

Multi-Agent Inference in Social Networks: A Finite Population Approach*

Jianqing Fan[†]

Xin Tong[‡]

Yao Zeng[§]

Princeton University

USC

Harvard University

Abstract

When people in a society want to make inference about some parameter, each person would potentially want to use data collected by other people. Information (data) exchange in social contexts is usually costly, so to make sound statistical decisions, people need to compromise between benefits and costs of information acquisition. Conflicts of interests and coordination will arise. Classical statistics does not consider people’s interaction in the data collection process. To address this ignorance, this work explores multi-agent Bayesian inference problems with a game theoretic social network model. Bearing our interest in aggregate inference at the societal level, we propose a new concept *finite population learning* to address whether with high probability, a large fraction of people can make “good” inferences. Serving as a foundation, this concept enables us to study the long run trend of aggregate inference quality as population grows.

KEYWORDS: Social networks, multi-agent inference, Bayesian learning, finite population learning, perfect learning, learning rates.

*We thank the Editor, the Associate Editor, anonymous referees, Daron Acemoglu, Sébastien Bubeck, John Campbell, Emmanuel Farhi, Drew Fudenberg, Matthew Jackson, Gareth James, Jingyi Jessica Li, Philip Reny, Philippe Rigollet, Andrei Shleifer, Alp Simsek, and Yiqing Xing for valuable comments.

[†]Department of Operations Research and Financial Engineering, Princeton University. His research is supported by NSF grant DMS-1206464 and the National Institute of General Medical Sciences of NIH through Grant Number R01-GM072611.

[‡]Marshall School of Business, University of Southern California.

[§]Department of Economics, Harvard University.

1 Introduction

Statistical inference framework usually adopts a single-person perspective without considering the potential impact of interaction between people. Take the Bayesian approach for example. Suppose an agent (person) is interested in a parameter θ . The Bayesian approach starts with a prior belief $\pi(\theta)$, a distribution on θ . After she observes data¹ $X_1, \dots, X_n \sim p(x|\theta)$, where $p(\cdot|\theta)$ is a conditional distribution given θ , the belief about θ is updated using the Bayes' Theorem,

$$\pi(\theta|X_1, \dots, X_n) = \frac{p(X_1, \dots, X_n|\theta)\pi(\theta)}{m(X_1, \dots, X_n)},$$

where $m(X_1, \dots, X_n) = \int p(X_1, \dots, X_n|\theta)\pi(\theta)d\theta$. Computation aside, such an update is clear once $\{X_1, \dots, X_n\}$ are given.

However, a more realistic but complicated situation is that the available data $\{X_1, \dots, X_n\}$ are scattered in the society. Then each agent has a choice between making inference based on her own data, and acquiring more data through costly exchange before inference. Given a fixed amount of information endowment to the society, information exchange could lead to more accurate inference results at both individual and societal levels. Boosted by the internet and particularly online social networks, this perspective is especially important for communication and decision making in the modern world. A subtle point is that the amount of information an agent is willing to share depends on other agents' actions, and such interactions call for tools beyond mainstream statistics. Recall that in clinical trials, an agent weighs more accurate statistical inference against higher monetary cost of data acquisition. Similarly, in social networks, agents balance inference accuracy and time cost of data acquisition. However, agents's inferences in a network are dependent in the sense that their decisions influence each other's data collection results. Therefore, aggregate inference at the societal level is not a trivial problem that simply adds up each agent's inference separately.

¹No distinction will be made among “data”, “information” and “signal” in the discussion.

Multi-agent inference with strategic data acquisition procedure introduces two impacts to the statistics field. First, it extends the “pipeline” of classical statistical inference, making agents’ access to data an endogenous outcome of their interactions. From an individual agent’s perspective, her inference results could be fundamentally different from those of classical statistical inference. Second, it provides a foundation to study aggregate inference at the societal level. More concretely, we address an important question with a new *finite population learning* concept: can a large fraction of agents in a given network make “good” inference about θ with high probability? Note that the network structure and agents’ inference are convoluted: social networks can influence agents’ decision on information acquisition; meanwhile, aggregates of individual agents’ inference comprise knowledge of the society.

This paper explores multi-agent Bayesian inference at the societal level using a game theoretic social network model. Our work differs from social network papers in existing statistical literature, in that the larger part of those papers focus on graphical models, which are ideal to model network structural formation, but not individual agents’ decision. We refer interested readers to [Kolaczyk \(2009\)](#) and [Newman \(2010\)](#) as a broad introduction to the existing literature. Our work supplements network structural modeling with an additional human behavior component using game theory. We demonstrate that communication mechanisms, strategic interactions, among other elements that sit outside graphical models are crucial to determine the aggregate inference quality at the societal level. We were inspired by the information exchange game in [Acemoglu et al. \(2012\)](#), which studies asymptotic and strategic learning behaviors when the network size (population in the society) goes to infinity. In contrast, our work focuses on networks of finite population, and predicts aggregate inference quality at the societal level. The finite population approach not only allows us to study explicit interplay of parameters regarding aggregate inference results, but also provides a more solid benchmark to assess inference quality when network size grows.

The rest of the paper is organized as follows. In Section 2, the game theoretic social

network model (Definition 1) and associated equilibria (Definition 2) is introduced and illustrated with examples. In Section 3, we devise the *finite population learning* criterion to capture the aggregate inference quality at the societal level (in an equilibrium), and derive necessary conditions and sufficient conditions for this criterion. These conditions involve only one equilibrium outcome—the number of signals each agent has acquired—in a clean and transparent manner. With the finite population learning criterion as a foundation, we study in Section 4 the long run trend of aggregate inference quality as population in a society grows. The trend will be captured by the *perfect learning* concept and associated *learning rates*. Two conditions for perfect learning are proposed: one involves equilibrium, and the other (stronger condition) bypasses equilibrium, and only relies on information precision, model parameters, and network structures. Learning rates measure the quality of perfect learning, and are demonstrated in typical examples. Section 5 makes a few remarks and suggests future research lines. Technical proofs and discussions on related social learning literature in other disciplines are relegated to the Supplementary Materials.

2 The Model

We develop a game theoretic social network model, and introduce its components below. Examples and illustrations are also provided to facilitate understanding.

Agents and Network Structure. In a directed graph $G_n = (\mathcal{N}^n, \mathcal{E}^n)$, each node $i \in \mathcal{N}^n = \{1, 2, \dots, n\}$ represents an agent, and an ordered pair $(j, i) \in \mathcal{E}^n$ means agent j can send information to agent i directly (i.e., agent j is agent i 's in-degree neighbor, and agent i is agent j 's out-degree neighbor). When both $(j, i) \in \mathcal{E}^n$ and $(i, j) \in \mathcal{E}^n$, these two agents can communicate directly with each other.

Inference Problem and Information. Each agent would like to make inference about a parameter of common interest $\theta \in \mathbb{R}$ (the exact criterion will be introduced later in the section). Agents' common prior knowledge on θ is modeled by a normally distributed prior $\theta \sim N(0, 1/\rho)$. At time $t = 0$, agent i is endowed with her own information, called

as private signal, $s_i = \theta + z_i$. All $z_i \sim N(0, 1/\bar{\rho})$ are independent with each other and also independent of θ .² Both ρ and $\bar{\rho}$ are assumed to be known parameters. The distributions of z_i 's are common knowledge to all agents and so is the network structure. Since the focus of our paper is not on technical issues related to Bayesian updates, we choose Gaussian distributions for simplicity.

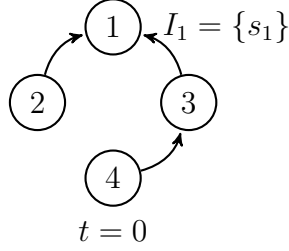
Information Exchange. Agents can make inference based on their own signal, or they might also exchange signals with other agents before making inferences. The information exchange process is described as follows. Suppose agents live in a world with continuous time $t \in [0, \infty)$. Waiting incurs a common exponential discount of the payoff with rate $r > 0$, i.e., $\exp(-rt)$. All agents communicate simultaneously at some time points following a homogeneous Poisson process with rate $\lambda > 0$, independent of θ and z_i . This Poisson process, common knowledge to all agents, defines some discrete communication rounds at which agents send off their private and acquired signals which are tagged with identities. After each communication round, agents update beliefs according to the Bayes' rule. For example, the posterior distribution of θ given k distinct signals is Gaussian with precision $\rho + k\bar{\rho}$. So more acquired signals, i.e., a larger k , will increase the precision and lead to better inference results. Given the above, there is a natural trade-off between acting earlier to reduce the payoff discount and waiting for more communication rounds to acquire more signals. This becomes an optimal stopping problem for each agent i : at any given time t , agent i either makes an estimate x_i of the parameter θ and “exits”, or “waits” for more signals. By “exiting”, we mean that agent i no longer receives new signals, but continues to transmit signal(s) she has so far when new communication rounds take place.³ In the following, we illustrate this information exchange scheme with an example.

An Example of Information Exchange. Suppose there are four agents in a

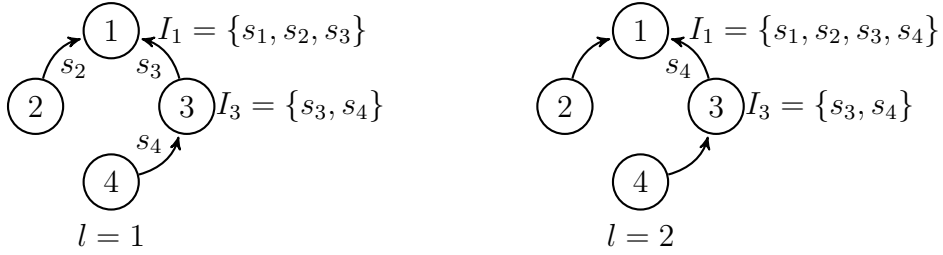
²Our results are not affected if θ and z_i have non-zero means.

³With this assumption, even if an agent decides to “exit” at time $t = 0$, she is obliged to send her private signal to neighbors. In general, when she decides to “exit” at communication round j , she is still obliged to send in future rounds the signals she collected up to round j . This assumption encourages information exchange, and it would affect none of the results but Proposition 3.

social network. At time $t = 0$, each agent i starts with her private signal s_i , and the total information endowment in the network is $\{s_1, s_2, s_3, s_4\}$. We only need to study communication rounds $l = 1, 2$, because the longest path in the network has length 2. We focus on the information set I_1 of agent 1.

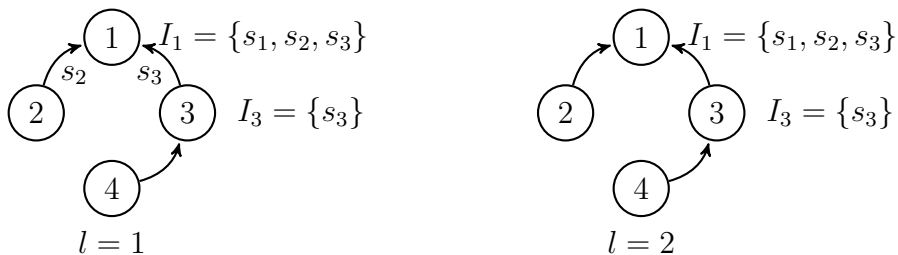


Two scenarios are studied. First, suppose no agent exits after time $t = 0$. The information flow is as follows:



After the first communication round, the information sets of agents 1 and 3 are changed: $I_1 = \{s_1, s_2, s_3\}$, $I_3 = \{s_3, s_4\}$. After $l = 2$, I_1 is updated to $\{s_1, s_2, s_3, s_4\}$, while I_2 , I_3 and I_4 are unchanged.

In the second scenario, suppose agent 3 exits after time $t = 0$. Although she is still obliged to send all her signals (in this case, only her private signal s_3) to out-degree neighbors, she will not collect signals from in-degree neighbors. The information flow is as follows.



As agent 3 does not receive signals from agent 4 at $l = 1$, she does not have new signals to send to agent 1 at $l = 2$. Therefore, $I_1 = \{s_1, s_2, s_3\}$ at $l = 2$.

Comparing the two scenarios, individual agents' decisions can affect others' information acquisition, thus inference. This interdependence differentiates aggregate inference at the societal level from the trivial sum of individual inferences, and motivates a game theoretical framework.

Payoff Function and Inference Decisions. We next specify the payoff function and agents' decision problems. Denote by $I_{i,t}^n$ the information set of agent i at time t . Suppose agent i estimate θ as x_i at time t , her instantaneous payoff is

$$u_i(x_i) = \psi - (x_i - \theta)^2,$$

where ψ is a real-valued constant. Note that the larger the ψ , the less sensitively the payoff depends on the squared error $(x_i - \theta)^2$, where x_i further depends on the agent's final information set. Hence, in what follows we call ψ the information sensitiveness. While ψ plays no role in classical statistical inference, it is important in individual agent's decision making due to the exponential discount of the payoff in time. Agent i 's optimal expected instantaneous payoff with estimate x_i given information set $I_{i,t}^n$ (without considering discounting) is

$$U_{i,t}^n(I_{i,t}^n) = \max_{x_i} \mathbb{E}(u_i(x_i) | I_{i,t}^n).$$

It is easy to see that agent i 's optimal estimate is $x_{i,t}^{n,*} = \mathbb{E}[\theta | I_{i,t}^n]$ if she decides to exit at time t . Thanks to the normality assumption on θ and signals $\{s_i\}_{i=1}^n$, agent i 's optimal expected instantaneous payoff based on k signals can be calculated explicitly:

$$\mathbb{E}[\psi - (x_{i,t}^{n,*} - \theta)^2 | I_{i,t}^n] = \psi - \frac{1}{\rho + \bar{\rho}k}. \quad (2.1)$$

At any time t with information set $I_{i,t}^n$, agent i has to make a decision about whether to

wait or to make an optimal estimate and exit. As a result, due to the common exponential discount in time, each agent should exit at some finite time point,⁴ and more precisely, right after a communication round.⁵ Therefore, to assess agents' inferences, we only need to consider the numbers of communication rounds before exiting. This simplification allows us to study the following game theoretic social network model, formally called a network game.

The Network Game. Denote l_i^n as agent i 's number of communication rounds before exiting, $i = 1, \dots, n$; $l^n = (l_1^n, \dots, l_n^n)$. Also denote by l_{-i}^n the vector l^n without the component l_i^n . Let τ_k be the occurring time of the k^{th} communication round. Agent i 's payoff for choosing l_i^n , i.e., making an optimal estimate and exiting after l_i^n communication rounds, is

$$U_i^n(l_i^n, l_{-i}^n) = \mathbb{E} \left\{ e^{-r\tau_{l_i^n}} \max_{x_i} \mathbb{E}[\psi - (x_i - \theta)^2 | I_i^n(l^n)] \right\},$$

where $I_i^n(l^n)$ is agent i 's information set upon exiting, which not only depends on her own l_i^n , but also depends on other agents' l_{-i}^n , as we have seen in the previous four-agent example. By (2.1) and the exponential waiting time of the Poisson process, it is easy to show that

$$U_i^n(l_i^n, l_{-i}^n) = \bar{r}^{l_i^n} \left(\psi - \frac{1}{\rho + \bar{\rho} k_i^{n, l^n}} \right),$$

where $\bar{r} = \lambda/(\lambda + r)$ and k_i^{n, l^n} is the number of signals agent i has upon exiting if agents in the network adopt l^n . With this reduction, the following game will be considered.

Definition 1 *The network game $\Gamma(G_n)$ is a triple $\{\mathcal{N}^n, \mathcal{L}^n, \mathcal{U}^n\}$, in which*

- (a) \mathcal{N}^n is the set of agents, i.e., $\mathcal{N}^n = \{1, 2, \dots, n\}$;
- (b) $\mathcal{L}^n = (L_1^n, \dots, L_n^n)$ is the collection of agents' strategy spaces. For every agent $i \in \mathcal{N}^n$,

⁴This is because, waiting towards infinity would incur a zero payoff due to discounting.

⁵Suppose an agent exits at a time between two communication rounds. Because she does not get new signals between two adjacent communication rounds, it is always better to exit right after the earlier communication round, due to continuous time discounting.

her strategy space L_i^n is a finite set

$$L_i^n = \{0, 1, 2, \dots, (l_i^n)_{max}\},$$

where $(l_i^n)_{max} = \max_{j \in \mathcal{N}^n \setminus \{i\}} \{\text{length of shortest path from } j \text{ to } i\}$;

(c) $U_i^n \in \mathcal{U}^n$ is the payoff function for agent i :

$$U_i^n(l_i^n, l_{-i}^n) = \bar{r}^{l_i^n} \left(\psi - \frac{1}{\rho + \bar{\rho} k_i^{n, l_i^n}} \right). \quad (2.2)$$

For brevity, we use G_n to refer to both the network and the associated network game $\Gamma(G_n)$ when there is no confusion.

We restrict ourselves to pure-strategy (no randomization) Nash equilibria of this game, which is defined below for the readers' convenience.

Definition 2 In the network game $\Gamma(G_n) = \{\mathcal{N}^n, \mathcal{L}^n, \mathcal{U}^n\}$, a pure-strategy Nash equilibrium⁷ $\sigma^{n,*}$ is a vector $l^{n,\sigma^*} = (l_1^{n,\sigma^*}, \dots, l_n^{n,\sigma^*}) \in \mathcal{L}^n$ such that for every $i \in \mathcal{N}^n$,

$$U_i^n(l_i^{n,\sigma^*}, l_{-i}^{n,\sigma^*}) \geq U_i^n(l_i^n, l_{-i}^{n,\sigma^*}), \text{ for every } l_i^n \in L_i^n.$$

In other words, agents' strategies are a *pure-strategy Nash equilibrium* (or *equilibrium* for brevity) if for every agent, her strategy is optimal given other players' strategies. Our network game is closely related to that in [Acemoglu et al. \(2012\)](#), which used a more complicated model to accommodate possible non-truthful communications and the resulting asymptotic learning behaviors. Their paper did not formally establish the existence or computability of an equilibrium. On the contrary, our goal is to study aggregate inference in a finite population network. We develop a simpler model to capture the in-

⁶If the signal precisions ρ and $\bar{\rho}$ are not known, we cannot simply replace the precisions by their estimates, as the estimates depend not only the number of signals, but also on the signals themselves. Generalization to the unknown precision cases would be interesting for further research.

⁷We refer interested readers to [Fudenberg and Tirole \(1991\)](#) that gives a textbook treatment of the equilibrium and the general proofs for its existence.

terdependence of agents' inference. Our model allows us to establish the existence of an equilibrium, which is also computationally feasible (Mckelvey and McLennan, 1996). The following lemma ensures the existence of equilibria in our network game.

LEMMA 1 *The network game $\Gamma(G_n)$ has at least one pure-strategy Nash equilibrium.*

We offer some intuition of this lemma here (its formal proof in the Supplementary Materials). When some agents stay longer, all other agents have (weakly) larger incentives to stay longer, because the amount of information that comes in the future (weakly) increases. This property is formally called “strategic complementarity” (Fudenberg and Tirole, 1991), and it essentially guarantees the existence of equilibria. This also plays an important role in making the equilibria easily computable ((Mckelvey and McLennan, 1996)).

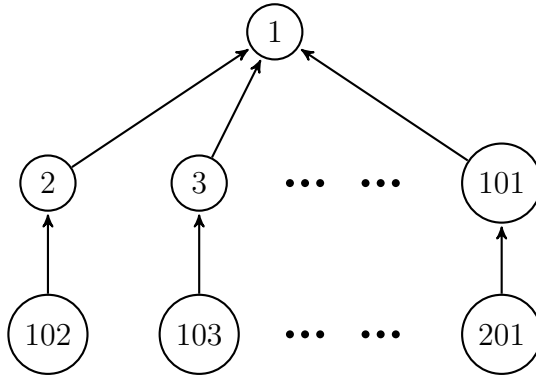
An Example of the Network Game and Its Equilibrium. In the four-agent network displayed previously, fix $\bar{r} = 0.9$, $\psi = 1$ and $\rho = \bar{\rho} = 0.5$. Agents 2 and 4 should exit at $t = 0$, because they will not get any new signals due to the network structure, while incurring discounting penalty should they not act promptly. The payoff matrix for agent 1 (rows) and 3 (columns) is as follows, in which the first and the second value in each cell are respectively the payoffs of agent 1 and agent 3 [see (2.2)].

		Agent 3	
		Round 0	Round 1
Agent 1	Round 0	0, 0	0, .3
	Round 1	.45, 0	.45, .3
	Round 2	.405, 0	.486, .3

There is only one equilibrium of the game. In this equilibrium, agents 2 and 4 exit immediately after they receive their own private signals in Round 0, while agent 3 exits after Round 1, and agent 1 exits after Round 2. Such an outcome is hard to rationalize with any single-agent decision models. No matter what other agents choose, agent 3 always prefers Round 1 to Round 0 ($0.3 > 0$). But without taking into account agent 3's

strategy, there is no clear best choice for agent 1: her best strategy is Round 1 should agent 3 choose Round 0, while her best strategy is Round 2 should agent 3 choose Round 1. Agent 1 is willing to wait longer in the equilibrium only because she believes that agent 3 would wait longer, and agent 3 will indeed do so.

Another Example of Strategic Interactions. In the equilibrium of the above example, all agents wait until their maximum rounds $(l_i^n)_{max}$. There would not be much difference were we to assume no exits (strategic interaction) to start with. To highlight the effect of strategic interactions, we provide another example. In the following network, 201 agents are organized in three layers. Agent 1 has 100 in-degrees, and agents 2 to 101 each has one in-degree. Fix $\bar{r} = 0.95$, $\psi = 10$, $\rho = 0.2$ and $\bar{\rho} = 0.003$. From (2.2) one can check that there is only one equilibrium, in which agent 1 exits after Round 1, while all other agents exit immediately at Round 0. In this equilibrium, agent 1 has 101 signals upon existing. However, if agents 2 to 101 were forced not to exit, they would get the signals from their in-degree neighbors at Round 1, and this encourages agent 1 to wait until Round 2 and has 201 signals upon exiting. The latter hypothetical scenario would not happen in an equilibrium since agents 2 to 101 want to exit earlier.



3 Finite Population Learning

In this section, we introduce *finite population learning* to assess aggregate inference quality at the societal level. This criterion answers whether with high probability, a large fraction of agents in the network can make “good” estimates in equilibrium. We derive necessary

conditions and sufficient conditions for this criterion, and discuss their implications. The finite population learning concept and its determination conditions echo the spirit of finite sample approach in the statistical learning literature, and it helps reveal the explicit interplay among the population size, parameters of the network game and the learning tolerances. This methodological point distinguishes our paper from previous works in the social networks literature that focus on the asymptotic effects of information aggregation (Acemoglu et al., 2011, 2012).

Definition 3 *Given a social network G_n with an equilibrium $\sigma^{n,*}$, we say G_n achieves $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning under $\sigma^{n,*}$ if*

$$\mathbb{P}_{\sigma^{n,*}} \left(\frac{1}{n} \sum_{i=1}^n (1 - M_i^{n,\varepsilon}) \geq \bar{\varepsilon} \right) \leq \delta,$$

in which $M_i^{n,\varepsilon} = \mathbf{1}(|x_i^* - \theta| \leq \varepsilon)$ where x_i^* is agent i 's optimal estimate upon exiting, and $\mathbb{P}_{\sigma^{n,*}}$ denotes the conditional probability given $\sigma^{n,*}$.

In this definition, the parameter ε defines a “good” estimate for individual agents, $1 - \bar{\varepsilon}$ represents the fraction of agents who make such good estimates, and $1 - \delta$ represents the probability at which such a high fraction of agents make such good estimates. $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning reflects a certain high quality of aggregate inference at the societal level, and these tolerance parameters can be tuned to different applications. In verbal discussions and when there is no confusion, we call $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning *finite population learning*.

A natural question to ask is whether such finite population learning occurs in a given network. If so, under what conditions? The following proposition provides a necessary condition and a sufficient condition. Denote by $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ the error function of the standard normal distribution.

Proposition 1 *For a given social network G_n under an equilibrium $\sigma^*(= \sigma^{n,*})$,*

(a) $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning does not occur if

$$\frac{1}{n} \sum_{i=1}^n \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n, \sigma^*}}{2}} \right) < (1 - \bar{\varepsilon})(1 - \delta). \quad (3.1)$$

(b) $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning occurs if

$$\frac{1}{n} \sum_{i=1}^n \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n, \sigma^*}}{2}} \right) \geq 1 - \bar{\varepsilon} \delta. \quad (3.2)$$

This proposition provides simple conditions for the occurrence of finite population learning: only one equilibrium outcome k_i^{n, σ^*} is involved. Hence, these conditions are more operative and transparent than their asymptotic counterparts in previous literature.

Conditions (3.1) and (3.2) also allow us to untangle the interplay among parameters. For example, we are able to answer the following question. Given the tolerances ε , $\bar{\varepsilon}$, δ and the signal precisions ρ and $\bar{\rho}$, how does the change in k_i^{n, σ^*} affect the occurrence of finite population learning in a given social network G_n ? When k_i^{n, σ^*} 's are sufficiently small to validate condition (3.1), finite population learning does not occur. Similarly, when some of k_i^{n, σ^*} 's are sufficiently large so that the condition (3.2) is satisfied, finite population learning occurs. Similar interpretations about unilateral changes also apply to parameters ε , $\bar{\varepsilon}$, δ , ρ and $\bar{\rho}$. As interplays among the parameters ε , $\bar{\varepsilon}$, δ , ρ , $\bar{\rho}$ and k_i^{n, σ^*} are clear through (3.1) and (3.2), the two conditions help us better understand the quality of inference in different circumstances. Also, a beauty of symmetry arises in our necessary conditions and sufficient conditions for finite population learning. The parameters $\bar{\varepsilon}$ and δ are completely interchangeable in these conditions, which was not expected as they capture tolerances in different categories. On the other hand, in these two conditions, the parameter ε stands in a position unchangeable with $\bar{\varepsilon}$ or δ , which hints that ε plays a different role in determining the aggregate inference quality.

Conditions (3.1) and (3.2) have powerful implications. For example, the next corollary establishes a necessary condition and a sufficient condition without equilibrium outcomes.

COROLLARY 1 For a given social network G_n ,

(a) $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning does not occur if

$$\operatorname{erf}\left(\varepsilon\sqrt{\frac{\rho + \bar{\rho}n}{2}}\right) < (1 - \bar{\varepsilon})(1 - \delta). \quad (3.3)$$

(b) $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning occurs if

$$\operatorname{erf}\left(\varepsilon\sqrt{\frac{\rho + \bar{\rho}}{2}}\right) \geq 1 - \bar{\varepsilon}\delta. \quad (3.4)$$

Corollary 1 follows from $1 \leq k_i^{n, \sigma^*} \leq n$. It is interesting because under some circumstances, we can determine an aggregate inference status without knowing either the structure of the social network or the equilibrium. Intuitively, if any tolerance parameter, information precision or the population size is too low, such that the condition (3.3) is satisfied, finite population learning does not occur no matter how effective the network is organized. Conversely, if any tolerance or information precision is sufficiently large such that condition (3.4) holds, finite population learning occurs even if all agents are isolated.

Finally, multiple equilibria emerge under some circumstances. In the following, we introduce a generalized (conservative) version of finite population learning to accommodate multiple equilibria without equilibrium selection.

Definition 4 Denote by $\Sigma^{n,*} = \{\sigma^*\}$ the set of equilibria of $\Gamma(G_n)$. $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning occurs if

$$\sup_{\sigma^* \in \Sigma^{n,*}} \mathbb{P}_{\sigma^*} \left(\frac{1}{n} \sum_{i=1}^n (1 - M_i^{n, \varepsilon}) \geq \bar{\varepsilon} \right) \leq \delta.$$

This definition offers a conservative standard in the sense that the least favorable equilibrium determines the aggregate inference status. When $\Sigma^{n,*}$ is a singleton, the above definition reduces to Definition 3. The proof of Proposition 1 can be recycled to deliver the next remark.

REMARK 1 For a given social network G_n ,

(a) $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning does not occur if

$$\min_{\sigma^* \in \Sigma^{n,*}} \frac{1}{n} \sum_{i=1}^n \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right) < (1 - \bar{\varepsilon})(1 - \delta).$$

(b) $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning occurs if

$$\min_{\sigma^* \in \Sigma^{n,*}} \frac{1}{n} \sum_{i=1}^n \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right) \geq 1 - \bar{\varepsilon}\delta.$$

4 Perfect Learning and the Rates

The finite population learning concept and Proposition 1 provide a solid foundation to investigate the aggregate inference quality as population in a network grows. This problem is important because the evolution of a society can impact the organization of information, the interdependence of individual inferences, and the aggregate inference at the societal level. A new concept, *perfect learning*, determines whether in a sequence of growing networks $\{G_n\}_{n=1}^\infty$ (population goes to infinity), $(\varepsilon, \bar{\varepsilon}, \delta_n)$ -learning can be achieved in G_n for all $n \in \mathbb{N}$ with $\delta_n \rightarrow 0$. This means with increasing population, a large fraction of the people in a society can make “good” inference almost surely. The sequence $\{\delta_n\}_{n=1}^\infty$ naturally induces a *learning rate*, which measures the speed towards perfect learning. This perfect learning concept and associated learning rates also apply to the tolerance parameters ε and $\bar{\varepsilon}$. Our approach is different from previous social networks literature on learning that follow a direct asymptotic approach (Acemoglu et al., 2011, 2012) or discuss other notions of learning rate under non-Bayesian inference context (Golub and Jackson, 2012a,b,c; Jadbabaie et al., 2013).

4.1 Perfect Learning

Recall that three tolerance parameters ε , $\bar{\varepsilon}$, and δ tune the learning status of a society. We call by *an evolution path* a sequence of growing networks $\{G_n\}_{n=1}^\infty$ (population goes to infinity), where existing links are kept when networks grow. To inquire the limiting behavior under an evolution path, we can focus on one parameter at a time. The following definition introduces δ -perfect learning on a given evolution path $\{G_n\}_{n=1}^\infty$.

Definition 5 *We say δ -perfect learning occurs on an evolution path $\{G_n\}_{n=1}^\infty$ under equilibria $\{\sigma^{n,*}\}_{n=1}^\infty$ ⁸ if there exists a vanishing positive sequence $\{\delta_n\}_{n=1}^\infty$ such that $(\varepsilon, \bar{\varepsilon}, \delta_n)$ -learning occurs in G_n under its associated $\sigma^{n,*}$ for all n .*

This definition conveys the idea that as a society becomes larger, eventually we know for sure that a pre-specified large fraction of the people can make “good” inference. In verbal discussions, we call δ -perfect learning just *perfect learning*. Compared to its counterpart in previous works, our definition of perfect learning is both stronger and more general for the following reasons. First, we require networks on an evolution path to achieve a certain quality of inference not only in the limit but also all along the path towards the limit. Second, by focusing on different parameters ε , $\bar{\varepsilon}$ and δ , we could potentially address three different kinds of limiting behaviors. As discussed in the previous section, these three parameters exhibit different impacts on finite population learning, so they play different roles in perfect learning as well. Third, this definition allows us to investigate learning rates (in the next subsection).

In the following, we derive two sufficient conditions for δ -perfect learning as Proposition 2 and Proposition 3, both taking Proposition 1 as a foundation. The first condition relies on the equilibrium outcome k_i^{n,σ^*} . The second condition relies only on the evolution path. To deliver the first sufficient condition, we define an *equilibrium informed agent*.

⁸Since we assume that the graph is evolved such that nodes/edges can only be added, for any agent i , the number of signals she gets increases in n . Concretely, if in a “smaller” graph i gets k_i signals in some equilibrium $\sigma^{n,*}$, then one can define a corresponding equilibrium $\sigma^{n',*}$ ($n' > n$), in which no agents exit earlier than in $\sigma^{n,*}$ and hence i gets (weakly) more signals. As an alternative, we can restrict the perfect learning definition and associated learning rates on such equilibria sequences.

Definition 6 (Equilibrium Informed Agent) For agent i on a given evolution path $\{G_n\}_{n=1}^\infty$, she is equilibrium informed with respect to $\{G_n\}_{n=1}^\infty$ under equilibria $\{\sigma^{n,*}\}_{n=1}^\infty$ if

$$\lim_{n \rightarrow \infty} k_i^{n,\sigma^*} = \infty.$$

Intuitively, an agent is equilibrium informed means that she enjoys increasing information advantage as population grows. The next proposition offers a sufficient condition for δ -perfect learning. In a similar spirit, a more general sufficient condition is derived as Lemma 3 in the Supplementary Materials. The proof of Proposition 2 is omitted as it is a corollary to Lemma 3.

Proposition 2 δ -perfect learning occurs on an evolution path $\{G_n\}_{n=1}^\infty$ under equilibria $\{\sigma^{n,*}\}_{n=1}^\infty$ if

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\text{EI}^{n,*}| = 1,$$

where $\text{EI}^{n,*}$ is the set of equilibrium informed agents in the network G_n with respect to $\{G_n\}_{n=1}^\infty$ under $\{\sigma^{n,*}\}_{n=1}^\infty$.

Proposition 2 states that perfect learning occurs when almost all agents are equilibrium informed. This reflects that most individuals should collect sufficient information for the society to achieve a high level of aggregate inference. We have such a transparent condition because our perfect learning concept is powered by finite population learning, a sufficient condition of which only involves one set of equilibrium variables: $\{k_i^{n,\sigma^*}\}$.

Next we consider the second sufficient condition that relies only on an evolution path. To streamline the presentation in the main texts, we assume that each agent enjoys a positive payoff even if she exits at $t = 0$. From (2.1), this is equivalent to the following assumption, which will be held for the rest of this section. In the Supplementary Materials, we relax this assumption and show that none of the following results is affected.

Assumption 1 $(\rho + \bar{\rho})\psi > 1$.

Before looking into the next sufficient condition for perfect learning, we point out an important observation. Although the number of signals an agent gets in equilibrium may diverge to infinity on an evolution path, the number of communication rounds she chooses will not increase unboundedly.

LEMMA 2 *Under Assumption 1, given a network G_n under equilibrium σ^* , agent i 's optimal communication round before exiting is bounded from above by a constant independent of n . Mathematically,*

$$l_i^{n,\sigma^*} \leq l_i^n < \ln \left[1 - \frac{1}{(\rho + \bar{\rho})\psi} \right] / \ln \bar{r}, \quad (4.1)$$

in which l_i^n stands for agent i 's optimal communication round given that other agents wait until their maximum rounds $(l_j^n)_{max}$, $j \neq i$.

From condition (4.1), we see that the upper bound is exclusively determined by parameters of the network game. A more general version (Lemma 4) of Lemma 2 is in the Supplementary Materials. A key idea behind this lemma is that an agent's attainable payoff is bounded from above by ψ , while waiting incurs a discounting of the payoff towards zero. Hence, when an agent gets sufficiently large number of signals within some finite communication rounds, even expecting infinite number of signals does not justify the discount of further waiting.

Lemma 2 plays an important role in shaping our next sufficient condition that directly links perfect learning status to formation of an evolution path. Recall that Proposition 2 states almost all agents' $k_i^{n,\sigma^*} \rightarrow \infty$ is sufficient for perfect learning. On the other hand, from Lemma 2 we know that no agent has an optimal unbounded communication round l_i^{n,σ^*} . Combining the two observations, the only possibility to validate the conditions in Proposition 2 is that almost all agents get unbounded number of signals within finite communication rounds. This consideration leads to the following definition of a *socially informed agent*.

Definition 7 (Socially Informed Agent) For each agent i on a given evolution path $\{G_n\}_{n=1}^\infty$, let $L_i = \min\{l_0 \in \mathbb{N} : \lim_{n \rightarrow \infty} |B_{i,l_0}^n| = \infty\}$, where $B_{i,l}^n$ is the set of agents in G_n whose shortest path to i has length at most l . Agent i is socially informed with respect to $\{G_n\}_{n=1}^\infty$ if L_i is finite, and if there exists $N \in \mathbb{N}$ such that for $n \geq N$, we have

$$\psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_i}^n|} > 0, \quad (4.2)$$

and

$$\bar{r}^{L_i} \left(\psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_i}^n|} \right) > \bar{r}^l \left(\psi - \frac{1}{\rho + \bar{\rho}|B_{i,l}^n|} \right) \text{ for all } 0 \leq l < L_i. \quad (4.3)$$

Moreover, we denote by SI^n the set of socially informed agents in the network G_n .

The definition of a socially informed agent does not require knowledge of any specific equilibrium. It only depends on the topological structure of the graph and on the parameters of the network game. In Definition 7, condition (4.2) is automatically satisfied in view of Assumption 1. Intuitively, a socially informed agent can be reached by a large number of neighbors after some finite communication rounds L_i . Furthermore, condition (4.3) ensures that this agent strictly prefers to wait at least until she collects all signals from in-degree neighbors up to distance L_i , given other agents do not exit. With the help of socially informed agents, we bypass equilibrium and state the following sufficient condition for perfect learning.

Proposition 3 δ -perfect learning occurs on an evolution path $\{G_n\}_{n=1}^\infty$ (under any equilibria $\{\sigma^{n,*}\}_{n=1}^\infty$) if

$$\lim_{n \rightarrow \infty} \frac{1}{n} |SI^n| = 1.$$

Proposition 3 is interesting because an agent's socially informed status only depends on network structures and model parameters alone. Since it is difficult to get closed form solutions for equilibria in general scenarios, Proposition 3 is of more value.

One important intuition behind this result is that agent i 's socially informed status guarantees that from a certain point on an evolution path, there is a "hub" within her

finite distance who can collect unbounded number of signals at the first communication round (i.e., in-degree goes to infinity). These hubs can collect their immediate in-degree neighbors' private signals regardless these neighbors' strategies, enabling us to bypass equilibrium. Moreover, conditions (4.2) and (4.3) ensure a chain of agents willing to pass hub's gathered signals to agent i . These hubs exist in important networks studied in literature. For example, a good representation of many real-world scenarios in politics and sociology is the island connection networks (Jackson, 2010; Easley and Kleinberg, 2010), which consist of nearly isolated subgraphs, but each subgraph is a nearly (two-way) complete graph.⁹ This may also represent more general social cliques or homophily as discussed in Golub and Jackson (2012a,b,c). Another typical class of networks with hubs is the (two-way) star networks, in which the star (hub) collects signals from all others and then send all signals together to every agent. In view of Proposition 3, a sequence of such networks with hubs may achieve perfect learning under any equilibria. On the negative side, the preferential attachment graphs, and Cayley trees where all nodes have k degrees, do not have such hubs.

4.2 Learning Rates

In this subsection we define learning rates for δ -perfect learning, and derive them for some typical network classes. Again, finite population learning serves as a foundation.

Definition 8 *If δ -perfect learning occurs on an evolution path $\{G_n\}_{n=1}^\infty$ under equilibria $\{\sigma^{n,*}\}_{n=1}^\infty$, we call the corresponding sequence of tolerances $\{\delta_n\}_{n=1}^\infty$ a learning rate.*

⁹On an evolution path, the populations of all but perhaps a finite number of subgraphs go to infinity. Real-world examples of island connection networks include the US Congress, in which the two major parties are well connected within themselves but there are only a few links between the two, and corporate email communication networks, in which internal email lists construct nearly complete graphs within individual companies, and a few connections are between companies. See Jackson (2010); Easley and Kleinberg (2010) for more examples.

Proposition 1 suggests a conservative approach to construct a learning rate. The sufficient condition part of this proposition prescribes that $(\varepsilon, \bar{\varepsilon}, \delta_n)$ -learning occurs if

$$\frac{1}{n} \sum_{i=1}^n \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n, \sigma^*}}{2}} \right) \geq 1 - \delta_n \bar{\varepsilon}.$$

Then we can solve the above inequalities with respect to δ_n , getting a sequence of lower bounds. This sequence, if converging to zero (δ -perfect learning achieved), can serve as a learning rate. However, without specific knowledge of network structures, it is hard in general to solve for k_i^{n, σ^*} in terms of n and other parameters. In the following, we will derive learning rates on some examples, and discuss general network classes when possible.

Example 1 (Isolated Agents) *When all agents are isolated from each other in a network G_n , we have $k_i^{n, \sigma^*} = 1$ for every agent i .*

In Example 1, the necessary condition (3.1) is reduced to

$$\operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho}}{2}} \right) < (1 - \bar{\varepsilon})(1 - \delta_n).$$

If parameters are such that $\operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho}}{2}} \right) < (1 - \bar{\varepsilon})$, the above inequality holds for large n for any vanishing sequence $\{\delta_n\}_{n=1}^\infty$. This tells us that in fairly general circumstances, an isolated evolution path cannot achieve δ -perfect learning.

Example 2 (Complete Graph) *When the network G_n is a (two-way) complete graph, and the benefit of getting $n - 1$ new signals justifies the discount of one communication round, $k_i^{n, \sigma^*} = n$ for every agent i .*

In Example 2,

$$\operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} n}{2}} \right) \geq 1 - \delta_n \bar{\varepsilon}, \quad \forall n \in \mathbb{N},$$

is a sufficient condition for δ -perfect learning, which translates to

$$\delta_n \geq \frac{1}{\bar{\varepsilon}} \left(1 - \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho}n}{2}} \right) \right). \quad (4.4)$$

The sequence of the right hand sides of inequality (4.4) can serve as a learning rate. We approximate the error function to get a conservative but more transparent rate. Note that the error function erf can be approximated by

$$1 - \operatorname{erf}(x) < \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}.$$

Therefore a sufficient condition for δ -perfect learning is

$$\delta_n \geq \frac{1}{\sqrt{\pi\bar{\varepsilon}} \varepsilon \sqrt{\rho + \bar{\rho}n}} \exp \left(-\frac{\varepsilon^2(\rho + \bar{\rho}n)}{4} \right).$$

Keep other parameters fixed, and focus on the relation between population size n and δ_n . We see that δ_n could decrease in the order of $\exp(-\bar{\rho}\varepsilon^2 n/5)$. This implies that when population grows, the probability that at least $\bar{\varepsilon}$ fraction of people make a “bad” inference decreases very quickly to zero.

Following the idea of error function approximations, we go beyond Example 2 to consider a more general case in which $k_i^{n,\sigma^*} \geq f(n)$ for every agent i where $f(n)$ is a deterministic sequence. A sufficient condition for δ -perfect learning is then

$$\delta_n \geq \frac{1}{\sqrt{\pi\bar{\varepsilon}} \varepsilon \sqrt{\rho + \bar{\rho}f(n)}} \exp \left(-\frac{\varepsilon^2(\rho + \bar{\rho}f(n))}{4} \right). \quad (4.5)$$

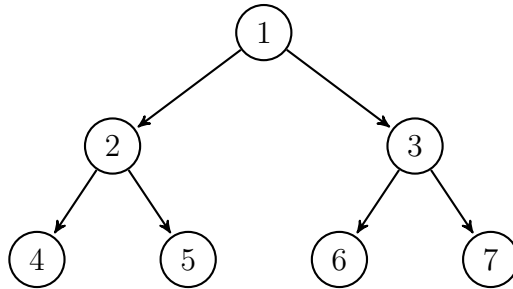
If $f(n)$ diverges to infinity as n goes to infinity, the right hand side of inequality (4.5) converges to 0. Keeping other parameters fixed, this implies δ_n could decrease in the order of $\exp(-\bar{\rho}\varepsilon^2 f(n)/5)$. Formally, we have the next proposition.

Proposition 4 *Suppose there exists a diverging sequence $f(n)$ such that $k_i^{n,\sigma^*} \geq f(n)$ for every agent i in network G_n with associated equilibrium $\sigma^{n,*}$, then δ -perfect learning could occur with learning rate $\{\delta_n\}_{n=1}^\infty$, where each δ_n is in the order of $\exp(-\bar{\rho}\varepsilon^2 f(n)/5)$.*

An interpretation of this proposition is that, even if each agent can only get a small proportion of information scattered in the network, perfect learning can still be reached at a fast rate. This speaks to the commonly observed class of island connection networks discussed above, and is also related to interesting results pertaining to social cliques or homophily as discussed in [Golub and Jackson \(2012a,b,c\)](#). Lemma 3 in the Supplementary Materials further renders Proposition 4 as a special case and provides more general implications on learning rates.

Next, we consider the binomial trees, which are an axiomatic representation of various hierarchical social structures ([Jackson, 2010](#)). In particular, as the information flow within a binomial tree can be either from the root to the leafs or from the leafs to the root, binomial trees can accommodate both the top-down and the bottom-up cases of information exchange in various real-world scenarios. Hence, it is instructive to analyze the binomial tree with a few different settings, where we generalize our network game by allowing the information sensitiveness $\psi = \psi_n$ to vary along an evolution path $\{G_n\}_{n=1}^\infty$.

Example 3 (Binomial Tree: Information Flow from Root to Leafs) *The agents in the communication network G_n form a binomial tree, where information can only flow from root to leafs. For simplicity, consider only the number of agents n such that $n = 1 + 2 + 4 + \dots + 2^{(m_n-1)}$, where m_n is the number of layers in the binomial tree. The following graph illustrates such a binomial tree with three layers.*



We will study two scenarios of this binomial tree, in both of which $\lambda = r$ so that $\bar{r} = 1/2$.

- i) $\psi_n = \rho = \bar{\rho} = 1$. For agent 1 on the top layer, he should exit right after round 0 because he does not have any chance to receive others' private information. For

agent 2 and 3, who are on the second top layer, they decide between round 0 and 1. A simple calculation on their pay off functions reveals that they should exit after round 0. Agents 4, 5, 6, 7 who are on the third layer potentially should decide between 0,1 and 2 rounds. But since agents 2 and 3 cannot not pass through agent 1's info, round 2 is eliminated before any calculation. So agents on the third layer actually faces same decision problems as agents on the second layer. Continue with the same argument till the m_n 'th layer, we learn that everyone in the network exits right after she gets her private signal. Therefore, this scenario is the same as *isolated agents* in terms of information exchange.

In general, as depicted in this subcase i), when the game is less information sensitive (i.e., ψ higher), the precision of the prior ρ is higher, or $\bar{\rho}$ is lower, it is less likely to achieve δ -perfect learning, even if the agents are well connected.

ii) $\psi_n < \frac{2}{\rho+(m_n-1)\bar{\rho}} - \frac{1}{\rho+m_n\bar{\rho}}$ and $\varepsilon^2 < -\frac{4}{\bar{\rho}} \log\left(\frac{1}{2}\sqrt{\frac{\rho+2\bar{\rho}}{\rho+\bar{\rho}}}\right)$. Same as subcase i), agent 1 does not have a choice. For agents on the second layer to choose exit at round 1, we need $\psi_n < \frac{2}{\rho+\bar{\rho}} - \frac{1}{\rho+2\bar{\rho}}$. For agents on the third layer to exit at round 2, we need

$$\psi_n < \min\left\{\frac{2}{\rho+\bar{\rho}} - \frac{1}{\rho+2\bar{\rho}}, \frac{2}{\rho+2\bar{\rho}} - \frac{1}{\rho+3\bar{\rho}}\right\} = \frac{2}{\rho+2\bar{\rho}} - \frac{1}{\rho+3\bar{\rho}}.$$

In general, an agent on layer j prefers to wait till the $j-1$ round if

$$\psi_n < \min\{g(1), \dots, g(j-1)\} = g(j-1).$$

where $g(x) = \frac{2}{\rho+x\bar{\rho}} - \frac{1}{\rho+(x+1)\bar{\rho}}$. The last equality holds because $g(x)$ is a decreasing function, thanks to $g'(x) < 0$. Hence under equilibrium, agents on layer j have j signals. In particular, agents in the last layer each has $m_n = \log_2(n+1)$ signals. Note that there are $\frac{n+1}{2}$ agents in this layer. Using (3.2), a similar derivation to

that in Example 2 leads to that a learning rate $\{\delta_n\}$ can be

$$\delta_n \geq \frac{1}{n\varepsilon\bar{\varepsilon}\sqrt{\pi}} \sum_{i=1}^{\log_2(n+1)} 2^{j-1} \frac{1}{\sqrt{\rho + \bar{\rho}j}} \exp\left(-\frac{\varepsilon^2(\rho + \bar{\rho}j)}{4}\right).$$

To unravel the right hand side of the above inequality, we let

$$h(x) = 2^{x-1} \frac{1}{\sqrt{\rho + \bar{\rho}x}} \exp\left(-\frac{\varepsilon^2(\rho + \bar{\rho}x)}{4}\right),$$

which is monotone increasing, because $h(x+1)/h(x) > 1$ under our condition.

Therefore, it is sufficient to have

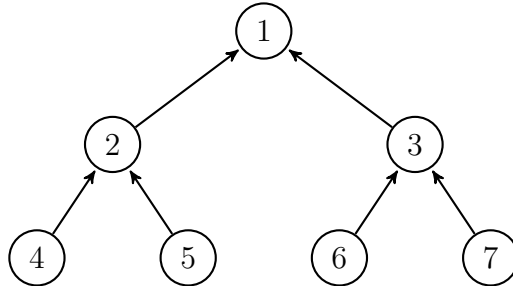
$$\delta_n \geq \frac{1}{n\varepsilon\bar{\varepsilon}\sqrt{\pi}} \log_2(n+1) h(\log_2(n+1)).$$

Therefore δ_n could decay in the order of $\sqrt{\log(n+1)} \cdot (n+1)^{-\varepsilon^2\bar{\rho}/4}$, which is a polynomial rate.

Compared to the complete graph, the binomial tree achieves perfect learning more slowly. The difference in learning rates arises not only from the physical network structures, but also from different strategic interactions among agents in the two environments.

Next, we consider two cases in which information flows in the opposite direction.

Example 4 (Binomial Tree: Information Flow from Leafs to Root) *Now let information flow from leafs to root, i.e., reverse all the directed edges in Example 3. The following graph illustrates such a binomial tree with three layers.*



We give the following results. The detailed analysis is similar to Example 3.

i) $\psi_n = \rho = \bar{\rho} = 1$. All agents exit after time 0.

ii) $\psi_n < \frac{2}{\rho + 2^{(m_n-1)\bar{\rho}}} - \frac{1}{\rho + 2^{m_n\bar{\rho}}}$. All agents get the maximum number of signals that they could possibly get, then δ_n can be such that

$$\delta_n \geq \frac{1}{n\varepsilon\bar{\varepsilon}\sqrt{\pi}} \sum_{j=1}^{\log_2(n+1)} 2^{j-1} \frac{1}{\sqrt{\rho + 2^{(m_n-j+1)\bar{\rho}}}} \exp\left(-\frac{\varepsilon^2(\rho + 2^{(m_n-j+1)\bar{\rho}})}{4}\right).$$

A conservative estimate on the summation on the right hand side would give $\delta_n \sim n^{-3/4}$, a much faster rate than that in Example 3 when $\varepsilon^2\bar{\rho} \ll 3$ (a typical case as we have in mind very small ε).

Note that information flow directions matter for learning rates. When parameters are in a comparable range, the bottom-up case exhibits a higher learning rate than the top-down case. In other words, the bottom-up organization of information flow within a binomial tree is more efficient. This result is consistent with early economics and sociology literature; Hayek (1945) for example, highlight the importance of effectively organizing dispersed information sources.

5 Remarks and Further Research

We have explored multi-agent Bayesian inference at the societal level with a game theoretic social network model. The inference quality is captured by a new concept, finite population learning, which answers the question: whether with high probability, a large fraction of people can make “good” inferences. Echoing the spirit of the finite sample method in statistics, our new concept helps reveal explicit interplays among people’s preferences, network characteristics, and tolerances on inference quality. With finite population learning as a foundation, we also provide conditions to determine the long term trend of aggregate inference quality as population in a society grows. Our work offers the statistics community a tractable framework to assess the inference quality at the societal level, taking strategic interdependence into account.

A further question to ask is: what specific topologies of social networks would improve inference quality at the societal level? To answer this question might open a new direction for statistical research. Our paper makes an initial attempt to inquire the interplay between network topology and aggregate inference, but a generic solution is difficult to reach. The difficulty lies in the effect of game parameters on the inference results; more concretely, slight perturbation of parameter realizations might lead to drastically different inference results in the same social network. The fundamental reason is that our approach calls for complete knowledge on the network structure to determine the quality of aggregate inference. It would be interesting to develop new criteria for aggregate inference on network classes with only some summary statistics regarding topologies. To achieve this, we need to look for novel statistical properties of networks. [Golub and Jackson \(2012a,b,c\)](#) are promising attempts towards this direction, but their results are currently limited to non-Bayesian decision problems.

References

- ACEMOGLU, D., BIMPIKIS, K. and OZDAGLAR, A. (2012). Dynamics of information exchange in endogenous social networks. *Theoretical Economics*.
- ACEMOGLU, D., DAHLEH, M., LOBEL, I. and OZDAGLAR, A. (2011). Bayesian learning in social networks. *Review of Economic Studies*, **78** 1201–1236.
- BANERJEE, A. (1992). A simple model of herd behavior. *Quarterly Journal of Economics*, **107** 797–817.
- BANERJEE, A. and FUDENBERG, D. (2004). Word-of-mouth learning. *Games and Economic Behavior*, **46** 1–2.
- BIKHCHANDANI, S., HIRSHLEIFER, D. and WELCH, I. (1992). A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of Political Economy*, **100** 992–1026.
- CRAWFORD, V. and SOBEL, J. (1982). Strategic information transmission. *Econometrica*, **50** 1431–1451.
- DEGROOT, M. (1974). Reaching a consensus. *Journal of the American Statistical Association*, **69** 118–121.
- DEGROOT, M. (2003). Persuasion bias, social influence, and unidimensional opinions. *Quarterly Journal of Economics*, **118** 909–968.
- DUFFIE, D., MALANUD, S. and MANSO, G. (2009). Information percolation with equilibrium search dynamics. *Econometrica*, **77** 1513–1574.
- EASLEY, D. and KLEINBERG, J. (2010). *Networks, Crowds, and Markets: Reasoning about a Highly Connected World*. Cambridge University Press.
- FUDENBERG, D. and TIROLE, J. (1991). *Game Theory*. MIT Press.

- GALEOTTI, A., GHIGLINO, C. and SQUINTANI, F. (2011). Strategic information transmission in networks. *working paper*.
- GOLUB, B. and JACKSON, M. (2010). Naive learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics*, **2** 112–149.
- GOLUB, B. and JACKSON, M. (2012a). Does homophily predict consensus times? testing a model of network structure via a dynamic process. *Review of Network Economics*.
- GOLUB, B. and JACKSON, M. (2012b). How homophily affects the speed of learning and best-response dynamics. *Quarterly Journal of Economics*.
- GOLUB, B. and JACKSON, M. (2012c). Network structure and the speed of learning: Measuring homophily based on its consequences. *Annals of Economics and Statistics*.
- HAGENBACH, J. and KOESSLER, F. (2010). Strategic communication networks. *Review of Economic Studies*, **77** 1072–1099.
- HAYEK, F. (1945). The use of knowledge in society. *The American Economic Review*, **35** 519–530.
- JACKSON, M. (2010). *An Overview of Social Networks and Economic Applications*. North Holland.
- JADBABAIE, A., MOLAVI, P. and TAHBAZ-SALEHI, A. (2013). Information heterogeneity and the speed of learning in social networks. *working paper*.
- KOLACZYK, E. (2009). *Statistical Analysis of Network Data: Methods and Models*. Springer.
- MCKELVEY, R. and MCLENNAN, A. (1996). *Computation of Equilibria in Finite Games*. Elsevier.
- MOBIUS, M., PHAN, T. and SZEIDL, A. (2010). Treasure hunt. *working paper*.

MULLER-FRANK, M. (2012). A general framework for rational learning in social networks.
Theoretical Economics.

NEWMAN, M. (2010). *Networks: an Introduction*. Oxford University Press.

SMITH, L. and SORENSEN, P. (2000). Pathological outcomes of observational learning.
Econometrica, **68** 371–398.

TOPKIS, D. (1979). Equilibrium points in non-zero sum n-person submodular games.
SIAM J. of Control and Optimization, **17** 773–787.

Supplementary Materials

The Supplementary Materials provide related literature, proofs and generalized results of corresponding parts in the main text.

Relation to the Literature of Learning in Social Networks. Our work lies in the category of Bayesian social learning in social networks, in which decision makers in a social network update their information according to the Bayes' rule. General Bayesian social learning is divided into two sub-categories, namely Bayesian observational learning and Bayesian communication learning. In Bayesian observational learning, agents observe past actions of their neighbors. From these observed actions, agents update their beliefs and make inferences. Herd behavior is a very typical consequence of observational learning. In literature, [Banerjee \(1992\)](#), [Bikhchandani et al. \(1992\)](#) and [Smith and Sorensen \(2000\)](#) are early attempts to model herd effects through Bayesian observational learning. [Banerjee and Fudenberg \(2004\)](#) relaxes the assumption of full observation network topology and study Bayesian observational learning with sampling of past actions. Recently, [Acemoglu et al. \(2011\)](#) and [Muller-Frank \(2012\)](#) investigate how detailed network structures could add new interesting insights.

Our work belongs to Bayesian communication learning, which means that agents cannot directly observe actions of others but can communicate with each other before making a decision. Consequently, agents update their beliefs and make inferences based on the information given by others. New interesting considerations arise in Bayesian communication learning; for example, agents may not want to truthfully reveal their information to others through communication. [Crawford and Sobel \(1982\)](#) pioneers the research in strategic communication, and [Acemoglu et al. \(2012\)](#) is an interesting piece that looks into how communication learning shapes information aggregation in social networks. Other works such as [Galeotti et al. \(2011\)](#) and [Hagenbach and Koessler \(2010\)](#) also study strategic Bayesian communication in social networks, but their focus is not on social learning.

Without investigating networks, [Duffie et al. \(2009\)](#) introduced a similar trade-off between better information and higher search costs in a social learning setting, and the information communication in their setup is isomorphic to our information exchange with identity-tagged signals.

There is a branch of literature that applies various non-Bayesian updating methods to investigate social learning. [DeGroot \(1974\)](#) develops a tractable non-Bayesian learning model which is frequently employed in research of social networks today. Essentially, the DeGroot model is pertaining to observational learning, in which agents make today's decisions by taking the average of neighbors' beliefs revealed in their decisions yesterday. [DeGroot \(2003\)](#) and [Golub and Jackson \(2010, 2012a,b,c\)](#) apply the DeGroot model to financial networks and general social networks, respectively. By a field experiment, [Mobius et al. \(2010\)](#) compares a non-Bayesian model of communication with a model in which agents communicate their signals and update information based on Bayes' rule. Their evidence is generally in favor of the Bayesian communication learning approach.

Proof of Lemma 1. In the network game, it is easy to check that an agent's payoff gain from waiting is weakly larger (i.e., no smaller than) when other agents also wait for more rounds. Formally, for every $i \in \mathcal{N}^n$, we have

$$U_i^n(l_i^n, l_{-i}^n) - U_i^n(l_i'^n, l_{-i}^n) \geq U_i^n(l_i^n, l_{-i}'^n) - U_i^n(l_i'^n, l_{-i}'^n), \quad (5.1)$$

for any $l_{-i}^n, l_{-i}'^n \in \mathcal{L}^n/L_i^n$ such that $l_{-i}^n \geq l_{-i}'^n$. Any complete information static game that satisfies condition (5.1) is a supermodular game. The Topkis Fixed-Point Theorem ([Topkis, 1979](#)) guarantees the existence of a pure-strategy Nash equilibrium in any supermodular game. This concludes the proof. We refer interested readers to [Fudenberg and Tirole \(1991\)](#) for a textbook treatment.

Proof of Proposition 1. To prevent $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning, it is enough to show that a lower bound of $\mathbb{P}_{\sigma^{n,*}} \left(\frac{1}{n} \sum_{i=1}^n (1 - M_i^{n,\varepsilon}) \geq \bar{\varepsilon} \right)$ is greater than δ . It follows from Markov inequality

ity,

$$\begin{aligned} \mathbb{P}_{\sigma^{n,*}} \left(\frac{1}{n} \sum_{i=1}^n M_i^{n,\varepsilon} > 1 - \bar{\varepsilon} \right) &\leq n^{-1}(1 - \bar{\varepsilon})^{-1} \sum_{i=1}^n \mathbb{E}_{\sigma^{n,*}} M_i^{n,\varepsilon} \\ &= n^{-1}(1 - \bar{\varepsilon})^{-1} \sum_{i=1}^n \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right). \end{aligned}$$

This implies that

$$\mathbb{P}_{\sigma^{n,*}} \left(\frac{1}{n} \sum_{i=1}^n (1 - M_i^{n,\varepsilon}) \geq \bar{\varepsilon} \right) > 1 - n^{-1}(1 - \bar{\varepsilon})^{-1} \sum_{i=1}^n \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right).$$

Therefore, it is enough to take

$$1 - n^{-1}(1 - \bar{\varepsilon})^{-1} \sum_{i=1}^n \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right) > \delta,$$

which concludes that condition (3.1) is necessary for $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning.

To ensure $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning, note that

$$\mathbb{P}_{\sigma^{n,*}} \left(\frac{1}{n} \sum_{i=1}^n (1 - M_i^{n,\varepsilon}) \geq \bar{\varepsilon} \right) \leq \frac{\mathbb{E}_{\sigma^{n,*}} (\sum_{i=1}^n (1 - M_i^{n,\varepsilon}))}{n\bar{\varepsilon}} = \frac{n - \sum_{i=1}^n \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right)}{n\bar{\varepsilon}}.$$

Demanding the right hand side of the above inequality no larger than δ is the same as assuming condition (3.2). This completes the proof.

The following provides a more general sufficient condition for δ -perfect learning. Given equilibria $\{\sigma^{n,*}\}_{n=1}^\infty$, let $f_1 \geq f_2 \geq \dots \geq f_J$, where each $f_j(n)$ is a monotone increasing function on n , and let $\{b_n^j, j = 1, \dots, J\}$ be such that

$$\frac{|\{i : k_i^{n,\sigma^*} \geq f_1(n)\}|}{n} \geq b_n^1,$$

$$\frac{|\{i : f_1(n) > k_i^{n,\sigma^*} \geq f_2(n)\}|}{n} \geq b_n^2,$$

and up until

$$\frac{|\{i : f_{J-1}(n) > k_i^{n,\sigma^*} \geq f_J(n)\}|}{n} \geq b_n^J.$$

Clearly, $b_n^1, \dots, b_n^J \in (0, 1)$ and $0 \leq b_n^1 + \dots + b_n^J \leq 1$. The rest agents i 's are such that $f_J(n) > k_i^{n,\sigma^*} \geq 1$. Their fraction is at most $1 - (b_n^1 + \dots + b_n^J)$.

LEMMA 3 *δ -perfect learning occurs if*

$$(a) \lim_{n \rightarrow \infty} \sum_{j=1}^J b_n^j = 1,$$

$$(b) \text{ for each } j \in \{1, \dots, J\}, \lim_{n \rightarrow \infty} b_n^j \left(1 - \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} f_j(n)}{2}} \right) \right) = 0.$$

Proof of Lemma 3. Recall that a sufficient condition for $(\varepsilon, \bar{\varepsilon}, \delta_n)$ -learning is

$$\frac{1}{n} \sum_{i=1}^n \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right) \geq 1 - \delta_n \varepsilon.$$

Then by the definition of b_n^j and f_j , it is enough to have

$$\sum_{j=1}^J \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} f_j(n)}{2}} \right) \cdot b_n^j + \left(1 - \sum_{j=1}^J b_n^j \right) \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho}}{2}} \right) \geq 1 - \delta_n \varepsilon.$$

This translates to

$$\delta_n \geq \frac{1}{\bar{\varepsilon}} \left(\sum_{j=1}^J b_n^j \left(1 - \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} f_j(n)}{2}} \right) \right) + \left(1 - \sum_{j=1}^J b_n^j \right) \left(1 - \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho}}{2}} \right) \right) \right).$$

To ensure the existence of $\{\delta_n\}$ such that $\lim_{n \rightarrow \infty} \delta_n = 0$, it is enough to have $\lim_{n \rightarrow \infty} \sum_{j=1}^J b_n^j = 1$ and

$$\lim_{n \rightarrow \infty} b_n^j \left(1 - \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} f_j(n)}{2}} \right) \right) = 0, \quad \text{for } j \leq J.$$

This completes the proof.

Note that if f_j does not increase strictly for $n \geq N^*$, b_n^j needs to decrease to 0. Also, allowing more than one tolerances among $\varepsilon, \bar{\varepsilon}, \delta$ to vary with population size n leads

to interesting learning results. In particular, from the proof of Lemma 3, a sufficient condition for $(\varepsilon, \bar{\varepsilon}_n, \delta_n)$ - learning is

$$\delta_n \bar{\varepsilon}_n \geq \sum_{j=1}^J b_n^j \left(1 - \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho} f_j(n)}{2}} \right) \right) + \left(1 - \sum_{j=1}^J b_n^j \right) \left(1 - \operatorname{erf} \left(\varepsilon \sqrt{\frac{\rho + \bar{\rho}}{2}} \right) \right).$$

In this condition, the role of δ_n and that of $\bar{\varepsilon}_n$ are completely interchangeable, which implies that we can trade in some probabilistic confidence for some fraction of agents who make wrong decisions.

The following provides a generalized version of Lemma 2 when Assumption 1 is relaxed.

LEMMA 4 (Generalized Lemma 2) *For any agent i , either the communication rounds she optimally experiences before taking an action in any social network G_n along an evolution path $\{G_n\}_{n=1}^\infty$ is bounded from above by a constant independent of n , or she waits until the maximum round. Specifically,*

(a) *If $(\rho + \bar{\rho})\psi > 1$, then for any agent i*

$$l_i^{n, \sigma^*} \leq l_i^n < \min \left\{ (l_i^n)_{max}, \ln \left(1 - \frac{1}{(\rho + \bar{\rho})\psi} \right) / \ln \bar{r} \right\},$$

where l_i^n stands for agent i 's optimal communication rounds given that other agents wait till the maximum round.

(b) *If $(\rho + \bar{\rho})\psi \leq 0$ (equivalently, $\psi \leq 0$), then for any agent i*

$$l_i^{n, \sigma^*} = l_i^n = (l_i^n)_{max}.$$

(c) *If $0 < (\rho + \bar{\rho})\psi \leq 1$, then there are two subcases.*

(c.1) *For agent i with*

$$\lim_{n \rightarrow \infty} |B_i^n| < \frac{1 - \rho\psi}{\bar{\rho}\psi},$$

where B_i^n is the set of agents whose signals agent i can get if no one exits before maximum round, we have

$$l_i^{n,\sigma^*} = l_i^n = (l_i^n)_{max}.$$

(c.2) For agent i with

$$\lim_{n \rightarrow \infty} |B_i^n| \geq \frac{1 - \rho\psi}{\bar{\rho}\psi},$$

we have either

$$l_i^{n,\sigma^*} \leq l_i^n \leq \min\left((l_i^n)_{max}, l_i^{\{G_n\}_{n=1}^\infty}\right),$$

or

$$l_i^{n,\sigma^*} = (l_i^n)_{max},$$

where $l_i^{\{G_n\}_{n=1}^\infty}$ is a constant that depends on the society and agent i 's position in the society, but does not change with n .

Proof of Lemma 4. We proceed case by case.

CASE (a), $(\rho + \bar{\rho})\psi > 1$.

In this case, agent i enjoys a positive payoff $\psi - \frac{1}{\rho + \bar{\rho}}$ if she exists at $t = 0$ and does not communicate with anyone else. Note that her expected payoff by taking l_i^n communication rounds is strictly upper bounded by $\bar{r}^{l_i^n} \psi$. Therefore, it is suboptimal for her to choose a l_i^n such that

$$\bar{r}^{l_i^n} \psi \leq \psi - \frac{1}{\rho + \bar{\rho}},$$

which implies

$$l_i^n < \ln\left(1 - \frac{1}{(\rho + \bar{\rho})\psi}\right) / \ln \bar{r}$$

is necessary for agent i 's optimality. It is obvious that $l_i^{n,\sigma^*} \leq l_i^n$, since other agents do not necessarily wait forever in an equilibrium, so that it may be optimal for agent i to exit earlier too. We get the result by combining these with the upper bound $l_i^n \leq (l_i^n)_{max}$.

CASE (b), $(\rho + \bar{\rho})\psi \leq 0$.

Now agent i always gets a negative payoff whenever she exits. Because waiting discounts the negative payoff, she optimally chooses to wait as long as possible, no matter what other agents do. Therefore, $l_i^{n,\sigma^*} = l_i^n = (l_i^n)_{max}$.

CASE (c.1), $0 < (\rho + \bar{\rho})\psi \leq 1$ and $\lim_{n \rightarrow \infty} |B_i^n| < \frac{1-\rho\psi}{\bar{\rho}\psi}$.

The maximum number of private signals agent i can get is $|B_i^n|$. Again, agent i always gets a negative payoff whenever she exists. Hence, $l_i^{n,\sigma^*} = l_i^n = (l_i^n)_{max}$.

CASE (c.2), $0 < (\rho + \bar{\rho})\psi \leq 1$ and $\lim_{n \rightarrow \infty} |B_i^n| \geq \frac{1-\rho\psi}{\bar{\rho}\psi}$.

For any G_n with $|B_i^n| \geq \frac{1-\rho\psi}{\bar{\rho}\psi}$, we consider the communication round $(l_i^n)_{max}$ when agent i obtains signals from all her sources B_i^n , provided others wait until maximum rounds. Note that $(l_i^n)_{max}$ is non-decreasing in n for any agent i (by the no deleting assumption), and $|B_{i,l}^n|$ is strictly monotone increasing in l when $l \leq (l_i^n)_{max}$.

Also for a given communication network G_n , there exists one communication round $l_i^{n'}$ such that after this round agent i gets positive payoff, given that other agents wait until maximum rounds. Hence, it is suboptimal for her to wait longer than $l_i^{n'}$ if

$$\bar{r}\psi \leq \psi - \frac{1}{\rho + \bar{\rho}|B_{i,l_i^{n'}}^n|},$$

which implies

$$|B_{i,l_i^n}^n| < \frac{\lambda + r - \rho r\psi}{\bar{\rho}r\psi} \quad (5.2)$$

is necessary for agent $l_i^{n'}$'s optimality.

Now we consider two sub-cases. First is when $\lim_{n \rightarrow \infty} |B_i^n| < \infty$. Then we must have $\lim_{n \rightarrow \infty} (l_i^n)_{max} < \infty$, since $(l_i^n)_{max} \leq |B_i^n|$. Hence,

$$l_i^n \leq \lim_{n \rightarrow \infty} (l_i^n)_{max} < \infty,$$

for all G_n satisfying $|B_i^n| \geq \frac{1-\rho\psi}{\bar{\rho}\psi}$ and $\lim_{n \rightarrow \infty} |B_i^n| < \infty$. We denote $\lim_{n \rightarrow \infty} (l_i^n)_{max}$ as $l_{1i}^{\{G_n\}_{n=1}^\infty}$, which is a constant that depends on the society and agent i 's position in the society and does not change with respect to n .

Second, we discuss the case when $\lim_{n \rightarrow \infty} |B_i^n| = \infty$. Now there should be either $\lim_{n \rightarrow \infty} (l_i^n)_{max} < \infty$ or $\lim_{n \rightarrow \infty} (l_i^n)_{max} = \infty$. In the former scenario, we have $l_i^n \leq \lim_{n \rightarrow \infty} (l_i^n)_{max} = l_{1i}^{\{G_n\}_{n=1}^\infty}$ for all G_n . In the latter case, as $(l_i^n)_{max}$ is non-decreasing in n for any given i and $|B_{i,l}^n|$ is strictly monotone increasing in l when $l \leq (l_i^n)_{max}$ for any G_n , there exists a largest G_N with its associated $(L_i^N)_{max}$ that satisfies condition (5.2). Hence, by (5.2) we obtain

$$l_i^n \leq (L_i^N)_{max},$$

for all G_n satisfying $|B_i^n| \geq \frac{1-\rho\psi}{\bar{\rho}\psi}$, $\lim_{n \rightarrow \infty} |B_i^n| = \infty$ and $\lim_{n \rightarrow \infty} (l_i^n)_{max} = \infty$. We denote such $(L_i^N)_{max}$ as $l_{2i}^{\{G_n\}_{n=1}^\infty}$, which is again a constant that depends on the society and agent i 's position in the society and does not change with respect to n . To sum up, we denote by $l_i^{\{G_n\}_{n=1}^\infty}$ either $l_{1i}^{\{G_n\}_{n=1}^\infty}$ or $l_{2i}^{\{G_n\}_{n=1}^\infty}$ in respective cases, and it follows $l_i^n \leq l_i^{\{G_n\}_{n=1}^\infty}$ for agent i in such G_n with $|B_i^n| \geq \frac{1-\rho\psi}{\bar{\rho}\psi}$, where $l_i^{\{G_n\}_{n=1}^\infty}$ is independent of n .

As for l_i^{m,σ^*} , since other agents play equilibrium strategies, agent i gets weakly fewer signals than that she can get when other agents wait until their maximum rounds. There can be two cases, either she gets positive payoff and takes an action weakly earlier, namely, $l_i^{m,\sigma^*} \leq l_i^n$, or she cannot get enough signals to ensure a positive payoff so that she optimally until the maximum round, i.e., $l_i^{m,\sigma^*} = (l_i^n)_{max}$. This concludes the proof.

Proof of Proposition 3. By Lemma 3, it suffices to show that $\lim_{n \rightarrow \infty} k_i^{n,\sigma^*} = \infty$ under any equilibria $\{\sigma^{n,*}\}_{n=1}^\infty$ for any socially informed agent i . In the following, we consider a fixed socially informed agent i . Recall that in Definition 7, L_i is defined as the smallest positive integer such that $\lim_{n \rightarrow \infty} |B_{i,L_i}^n| = \infty$. Denote by $B_{i,l}^{n,\sigma^*}$ the set of agents whose signals can reach i in the first l rounds of communication under equilibrium $\sigma^{n,*}$.

The problem is simple when $L_i = 1$. Clearly, $B_{i,1}^{n,\sigma^*} = B_{i,1}^n$ under any equilibrium $\sigma^{n,*}$. As agent i is socially informed, we have for sufficiently large n

$$\psi - \frac{1}{\rho + \bar{\rho}|B_{i,1}^{n,\sigma^*}|} > 0 \text{ under any } \sigma^{n,*}$$

and

$$\bar{r} \left(\psi - \frac{1}{\rho + \bar{\rho} |B_{i,1}^{n,\sigma^*}|} \right) > \psi - \frac{1}{\rho + \bar{\rho}}$$
 under any $\sigma^{n,*}$.

The above display implies that agent i should at least wait for one communication round. Hence, $k_i^{n,\sigma^*} \geq |B_{i,1}^{n,\sigma^*}|$ under any $\sigma^{n,*}$ for sufficiently large n . As a consequence, $\lim_{n \rightarrow \infty} k_i^{n,\sigma^*} \geq \lim_{n \rightarrow \infty} |B_{i,1}^{n,\sigma^*}| = \lim_{n \rightarrow \infty} |B_{i,1}^n| = \infty$ under any $\{\sigma^{n,*}\}_{n=1}^\infty$.

The following discussion is on the cases when $L_i \geq 2$. We proceed through three steps.

STEP 1. We claim when $L_i \geq 2$, for sufficiently large n , there exists at least one path $\{j_{L_i-1}, j_{L_i-2}, \dots, j_1, i\}$ from j_{L_i-1} to i such that

$$\lim_{n \rightarrow \infty} |B_{j_{L_i-l}, l}^n| = \infty \text{ for all } l \in \{1, \dots, L_i - 1\}. \quad (5.3)$$

Now we construct such a path. Because L_i is the smallest integer j such that $\lim_{n \rightarrow \infty} |B_{i,j}^n| = \infty$, $B_{i,L_i-1}^n \setminus B_{i,L_i-2}^n$, the set of agents that are of distance $L_i - 1$ to i , must be finite in the limit, i.e., $\lim_{n \rightarrow \infty} |B_{i,L_i-1}^n \setminus B_{i,L_i-2}^n| < \infty$. Therefore, there is at least one agent j of distance $L_i - 1$ to i , such that $\lim_{n \rightarrow \infty} |B_{j,1}^n| = \infty$. We denote one of such agents j as j_{L_i-1} . If $L_i = 2$, the desired path has been constructed. When $L_i \geq 3$, choose any path $\{j_{L_i-1}, j_{L_i-2}, \dots, j_1, i\}$ from the chosen j_{L_i-1} to i . Clearly, $j_{L_i-l} \in B_{i,L_i-l}^n$. Moreover, condition (5.3) is satisfied in view of $\lim_{n \rightarrow \infty} |B_{j_{L_i-1}, 1}^n| = \infty$.

STEP 2. We next argue that when $L_i \geq 2$, agent j_{L_i-l} on the path $\{j_{L_i-1}, j_{L_i-2}, \dots, j_1, i\}$ will not exit before she experiences l communication rounds under any equilibrium $\sigma^{n,*}$ provided that n is sufficiently large. It is worth noting that agent j_{L_i-l} does not necessarily get a positive payoff when she experiences l communication rounds in equilibrium.

We will see this by induction from j_{L_i-1} to j_1 sequentially. We first show that agent j_{L_i-1} will not exit before she experiences her first communication round in any equilibrium $\sigma^{n,*}$ provided that n is sufficiently large. It requires that there exists N such that for all

social networks $G_n \in \{G_n\}_{n=1}^\infty$ and its associated equilibrium $\sigma^{n,*}$ with $n \geq N$,

$$\bar{r} \left(\psi - \frac{1}{\rho + \bar{\rho} |B_{j_{L_i-1},1}^{n,\sigma^*}|} \right) > \psi - \frac{1}{\rho + \bar{\rho}}. \quad (5.4)$$

To validate condition (5.4), recall condition (4.3) from Definition 7 for $l = L_i - 1$, which states that there exists N such that for all social networks $G_n \in \{G_n\}_{n=1}^\infty$ with $n \geq N$ it holds

$$\bar{r} \left(\psi - \frac{1}{\rho + \bar{\rho} |B_{i,L_i}^n|} \right) > \psi - \frac{1}{\rho + \bar{\rho} |B_{i,L_i-1}^n|}. \quad (5.5)$$

By the definition of L_i , the construction of j_{L_i-1} and the fact that $B_{j_{L_i-1},1}^{n,\sigma^*} = B_{j_{L_i-1},1}^n$ under any equilibrium $\sigma^{n,*}$ with any n , we know that $\lim_{n \rightarrow \infty} |B_{j_{L_i-1},1}^{n,\sigma^*}| = \lim_{n \rightarrow \infty} |B_{j_{L_i-1},1}^n| = \infty$ under any $\sigma^{n,*}$ and $\lim_{n \rightarrow \infty} |B_{i,L_i-1}^n| < \infty$. Also we have $|B_{i,L_i-1}^n| \geq 1$. Note that the right hand side of condition (5.5) is greater than or equal to the right hand side of condition (5.4), we obtain easily that (5.4) holds for sufficiently large n . Hence we get that agent j_{L_i-1} will not exit before she experiences her first communication round under any $\sigma^{n,*}$ provided that n is sufficiently large.

We then show that agent j_{L_i-2} (for $L_i \geq 3$) will not exit before she experiences her second communication round under any equilibrium for sufficiently large n . It requires that there exists N such that for all social networks $G_n \in \{G_n\}_{n=1}^\infty$ and its associated equilibrium $\sigma^{n,*}$ with $n \geq N$,

$$\bar{r}^2 \left(\psi - \frac{1}{\rho + \bar{\rho} |B_{j_{L_i-2},2}^{n,\sigma^*}|} \right) > \psi - \frac{1}{\rho + \bar{\rho}}, \quad (5.6)$$

and

$$\bar{r}^2 \left(\psi - \frac{1}{\rho + \bar{\rho} |B_{j_{L_i-2},2}^{n,\sigma^*}|} \right) > \bar{r} \left(\psi - \frac{1}{\rho + \bar{\rho} |B_{j_{L_i-2},1}^{n,\sigma^*}|} \right). \quad (5.7)$$

To validate (5.6) and (5.7), we use again the condition (4.3) from Definition 7 for $l = L_i - 2$ and $l = L_i - 1$, which state that there exists N such that for all social networks

$G_n \in \{G_n\}_{n=1}^\infty$ with $n \geq N$ we have

$$\bar{r}^2 \left(\psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_i}^n|} \right) > \psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_{i-2}}^n|}, \quad (5.8)$$

and

$$\bar{r}^2 \left(\psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_i}^n|} \right) > \bar{r} \left(\psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_{i-1}}^n|} \right). \quad (5.9)$$

Similarly, by the definition of L_i and the construction of $j_{L_{i-1}}$ and $j_{L_{i-2}}$, we know that $\lim_{n \rightarrow \infty} |B_{j_{L_{i-2}},2}^n| = \lim_{n \rightarrow \infty} |B_{i,L_i}^n| = \infty$, $\lim_{n \rightarrow \infty} |B_{i,L_{i-1}}^n| < \infty$, $\lim_{n \rightarrow \infty} |B_{i,L_{i-2}}^n| < \infty$, and $\lim_{n \rightarrow \infty} |B_{j_{L_{i-2}},1}^{n,\sigma^*}| \leq \lim_{n \rightarrow \infty} |B_{j_{L_{i-2}},1}^n| < \infty$ under any equilibrium $\sigma^{n,*}$. Also we have $|B_{i,L_{i-2}}^n| \geq 1$ and $B_{j_{L_{i-2}},1}^{n,\sigma^*} \subseteq B_{j_{L_{i-2}},1}^n \subseteq B_{i,L_{i-1}}^n$ (and thus $|B_{i,L_{i-1}}^n| \geq |B_{j_{L_{i-2}},1}^n| \geq |B_{j_{L_{i-2}},1}^{n,\sigma^*}|$) for any n under any equilibrium $\sigma^{n,*}$. Note that the right hand side of condition (5.8) is greater than or equal to the right hand side of condition (5.6), and the right hand side of condition (5.9) is greater than or equal to the right hand side of condition (5.7). Then it can be verified that the next two inequalities hold for sufficiently large n , the right hand sides of which are the same as those in conditions (5.6) and (5.7):

$$\bar{r}^2 \left(\psi - \frac{1}{\rho + \bar{\rho}|B_{j_{L_{i-2}},2}^n|} \right) > \psi - \frac{1}{\rho + \bar{\rho}}, \quad (5.10)$$

and

$$\bar{r}^2 \left(\psi - \frac{1}{\rho + \bar{\rho}|B_{j_{L_{i-2}},2}^n|} \right) > \bar{r} \left(\psi - \frac{1}{\rho + \bar{\rho}|B_{j_{L_{i-2}},1}^{n,\sigma^*}|} \right). \quad (5.11)$$

Furthermore, recall that we have already shown that agent $j_{L_{i-1}}$ will not exit before she experiences her first communication round under any equilibrium $\sigma^{n,*}$ provided that n is sufficiently large, which implies that $B_{j_{L_{i-1}},1}^{n,\sigma^*} \subseteq B_{j_{L_{i-2}},2}^{n,\sigma^*}$ under any $\sigma^{n,*}$ for sufficiently large n , and thus $\lim_{n \rightarrow \infty} |B_{j_{L_{i-2}},2}^{n,\sigma^*}| \geq \lim_{n \rightarrow \infty} |B_{j_{L_{i-1}},1}^{n,\sigma^*}| = \infty$ under any equilibrium $\sigma^{n,*}$. Also we know that $\lim_{n \rightarrow \infty} |B_{j_{L_{i-2}},1}^{n,\sigma^*}| < \infty$. Together with conditions (5.10) and (5.11), these facts validate conditions (5.6) and (5.7). Hence we get that agent $j_{L_{i-2}}$ will not exit before she experiences her second communication round in any $\sigma^{n,*}$ provided that n is

sufficiently large.

The arguments above for j_{L_i-2} can be extended successively to j_1 . Hence, under any equilibrium $\sigma^{n,*}$, no j_{L_i-l} in the established path $\{j_{L_i-1}, j_{L_i-2}, \dots, j_1, i\}$ will exit before she experiences l communication rounds under any equilibrium $\sigma^{n,*}$ provided that n is sufficiently large. A byproduct is that $\lim_{n \rightarrow \infty} |B_{j_{L_i-l}, l}^{n, \sigma^*}| = \infty$ under any $\sigma^{n,*}$, for $l \in \{1, 2, \dots, L_i - 1\}$.

STEP 3. Finally, we argue that the socially informed agent i will not exit before she experiences L_i communication rounds under any equilibrium $\sigma^{n,*}$ when n is sufficiently large. It requires that there exists $N \in \mathbb{N}$ such that for all social networks $G_n \in \{G_n\}_{n=1}^\infty$ with $n \geq N$, we have

$$\psi - \frac{1}{\rho + \bar{\rho}|B_{i, L_i}^{n, \sigma^*}|} > 0, \quad (5.12)$$

and

$$\bar{r}^{L_i} \left(\psi - \frac{1}{\rho + \bar{\rho}|B_{i, L_i}^{n, \sigma^*}|} \right) > \bar{r}^l \left(\psi - \frac{1}{\rho + \bar{\rho}|B_{i, l}^{n, \sigma^*}|} \right), \quad (5.13)$$

for all $l < L_i$.

Recall that we have already shown that agent j_{L_i-l} in the constructed path will not exit before she experiences $L_i - l$ communication rounds for $l \in \{1, 2, \dots, L_i - 1\}$, under any equilibrium $\sigma^{n,*}$ provided that n is sufficiently large, which implies that $B_{j_{L_i-1}, 1}^{n, \sigma^*} \subseteq B_{j_2, L_i-2}^{n, \sigma^*} \subseteq \dots \subseteq B_{j_1, L_i-1}^{n, \sigma^*} \subseteq B_{i, L_i}^{n, \sigma^*}$ under any $\sigma^{n,*}$ for sufficiently large n , and thus $\lim_{n \rightarrow \infty} |B_{i, L_i}^{n, \sigma^*}| \geq \lim_{n \rightarrow \infty} |B_{j_1, L_i-1}^{n, \sigma^*}| \geq \dots \geq \lim_{n \rightarrow \infty} |B_{j_{L_i-2}, 2}^{n, \sigma^*}| \geq \lim_{n \rightarrow \infty} |B_{j_{L_i-1}, 1}^{n, \sigma^*}| = \infty$ under any $\sigma^{n,*}$. Also, we have $B_{i, l}^{n, \sigma^*} \subseteq B_{i, l}^n$ and thus $|B_{i, l}^{n, \sigma^*}| \leq |B_{i, l}^n|$, under any $\sigma^{n,*}$ for $l \in \{1, 2, \dots, L_i - 1\}$, which implies the right hand sides of condition (4.3) are greater than or equal to the right hand sides of condition (5.13), for $l \in \{1, 2, \dots, L_i - 1\}$. Moreover, we know that $\lim_{n \rightarrow \infty} |B_{i, l}^{n, \sigma^*}| \leq \lim_{n \rightarrow \infty} |B_{i, l}^n| < \infty$ for $l \in \{1, 2, \dots, L_i - 1\}$ by the definition of L_i . Together with conditions (4.2) and (4.3) in Definition 7, these facts validate conditions (5.12) and (5.13). Hence the socially informed agent i will not exit before she experiences L_i communication rounds and she can enjoy a positive payoff when she experiences L_i communication rounds, under any $\sigma^{n,*}$ provided that n is sufficiently large. This further

implies $k_i^{n,\sigma^*} \geq |B_{i,L_i}^{n,\sigma^*}|$ under any $\sigma^{n,*}$ with sufficiently large n , which finally leads to $\lim_{n \rightarrow \infty} |k_i^{n,\sigma^*}| \geq \lim_{n \rightarrow \infty} |B_{i,L_i}^n| = \lim_{n \rightarrow \infty} |B_{i,L_i}^n| = \infty$ under any $\sigma^{n,*}$ when $L_i \geq 2$. This concludes the proof.