lambda_1 = 50$, $\lambda_2 = 20$, $\lambda_3 = 10$ and corresponding constants $c_1 = 0.2$, $c_2 = 0.5$, $c_3 = 1$. Data was generated from multivariate Gaussian. The number of simulations is 1000. The histograms of the standardized empirical eigenvalues $\sqrt{n/2}(\hat{\lambda}_j/\lambda_j - 1 - c_j)$, and their associated asymptotic distributions (standard normal) are plotted in Figure 1. The approximations are very good even for this low sample size $n = 50$.

Figure 2 shows the histograms of $\sqrt{n}(\hat{\boldsymbol{\xi}}_{jA} - \mathbf{e}_{jA}/\sqrt{1+c_j})$ for the first three elements (the spiked part) of the first three eigenvectors. According to the asymptotic result, the values in the diagonal position should stochastically converge to 0 as observed. On the other hand, plots in the off-diagonal position should converge in distribution to $N(0, 1)$ for $k \neq j$, which is indeed the case. We also report the correlations between the first three elements for the three eigenvectors based on those 1000 repetitions in Table 1. The correlations are all quite close to 0, which is consistent with the theory.

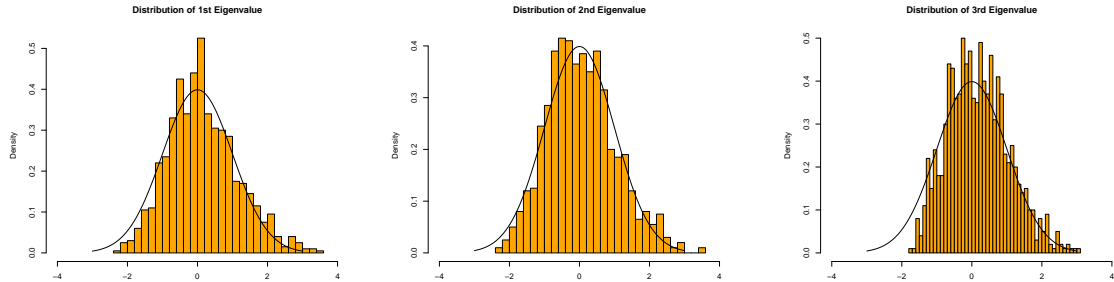


Figure 1: Behaviors of empirical eigenvalues. The empirical distributions of $\sqrt{n/2}(\hat{\lambda}_j/\lambda_j - 1 - c_j)$ for $j = 1, 2, 3$ are compared with their asymptotic distributions $N(0, 1)$.

Table 1: The correlations between the first three elements for each of the three empirical eigenvectors based on 1000 repetitions

	1st & 2nd elements	1st & 3rd elements	2nd & 3rd elements
1st Eigenvector	0.00156	-0.00192	-0.04112
2nd Eigenvector	-0.02318	-0.00403	0.01483
3rd Eigenvector	-0.02529	-0.04004	0.12524

For the normalized nonspiked part $\hat{\xi}_{jB}/\|\hat{\xi}_{jB}\|$, it should be distributed uniformly over the unit sphere. This can be tested by the recent results of Cai et al. (2013a). For any n data points $\mathbf{X}_1, \dots, \mathbf{X}_n$ on p -dimensional sphere, define the normalized empirical distribution of angles of each pair of vectors as

$$\mu_{n,p} = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \delta_{\sqrt{p-2}(\pi/2 - \Theta_{ij})},$$

where $\Theta_{ij} \in [0, \pi]$ is the angle between vectors \mathbf{X}_i and \mathbf{X}_j . When the data are generated uniformly from a sphere, $\mu_{n,p}$ converges to the standard normal distribution with probability 1 (Cai et al., 2013a). Figure 3 shows the empirical distributions of all pairwise angles of the realized $\hat{\xi}_{jB}/\|\hat{\xi}_{jB}\|$ ($j = 1, 2, 3$) in 1000 simulations. Since number of such pairwise angles is $\binom{1000}{2}$, the empirical distributions and the asymptotic distributions $N(0, 1)$ are almost identical. The normality holds even for a small subset of the angles.

In short, all the simulation results match well with the theoretical results for the ultra high dimensional regime.

5.2 Performance of S-POET

We demonstrate the effectiveness of S-POET in comparison with the POET. A similar covariance setting was used, i.e. $m = 3$ and $c_1 = 0.2, c_2 = 0.5, c_3 = 1$. The sample size T ranges from 50 to 150 and $p = \lceil T^{3/2} \rceil$. Note that when $T = 150$, $p \approx 1800$. The

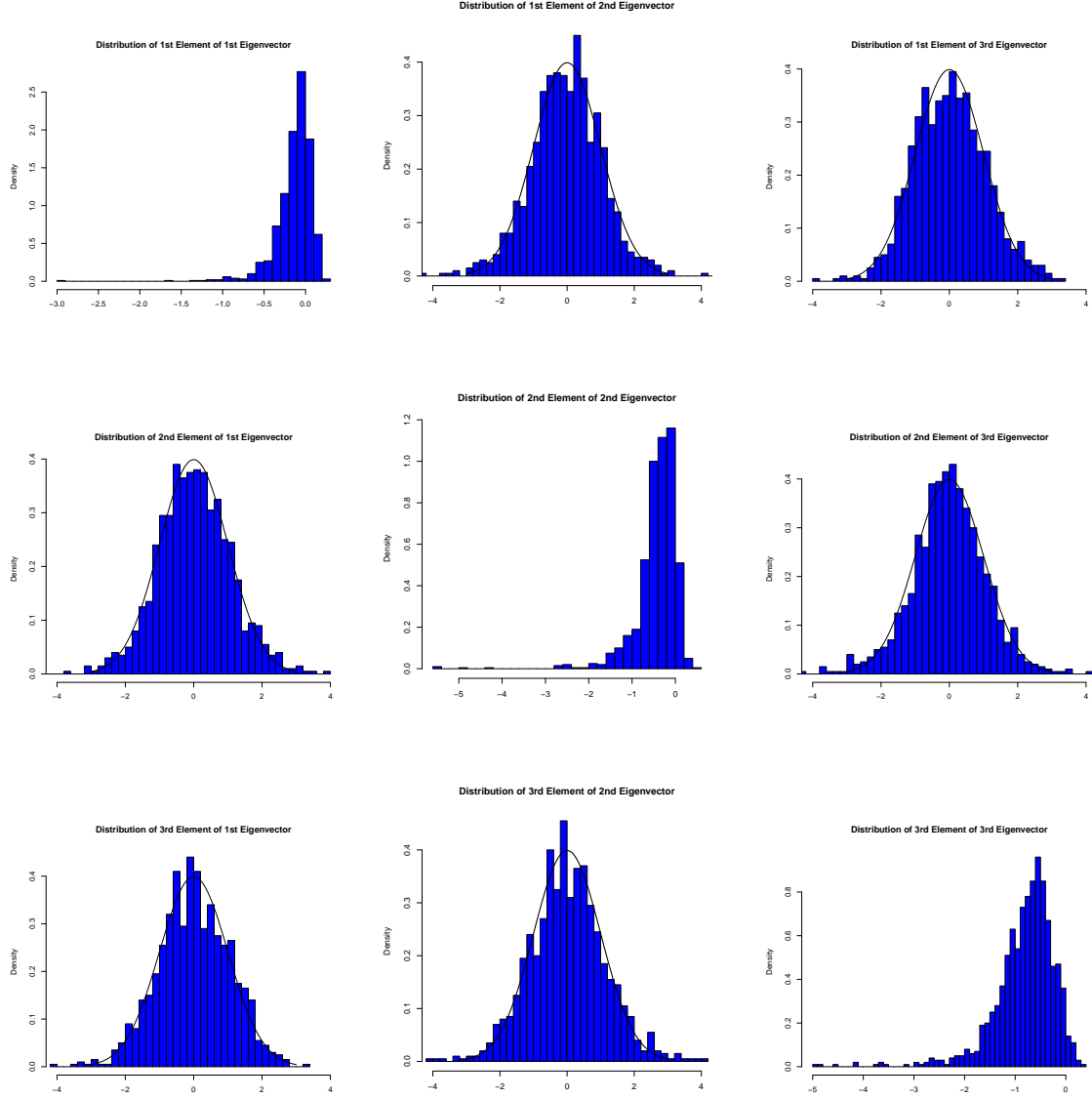


Figure 2: Behaviors of empirical eigenvectors. The histograms of the k th elements of the j th empirical vectors are depicted in the location (k, j) for $k, j \leq 3$. Off-diagonal plots of values $\sqrt{n}\hat{\xi}_{jA,k}/\sqrt{\frac{c_j c_k}{(1+c_j)(c_j-c_k)^2}}$ are compared to their asymptotic distributions $N(0, 1)$ for $k \neq j$ while diagonal plots of values $\sqrt{n}(\hat{\xi}_{jA,k} - 1/\sqrt{1+c_j})$ are compared to stochastically 0 for $k = j$.

spiked eigenvalues are determined from $p/(T\lambda_j) = c_j$ so that λ_j is of order \sqrt{T} , which is much smaller than p . For each pair of T and p , the following steps are used to generate observed data through factor model for 200 times.

- (1) Each row of \mathbf{B} is simulated from the standard multivariate normal distribution and the j th column is normalized to have norm λ_j for $j = 1, 2, 3$.
- (2) Each row of \mathbf{F} is simulated from standard multivariate normal distribution.

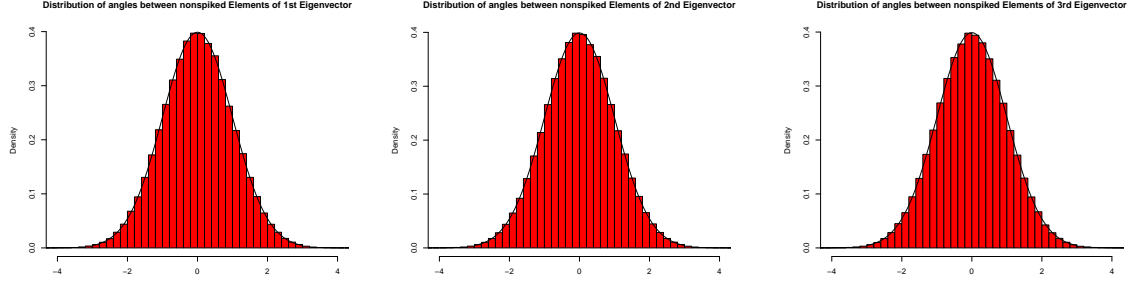


Figure 3: The empirical distributions of all pairwise angles of the 1000 realized $\hat{\xi}_{jB}/\|\hat{\xi}_{jB}\|$ ($j = 1, 2, 3$) compared with their asymptotic distributions $N(0, 1)$.

- (3) Set $\Sigma_u = \text{diag}(\sigma_1^2, \dots, \sigma_p^2)$ where σ_i 's are generated from $\text{Gamma}(\alpha, \beta)$ with $\alpha = \beta = 100$ (mean 1, standard deviation 0.1). The idiosyncratic error \mathbf{U} is simulated from $N(\mathbf{0}, \Sigma_u)$.
- (4) Compute the observed data $\mathbf{Y} = \mathbf{B}\mathbf{F}' + \mathbf{U}$.

Both S-POET and POET are applied to estimate the covariance matrix $\Sigma = \mathbf{B}\mathbf{B}' + \Sigma_u$. Their mean estimation errors over 200 simulations, measured in relative spectral norm $\|\hat{\Sigma} - \Sigma\|_{\Sigma}$, relative Frobenius norm $\|\hat{\Sigma} - \Sigma\|_{\Sigma, F}$, spectral norm $\|\hat{\Sigma} - \Sigma\|$ and max norm $\|\hat{\Sigma} - \Sigma\|_{\max}$, are reported in Figure 4. The errors for sample covariance matrix are also depicted for comparison. First notice that no matter in what norm, S-POET uniformly outperforms POET and sample covariance. It affirms the claim that shrinkage of spiked eigenvalues is necessary to maintain good performance when the spikeness is not sufficiently large. Since the low rank part is not shrunk for POET, its error under the spectral norm is comparable and even slightly larger than that of the sample covariance matrix. The error under max norm and relative Frobenius norm as expected decreases as T and p increase. However the relative error under the spectral norm does not converge: our theory shows it should increase in the order $p/T = \sqrt{T}$.

5.3 FDP estimation

In this section, we report simulation results on FDP estimation by using both POET and S-POET. The data are simulated in a similar way as in Section 5.2 with $p = 1000$ and $n = 100$. The first $m = 3$ eigenvalues have spike size proportional to p/\sqrt{n} which corresponding to $\alpha = \beta = 2/3$ in Theorem 4.3. The true FDP is calculated by using $\text{FDP}(t) = V(t)/R(t)$ with $t = 0.01$. The approximate FDP, $\text{FDP}_A(t)$, is calculated as in (4.13) with known \mathbf{B} but estimated \mathbf{W} given by $\hat{\mathbf{W}} = (\mathbf{B}\mathbf{B}')^{-1}\mathbf{B}'\mathbf{Z}$. This $\text{FDP}_A(t)$ based on a known sample covariance matrix serves as a benchmark for our estimated covariance matrix to compare with. We employ POET and S-POET to get $\widehat{\text{FDP}}_{U, \text{POET}}(t)$ and $\widehat{\text{FDP}}_{U, \text{SPOET}}(t)$.

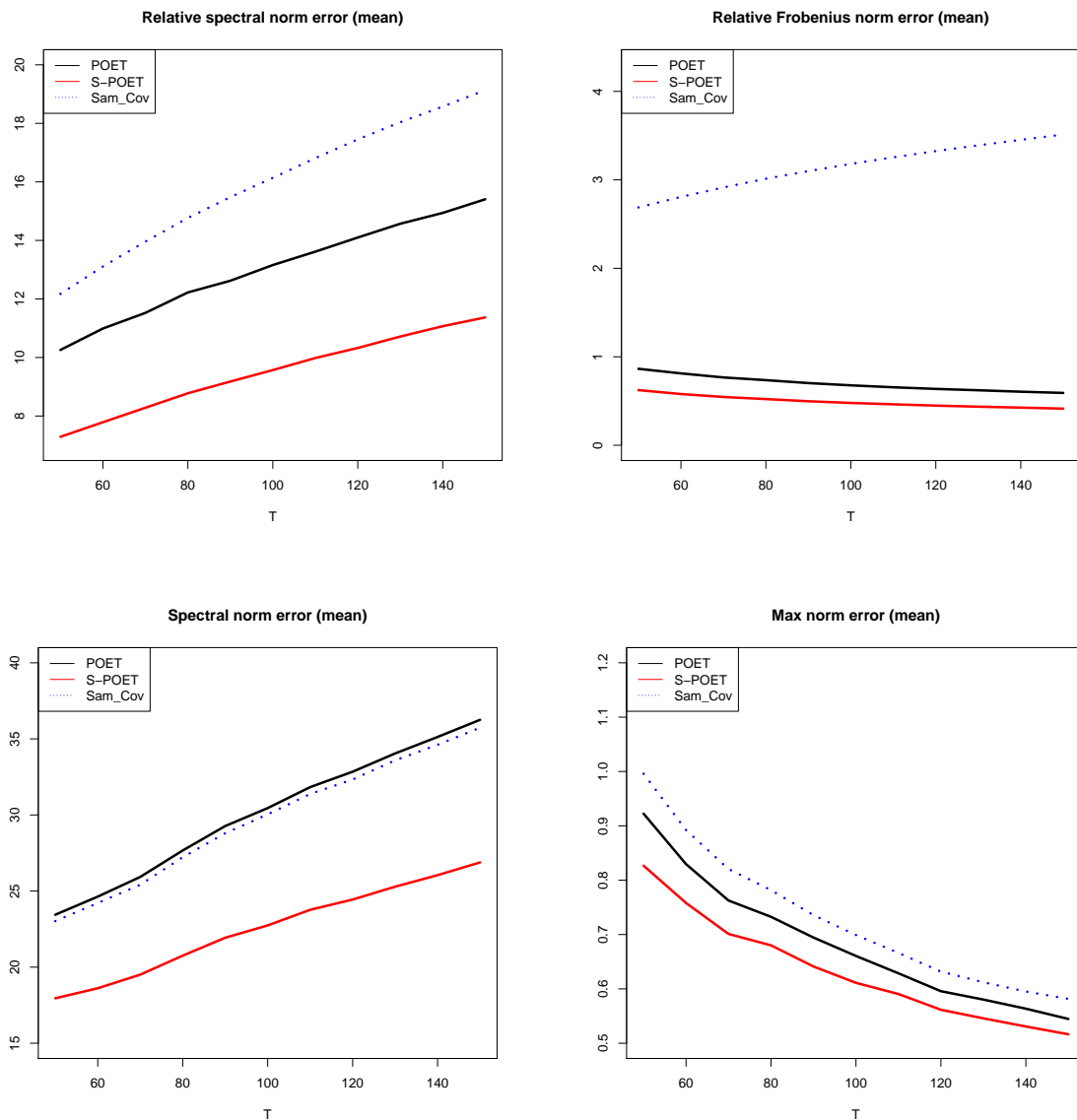


Figure 4: Estimation error of covariance matrix under respectively relative spectral, relative Frobenius, spectral and max norms using S-POET (red), POET (black) and sample covariance (blue).

In Figure 5, three scatter plots are drawn to compare $\widehat{\text{FDP}}_A(t)$, $\widehat{\text{FDP}}_{U,POET}(t)$ and $\widehat{\text{FDP}}_{U,SPOET}(t)$ with the true $\text{FDP}(t)$. The points are basically aligned along the 45 degree line, meaning that all of them are quite close to the true FDP. With the semi-strong signal $\lambda \propto p/\sqrt{n}$, although much weaker than order p , POET accomplishes the task as well as S-POET. But shrinkage definitely does not hurt. Both estimators performs as well as if we know the covariance matrix Σ , the benchmark.

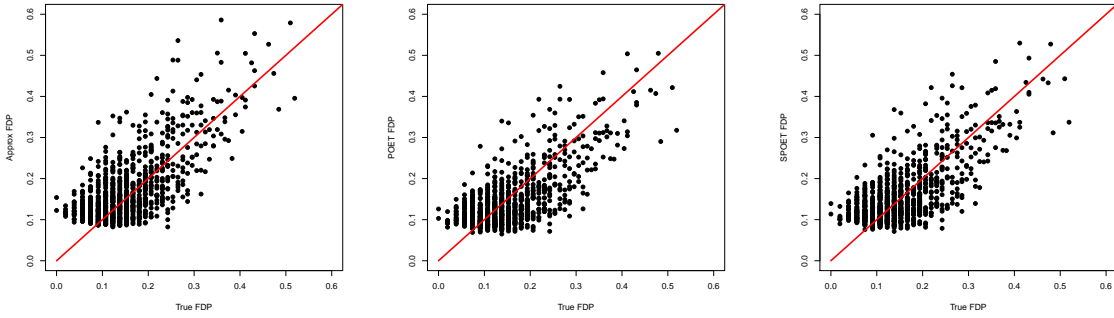


Figure 5: Comparison of estimated FDP's with true values. The left plot assumes knowledge of \mathbf{B} , the middle and right ones are corresponding to POET and S-POET methods respectively. The results are aligned along the 45-degree line, indicating the accuracy of the estimation of FDP.

6 Proofs for Section 3

6.1 Proof of Theorem 3.1

We first provide three lemmas that we will use in the proof. Lemma 6.1 is key to our proofs. It provides useful non-asymptotic upper and lower bound for the eigenvalues of weighted Wishart matrix for sub-Gaussian distributions.

Lemma 6.1. *Let $\mathbf{A}_1, \dots, \mathbf{A}_n$'s be n independent p dimensional sub-Gaussian random vectors with zero mean and identity variance with the sub-Gaussian norms bounded by a constant C_0 . Then for every $t \geq 0$, with probability at least $1 - 2 \exp(-ct^2)$, one has*

$$\bar{w} - \max\{\delta, \delta^2\} \leq \lambda_p \left(\frac{1}{n} \sum_{i=1}^n w_i \mathbf{A}_i \mathbf{A}_i' \right) \leq \lambda_1 \left(\frac{1}{n} \sum_{i=1}^n w_i \mathbf{A}_i \mathbf{A}_i' \right) \leq \bar{w} + \max\{\delta, \delta^2\}.$$

where $\delta = C\sqrt{p/n} + t/\sqrt{n}$ for constants $C, c > 0$, depending on C_0 . Here $|w_i|$'s is bounded from above and below for all i and $\bar{w} = n^{-1} \sum_{i=1}^n w_i$.

The above lemma is the extension of the classical Davidson-Szarek bound [Theorem II.7 of Davidson and Szarek (2001)] to the weighted sample covariance with sub-Gaussian distribution. It was shown by Vershynin (2010) that the conclusion holds with $w_i = 1$ for all i . With similar techniques to those developed in Vershynin (2010) we can obtain the above lemma for general bounded weights. The details are omitted.

Now in order to prove the theorem, let us define two quantities and treat them separately in the following two lemmas. Let

$$\mathbf{A} = n^{-1} \sum_{j=1}^m \lambda_j \mathbf{Z}_j \mathbf{Z}_j', \quad \text{and} \quad \mathbf{B} = n^{-1} \sum_{j=m+1}^p \lambda_j \mathbf{Z}_j \mathbf{Z}_j',$$

where \mathbf{Z}_j is columns of $\mathbf{X}\mathbf{\Lambda}^{-\frac{1}{2}}$. Then,

$$\tilde{\Sigma} = \frac{1}{n} \sum_{j=1}^p \lambda_j \mathbf{Z}_j \mathbf{Z}_j' = \mathbf{A} + \mathbf{B}. \quad (6.1)$$

Lemma 6.2. *Under Assumptions 2.1 - 2.3, as $n \rightarrow \infty$,*

$$\sqrt{n} \left(\lambda_j(\mathbf{A}) / \lambda_j - 1 \right) \stackrel{d}{\Rightarrow} N(0, \kappa_j - 1), \quad \text{for } j = 1, \dots, m.$$

In addition, they are asymptotically independent.

Proof. First assume all the λ_j 's for $1 \leq j \leq m$ have the same order. Note that $\lambda_j^{-1} \mathbf{A} = n^{-1} \sum_{i=1}^m (\lambda_i / \lambda_j) \mathbf{Z}_i \mathbf{Z}_i'$ has the same eigenvalues as matrix $\lambda_j^{-1} \tilde{\mathbf{A}} = n^{-1} \tilde{\mathbf{Z}}' \tilde{\mathbf{Z}}$, where $\tilde{\mathbf{Z}}$ is an $n \times m$ matrix with i.i.d. rows, which are sub-Gaussian distributed with mean $\mathbf{0}$ and variance $\mathbf{\Lambda} / \lambda_j$. We normalize matrices by a factor of λ_j , since spiked eigenvalues diverge. Therefore, we are in the low dimensional situation as Theorem 1 of Anderson (1963). The result therein can be easily extended from the Gaussian distribution to sub-Gaussian case. The limiting distribution of $\lambda_j(\tilde{\mathbf{A}})$ for $j = 1, \dots, m$ are still independent and

$$\sqrt{n} (\lambda_j(\mathbf{A}) / \lambda_j - 1) = \sqrt{n} (\lambda_j(\tilde{\mathbf{A}}) / \lambda_j - 1) \stackrel{d}{\Rightarrow} N(0, (\kappa_j - 1) \lambda_j^2 / \lambda_j^2).$$

The only difference is that the kurtosis of Gaussian is replaced by that of sub-Gaussian distribution. The lemma follows easily.

If the spikes are not of the same order, assume for $k > m_1$, $\lambda_k / \lambda_j = o(1)$. By Weyl's inequality, $\lambda_j(\lambda_j^{-1} \mathbf{A}) = \lambda_j(n^{-1} \sum_{i=1}^{m_1} (\lambda_i / \lambda_j) \mathbf{Z}_i \mathbf{Z}_i') + o_P(n^{-1/2})$, so that the terms $k > m_1$ does not contribute to the asymptotic distribution. According to Anderson (1963), the limiting distribution of the j^{th} eigenvalue of $\lambda_j^{-1} \mathbf{A}$ does not depend on the order of other eigenvalues. Therefore its limiting distribution is well defined and as derived above. \square

Lemma 6.3. *Under Assumptions 2.1 - 2.3, for $j = 1, \dots, m$, we have*

$$\lambda_k(\mathbf{B}) / \lambda_j = \bar{c} c_j + O_P(c_j \sqrt{n/p}) + o_P(n^{-\frac{1}{2}}), \quad \text{for } k = 1, 2, \dots, n.$$

Proof. By definition of \mathbf{B} , $\mathbf{B} = n^{-1} \mathbf{Z}_B \mathbf{\Lambda}_B \mathbf{Z}_B'$ where \mathbf{Z}_B is $n \times (p - m)$ random matrix with independent sub-Gaussian entries of zero mean and unit variance and $\mathbf{\Lambda}_B$ is the diagonal matrix with entries $\lambda_{m+1}, \dots, \lambda_p$. By Lemma 6.1 with $t = \sqrt{n}$, for any $k \leq n$,

$$\frac{n}{p - m} \lambda_k(\mathbf{B}) = \frac{1}{p - m} \sum_{j=m+1}^p \lambda_j + O_P\left(\sqrt{\frac{n}{p}}\right) = \bar{c} + O_P\left(\sqrt{\frac{n}{p}}\right) + o_P(n^{-1/2}).$$

Therefore,

$$\frac{\lambda_k(\mathbf{B})}{\lambda_j} = \frac{n\lambda_k(\mathbf{B})}{p-m} \frac{p-m}{n\lambda_j} = \bar{c}c_j + O_P\left(c_j\sqrt{\frac{n}{p}}\right) + o_P(n^{-\frac{1}{2}}).$$

□

Proof of Theorem 3.1. By Wely's Theorem, $\lambda_j(\mathbf{A}) + \lambda_n(\mathbf{B}) \leq \hat{\lambda}_j \leq \lambda_j(\mathbf{A}) + \lambda_1(\mathbf{B})$. Therefore from Lemma 6.3,

$$\frac{\hat{\lambda}_j}{\lambda_j} = \frac{\lambda_j(\mathbf{A})}{\lambda_j} + \bar{c}c_j + O_P\left(c_j\sqrt{\frac{n}{p}}\right) + o_P(n^{-1/2}),$$

By Lemma 6.2 and Slutsky's theorem, we conclude that $\sqrt{n}\left(\frac{\hat{\lambda}_j}{\lambda_j} - (1 + \bar{c}c_j + O_P(c_j\sqrt{n/p}))\right)$ converges in distribution to $N(0, \kappa_j - 1)$ and the limiting distributions of the first m eigenvalues are independent. □

6.2 Proofs of Theorem 3.2

The proof of Theorem 3.2 is quite involved. The basic idea for proving part (i) is outlined in Section 2. We first give the main idea of the proof and relegate less important technical lemmas to the end of the proof. The proof of part (ii) utilizes the invariance of standard Gaussian distribution under orthogonal transformations.

Proof of Theorem 3.2. (i) Let us start by proving for the asymptotic normality of $\hat{\boldsymbol{\xi}}_{jA}$. Write

$$\mathbf{X} = (\mathbf{Z}_A \boldsymbol{\Lambda}_A^{\frac{1}{2}}, \mathbf{Z}_B \boldsymbol{\Lambda}_B^{\frac{1}{2}}) = (\sqrt{\lambda_1} \mathbf{Z}_1, \dots, \sqrt{\lambda_m} \mathbf{Z}_m, \sqrt{\lambda_{m+1}} \mathbf{Z}_{m+1}, \dots, \sqrt{\lambda_p} \mathbf{Z}_p),$$

where each \mathbf{Z}_j follows a sub-Gaussian distribution with mean $\mathbf{0}$ and identity variance \mathbf{I}_n . Then by the eigenvalue relationship of Equation (2.2), we have

$$\hat{\boldsymbol{\xi}}_{jA} = \frac{\boldsymbol{\Lambda}_A^{\frac{1}{2}} \mathbf{Z}'_A \mathbf{u}_j}{\sqrt{n\hat{\lambda}_j}} \quad \text{and} \quad \mathbf{u}_j = \frac{\mathbf{X} \hat{\boldsymbol{\xi}}_j}{\sqrt{n\hat{\lambda}_j}} = \frac{\mathbf{Z}_A \boldsymbol{\Lambda}_A^{\frac{1}{2}} \hat{\boldsymbol{\xi}}_{jA}}{\sqrt{n\hat{\lambda}_j}} + \frac{\mathbf{Z}_B \boldsymbol{\Lambda}_B^{\frac{1}{2}} \hat{\boldsymbol{\xi}}_{jB}}{\sqrt{n\hat{\lambda}_j}}. \quad (6.2)$$

Recall \mathbf{u}_j is the eigenvector of the matrix $\tilde{\boldsymbol{\Sigma}}$, that is, $\frac{1}{n} \mathbf{X} \mathbf{X}' \mathbf{u}_j = \hat{\lambda}_j \mathbf{u}_j$. Using $\mathbf{X} = (\mathbf{Z}_A \boldsymbol{\Lambda}_A^{\frac{1}{2}}, \mathbf{Z}_B \boldsymbol{\Lambda}_B^{\frac{1}{2}})$, we obtain

$$\left(\mathbf{I}_n - \frac{1}{n} \mathbf{Z}_A \frac{\boldsymbol{\Lambda}_A}{\lambda_j} \mathbf{Z}'_A\right) \mathbf{u}_j = \mathbf{D} \mathbf{u}_j - \Delta \mathbf{u}_j, \quad (6.3)$$

where we denote $\mathbf{D} = (n\lambda_j)^{-1} \mathbf{Z}_B \boldsymbol{\Lambda}_B \mathbf{Z}'_B - \bar{c}c_j \mathbf{I}_n$, $\Delta = \hat{\lambda}_j/\lambda_j - (1 + \bar{c}c_j)$. We then left-multiply Equation (6.3) by $\boldsymbol{\Lambda}_A^{\frac{1}{2}} \mathbf{Z}'_A / \sqrt{n\hat{\lambda}_j}$ and employ relationship (6.2) to replace \mathbf{u}_j by

$\hat{\boldsymbol{\xi}}_{jA}$ and $\hat{\boldsymbol{\xi}}_{jB}$ as follows:

$$\begin{aligned} \left(\mathbf{I}_n - \frac{\boldsymbol{\Lambda}_A}{\lambda_j}\right) \hat{\boldsymbol{\xi}}_{jA} &= \frac{\boldsymbol{\Lambda}_A^{\frac{1}{2}} \left(\frac{1}{n} \mathbf{Z}'_A \mathbf{Z}_A - \mathbf{I}_n\right) \boldsymbol{\Lambda}_A^{\frac{1}{2}}}{\lambda_j} \hat{\boldsymbol{\xi}}_{jA} + \frac{\boldsymbol{\Lambda}_A^{\frac{1}{2}} \mathbf{Z}'_A \mathbf{D} \mathbf{Z}_A \boldsymbol{\Lambda}_A^{\frac{1}{2}}}{n \hat{\lambda}_j} \hat{\boldsymbol{\xi}}_{jA} \\ &\quad + \frac{\boldsymbol{\Lambda}_A^{\frac{1}{2}} \mathbf{Z}'_A \mathbf{D} \mathbf{Z}_B \boldsymbol{\Lambda}_B^{\frac{1}{2}}}{n \hat{\lambda}_j} \hat{\boldsymbol{\xi}}_{jB} - \Delta \hat{\boldsymbol{\xi}}_{jA}. \end{aligned} \quad (6.4)$$

Further define

$$\mathbf{R} = \sum_{k \in [m] \setminus j} \frac{\lambda_j}{\lambda_j - \lambda_k} \mathbf{e}_{kA} \mathbf{e}'_{kA}.$$

Then we have $\mathbf{R}(\mathbf{I} - \boldsymbol{\Lambda}_A/\lambda_j) = \mathbf{I}_m - \mathbf{e}_{jA} \mathbf{e}'_{jA}$. Therefore, by left multiplying \mathbf{R} to Equation (6.4),

$$\begin{aligned} \hat{\boldsymbol{\xi}}_{jA} - \langle \hat{\boldsymbol{\xi}}_{jA}, \mathbf{e}_{jA} \rangle \mathbf{e}_{jA} &= \mathbf{R} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} \mathbf{K} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} \hat{\boldsymbol{\xi}}_{jA} \\ &\quad + \mathbf{R} \frac{\boldsymbol{\Lambda}_A^{\frac{1}{2}} \mathbf{Z}'_A \mathbf{D} \mathbf{Z}_B \boldsymbol{\Lambda}_B^{\frac{1}{2}}}{n \hat{\lambda}_j} \hat{\boldsymbol{\xi}}_{jB} - \Delta \mathbf{R} \hat{\boldsymbol{\xi}}_{jA}, \end{aligned} \quad (6.5)$$

where $\mathbf{K} = n^{-1} \mathbf{Z}'_A \mathbf{Z}_A - \mathbf{I}_n + \lambda_j (n \hat{\lambda}_j)^{-1} \mathbf{Z}'_A \mathbf{D} \mathbf{Z}_A$. Dividing both side by $\|\hat{\boldsymbol{\xi}}_{jA}\|$, we are able to write

$$\frac{\hat{\boldsymbol{\xi}}_{jA}}{\|\hat{\boldsymbol{\xi}}_{jA}\|} - \mathbf{e}_{jA} = \mathbf{R} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} \mathbf{K} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} \mathbf{e}_{jA} + \mathbf{r}_n, \quad (6.6)$$

where

$$\begin{aligned} \mathbf{r}_n &= \left(\left\langle \frac{\hat{\boldsymbol{\xi}}_{jA}}{\|\hat{\boldsymbol{\xi}}_{jA}\|}, \mathbf{e}_{jA} \right\rangle - 1 \right) \mathbf{e}_{jA} + \mathbf{R} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} \mathbf{K} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} \left(\frac{\hat{\boldsymbol{\xi}}_{jA}}{\|\hat{\boldsymbol{\xi}}_{jA}\|} - \mathbf{e}_{jA} \right) \\ &\quad + \mathbf{R} \frac{\boldsymbol{\Lambda}_A^{\frac{1}{2}} \mathbf{Z}'_A \mathbf{D} \mathbf{Z}_B \boldsymbol{\Lambda}_B^{\frac{1}{2}}}{n \hat{\lambda}_j} \frac{\hat{\boldsymbol{\xi}}_{jB}}{\|\hat{\boldsymbol{\xi}}_{jA}\|} - \Delta \mathbf{R} \left(\frac{\hat{\boldsymbol{\xi}}_{jA}}{\|\hat{\boldsymbol{\xi}}_{jA}\|} - \mathbf{e}_{jA} \right). \end{aligned} \quad (6.7)$$

We will show in Lemma 6.4 below that \mathbf{r}_n is a smaller order term. By Lemma 6.4, noticing that $(\boldsymbol{\Lambda}_A/\lambda_j)^{\frac{1}{2}} \mathbf{e}_{jA} = \mathbf{e}_{jA}$,

$$\sqrt{n} \left(\frac{\hat{\boldsymbol{\xi}}_{jA}}{\|\hat{\boldsymbol{\xi}}_{jA}\|} - \mathbf{e}_{jA} + O_P \left(c_j \sqrt{\frac{\bar{n}}{p}} \right) \right) = \sqrt{n} \mathbf{R} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} \mathbf{K} \mathbf{e}_{jA} + o_P(1). \quad (6.8)$$

Now let us derive normality of the right hand side of (6.8). According to definition of \mathbf{R} ,

$$\mathbf{R} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} = \sum_{k \in [m] \setminus j} \frac{\sqrt{\lambda_j \lambda_k}}{\lambda_j - \lambda_k} \mathbf{e}_{kA} \mathbf{e}'_{kA} \rightarrow \sum_{k \in [m] \setminus j} a_{jk} \mathbf{e}_{kA} \mathbf{e}'_{kA}. \quad (6.9)$$

Let $\mathbf{W} = \sqrt{n} \mathbf{K} \mathbf{e}_{jA} = (W_1, \dots, W_m)$ and $\mathbf{W}^{(-j)}$ be the $m-1$ dimension vector without the j th element in \mathbf{W} . Since the j th diagonal element of \mathbf{R} is zero, $\mathbf{R}(\boldsymbol{\Lambda}_A/\lambda_j)^{\frac{1}{2}} \mathbf{W}$ depends

only on $\mathbf{W}^{(-j)}$. Therefore, by Lemma 6.5 below and Slutsky's theorem,

$$\sqrt{n}\mathbf{R}\left(\frac{\Lambda_A}{\lambda_j}\right)^{\frac{1}{2}}\mathbf{K}\mathbf{e}_{jA} = \mathbf{R}\left(\frac{\Lambda_A}{\lambda_j}\right)^{\frac{1}{2}}\mathbf{W} \xrightarrow{d} N_m\left(\mathbf{0}, \sum_{k \in [m] \setminus j} a_{jk}^2 \mathbf{e}_{kA} \mathbf{e}'_{kA}\right).$$

Similar to Shen et al. (2013), it is not hard to show that $\|\hat{\boldsymbol{\xi}}_{jA}\| \xrightarrow{P} (1 + \bar{c}c_j)^{-1/2}$, which is provided in Lemma 6.6 for the sake of completeness. Hence,

$$\sqrt{n}\left(\hat{\boldsymbol{\xi}}_{jA} - \frac{1}{\sqrt{1 + \bar{c}c_j}}\mathbf{e}_{jA} + O_P\left(c_j\sqrt{\frac{n}{p}}\right)\right) \xrightarrow{d} N_m\left(\mathbf{0}, \frac{1}{1 + \bar{c}c_j} \sum_{k \in [m] \setminus j} a_{jk}^2 \mathbf{e}_{kA} \mathbf{e}'_{kA}\right).$$

This concludes Equation (3.2). Since the j^{th} diagonal element of $\boldsymbol{\Sigma}_j$ is zero, (3.2) implies (3.3) easily. This completes the first part of the proof.

(ii) We now prove conclusion for the non-spiked part $\hat{\boldsymbol{\xi}}_{jB}$. Recall that \mathbf{X}_i follows $N(\mathbf{0}, \Lambda)$. Consider $\mathbf{X}_i^{(R)} = \text{diag}(\mathbf{I}_m, \Omega)\mathbf{X}_i$ where as defined in the theorem $\Omega = \text{diag}(\sqrt{\bar{c}/\lambda_{m+1}}, \dots, \sqrt{\bar{c}/\lambda_p})$. Here the index (R) means rescaled data by $\text{diag}(\mathbf{I}_m, \Omega)$. After rescaling, we have $\mathbf{X}_i^{(R)} \sim N(\mathbf{0}, \text{diag}(\Lambda_A, \bar{c}\mathbf{I}_{p-m}))$. Correspondingly, the $n \times p$ data matrix $\mathbf{X}^{(R)} = \mathbf{X}\text{diag}(\mathbf{I}_m, \Omega) = (\mathbf{X}_A, \mathbf{X}_B\Omega)$ where $\mathbf{X}_A = \mathbf{Z}_A\Lambda_A^{\frac{1}{2}}$ and $\mathbf{X}_B = \mathbf{Z}_B\Lambda_B^{\frac{1}{2}}$ as the notations before. Assume $\hat{\boldsymbol{\xi}}_j^{(R)}$ and $\mathbf{u}_j^{(R)}$ are eigenvectors given by $\hat{\boldsymbol{\Sigma}}^{(R)}$ and $\tilde{\boldsymbol{\Sigma}}^{(R)}$ of the rescaled data $\mathbf{X}^{(R)}$ and $\hat{\boldsymbol{\xi}}_j^{(R)} = (\hat{\boldsymbol{\xi}}_{jA}^{(R)}, \hat{\boldsymbol{\xi}}_{jB}^{(R)})'$. It has been proved by Paul (2007) that $\hat{\boldsymbol{\xi}}_{jB}^{(R)} / \|\hat{\boldsymbol{\xi}}_{jB}^{(R)}\|$ is distributed uniformly over the unit sphere and is independent of $\|\hat{\boldsymbol{\xi}}_{jB}^{(R)}\|$ due to the orthogonal invariance of the non-spiked part of $\hat{\boldsymbol{\xi}}_{jB}^{(R)}$. Hence it only remains to link $\hat{\boldsymbol{\xi}}_{jB}$ with $\hat{\boldsymbol{\xi}}_{jB}^{(R)}$.

Note that $\tilde{\boldsymbol{\Sigma}} = n^{-1}\mathbf{X}\mathbf{X}'$ and $\tilde{\boldsymbol{\Sigma}}^{(R)} = n^{-1}\mathbf{X}^{(R)}\mathbf{X}^{(R)'}$, so

$$\|\tilde{\boldsymbol{\Sigma}} - \tilde{\boldsymbol{\Sigma}}^{(R)}\| = \left\| \frac{1}{n}\mathbf{X}_B(\mathbf{I} - \Omega^2)\mathbf{X}_B' \right\| = \frac{1}{n} \sum_{j=m+1}^p (\lambda_j - \bar{c})\mathbf{Z}_j\mathbf{Z}_j',$$

where the last term is of order $O_P(\lambda_j c_j \sqrt{n/p})$ by Lemma 6.1. Thus by the sin θ theorem of Davis and Kahan (1970) we know $\|\mathbf{u}_j - \mathbf{u}_j^{(R)}\| = O_P(c_j \sqrt{n/p})$. Next we convert from \mathbf{u}_j to $\hat{\boldsymbol{\xi}}_{jB}$ using the basic relationship (2.2). We have,

$$\|\Omega\hat{\boldsymbol{\xi}}_{jB} - \hat{\boldsymbol{\xi}}_{jB}^{(R)}\| = \left\| \frac{\Omega\mathbf{X}_B'\mathbf{u}_j}{\sqrt{n\hat{\lambda}_j}} - \frac{\mathbf{X}_B^{(R)'}\mathbf{u}_j^{(R)}}{\sqrt{n\hat{\lambda}_j^{(R)}}} \right\| \leq \|\Omega\| \left\| \frac{\mathbf{X}_B'\mathbf{u}_j}{\sqrt{n\hat{\lambda}_j}} - \frac{\mathbf{X}_B^{(R)'}\mathbf{u}_j^{(R)}}{\sqrt{n\hat{\lambda}_j^{(R)}}} \right\|.$$

It is easy to show that the right hand side is $O_P(c_j \sqrt{n/p})$ since $\|\mathbf{u}_j - \mathbf{u}_j^{(R)}\| = O_P(c_j \sqrt{n/p})$, both $\hat{\lambda}_j$ and $\hat{\lambda}_j^{(R)}$ converges to $\lambda_j(1 + \bar{c}c_j)$ with identical rates and $\|\mathbf{X}_B'/\sqrt{n\hat{\lambda}_j}\| = O_P(1)$. Hence we conclude

$$\Omega\hat{\boldsymbol{\xi}}_{jB} - \hat{\boldsymbol{\xi}}_{jB}^{(R)} = O_P(c_j \sqrt{n/p}).$$

In addition, by Lemma 6.6, $\|\hat{\boldsymbol{\xi}}_{jB}^{(R)}\| \xrightarrow{P} r := \sqrt{\bar{c}_j/(1 + \bar{c}_j)}$. Since $\hat{\boldsymbol{\xi}}_{jB}^{(R)}/\|\hat{\boldsymbol{\xi}}_{jB}^{(R)}\|$ is uniformly distributed over a unit sphere, we have

$$\hat{\boldsymbol{\xi}}_{jB}^{(R)} = \hat{\boldsymbol{\xi}}_{jB}^{(R)}/\|\hat{\boldsymbol{\xi}}_{jB}^{(R)}\| r + o_P(1).$$

The first term is uniformly distributed over a centered ball of radius r . The conclusion (3.4) follows.

From the above derivation, to prove the max norm bound (3.5) of $\|\hat{\boldsymbol{\xi}}_{jB}\|_{\max}$, it suffices to show $\|\hat{\boldsymbol{\xi}}_{jB}^{(R)}\|_{\max} = O_P(\sqrt{c_j \log p/p})$. Since its norm $\|\hat{\boldsymbol{\xi}}_{jB}^{(R)}\| = O_P(\sqrt{c_j})$, we only need to show the fact that $\|\mathbf{v}\|_{\max} = O_P(\sqrt{\log p/p})$ for any random variable \mathbf{v} that is uniformly distributed on unit sphere of dimension p . This follows easily from its normal representation. Let $Z_i, i = 1, \dots, p$ to be iid standard normal distributed and $v_i = Z_i/\|\mathbf{Z}\|$, then \mathbf{v} is uniformly distributed on the unit sphere of dimension p . It then follows

$$\max_{i \leq p} |v_i| = \max_{i \leq p} |Z_i|/\|\mathbf{Z}\| = O_P(\sqrt{\log p/p}).$$

This completes the proof. \square

Lemma 6.4. *As $n \rightarrow \infty$, $\|\mathbf{r}_n\| = o_P(n^{-\frac{1}{2}}) + O_P(c_j \sqrt{n/p})$.*

Proof. Define $\mathbf{v}_j = \hat{\boldsymbol{\xi}}_{jA}/\|\hat{\boldsymbol{\xi}}_{jA}\| - \langle \hat{\boldsymbol{\xi}}_{jA}/\|\hat{\boldsymbol{\xi}}_{jA}\|, \mathbf{e}_{jA} \rangle \mathbf{e}_{jA}$ and α_j, β_j and γ_j as follows:

$$\begin{aligned} \alpha_j &= \left\| \mathbf{R} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} \mathbf{K} \mathbf{e}_{jA} \right\|, \\ \beta_j &= \left\| \mathbf{R} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} \mathbf{K} \left(\frac{\boldsymbol{\Lambda}_A}{\lambda_j} \right)^{\frac{1}{2}} \right\| + \Delta \|\mathbf{R}\|, \\ \gamma_j &= \left\| \mathbf{R} \frac{\boldsymbol{\Lambda}_A^{\frac{1}{2}} \mathbf{Z}'_A \mathbf{D} \mathbf{Z}_B \boldsymbol{\Lambda}_B^{\frac{1}{2}}}{n \hat{\lambda}_j} \frac{\hat{\boldsymbol{\xi}}_{jB}}{\|\hat{\boldsymbol{\xi}}_{jA}\|} \right\|. \end{aligned}$$

We claim that $\alpha_j, \beta_j = O_P(c_j \sqrt{n/p} + n^{-\frac{1}{2}})$, $\gamma_j = O_P(c_j \sqrt{n/p}) + o_P(n^{-\frac{1}{2}})$ and $\|\mathbf{v}_j\| = O_P(c_j \sqrt{n/p} + n^{-\frac{1}{2}})$. Then the rate of $\|\mathbf{r}_n\|$ could be easily derived from its definition (6.7) and the above results. To be specific, first notice the following two inequalities: by (6.5), $\|\mathbf{v}_j\| \leq \beta_j + \gamma_j$; by orthogonal decomposition $\hat{\boldsymbol{\xi}}_{jA}/\|\hat{\boldsymbol{\xi}}_{jA}\| = \langle \hat{\boldsymbol{\xi}}_{jA}/\|\hat{\boldsymbol{\xi}}_{jA}\|, \mathbf{e}_{jA} \rangle \mathbf{e}_{jA} + \mathbf{v}_j$, we have $1 - \langle \hat{\boldsymbol{\xi}}_{jA}/\|\hat{\boldsymbol{\xi}}_{jA}\|, \mathbf{e}_{jA} \rangle = 1 - \sqrt{1 - \|\mathbf{v}_j\|^2} \leq \|\mathbf{v}_j\|^2$. Note that we always choose $\hat{\boldsymbol{\xi}}$ so that $\langle \hat{\boldsymbol{\xi}}_{jA}/\|\hat{\boldsymbol{\xi}}_{jA}\|, \mathbf{e}_{jA} \rangle$ is positive. Therefore

$$\begin{aligned} \left\| \frac{\hat{\boldsymbol{\xi}}_{jA}}{\|\hat{\boldsymbol{\xi}}_{jA}\|} - \mathbf{e}_{jA} \right\| &= \left\| \mathbf{v}_j + \left(\langle \frac{\hat{\boldsymbol{\xi}}_{jA}}{\|\hat{\boldsymbol{\xi}}_{jA}\|}, \mathbf{e}_{jA} \rangle - 1 \right) \mathbf{e}_{jA} \right\| \\ &\leq \|\mathbf{v}_j\| (1 + \|\mathbf{v}_j\|) \leq \|\mathbf{v}_j\| (1 + \beta_j + \gamma_j). \end{aligned} \tag{6.10}$$

Hence, by (6.7),

$$\begin{aligned}\|\mathbf{r}_n\| &\leq \|\mathbf{v}_j\|^2 + \beta_j \left\| \frac{\hat{\boldsymbol{\xi}}_{jA}}{\|\hat{\boldsymbol{\xi}}_{jA}\|} - \mathbf{e}_{jA} \right\| + \gamma_j \\ &\leq \|\mathbf{v}_j\|(\|\mathbf{v}_j\| + \beta_j(1 + \beta_j + \gamma_j)) + \gamma_j = O_P(c_j\sqrt{n/p}) + o_P(n^{-\frac{1}{2}}).\end{aligned}$$

It remains to show the claims above. Let us first show that $\gamma_j = O_P(\sqrt{n/p}) + o_P(n^{-\frac{1}{2}})$. In order to prove this, we need the rate of $\|\mathbf{D}\|$. By Lemma 6.1, $\|(p-m)^{-1}\mathbf{Z}_B\boldsymbol{\Lambda}_B\mathbf{Z}'_B - \bar{c}\mathbf{I}\| = O_P(\sqrt{n/p})$, so we have

$$\begin{aligned}\|\mathbf{D}\| &= \left\| \frac{1}{n} \frac{\mathbf{Z}_B\boldsymbol{\Lambda}_B\mathbf{Z}'_B}{\lambda_j} - \bar{c}_j\mathbf{I}_n \right\| \\ &\leq \frac{p-m}{n\lambda_j} \left\| \frac{1}{p-m} \mathbf{Z}_B\boldsymbol{\Lambda}_B\mathbf{Z}'_B - \bar{c}\mathbf{I} \right\| + \bar{c} \left| \frac{p-m}{n\lambda_j} - c_j \right| \\ &= O_P(c_j\sqrt{n/p}) + o_P(n^{-\frac{1}{2}}).\end{aligned}$$

Hence

$$\gamma_j \leq \left\| \frac{\mathbf{R}\boldsymbol{\Lambda}_A^{\frac{1}{2}}}{\sqrt{\lambda_j}} \right\| \left\| \frac{\lambda_j}{\hat{\lambda}_j} \right\| \left\| \frac{\mathbf{Z}'_A}{\sqrt{n}} \right\| \|\mathbf{D}\| \left\| \frac{\mathbf{Z}_B\boldsymbol{\Lambda}_B^{\frac{1}{2}}}{\sqrt{n\lambda_j}} \right\| \frac{\|\hat{\boldsymbol{\xi}}_{jB}\|}{\|\hat{\boldsymbol{\xi}}_{jA}\|} = O_P(c_j\sqrt{n/p}) + o_P(n^{-\frac{1}{2}}),$$

since the other terms except $\|\mathbf{D}\|$ are all $O_P(1)$. Indeed, (6.9) says the first term is asymptotically bounded. We have shown in the proofs of Lemmas 6.2 and 6.3 that the second, third and fifth terms are $O_P(1)$. In addition, the facts that $\|\hat{\boldsymbol{\xi}}_{jB}\| \leq 1$, $\|\hat{\boldsymbol{\xi}}_{jA}\| \xrightarrow{P} (1 + \bar{c}_j)^{-\frac{1}{2}}$ imply the last term is $O_P(1)$.

Then let us show that α_j and β_j are both of order $O_P(c_j\sqrt{n/p} + n^{-\frac{1}{2}})$. The rate of $\|\mathbf{K}\|$ is needed. By Lemma 6.1, $\|\frac{1}{n}\mathbf{Z}'_A\mathbf{Z}_A - \mathbf{I}\| = O_P(\sqrt{m/n}) = O_P(n^{-\frac{1}{2}})$. Thus,

$$\begin{aligned}\|\mathbf{K}\| &= \left\| \frac{1}{n} \mathbf{Z}'_A\mathbf{Z}_A - \mathbf{I}_n + \frac{\lambda_j}{\hat{\lambda}_j} \frac{1}{n} \mathbf{Z}'_A\mathbf{D}\mathbf{Z}_A \right\| \\ &\leq \left\| \frac{1}{n} \mathbf{Z}'_A\mathbf{Z}_A - \mathbf{I}_n \right\| + \left| \frac{\lambda_j}{\hat{\lambda}_j} \right| \|\mathbf{D}\| \left\| \frac{1}{n} \mathbf{Z}'_A\mathbf{Z}_A \right\| = O_P(c_j\sqrt{n/p} + n^{-\frac{1}{2}}).\end{aligned}$$

Then easily we get $\alpha_j = O_P(c_j\sqrt{n/p} + n^{-\frac{1}{2}})$ since $\|\mathbf{R}\|$ is bounded asymptotically. Note that from Theorem 3.1 that $\Delta = \hat{\lambda}_j/\lambda_j - (1 + \bar{c}_j) = O_P(c_j\sqrt{n/p} + n^{-\frac{1}{2}})$, so

$$\beta_j \leq \|\mathbf{K}\| \|\mathbf{R}\boldsymbol{\Lambda}_A/\lambda_j\| + \Delta \|\mathbf{R}\| = O_P(c_j\sqrt{n/p} + n^{-\frac{1}{2}}),$$

where similar to (6.9), $\|\mathbf{R}\boldsymbol{\Lambda}_A/\lambda_j\|$ and $\|\mathbf{R}\|$ are $O_P(1)$.

Finally, $\|\mathbf{v}_j\| \leq \beta_j + \gamma_j = O_P(c_j\sqrt{n/p} + n^{-\frac{1}{2}})$. The proof is now complete. \square

Lemma 6.5. $\mathbf{W}^{(-j)} \xrightarrow{d} N(\mathbf{0}, \mathbf{I}_{m-1})$.

Proof. Recall $\mathbf{W} = \sqrt{n}\mathbf{K}\mathbf{e}_{jA}$. Then, by the definition of \mathbf{K} ,

$$\mathbf{W} = \frac{1}{\sqrt{n}}\mathbf{Z}'_A\mathbf{Z}_j - \sqrt{n}\mathbf{e}_{jA} + \frac{\lambda_j}{\hat{\lambda}_j}\frac{1}{\sqrt{n}}\mathbf{Z}'_A\mathbf{D}\mathbf{Z}_j.$$

Its t^{th} component is $W_t = n^{-1/2}\mathbf{Z}'_t\mathbf{Z}_j + \delta_{nt}$ for $t \in [m] \setminus j$ where $\delta_{nt} = (\lambda_j/\hat{\lambda}_j) \cdot n^{-1/2}\mathbf{Z}'_t\mathbf{D}\mathbf{Z}_j$. Denote $\tilde{\mathbf{W}} = (n^{-1/2}\mathbf{Z}'_t\mathbf{Z}_j)_{t \in [m] \setminus j}$ and $\boldsymbol{\delta}_n = (\delta_{nt})_{t \in [m] \setminus j}$. We claim as $n \rightarrow \infty$, $\|\boldsymbol{\delta}_n\| = o_P(1)$. So $\mathbf{W}^{(-j)} = \tilde{\mathbf{W}} + o_P(1)$. In order to prove the lemma, it suffices to show that $\tilde{\mathbf{W}}$ follows $N(\mathbf{0}, \mathbf{I}_{m-1})$. That is, for any vector \mathbf{a} of $m-1$ dimension, $\mathbb{E}[\exp(i\mathbf{a}'\tilde{\mathbf{W}})] \rightarrow \exp(-\|\mathbf{a}\|^2/2)$ almost surely.

$$\mathbb{E}\left[e^{i\mathbf{a}'\tilde{\mathbf{W}}}\right] = \mathbb{E}\left[\mathbb{E}\left[\prod_{t \in [m] \setminus j} e^{ia_t \mathbf{Z}'_t \mathbf{Z}_j / \sqrt{n}} \mid \mathbf{Z}_j\right]\right] = \mathbb{E}\left[\prod_{t \in [m] \setminus j} \prod_{k=1}^n f_t\left(\frac{1}{\sqrt{n}}a_t Z_{kj}\right)\right],$$

where $f_j(u) = \mathbb{E}[\exp(iuZ_{kj})]$ is the characteristic function of each element of \mathbf{Z}_j . The sub index j means we actually allow different characteristic functions for the columns of \mathbf{Z}_A and \mathbf{Z}_B .

By Taylor expansion, we can easily derive

$$|e^{ix} - 1 - ix + x^2/2| \leq (|x|^3/6) \wedge x^2,$$

from which it holds that

$$|f_j(u) - 1 - iu\mathbb{E}[Z_{kj}] + \frac{u^2}{2}\mathbb{E}[Z_{kj}^2]| \leq u^2\mathbb{E}\left[\frac{|u|}{6}|Z_{kj}|^3 \wedge Z_{kj}^2\right].$$

$\mathbb{E}\left[\frac{|u|}{6}|Z_{kj}|^3 \wedge Z_{kj}^2\right]$ goes to 0 as $u \rightarrow 0$ and is dominated by the integrable function Z_{kj}^2 . So by Dominated Convergence Theorem the right hand side is $o(u^2)$. Therefore, $f_j(u) = 1 - u^2/2 + o(u^2)$. Using this result, we have

$$\begin{aligned} \mathbb{E}\left[e^{i\mathbf{a}'\tilde{\mathbf{W}}}\right] &= \mathbb{E}\left[\prod_{t \in [m] \setminus j} \prod_{k=1}^n \left(1 - \frac{a_t^2}{2n} Z_{kj}^2\right)\right] + o(1) \\ &= \mathbb{E}\left[\prod_{k=1}^n \left(1 - \frac{\|\mathbf{a}\|^2}{2n} Z_{kj}^2\right)\right] + o(1) \\ &= \prod_{k=1}^n \mathbb{E}\left[1 - \frac{\|\mathbf{a}\|^2}{2n} Z_{kj}^2\right] + o(1) \\ &= \left(1 - \frac{\|\mathbf{a}\|^2}{2n}\right)^n + o(1) \xrightarrow{a.s.} \exp(-\|\mathbf{a}\|^2/2). \end{aligned}$$

which means that $\tilde{\mathbf{W}}$ follows $N(0, \mathbf{I}_{m-1})$.

Now let us validate $\|\boldsymbol{\delta}_n\| = o_P(1)$. Clearly

$$|\delta_{nt}| \leq |\lambda_j/\hat{\lambda}_j| \left| \frac{1}{\sqrt{n}} \mathbf{Z}'_t \mathbf{Z}_j \right| \|\mathbf{D}\|.$$

We have shown that $|\lambda_j/\hat{\lambda}_j| = O_P(1)$ and $\|\mathbf{D}\| = O_P(\sqrt{n/p}) + o_P(n^{-1/2}) = o_P(1)$. In order to prove $\delta_{nt} = o_P(1)$, it suffice to show $\frac{1}{\sqrt{n}} \mathbf{Z}'_t \mathbf{Z}_j = O_P(1)$.

$$\mathbb{E} \left[\left| \frac{1}{\sqrt{n}} \mathbf{Z}'_t \mathbf{Z}_j \right|^2 \right] = \frac{1}{n} \mathbb{E} \left[\mathbb{E} \left[(\mathbf{Z}'_t \mathbf{Z}_j)^2 \mid \mathbf{Z}_j \right] \right] = \frac{1}{n} \mathbb{E} \left[\mathbf{Z}'_j \mathbf{Z}_j \right] = 1.$$

So by Markov inequality, $|n^{-1/2} \mathbf{Z}'_t \mathbf{Z}_j|$ is $O_P(1)$, which gives $\delta_{nt} = o_P(1)$. Finally $\|\boldsymbol{\delta}_n\| = o_P(1)$ since $\boldsymbol{\delta}_n$ is of fixed length $m - 1$. The proof is complete. \square

Lemma 6.6. $\|\hat{\boldsymbol{\xi}}_{jA}\| \xrightarrow{P} (1 + \bar{c}c_j)^{-1/2}$.

Proof. Recall that $\mathbf{X} = (\mathbf{Z}_A \boldsymbol{\Lambda}_A^{\frac{1}{2}}, \mathbf{Z}_B \boldsymbol{\Lambda}_B^{\frac{1}{2}})$. Let $\mathbf{Z} = (\mathbf{Z}_A, \mathbf{Z}_B)$, then

$$\mathbf{Z} = \mathbf{X} \boldsymbol{\Lambda}^{-\frac{1}{2}} = \sqrt{n} \hat{\boldsymbol{\Lambda}}^{\frac{1}{2}} (\hat{\boldsymbol{\xi}}_1, \dots, \hat{\boldsymbol{\xi}}_p)' \boldsymbol{\Lambda}^{-\frac{1}{2}},$$

where $\boldsymbol{\Lambda} = \text{diag}(\boldsymbol{\Lambda}_A, \boldsymbol{\Lambda}_B)$. Define $\bar{\boldsymbol{\Lambda}} = \text{diag}(1, \dots, 1, \lambda_{m+1}, \dots, \lambda_p)$ and consider the eigenvalue of the matrix $n^{-1} \mathbf{Z} \bar{\boldsymbol{\Lambda}} \mathbf{Z}'$. The j -th diagonal element of the matrix must lie in between its minimum and maximum eigenvalues. That is

$$\lambda_n \left(\frac{1}{n} \mathbf{Z} \bar{\boldsymbol{\Lambda}} \mathbf{Z}' \right) \leq \left(\frac{1}{n} \mathbf{Z} \bar{\boldsymbol{\Lambda}} \mathbf{Z}' \right)_{jj} = \hat{\lambda}_j \sum_{k=1}^p \hat{\xi}_{jk}^2 \frac{\bar{\lambda}_k}{\lambda_k} \leq \lambda_1 \left(\frac{1}{n} \mathbf{Z} \bar{\boldsymbol{\Lambda}} \mathbf{Z}' \right),$$

where $\hat{\xi}_{jk}$ is the k -th element of the j -th empirical eigenvector for $j \leq m$. Divided by $\hat{\lambda}_j$, then both the left and right hand side converge in probability to $c_j(m + \sum_{j=m+1}^p \lambda_j)/(p(1 + \bar{c}c_j))$ by Lemma 6.1, thus to $\bar{c}c_j/(1 + \bar{c}c_j)$. So $\sum_{k=1}^p \hat{\xi}_{jk}^2 \bar{\lambda}_k/\lambda_k \xrightarrow{P} \bar{c}c_j/(1 + \bar{c}c_j)$. Also, by definition, $\bar{\lambda}_k/\lambda_k \rightarrow 0$ for $k \leq m$ while the ratio is 1 for $k > m$. Hence, $\sum_{k=m+1}^p \hat{\xi}_{jk}^2 \xrightarrow{P} \bar{c}c_j/(1 + \bar{c}c_j)$, which implies that $\|\hat{\boldsymbol{\xi}}_{jA}\| = \sqrt{1 - \sum_{k=m+1}^p \hat{\xi}_{jk}^2} \xrightarrow{P} (1 + \bar{c}c_j)^{-1/2}$. \square

7 Proofs for Section 4

7.1 Comparison on assumptions

The following assumptions are from Fan et al. (2013a), where the results were established for the mixing sequence. But we only consider i.i.d. data in this paper. The assumptions are listed for completeness and comparison with Assumptions 4.1 and 4.2.

Assumption 7.1. $\|p^{-1} \mathbf{B}' \mathbf{B} - \boldsymbol{\Omega}\| = o(1)$ for some $m \times m$ symmetric positive definite matrix $\boldsymbol{\Omega}$ such that $\boldsymbol{\Omega}$ has m distinct eigenvalues and that $\lambda_{\min}(\boldsymbol{\Omega})$ and $\lambda_{\max}(\boldsymbol{\Omega})$ are

bounded away from both zero and infinity.

Assumption 7.2. (i) $\{\mathbf{u}_t, \mathbf{f}_t\}_{t \geq 1}$ is strictly stationary. In addition, $\mathbb{E}[u_{it}] = \mathbb{E}[u_{it}f_{jt}] = 0$ for all $i \leq p, j \leq m$ and $t \leq T$.

(ii) There exist positive constants c_1 and c_2 such that $\lambda_{\min}(\boldsymbol{\Sigma}_u) > c_1$, $\|\boldsymbol{\Sigma}_u\|_\infty < c_2$, and $\min_{i,j} \text{Var}(u_{it}u_{jt}) > c_1$.

(iii) There exist positive constants r_1, r_2, b_1 and b_2 such that for $s > 0, i \leq p, j \leq m$,

$$\mathbb{P}(|u_{it}| > s) \leq \exp(-(s/b_1)^{r_1}) \quad \text{and} \quad \mathbb{P}(|f_{jt}| > s) \leq \exp(-(s/b_2)^{r_2}).$$

We introduce the strong mixing conditions. Let $\mathcal{F}_{-\infty}^0$ and \mathcal{F}_n^∞ denote the σ -algebras generated by $\{(\mathbf{f}_s, \mathbf{u}_s) : -\infty \leq s \leq 0\}$ and $\{(\mathbf{f}_s, \mathbf{u}_s) : n \leq s \leq \infty\}$ respectively. In addition, define the mixing coefficient

$$\alpha(n) = \sum_{A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_n^\infty} |\mathbb{P}(A)\mathbb{P}(B) - \mathbb{P}(AB)|.$$

Assumption 7.3. There exists $r_3 > 0$ such that $3r_1^{-1} + 1.5r_2^{-1} + r_3^{-1} > 1$ and $C > 0$ satisfying $\alpha(n) \leq \exp(-Cn^{r_3})$ for all n .

Note that for the independence case, Assumption 7.3 is trivially satisfied since $\alpha(n) = 0$ for all n .

Assumption 7.4. There exists $M > 0$ such that for all $i \leq p$ and $s, t \leq T$,

(i) $\|\mathbf{b}_i\|_{\max} \leq M$,

(ii) $\mathbb{E}[p^{-1/2}(\mathbf{u}'_s \mathbf{u}_t - \mathbb{E}\mathbf{u}'_s \mathbf{u}_t)]^4 \leq M$,

(iii) $\mathbb{E}\|p^{-1/2} \sum_{i=1}^p \mathbf{b}_i u_{it}\|^4 \leq M$.

7.2 Convergence rate of error matrix

In order to achieve convergence rate for the covariance matrix of idiosyncratic error, we employ the following lemma from Fan et al. (2013a).

Lemma 7.1. Suppose that $(\log p)^{6\alpha} = o(T)$ where $\alpha = 3r_1^{-1} + 1$ and Assumption 4.2 holds. In addition, suppose that there is a sequence $a_T = o(1)$ so that $\max_{i \leq p} T^{-1} \sum_{t=1}^T |\hat{u}_{it} - u_{it}|^2 = O_P(a_T^2)$ and $\max_{i \leq p, t \leq T} |\hat{u}_{it} - u_{it}| = o_P(1)$. Then there is a constant $C > 0$ in the adaptive thresholding estimator (4.6) with $\tau_{ij} = C\omega_T(\hat{\sigma}_{u,ii}\hat{\sigma}_{u,jj})^{1/2}$ and

$$\omega_T = \sqrt{\frac{\log p}{T}} + a_T,$$

such that

$$\|\hat{\boldsymbol{\Sigma}}_u^\top - \boldsymbol{\Sigma}_u\| = O_P(\omega_T^{1-q} m_p).$$

The essential step of applying the previous lemma is to find a_T . We start by getting the convergence rate of $\hat{\mathbf{F}}$ and $\hat{\mathbf{B}}$. Let \mathbf{V} denote the $m \times m$ diagonal matrix of the first m largest eigenvalues of the sample covariance matrix in decreasing order. Recall that

$$\frac{1}{T} \mathbf{Y}' \mathbf{Y} \hat{\mathbf{F}} = \hat{\mathbf{F}} \mathbf{V}.$$

Define

$$\mathbf{H} = \frac{1}{T} \mathbf{V}^{-1} \hat{\mathbf{F}}' \mathbf{F} \mathbf{B}' \mathbf{B}.$$

Lemma 7.2. *The rates of convergence of $\hat{\mathbf{F}}$ are as follows:*

$$(i) \quad \|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_F = O_P\left(\frac{p}{\lambda_m \sqrt{T}} + \sqrt{\frac{T}{\lambda_m}}\right),$$

$$(ii) \quad \|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_{\max} = O_P\left(\left(\frac{1}{\sqrt{\lambda_m}} + \frac{p}{\lambda_m T} + \frac{\sqrt{p}}{\lambda_m}\right) (\log T)^{\frac{2}{r_2}}\right),$$

Proof. (i) By definition of $\hat{\mathbf{F}}$ and \mathbf{H}

$$\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}' = \frac{1}{T} (\mathbf{Y}' \mathbf{Y} - \mathbf{F} \mathbf{B}' \mathbf{B} \mathbf{F}') \hat{\mathbf{F}} \mathbf{V}^{-1}.$$

Since $\|\hat{\mathbf{F}}\|_F = O_P(\sqrt{T})$, $\|\mathbf{V}^{-1}\| = O_P(1/\lambda_m)$ from Theorem 3.1, we have

$$\|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_F \leq O_P\left(\frac{1}{\lambda_m \sqrt{T}}\right) \|\mathbf{U}' \mathbf{U} + \mathbf{F} \mathbf{B}' \mathbf{U} + \mathbf{U}' \mathbf{B} \mathbf{F}'\|,$$

where we used the fact $\|\mathbf{A} \mathbf{B}\|_F \leq \|\mathbf{A}\| \|\mathbf{B}\|_F$. By Lemma 6.1,

$$\|\frac{1}{T} \mathbf{U}' \mathbf{U}\| = \|\frac{1}{T} \mathbf{U} \mathbf{U}'\| \leq \|\frac{1}{T} \mathbf{U} \mathbf{U}' - \boldsymbol{\Sigma}_u\| + \|\boldsymbol{\Sigma}_u\| = O_P\left(\frac{p}{T}\right),$$

and since $\|\mathbf{B}\|_{\max} = O_P(\sqrt{\lambda_1/p})$ from Assumption 4.2,

$$\mathbb{E} \|\mathbf{B}' \mathbf{U}\|_F^2 = \sum_{t=1}^T \sum_{j=1}^m \mathbb{E} \left(\sum_{i=1}^p u_{it} b_{ij} \right)^2 \leq \sum_{t=1}^T \sum_{j=1}^m \sum_{i_1=1}^p \sum_{i_2=1}^p |\sigma_{u, i_1 i_2}| O\left(\frac{\lambda_1}{p}\right) = O(T \lambda_1).$$

Therefore by Markov inequality,

$$\|\mathbf{F} \mathbf{B}' \mathbf{U}\| \leq \|\mathbf{F}\|_F \|\mathbf{B}' \mathbf{U}\| = O_P(T \sqrt{\lambda_1}).$$

Hence,

$$\|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_F = O_P\left(\frac{p}{\lambda_m \sqrt{T}} + \sqrt{\frac{T}{\lambda_m}}\right).$$

(ii) From (i) we conclude

$$\|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_{\max} \leq O_P\left(\frac{1}{\lambda_m T}\right) \|\mathbf{U}' \mathbf{U} \hat{\mathbf{F}} + \mathbf{F} \mathbf{B}' \mathbf{U} \hat{\mathbf{F}} + \mathbf{U}' \mathbf{B} \mathbf{F}' \hat{\mathbf{F}}\|_{\max}.$$

Let us bound each term separately. For the first term, $\|\mathbf{U}'\mathbf{U}\hat{\mathbf{F}}\|_{\max} \leq \|\mathbf{U}'\mathbf{U}\|_{\infty}\|\hat{\mathbf{F}}\|_{\max}$ and

$$\|\mathbf{U}'\mathbf{U}\|_{\infty} = \max_t \sum_{s=1}^T |\mathbf{u}'_t \mathbf{u}_s| = \max_t \sum_{s=1}^T \left| \mathbf{u}'_t \mathbf{u}_s - \mathbb{E}[\mathbf{u}'_t \mathbf{u}_s] \right| + \mathbb{E}[\mathbf{u}'_t \mathbf{u}_t] = O_P(T\sqrt{p} + p).$$

The second term is bounded as $\|\mathbf{F}\mathbf{B}'\mathbf{U}\hat{\mathbf{F}}\|_{\max} \leq m\|\mathbf{F}\|_{\max}\|\mathbf{B}'\mathbf{U}\|_{\infty}\|\hat{\mathbf{F}}\|_{\max}$ and

$$\|\mathbf{B}'\mathbf{U}\|_{\infty} = \max_{k \leq m} \sum_{t=1}^T |\tilde{\mathbf{b}}'_k \mathbf{u}_t| = O(T\sqrt{\lambda_1}),$$

since $\text{var}(\tilde{\mathbf{b}}'_k \mathbf{u}_t) = \tilde{\mathbf{b}}'_k \boldsymbol{\Sigma}_u \tilde{\mathbf{b}}_k = O(\lambda_1)$. The third term can be bounded similarly. Together with the fact that $\|\mathbf{B}\|_{\max} = O_P(\sqrt{\lambda_1/p})$ and $\|\mathbf{F}\|_{\max} = O_P((\log T)^{1/r_2})$ from Assumption 4.2, we obtain

$$\begin{aligned} \|\hat{\mathbf{F}} - \mathbf{F}\mathbf{H}'\|_{\max} &\leq O_P\left(\frac{1}{\lambda_m T}\right) \left((p + T\sqrt{p})(\log T)^{\frac{1}{r_2}} + T\sqrt{\lambda_1}(\log T)^{\frac{2}{r_2}} \right) \\ &= O_P\left(\left(\frac{1}{\sqrt{\lambda_m}} + \frac{p}{\lambda_m T} + \frac{\sqrt{p}}{\lambda_m} \right) (\log T)^{\frac{2}{r_2}} \right). \end{aligned}$$

□

Lemma 7.3. *The rates of convergence for $\hat{\mathbf{B}}$ are as follows. Two regimes are considered.*

If $\lambda_m > C_1 p$ for constant $C_1 > 0$, we have

- (i) $\|\mathbf{H}^{-1}\| = O_P(1)$,
- (ii) $\|\hat{\mathbf{B}} - \mathbf{B}\mathbf{H}^{-1}\|_{\max} = O_P(\sqrt{\log p/T})$.

If $C_2\sqrt{p}(\log T)^{1/r_2} \leq \lambda_m \leq C_1 p$ for constant $C_2 > 0$, we have

- (i') $\|\mathbf{H}'\mathbf{H} - \mathbf{I}_m\| = O_P(c_m + 1/\sqrt{\lambda_m} + 1/\sqrt{T})$,
- (ii') $\|\hat{\mathbf{B}} - \mathbf{B}\mathbf{H}'\|_{\max} = O_P(\sqrt{\log p/T})$.

Proof. (i') From Lemma 7.2 (i) we have

$$\|\mathbf{F}'(\hat{\mathbf{F}} - \mathbf{F}\mathbf{H}')\|_F \leq O_P\left(\frac{1}{\lambda_m}\right) \left(\frac{1}{T}\|\mathbf{F}'\mathbf{U}'\mathbf{U}\mathbf{F}\| + \frac{1}{T}\|\mathbf{F}'\mathbf{U}'\mathbf{U}\| \|\hat{\mathbf{F}} - \mathbf{F}\mathbf{H}\| + 2\|\mathbf{F}\mathbf{B}'\mathbf{U}\| \right).$$

We claim $\|\mathbf{F}'\mathbf{U}'\| = O_P(\sqrt{Tp})$. Hence, $\|\mathbf{F}'(\hat{\mathbf{F}} - \mathbf{F}\mathbf{H}')\|_F = O_P(p/\lambda_m) + O_P(T/\sqrt{\lambda_m})$. With this, we bound $\|\mathbf{H}'\mathbf{H} - \mathbf{I}_m\|$. First obviously $\|\mathbf{H}\| = O_P(1)$ since λ_1/λ_m is bounded. Then from Fan et al. (2013a), we know

$$\begin{aligned} \|\mathbf{H}'\mathbf{H} - \mathbf{I}_m\| &\leq \frac{1}{T}\|\mathbf{F}'(\hat{\mathbf{F}} - \mathbf{F}\mathbf{H}')\| (1 + \|\mathbf{H}\|) + \|\mathbf{H}\|^2 \|\mathbf{F}'\mathbf{F}/T - \mathbf{I}_m\| \\ &= O_P(c_m + 1/\sqrt{\lambda_m} + 1/\sqrt{T}). \end{aligned}$$

It remains to show that $\|\mathbf{F}'\mathbf{U}'\| = O_P(\sqrt{Tp})$. By definition,

$$\|\mathbf{F}'\mathbf{U}'\mathbf{U}\mathbf{F}\| = \|\mathbf{U}\mathbf{F}\mathbf{F}'\mathbf{U}'\| = \sup_{\mathbf{x} \in S^{p-1}} \|\mathbf{F}'\mathbf{U}'\mathbf{x}\|^2 \leq 2 \sup_{\mathbf{x} \in \mathcal{N}} \|\mathbf{F}'\mathbf{U}'\mathbf{x}\|^2,$$

where \mathcal{N} is a $1/4$ -net of the unit sphere S^{p-1} and $|\mathcal{N}| \leq 9^p$. Since $\|\mathbf{F}'\mathbf{U}'\mathbf{x}\|^2 = \sum_{k=1}^m (\sum_{t \leq T} f_{kt} \mathbf{u}'_t \mathbf{x})^2 \leq mCT \sum_{t \leq T} (\mathbf{u}'_t \mathbf{x})^2$, using Chernoff bound, we have

$$\mathbb{P}\left(\|\mathbf{F}'\mathbf{U}'\mathbf{U}\mathbf{F}\| \geq t\right) \leq 9^p \cdot e^{-\frac{\theta t}{2CmT}} (\mathbb{E}[e^{\theta(\mathbf{u}'_t \mathbf{x})^2}])^T.$$

$\mathbf{u}'_t \mathbf{x}$ is sub-Gaussian, so choosing $t \asymp Tp$, we obtain that $\|\mathbf{U}\mathbf{F}\| = O_P(\sqrt{Tp})$.

(i) In (i'), we showed $\|\mathbf{H}'\mathbf{H} - \mathbf{I}_m\| = O_P(c_m + 1/\sqrt{\lambda_m} + 1/\sqrt{T})$. If in addition, we know $\lambda_m \geq C_1 p$, then $c_m = o(1)$ so that $\|\mathbf{H}'\mathbf{H} - \mathbf{I}_m\| = o_P(1)$. So we conclude $\lambda_{\min}(\mathbf{H}'\mathbf{H}) > 1/2$ with probability approaching one according to Weyl's Theorem. Thus $\|\mathbf{H}^{-1}\| = O_P(1)$.

(ii) Decompose $\hat{\mathbf{B}} - \mathbf{B}\mathbf{H}^{-1}$ as follows:

$$\begin{aligned} \hat{\mathbf{B}} - \mathbf{B}\mathbf{H}^{-1} &= \frac{1}{T} \mathbf{Y}\hat{\mathbf{F}} - \mathbf{B}\mathbf{H}^{-1} \\ &= \frac{1}{T} \mathbf{B}\mathbf{H}^{-1}(\mathbf{H}\mathbf{F}' - \hat{\mathbf{F}}')\hat{\mathbf{F}} + \frac{1}{T} \mathbf{U}(\hat{\mathbf{F}} - \mathbf{F}\mathbf{H}') + \frac{1}{T} \mathbf{U}\mathbf{F}\mathbf{H}'. \end{aligned}$$

Fan et al. (2013a) showed that

$$\frac{1}{T} \|\mathbf{U}\mathbf{F}\|_{\max} = \max_{i \leq p, k \leq m} \left| \frac{1}{T} \sum_{t=1}^T u_{it} f_{tk} \right| = O_P\left(\sqrt{\frac{\log p}{T}}\right).$$

Thus the max norm of the last term is $O_P(\sqrt{\log p/T})$. The max norms of the first and second terms are bounded respectively by

$$\frac{m}{T} \|\mathbf{B}\|_{\max} \|\mathbf{H}^{-1}\| \|\mathbf{H}\mathbf{F}' - \hat{\mathbf{F}}'\|_{\max} \cdot \sqrt{T} \|\hat{\mathbf{F}}\|_F = O_P\left(\left(\frac{1}{\sqrt{\lambda_m}} + \frac{1}{\sqrt{p}} + \sqrt{\frac{c_m}{T}}\right) (\log T)^{\frac{2}{r_2}}\right),$$

and by

$$\begin{aligned} &O_P\left(\frac{1}{\lambda_m T^2}\right) \|\mathbf{U}\mathbf{U}'\mathbf{U}\hat{\mathbf{F}} + \mathbf{U}\mathbf{F}\mathbf{B}'\mathbf{U}\hat{\mathbf{F}} + \mathbf{U}\mathbf{U}'\mathbf{B}\mathbf{F}'\hat{\mathbf{F}}\|_{\max} \\ &\leq O_P\left(\frac{1}{\lambda_m T^2}\right) \left(\|\mathbf{U}\mathbf{U}'\mathbf{U}\|_{\infty} \|\hat{\mathbf{F}}\|_{\max} + m \|\mathbf{U}\mathbf{F}\|_{\max} \|\mathbf{B}'\mathbf{U}\|_{\infty} \|\hat{\mathbf{F}}\|_{\max} + T \|\mathbf{B}'\mathbf{U}\|_{\infty} \|\mathbf{U}\|_{\max}\right) \\ &= O_P\left(\frac{1}{\lambda_m T^2}\right) \left(T \sqrt{pT} (\log T)^{\frac{1}{r_2}} + \sqrt{T \log p} (T \sqrt{\lambda_1}) (\log T)^{\frac{1}{r_2}} + T^2 \sqrt{\lambda_1} (\log(pT))^{\frac{1}{r_1}}\right). \end{aligned}$$

Simplify and Combine the rates together, and note $\lambda_m > C_1 p$ in this case and

$\sqrt{p}(\log T)^{1/r_2} = o(\lambda_m)$, we obtain,

$$\|\hat{\mathbf{B}} - \mathbf{B}\mathbf{H}^{-1}\|_{\max} = O_P\left(\frac{1}{\sqrt{p}}\left((\log T)^{\frac{2}{r_2}} + (\log(pT))^{\frac{1}{r_1}}\right) + \sqrt{\frac{\log p}{T}}\right) = O_P\left(\sqrt{\frac{\log p}{T}}\right).$$

(ii') Now let us consider the other situation. We have a different decomposition of $\hat{\mathbf{B}} - \mathbf{B}\mathbf{H}'$:

$$\begin{aligned}\hat{\mathbf{B}} - \mathbf{B}\mathbf{H}' &= \frac{1}{T}\mathbf{Y}\hat{\mathbf{F}} - \mathbf{B}\mathbf{H}' \\ &= \frac{1}{T}\mathbf{B}\mathbf{F}'(\hat{\mathbf{F}} - \mathbf{F}\mathbf{H}') + \mathbf{B}\left(\frac{1}{T}\mathbf{F}'\mathbf{F} - \mathbf{I}_m\right)\mathbf{H}' + \frac{1}{T}\mathbf{U}(\hat{\mathbf{F}} - \mathbf{F}\mathbf{H}') + \frac{1}{T}\mathbf{U}\mathbf{F}\mathbf{H}'.\end{aligned}$$

As before, the max norm of the last term is $O_P(\sqrt{\log p/T})$. The max norms of the first three terms are bounded respectively by

$$\frac{\sqrt{m}}{T}\|\mathbf{B}\|_{\max}\|\mathbf{F}'(\hat{\mathbf{F}} - \mathbf{F}\mathbf{H}')\|_F = O_P(\sqrt{c_m/T} + 1/\sqrt{p});$$

$$\sqrt{m}\|\mathbf{B}\|_{\max}\left\|\frac{1}{T}\mathbf{F}'\mathbf{F} - \mathbf{I}_m\right\|\|\mathbf{H}'\| = O_P(\sqrt{\lambda_1/(pT)});$$

and

$$\begin{aligned}O_P\left(\frac{1}{\lambda_m T^2}\right)\|\mathbf{U}\mathbf{U}'\mathbf{U}\hat{\mathbf{F}} + \mathbf{U}\mathbf{F}\mathbf{B}'\mathbf{U}\hat{\mathbf{F}} + \mathbf{U}\mathbf{U}'\mathbf{B}\mathbf{F}'\hat{\mathbf{F}}\|_{\max} \\ \leq O_P\left(\frac{1}{\lambda_m T^2}\right)\left(T\sqrt{pT}(\log T)^{\frac{1}{r_2}} + \sqrt{T\log p}(T\sqrt{\lambda_1})(\log T)^{\frac{1}{r_2}} + T\|\mathbf{B}'\mathbf{U}\mathbf{U}'\|_{\max}\right),\end{aligned}$$

where $\|\mathbf{B}'\mathbf{U}\mathbf{U}'\|_{\max} = O_P(T\sqrt{\lambda_1/p} + \sqrt{\lambda_1\log p})$ is quite small.

Simplify and Combine the rates together, we obtain,

$$\|\hat{\mathbf{B}} - \mathbf{B}\mathbf{H}'\|_{\max} = O_P\left(\frac{\sqrt{p}(\log T)^{1/r_2}}{\lambda_m\sqrt{T}} + \sqrt{\frac{\lambda_1}{pT}} + \sqrt{\frac{\log p}{T}}\right) = O_P\left(\sqrt{\frac{\log p}{T}}\right).$$

□

Theorem 7.1. *Under the assumptions of Theorem 4.1, by applying adaptive thresholding estimator (4.6) with $\tau_{ij} = C\omega_T(\hat{\sigma}_{u,ii}\hat{\sigma}_{u,jj})^{1/2}$ and $\omega_T = \sqrt{\log p/T} + \sqrt{1/p}$, we have*

$$\|\hat{\boldsymbol{\Sigma}}_u^\top - \boldsymbol{\Sigma}_u\| = O_P(\omega_T^{1-q}m_p)$$

Proof. Recall that $\hat{u}_{it} = y_{it} - \hat{\mathbf{b}}_i'\hat{\mathbf{f}}_t$. We separately consider the two cases in Lemma 7.3. If $\lambda_m > C_1p$, so \mathbf{H}^{-1} is well defined. We have

$$u_{it} - \hat{u}_{it} = (\hat{\mathbf{b}}_i' - \mathbf{b}_i'\mathbf{H}^{-1})(\hat{\mathbf{f}}_t - \mathbf{H}\mathbf{f}_t) + \mathbf{b}_i'\mathbf{H}^{-1}(\hat{\mathbf{f}}_t - \mathbf{H}\mathbf{f}_t) + (\hat{\mathbf{b}}_i' - \mathbf{b}_i'\mathbf{H}^{-1})\mathbf{H}\mathbf{f}_t.$$

Therefore by Cauchy-Schwarz,

$$\begin{aligned} \max_{i \leq p} T^{-1} \sum_{t=1}^T |\hat{u}_{it} - u_{it}|^2 &\leq 3 \max_i \|\mathbf{b}'_i \mathbf{H}^{-1}\|^2 \frac{1}{T} \|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_F^2 \\ &\quad + 3m \|\hat{\mathbf{B}} - \mathbf{B} \mathbf{H}^{-1}\|_{\max}^2 \frac{1}{T} \|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_F^2 \\ &\quad + 3m \|\hat{\mathbf{B}} - \mathbf{B} \mathbf{H}^{-1}\|_{\max}^2 \frac{1}{T} \sum_{t=1}^T \|\mathbf{H} \mathbf{f}_t\|^2. \end{aligned}$$

It follows from Lemma 7.2 and 7.3 (ii) that

$$\max_{i \leq p} T^{-1} \sum_{t=1}^T |\hat{u}_{it} - u_{it}|^2 = O_P \left(\frac{\lambda_1}{pT} \left(\frac{p^2}{\lambda_m^2 T} + \frac{T}{\lambda_m} \right) + \frac{\log p}{T} \right) = O_P \left(\frac{\log p}{T} + \frac{1}{p} \right).$$

Replacing the average over t in the above inequality with maximum over t and $T^{-1} \|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_F^2$ with $m \|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_{\max}^2$, we can also derive bound for $\max_{i \leq p, t \leq T} |\hat{u}_{it} - u_{it}|$. Since from Assumption 4.2 we have $\max_{t \leq T} \|\mathbf{f}_t\| = O_P((\log T)^{1/r_2})$, we get $\max_{i \leq p, t \leq T} |\hat{u}_{it} - u_{it}| = o_P(1)$.

Now if $C_2 \sqrt{p} (\log T)^{1/r_2} \leq \lambda_m \leq C_1 p$, we apply a different way of decomposing $u_{it} - \hat{u}_{it}$.

$$u_{it} - \hat{u}_{it} = (\hat{\mathbf{b}}'_i - \mathbf{b}'_i \mathbf{H}')(\hat{\mathbf{f}}_t - \mathbf{H} \mathbf{f}_t) + \mathbf{b}'_i \mathbf{H}'(\hat{\mathbf{f}}_t - \mathbf{H} \mathbf{f}_t) + (\hat{\mathbf{b}}'_i - \mathbf{b}'_i \mathbf{H}') \mathbf{H} \mathbf{f}_t + \mathbf{b}'_i (\mathbf{H}' \mathbf{H} - \mathbf{I}_m) \mathbf{f}_t.$$

Therefore by Cauchy-Schwarz,

$$\begin{aligned} \max_{i \leq p} T^{-1} \sum_{t=1}^T |\hat{u}_{it} - u_{it}|^2 &\leq 4 \max_i \|\mathbf{b}'_i \mathbf{H}'\|^2 \frac{1}{T} \|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_F^2 \\ &\quad + 4m \|\hat{\mathbf{B}} - \mathbf{B} \mathbf{H}'\|_{\max}^2 \frac{1}{T} \|\hat{\mathbf{F}} - \mathbf{F} \mathbf{H}'\|_F^2 \\ &\quad + 4m \|\hat{\mathbf{B}} - \mathbf{B} \mathbf{H}'\|_{\max}^2 \frac{1}{T} \sum_{t=1}^T \|\mathbf{H} \mathbf{f}_t\|^2 \\ &\quad + 4 \max_i \|\mathbf{b}_i\|^2 \|\mathbf{H}' \mathbf{H} - \mathbf{I}\| \frac{1}{T} \sum_{t=1}^T \|\mathbf{f}_t\|^2. \end{aligned}$$

It follows from Lemma 7.2 and 7.3 (ii') that

$$\max_{i \leq p} T^{-1} \sum_{t=1}^T |\hat{u}_{it} - u_{it}|^2 = O_P \left(\frac{\lambda_1}{pT} \left(\frac{p^2}{\lambda_m^2 T} + \frac{T}{\lambda_m} \right) + \frac{\log p}{T} + \frac{\lambda_1}{pT} \right) = O_P \left(\frac{\log p}{T} + \frac{1}{p} \right).$$

Again, it is not hard to show $\max_{i \leq p, t \leq T} |\hat{u}_{it} - u_{it}| = o_P(1)$.

Finally, Lemma 7.1 concludes the theorem by choosing $a_T = \sqrt{\log p/T} + \sqrt{1/p}$ in both cases. \square

7.3 Proofs of Theorems in Section 4

Given Theorem 7.1, we are ready to start showing theorems in this section. The proofs were built based on conclusions in Section 3.

Proof of Theorem 4.1. We first prove the theorem for term Δ_{L1} . Write $\mathbf{B} = (\tilde{\mathbf{b}}_1, \dots, \tilde{\mathbf{b}}_m)$ and the minimizer of (4.5) as $\hat{\mathbf{B}} = (\hat{\mathbf{b}}_1, \dots, \hat{\mathbf{b}}_m)$. Since $\hat{\mathbf{B}}$ is just the eigenvectors (unnormalized) of $\hat{\Sigma}$, we have:

$$\hat{\lambda}_j = \|\hat{\mathbf{b}}_j\|^2 \text{ and } \hat{\xi}_j = \hat{\mathbf{b}}_j / \|\hat{\mathbf{b}}_j\|.$$

Then $\hat{\lambda}_j^S = \|\hat{\mathbf{b}}_j\|^2 / (1 + \bar{c}c_j)$ or $\hat{\lambda}_j^S = \|\hat{\mathbf{b}}_j\|^2 - \bar{c}p/n$ if c_j is unknown. Let $\hat{\Lambda} = \text{diag}(\|\hat{\mathbf{b}}_1\|^2, \dots, \|\hat{\mathbf{b}}_m\|^2)$ be the diagonal matrix of the first m empirical eigenvalues and $\hat{\Gamma} = (\hat{\mathbf{b}}_1 / \|\hat{\mathbf{b}}_1\|, \dots, \hat{\mathbf{b}}_m / \|\hat{\mathbf{b}}_m\|)$ be the empirical eigenvector matrix. In Sections 3.1 and 3.2, our results for empirical eigenvalues and eigenvectors imply the following:

$$\|\Lambda^{-1/2}(\hat{\Lambda}^S - \Lambda)\Lambda^{-1/2}\| = O_P(c_m\sqrt{T/p} + T^{-1/2}); \quad (7.1)$$

and

$$\|\hat{\Gamma}'\Gamma - \mathbf{D}\| = O_P(c_m\sqrt{T/p} + T^{-1/2}), \quad (7.2)$$

where $\mathbf{D} = \text{diag}((1 + \bar{c}c_1)^{-1/2}, \dots, (1 + \bar{c}c_m)^{-1/2})$ and $\hat{\Lambda}^S = \mathbf{D}\hat{\Lambda}\mathbf{D}$. Now let us start to bound Δ_{L1} and Δ_{L2} .

$$\begin{aligned} \Delta_{L1} &\leq \|\Lambda^{-1/2}\Gamma'(\hat{\Gamma}\hat{\Lambda}^S\hat{\Gamma}' - \Gamma\Lambda\Gamma')\Gamma\Lambda^{-1/2}\| + \|\Lambda^{-1/2}\Gamma'(\Gamma\Lambda\Gamma' - \mathbf{B}\mathbf{B}')\Gamma\Lambda^{-1/2}\| \\ &=: \Delta_{L1}^{(1)} + \Delta_{L1}^{(2)}. \end{aligned}$$

We handle the two terms separately.

$$\Delta_{L1}^{(1)} \leq \|\Lambda^{-1/2}\Gamma'(\hat{\Gamma}\hat{\Lambda}^S\hat{\Gamma}' - \Gamma\mathbf{D}\Lambda\mathbf{D}\Gamma')\Gamma\Lambda^{-1/2}\|,$$

where we used $\mathbf{D}^2 \preceq \mathbf{I}$. The right hand side is further bounded by $I + 2II + III$ with

$$\begin{aligned} I &= \|\Lambda^{-1/2}(\Gamma'\hat{\Gamma} - \mathbf{D})\hat{\Lambda}^S(\hat{\Gamma}'\Gamma - \mathbf{D})\Lambda^{-1/2}\|, \\ II &= \|\Lambda^{-1/2}(\Gamma'\hat{\Gamma} - \mathbf{D})\hat{\Lambda}^S\mathbf{D}\Lambda^{-1/2}\|, \quad III = \|\Lambda^{-1/2}\mathbf{D}(\hat{\Lambda}^S - \Lambda)\mathbf{D}\Lambda^{-1/2}\|. \end{aligned}$$

By Equations (7.1) and (7.2), we conclude that II and III are of order $O_P(c_m\sqrt{T/p} + T^{-1/2})$ and I is of smaller order. Thus $\Delta_{L1}^{(1)} = O_P(c_m\sqrt{T/p} + T^{-1/2})$. In order to derive rate of $\Delta_{L1}^{(2)}$, denote $\tilde{\Lambda} = \text{diag}(\|\tilde{\mathbf{b}}_1\|^2, \dots, \|\tilde{\mathbf{b}}_m\|^2)$ and $\tilde{\Gamma} = (\tilde{\mathbf{b}}_1 / \|\tilde{\mathbf{b}}_1\|, \dots, \tilde{\mathbf{b}}_m / \|\tilde{\mathbf{b}}_m\|)$ so that $\mathbf{B}\mathbf{B}' = \tilde{\Gamma}\tilde{\Lambda}\tilde{\Gamma}'$. We could treat $\Delta_{L1}^{(2)}$ similar to $\Delta_{L1}^{(1)}$. $\Delta_{L1}^{(2)}$ could be bounded by

$I' + 2II' + III'$ with

$$\begin{aligned} I' &= \|\Lambda^{-\frac{1}{2}}(\Gamma'\tilde{\Gamma} - \mathbf{I})\tilde{\Lambda}(\tilde{\Gamma}'\Gamma - \mathbf{I})\Lambda^{-\frac{1}{2}}\|, \\ II' &= \|\Lambda^{-\frac{1}{2}}(\Gamma'\tilde{\Gamma} - \mathbf{I})\tilde{\Lambda}\Lambda^{-\frac{1}{2}}\|, \quad III' = \|\Lambda^{-\frac{1}{2}}(\tilde{\Lambda} - \Lambda)\Lambda^{-\frac{1}{2}}\|. \end{aligned}$$

By Weyl's theorem, $|\lambda_j - \|\tilde{\mathbf{b}}_j\|^2| \leq \|\Sigma_u\| \leq C$, so $III' = O(1/\lambda_m) = O(c_m T/p)$. By *sin* θ theorem,

$$\|\Gamma'\tilde{\Gamma} - \mathbf{I}\| = \|\Gamma'(\tilde{\Gamma} - \Gamma)\| \leq \|\tilde{\Gamma} - \Gamma\| \leq C\|\Sigma_u\|/\lambda_m = O(c_m T/p),$$

so is II' . Since I' is of smaller order, we conclude $\Delta_{L1}^{(2)} = O(c_m T/p)$. Therefore, $\Delta_{L1} \leq \Delta_{L1}^{(1)} + \Delta_{L1}^{(2)} = O_P(c_m \sqrt{T/p} + T^{-1/2})$.

The bound for term Δ_{L2} is derived in the following. Recall that

$$\Delta_{L2} = \|\Theta^{-\frac{1}{2}}\Omega'(\hat{\Gamma}\hat{\Lambda}^S\hat{\Gamma}' - \mathbf{B}\mathbf{B}')\Omega\Theta^{-\frac{1}{2}}\|,$$

which is bounded by

$$\|\Theta^{-\frac{1}{2}}\Omega'\hat{\Gamma}\hat{\Lambda}^S\hat{\Gamma}'\Omega\Theta^{-\frac{1}{2}}\| + \|\Theta^{-\frac{1}{2}}\Omega'\tilde{\Gamma}\tilde{\Lambda}\tilde{\Gamma}'\Omega\Theta^{-\frac{1}{2}}\| =: \Delta_{L2}^{(1)} + \Delta_{L2}^{(2)}.$$

$$\Delta_{L2}^{(1)} \leq \|\Theta^{-1}\|\|\Omega'\hat{\Gamma}\|^2\|\hat{\Lambda}^S\| = O_P(p/T),$$

because by Lemma 6.6, $\|\Omega'\hat{\Gamma}\| = O_P(\sqrt{c_m}) = O_P(\sqrt{p/(T\lambda_m)})$.

$$\Delta_{L2}^{(2)} \leq \|\Theta^{-1}\|\|\Omega'\tilde{\Gamma}\|^2\|\tilde{\Lambda}\| = O_P(1/\lambda_m),$$

as $\|\Omega'\tilde{\Gamma}\| = \|\Gamma\Gamma' - \tilde{\Gamma}\tilde{\Gamma}'\| = O(\|\Sigma_u\|/\lambda_m) = O_P(1/\lambda_m)$ by *sin* θ Theorem. Finally, $\Delta_{L2}^{(1)} = O_P(p/T + 1/\lambda_m)$.

Finally let us look at term Δ_S . Since $\Delta_S \leq \|\Sigma^{-1}\|\|\hat{\Sigma}_u^\top - \Sigma_u\|$, it suffices to bound $\|\hat{\Sigma}_u^\top - \Sigma_u\|$, which has already been done in Theorem 7.1. So

$$\Delta_S = O_P\left(m_p\left(\frac{\log p}{T} + \frac{1}{p}\right)^{(1-q)/2}\right).$$

□

Proof of Theorem 4.2. The numerator of the relative risk is bounded by

$$|\mathbf{w}'(\hat{\Gamma}\hat{\Lambda}^S\hat{\Gamma}' - \mathbf{B}\mathbf{B}')\mathbf{w}| + |\mathbf{w}'(\hat{\Sigma}_u^\top - \Sigma_u)\mathbf{w}|.$$

The second term is bounded by $\|\hat{\Sigma}_u^\top - \Sigma_u\|\|\mathbf{w}\|^2$, thus is $O_P(\Delta_S\|\mathbf{w}\|^2)$. By using $\mathbf{w} =$

$(\mathbf{\Gamma}, \mathbf{\Omega})\boldsymbol{\eta}$, the first term can be written as

$$|(\boldsymbol{\eta}'_A \mathbf{\Gamma}' + \boldsymbol{\eta}'_B \mathbf{\Omega}') \mathbf{E}(\mathbf{\Gamma} \boldsymbol{\eta}_A + \mathbf{\Omega} \boldsymbol{\eta}_B)| = O_P(\boldsymbol{\eta}'_A \mathbf{\Gamma}' \mathbf{E} \mathbf{\Gamma} \boldsymbol{\eta}_A + \boldsymbol{\eta}'_B \mathbf{\Omega}' \mathbf{E} \mathbf{\Omega} \boldsymbol{\eta}_B),$$

where $\mathbf{E} = \hat{\mathbf{\Gamma}} \hat{\mathbf{\Lambda}}^S \hat{\mathbf{\Gamma}}' - \mathbf{B} \mathbf{B}'$. It is easy to see from the proof of Theorem 4.1 that

$$\boldsymbol{\eta}'_A \mathbf{\Gamma}' \mathbf{E} \mathbf{\Gamma} \boldsymbol{\eta}_A = O_P(\Delta_{L1} \lambda_1 \|\boldsymbol{\eta}_A\|^2),$$

By Theorem 3.2, $\|\boldsymbol{\Omega}' \hat{\mathbf{\Gamma}}\|_{\max} = \max_{j \leq m} \|\hat{\boldsymbol{\xi}}_{jB}\|_{\max} = O_P(c_m \sqrt{T/p} + \sqrt{c_m \log p/p})$. From proof of Theorem 4.1, we know that $\|\boldsymbol{\Omega}' \tilde{\mathbf{\Gamma}}\|_{\max} \leq \|\boldsymbol{\Omega}' \tilde{\mathbf{\Gamma}}\| = O_P(1/\lambda_m)$. Therefore,

$$\boldsymbol{\eta}'_B \mathbf{\Omega}' \mathbf{E} \mathbf{\Omega} \boldsymbol{\eta}_B \leq \|\boldsymbol{\eta}_B\|_1^2 (\|\boldsymbol{\Omega}' \hat{\mathbf{\Gamma}}\|_{\max}^2 \|\hat{\mathbf{\Lambda}}^S\| + \|\boldsymbol{\Omega}' \tilde{\mathbf{\Gamma}}\|_{\max}^2 \|\tilde{\mathbf{\Lambda}}\|),$$

which gives $\boldsymbol{\eta}'_B \mathbf{\Omega}' \mathbf{E} \mathbf{\Omega} \boldsymbol{\eta}_B = O_P(c_m + \log p/T + \lambda_m^{-1})$.

The denominator is lower bounded by $\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w} \geq \lambda_m \|\boldsymbol{\eta}_A\|^2 + c \|\boldsymbol{\eta}_B\|^2$. Thus the relative risk is of order

$$O_P\left(\frac{\Delta_{L1} \lambda_1 \|\boldsymbol{\eta}_A\|^2 + c_m + \log p/T + \lambda_m^{-1}}{\lambda_m \|\boldsymbol{\eta}_A\|^2 + c \|\boldsymbol{\eta}_B\|^2} + \Delta_S\right) = O_P\left(T^{-\min\{\frac{\alpha+\beta-1}{\beta}, \frac{1}{2}\}} + m_p w_T^{1-q}\right),$$

according to the convergence rate of Δ_{L1} and Δ_S in Theorem 4.1. Note the rate $O_P(T^{-(\alpha+\beta-1)/\beta})$ comes from c_m in the numerator. If we further assume $\|\boldsymbol{\eta}_A\| \geq C_2$, this rate becomes c_m/λ_m dominated by $T^{-1/2}$, thus the relative risk is of order $O_P(T^{-1/2} + m_p w_T^{1-q})$. \square

Proof of Theorem 4.3. The proof follows Theorem 1 of Fan and Han (2013). Using their notation, we have

$$\widehat{\text{FDP}}_U(t) - \text{FDP}_A(t) = (\Delta_1 + \Delta_2)/R(t) + O(p^{\theta-1/2} \|\boldsymbol{\mu}^*\|),$$

where with $\widetilde{\mathbf{W}} = (\mathbf{B}'\mathbf{B})^{-1}(\mathbf{B}'\mathbf{Z})$,

$$\begin{aligned} \Delta_1 &= \sum_{i=1}^p \left[\Phi(\hat{a}_i(z_{t/2} + \hat{\mathbf{b}}'_i \widetilde{\mathbf{W}})) - \Phi(a_i(z_{t/2} + \mathbf{b}'_i \widetilde{\mathbf{W}})) \right], \\ \Delta_2 &= \sum_{i=1}^p \left[\Phi(\hat{a}_i(z_{t/2} - \hat{\mathbf{b}}'_i \widetilde{\mathbf{W}})) - \Phi(a_i(z_{t/2} - \mathbf{b}'_i \widetilde{\mathbf{W}})) \right]. \end{aligned}$$

We just need to bound Δ_1 , then Δ_2 can be bound similarly. As shown in Fan and Han (2013),

$$|\Delta_1| \leq C \left(\sum_{j=1}^m |\hat{\lambda}_j^S - \lambda_j| + \lambda_j \|\hat{\boldsymbol{\xi}}_j^S - \boldsymbol{\xi}_j\| + \sqrt{p} (\|\boldsymbol{\mu}^*\| + \sqrt{p}) \|\hat{\boldsymbol{\xi}}_j^S - \boldsymbol{\xi}_j\| \right),$$

where $\hat{\lambda}_j^S$ and $\hat{\xi}_j^S$ are the j th eigenvalue and eigenvector of $\hat{\Sigma}^S$ defined in (4.8). So by Weyl's theorem and Theorem 4.1,

$$\begin{aligned} |\hat{\lambda}_j^S - \lambda_j| &\leq \|\hat{\Sigma}^S - \Sigma\| = O_P(\Delta_{L1}\lambda_1 + \Delta_{L2} + \Delta_S) \\ &= O_P\left(\lambda_1\left(c_m\sqrt{\frac{T}{p}} + \frac{1}{\sqrt{T}}\right) + \sqrt{\frac{\log p}{T}} + \frac{p}{T}\right) \\ &= O_P\left(\frac{p}{T} + \frac{p^\alpha}{\sqrt{T}}\right), \end{aligned}$$

since $\lambda_m \propto p^\alpha$. By *sin θ* theorem, we also have $\|\hat{\xi}_j^S - \xi_j\| \leq O_P(\|\hat{\Sigma}^S - \Sigma\|/\lambda_j)$. So finally

$$|\Delta_1/R(t)| = O_P\left(p^\theta\left(\frac{1}{T} + \frac{p^{\alpha-1}}{\sqrt{T}} + \left(\frac{\|\mu^*\|}{\sqrt{p}} + 1\right)\left(\frac{p^{1-\alpha}}{T} + \frac{1}{\sqrt{T}}\right)\right)\right).$$

Since $Cp^{1-\alpha} < Cp^\beta \leq T$, so

$$|\widehat{\text{FDP}}_U(t) - \text{FDP}_A(t)| = O_P\left(p^\theta(\|\mu^*\|p^{-\frac{1}{2}} + T^{-\min\{\frac{\alpha+\beta-1}{\beta}, \frac{1}{2}\}})\right).$$

□

References

- ANDERSON, T. W. (1963). Asymptotic theory for principal component analysis. *The Annals of Mathematical Statistics* **34** 122–148.
- ANTONIADIS, A. and FAN, J. (2001). Regularization of wavelet approximations. *Journal of the American Statistical Association* **96**.
- BAI, Z. (1999). Methodologies in spectral analysis of large-dimensional random matrices, a review. *Statist. Sinica* **9** 611–677.
- BAI, Z. and SILVERSTEIN, J. W. (2009). *Spectral analysis of large dimensional random matrices (2nd ed.)*. Springer.
- BAI, Z. and YIN, Y. (1993). Limit of the smallest eigenvalue of a large dimensional sample covariance matrix. *The annals of Probability* 1275–1294.
- BAIK, J., BEN AROUS, G. and PÉCHÉ, S. (2005). Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Annals of Probability* 1643–1697.
- BICKEL, P. J. and LEVINA, E. (2008). Covariance regularization by thresholding. *The Annals of Statistics* 2577–2604.

- CAI, T., FAN, J. and JIANG, T. (2013a). Distributions of angles in random packing on spheres. *arXiv preprint arXiv:1306.0256* .
- CAI, T., MA, Z. and WU, Y. (2013b). Optimal estimation and rank detection for sparse spiked covariance matrices .
URL <http://arxiv.org/abs/1305.3235>
- CANDÈS, E. J., LI, X., MA, Y. and WRIGHT, J. (2011). Robust principal component analysis? *J. ACM* **58** 11:1–11:37.
URL <http://doi.acm.org/10.1145/1970392.1970395>
- CHANDRASEKARAN, V., SANGHAVI, S., PARRILO, P. A. and WILLSKY, A. S. (2011). Rank-sparsity incoherence for matrix decomposition. *SIAM Journal on Optimization* **21** 572–596.
- DAVIDSON, K. R. and SZAREK, S. J. (2001). Chapter 8 local operator theory, random matrices and banach spaces. In *Handbook of the Geometry of Banach Spaces* (W. Johnson and J. Lindenstrauss, eds.), vol. 1. Elsevier Science B.V., 317 – 366.
URL <http://www.sciencedirect.com/science/article/pii/S1874584901800103>
- DAVIS, C. and KAHAN, W. M. (1970). The rotation of eigenvectors by a perturbation. iii. *SIAM Journal on Numerical Analysis* **7** 1–46.
- DONOHO, D. L., GAVISH, M. and JOHNSTONE, I. M. (2014). Optimal shrinkage of eigenvalues in the spiked covariance model. *arXiv preprint arXiv:1311.0851* .
- FAN, J., FAN, Y. and LV, J. (2008). High dimensional covariance matrix estimation using a factor model. *Journal of Econometrics* **147** 186–197.
- FAN, J. and HAN, X. (2013). Estimation of false discovery proportion with unknown dependence. *arXiv preprint arXiv:1305.7007* .
- FAN, J., LIAO, Y. and MINCHEVA, M. (2013a). Large covariance estimation by thresholding principal orthogonal complements. *Journal of the Royal Statistical Society: Series B* **75** 1–44.
- FAN, J., LIAO, Y. and SHI, X. (2013b). Risks of large portfolios. *arXiv preprint arXiv:1302.0926* .
- FAN, J., LIAO, Y. and WANG, W. (2014). Projected principal component analysis in factor models. *arXiv preprint arXiv:1406.3836* .
- FAN, J., ZHANG, J. and YU, K. (2012). Vast portfolio selection with gross-exposure constraints. *Journal of the American Statistical Association* **107** 592–606.

- JOHNSTONE, I. M. and LU, A. Y. (2009). On consistency and sparsity for principal components analysis in high dimensions. *Journal of the American Statistical Association* **104** 682–693.
URL <http://amstat.tandfonline.com/doi/abs/10.1198/jasa.2009.0121>
- MARKOWITZ, H. (1952). Portfolio selection*. *The journal of finance* **7** 77–91.
- ONATSKI, A. (2012). Asymptotics of the principal components estimator of large factor models with weakly influential factors. *Journal of Econometrics* **168** 244–258.
- PAUL, D. (2007). Asymptotics of sample eigenstructure for a large dimensional spiked covariance model. *Statistica Sinica* **17** 1617–1642.
- ROTHMAN, A. J., LEVINA, E. and ZHU, J. (2009). Generalized thresholding of large covariance matrices. *Journal of the American Statistical Association* **104** 177–186.
- SHEN, D., SHEN, H., ZHU, H. and MARRON, J. S. (2013). Surprising asymptotic conical structure in critical sample eigen-directions .
URL <http://arxiv.org/abs/1303.6171>
- VERSHYNIN, R. (2010). Introduction to the non-asymptotic analysis of random matrices. *arXiv preprint arXiv:1011.3027* .