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DOUBLE RAMIFICATION CYCLES ON THE MODULI SPACES OF ADMISSIBLE COVERS

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Abstract. We derive a formula for the virtual class of the moduli space of rubber maps to $[\mathbb{P}^1/G]$ pushed forward to the moduli space of stable maps to BG. As an application, we show that the Gromov-Witten theory of $[\mathbb{P}^1/G]$ relative to 0 and ∞ are determined by known calculations.

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1. Introduction

1.1. $\mathbb{P}^1$**-stacks.** This paper is motivated by the study of Gromov-Witten theory of $\mathbb{P}^1$-stacks of the following form:

$$[\mathbb{P}^1/G].$$

Here, G is a finite group. The G-action on $\mathbb{P}^1$ is given by the one-dimensional representation $L = \mathbb{C}$,

$$\varphi : G \longrightarrow \mu_a = \operatorname{Im}\varphi \subset \mathbb{C}^* = GL(L),$$

together with the trivial one-dimensional representation $\mathbb{C}$ via

$$\mathbb{P}^1 = \mathbb{P}(L \oplus \mathbb{C}).$$

The $\mathbb{C}^*$-action on $\mathbb{P}^1$ given by

$$\lambda \cdot [z_0, z_1] := [z_0, \lambda z_1], \quad \lambda \in \mathbb{C}^*, [z_0, z_1] \in \mathbb{P}^1$$

commutes with this G-action and induces a $\mathbb{C}^*$-action on $[\mathbb{P}^1/G]$.

1.2. **Stacky rubbers.** The *relative* Gromov-Witten theory of the pairs

$$([\mathbb{P}^1/G], [0/G]), ([\mathbb{P}^1/G], [0/G] \cup [\infty/G]) \tag{1}$$

arise naturally in the pursue of Leray-Hirsch type results in orbifold Gromov-Witten theory, see [11]. Indeed,

$$[\mathbb{P}^1/G] = [\mathbb{P}(L \oplus \mathbb{C})/G] \longrightarrow BG$$

can be viewed as the stacky $\mathbb{P}^1$-bundle associated to the line bundle

$$L \longrightarrow BG.$$

The relative Gromov-Witten theory of the pairs (1) may be studied using virtual localization with respect to the $\mathbb{C}^*$-action on $[\mathbb{P}^1/G]$. Rubber invariants naturally arise in this approach. Let

$$\overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^\sim$$

be the moduli space of rubber maps, see Section 2 for precise definitions. Post-composition with $[\mathbb{P}^1/G] \to BG$ defines a map

$$\epsilon : \overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^\sim \longrightarrow \overline{M}_{g,l(\mu_0)+l(\mu_\infty)+\#I}(BG).$$

The cycle

$$DR_g^G(\mu_0, \mu_\infty, I) := \epsilon_*[\overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^\sim]^{\mathrm{vir}} \in A^g(\overline{M}_{g,l(\mu_0)+l(\mu_\infty)+\#I}(BG))$$

is termed *stacky double-ramification cycle.* The main result of this paper is a formula for $DR_g^G(\mu_0, \mu_\infty, I)$. The formula, which involves complicated notations, is given in Theorem 3.9 below.

When $G = \{1\}$, our formula reduces to Pixton's formula for double ramification cycles, proven in [2]. Our proof, given in the bulk of this paper, closely follows that of [2].

The main application of the formula for $DR_g^G(\mu_0, \mu_\infty, I)$ is the following

Theorem 1.1. *The relative Gromov-Witten theory of*

$$([\mathbb{P}^1/G], [0/G]) \text{ and } ([\mathbb{P}^1/G], [0/G] \cup [\infty/G])$$

are completely determined.

Proof. Since evaluation maps on $\overline{M}_{g,I}([\mathbb{P}^1/G],\mu_0,\mu_\infty)^\sim$ factor through ϵ, rubber invariants[1] are all determined by the formula for $DR_g^G(\mu_0,\mu_\infty,I)$, together with the Gromov-Witten theory of BG solved by [3].

Virtual localization reduces the calculation of relative Gromov-Witten invariants to calculating rubber invariants with target descendants. By rubber calculus in the fiber class case [5], rubber invariants with target descendants are determined by those without target descendants. The proof is complete. □

Theorem 1.1 is an evidence supporting [11, Conjecture 2.2], and we expect that Theorem 1.1 plays an important role in the general case.

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2. Stacky double ramification cycle

Let G be a finite group and $L=\mathbb{C}$ a one dimensional G-representation given by the map

$$G \xrightarrow{\varphi} \mu_a = \operatorname{Im}\varphi \subset GL(L)=\mathbb{C}^*.$$

Let $K := \ker\varphi$, we obtain the exact sequence

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} \mu_a \longrightarrow 1$$

Definition 2.1. For a conjugacy class $\mathfrak{c}\subset G$, define

$$r(\mathfrak{c})\in\mathbb{N}$$

to be the order of any element of $\mathfrak{c}$. Define[2]

$$a_{\mathfrak{c}}(L)\in\{0,\dots,r(\mathfrak{c})-1\}$$

to be the unique integer such that each element of $\mathfrak{c}$ acts on L by multiplication by $\exp\left(\frac{2\pi\sqrt{-1}a_{\mathfrak{c}}(L)}{r(\mathfrak{c})}\right)$. In other words, the representation $\varphi: G\to GL(L)=\mathbb{C}^*$ maps $\mathfrak{c}$ to $\exp\left(\frac{2\pi\sqrt{-1}a_{\mathfrak{c}}(L)}{r(\mathfrak{c})}\right)$.

Consider the quotient stack $[\mathbb{P}^1/G]$, where the G-action on $\mathbb{P}^1$ is given by the 1-dimensional representation φ together with the trivial one-dimensional representation $\mathbb{C}$ via

$$\mathbb{P}^1=\mathbb{P}(L\oplus\mathbb{C}).$$

Definition 2.2. Let A denote the following data:

$$\mu_0=\{(c_{0i},f_{0i},\mathfrak{c}_{0i})\}_i \quad \mu_\infty=\{(c_{\infty i},f_{\infty i},\mathfrak{c}_{\infty i})\}_i,\quad I=\{\mathfrak{c}_1,\dots,\mathfrak{c}_k\},$$

where $c_{0i},c_{\infty i}\in\mathbb{Z}_{\geq 0}$, $f_{0i},f_{\infty i}\in\mathbb{N}$, $\mathfrak{c}_{0i},\mathfrak{c}_{\infty i},\mathfrak{c}_1,\dots,\mathfrak{c}_k\in\operatorname{Conj}(G)$ such that

(i) f_{0i} (resp. $f_{\infty i}$) is the order of any element in $\mathfrak{c}_{0i}$ (resp. $\mathfrak{c}_{\infty i}$).
(ii) $\sum_i \frac{c_{0i}}{f_{0i}}=\sum_j\frac{c_{\infty j}}{f_{\infty j}}$.
(iii) $\operatorname{age}_{\mathfrak{c}_{0i}}(L)=\langle\frac{c_{0i}}{f_{0i}}\rangle$, $\quad\operatorname{age}_{\mathfrak{c}_{\infty j}}(L)=\langle\frac{c_{\infty j}}{f_{\infty j}}\rangle$.

[1]Following [5], we treat disconnected invariants as products of connected ones.
[2]$a_{\mathfrak{c}}(L)$ is well-defined because L is 1-dimensional.

(iv) $\mathrm{age}_{\mathfrak{c}_i}(L) = 0$, $1 \le i \le k$. So $\mathfrak{c}_i \in \mathrm{Conj}(K)$.
(v) Monodromy condition[3] in genus g holds for $\{\mathfrak{c}_{0i}\} \cup \{\mathfrak{c}_{\infty j}^{-1}\} \cup \{\mathfrak{c}_1, \dots, \mathfrak{c}_k\}$.

Here the monodromy condition in genus g means the following.

Definition 2.3 (Monodromy condition)**.** Let H be a finite group. We say that the collection of conjugacy classes $\mathfrak{c}_1, ..., \mathfrak{c}_n$ of H satisfy monodromy condition in genus g if there exist

$$h_i \in \mathfrak{c}_i, 1 \le i \le n, \quad a_j, b_j \in H, 1 \le j \le g,$$

such that

$$\prod_{i=1}^{n} h_i = \prod_{j=1}^{g} [a_j, b_j].$$

Remark 2.4. The data μ_0, μ_∞ are referred to as *stacky partitions.* The length of μ_0, denoted by $l(\mu_0)$, is the number of triples in the partition μ_0.

The moduli space

$$\overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^\sim$$

parametrizes stable relative maps of connected twisted curves of genus g to rubber with ramification profiles μ_0, μ_∞ over $[0/G]$ and $[\infty/G]$ respectively, and additional marked points whose stack structures are described by I. As noted in [11, Appendix A. 2], rubber theory in the stack setting may be defined in the same way as e.g. [5, Section 1.5].

Set $n = l(\mu_0) + l(\mu_\infty) + \#I$. A Riemann-Roch calculation[4] shows that the virtual dimension of $\overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^\sim$ is

$$\mathrm{vdim}\, \overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^\sim = 2g - 3 + n.$$

The moduli space $\overline{M}_{g,n}(BG)$ of n-pointed genus g stable maps to BG is smooth of dimension $3g - 3 + n$. There is a morphism

$$\epsilon : \overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^\sim \longrightarrow \overline{M}_{g,n}(BG)$$

defined by post-composition with $[\mathbb{P}^1/G] \to BG$.

Definition 2.5. The stacky double ramification cycle is defined to be the push-forward

$$DR_g^G(A) = \epsilon_*[\overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^\sim]^{\mathrm{vir}} \in A^g(\overline{M}_{g,n}(BG))$$

Remark 2.6. The cycle $DR_g^G(A)$ is supported on the component of $\overline{M}_{g,n}(BG)$ parametrizing stable maps with orbifold structures at marked points given by $\{\mathfrak{c}_{0i}\} \cup \{\mathfrak{c}_{\infty j}^{-1}\} \cup I$.

[3]For a conjugacy class (g), $(g)^{-1}$ stands for the conjugacy class (g^{-1}).

[4]Note that the relative tangent bundle $T_{[\mathbb{P}^1/G]}(-[0/G] - [\infty/G])$ entering the Riemann-Roch formula is in fact trivial.

3. Total Chern class

3.1. **General case.** Let H be a finite group and $V = \mathbb{C}$ a one-dimensional H-representation. The representation $H \to GL(V) = \mathbb{C}^*$ maps a conjugacy class $\mathfrak{c}$ to $\exp\left(2\pi\sqrt{-1}\frac{a_{\mathfrak{c}}(V)}{r(\mathfrak{c})}\right)$, where $a_{\mathfrak{c}}(V) \in \{0, \dots, r(\mathfrak{c}) - 1\}$.

We write $\mathrm{Conj}(H)$ for the set of conjugacy classes of H. The inertia stack IBH is decomposed as

$$IBH = \coprod_{\mathfrak{c}=(h)\in\mathrm{Conj}(H)} BC_H(h)$$

where $C_H(h) \subseteq H$ is the centralizer of $h \in H$.

We write $\overline{M}_{g,n}(BH)$ for the moduli stack of stable n-pointed genus g maps to BH. For $1 \le i \le n$, there is the i-th evaluation map

$$\mathrm{ev}_i : \overline{M}_{g,n}(BH) \longrightarrow IBH.$$

Pick $\mathfrak{c}_1, \dots, \mathfrak{c}_n \in \mathrm{Conj}(H)$, let

$$\overline{M}_{g,n}(BH; \mathfrak{c}_1, \dots, \mathfrak{c}_n) := \bigcap_{i=1}^{n} \mathrm{ev}_i^{-1}(BC_H(h_i)),$$

where $\mathfrak{c}_i = (h_i)$. Denote the universal family as follows:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ f\ } & BH \\ \big\downarrow{\scriptstyle \pi} & & \\ \overline{M}_{g,n}(BH; \mathfrak{c}_1, \dots, \mathfrak{c}_n) & & \end{array}$$

Consider the virtual bundle

$$V_{g,n} := \mathbf{R}\pi_* f^* V,$$

where V is viewed as a line bundle on BH. The Chern character $ch(V_{g,n})$ was calculated in much greater generality in [10], by using Toën's Grothendieck-Riemann-Roch formula for stacks [9]. Applied to the present situation, we find

$$\begin{aligned} ch(V_{g,n}) = &\pi_*(ch(f^*V)Td^\vee(\bar{L}_{n+1})) \\ &- \sum_{i=1}^{n}\sum_{m\ge1} \frac{\mathrm{ev}_i^* A_m}{m!}\bar{\psi}_i^{m-1} \\ &+ \frac{1}{2}(\pi\circ\iota)_* \sum_{m\ge2}\frac{1}{m!} r_{\mathrm{node}}^2(\mathrm{ev}_{\mathrm{node}}^* A_m)\frac{\bar{\psi}_+^{m-1} + (-1)^m\bar{\psi}_-^{m-1}}{\bar{\psi}_+ + \bar{\psi}_-} \end{aligned} \tag{2}$$

The formula is explained and further processed as follows.

- r_{node} is the order of the orbifold structure at the node.
- $\mathrm{ev}_{\mathrm{node}}$ is the evaluation map at the node defined in [10, Appendix B].
- $\bar{\psi}_+$ and $\bar{\psi}_-$ are the $\bar{\psi}$-classes associated the the branches of the node.
- Since $\dim BH = 0$, we have

$$ch(f^*V) = ch_0(f^*V) = \mathrm{rank}\, V = 1.$$

- By definition, the Todd class is
$$Td^\vee(\bar{L}_{n+1}) = \frac{\bar{\psi}_{n+1}}{e^{\bar{\psi}_{n+1}}-1} = \sum_{r\geq 0}\frac{B_r}{r!}\bar{\psi}_{n+1}^r,$$
where B_r's are the Bernoulli numbers. Therefore,
$$\pi_*(ch(f^*V)Td^\vee(\bar{L}_{n+1})) = \sum_{r\geq 0}\frac{B_r}{r!}\pi_*(\bar{\psi}_{n+1}^r)$$
- A_m is defined in [10, Definition 4.1.2]. We have $A_m \in H^*(IBH)$. For $\mathfrak{c} = (h) \in \mathrm{Conj}(H)$, the component of A_m in $H^0(BC_H(h)) \subset H^*(IBH)$ is $B_m(\frac{a_\mathfrak{c}(V)}{r(\mathfrak{c})})$. Here $B_m(x)$ are Bernoulli polynomials, defined by
$$\frac{te^{tx}}{e^t-1} = \sum_{m\geq 0}\frac{B_m(x)}{m!}t^m.$$
- The map $\iota : \mathcal{Z}_{\text{node}} \hookrightarrow \mathcal{C}$ is the inclusion of the locus of the nodes. The last term of the right hand side of (2) may be rewritten using the map
$$B_{\text{node}} \xrightarrow{\mathfrak{i}} \overline{M}_{g,n}(BH;\mathfrak{c}_1,\ldots,\mathfrak{c}_n),$$
whose image is the locus of nodal curves. The map $\mathfrak{i}$ exhibits B_{node} as the universal gerbe at the node, and hence degree of $\mathfrak{i}$ is $\frac{1}{r_{\text{node}}}$.

Given the above, we can write $ch_m(V_{g,n})$, the degree-$2m$ component of $ch(V_{g,n})$, as
$$\begin{aligned} ch_m(V_{g,n}) =& \frac{B_{m+1}}{(m+1)!}\pi_*(\bar{\psi}_{n+1}^{m+1}) \\ &+ \sum_{i=1}^n \frac{1}{(m+1)!}B_{m+1}\left(\frac{a_{\mathfrak{c}_i}(V)}{r(\mathfrak{c}_i)}\right)\bar{\psi}_i^m \\ &+ \frac{1}{2}\sum_{\mathfrak{c}\in\mathrm{Conj}(H)}\frac{r(\mathfrak{c})}{(m+1)!}B_{m+1}\left(\frac{a_\mathfrak{c}(V)}{r(\mathfrak{c})}\right)\zeta_{\mathfrak{c}*}\left(\frac{\bar{\psi}_+^m-(-\bar{\psi}_-)^m}{\bar{\psi}_++\bar{\psi}_-}\right)\end{aligned}$$
where $\zeta_\mathfrak{c} : B_{\text{node},\mathfrak{c}} \to \overline{M}_{g,n}(BH;\mathfrak{c}_1,\ldots,\mathfrak{c}_n)$ is the universal gerbe at the node whose orbifold structure is given by $\mathfrak{c}$.

Using the formula
$$c(-E^\bullet) = \exp\left(\sum_{m\geq 1}(-1)^m(m-1)!ch_m(E^\bullet)\right), \quad E^\bullet \in D^b,$$
we can derive a formula for $c(-V_{g,n})$. To write this down we need more notations.

As in [2], the strata of $\overline{M}_{g,n}$ are indexed by stable graphs. The strata of $\overline{M}_{g,n}(BH;\mathfrak{c}_1,\ldots,\mathfrak{c}_n)$ are indexed by stable graphs together with choices of conjugacy classes of H describing orbifold structures.

Let $G_{g,n}$ be the set of stable graphs of genus g with n legs. Following [2], a stable graph is denoted by
$$\Gamma = (\mathrm{V},\mathrm{H},\mathrm{L}, g:\mathrm{V}\to\mathbb{Z}_{\geq 0}, v:\mathrm{H}\to V, \iota:\mathrm{H}\to\mathrm{H}) \in G_{g,n}.$$
Properties in [2, Section 0.3.2] are required for Γ.

Remark 3.1. The set of legs $\mathrm{L}(\Gamma)$ corresponds to the set of markings. The set of half edges $\mathrm{H}(\Gamma)$ corresponds to the union of the set of a side of an edge and the set of legs. Each half edge is labelled with a vertex $v \in \mathrm{V}(\Gamma)$. Each vertex $v \in \mathrm{V}(\Gamma)$ is labelled with a nonnegative integer $g(v)$, called the genus.

Definition 3.2. We define $\chi_{\Gamma,H}$ to be the set of maps

$$\chi : \mathrm{H}(\Gamma) \to \mathrm{Conj}(H)$$

such that,

- χ maps the i-th leg h_i to $\mathfrak{c}_i$, $1 \le i \le n$;
- for a vertex $v \in \mathrm{V}(\Gamma)$, there exists $(\alpha_j), (\beta_j) \in \mathrm{Conj}(H)$, for $1 \le j \le g(v)$, and $k_{\mathrm{h}} \in \chi(\mathrm{h})$, for $\mathrm{h} \in v$, such that

$$\prod_{\mathrm{h} \in v} k_{\mathrm{h}} = \prod_{j=1}^{g(v)} [\alpha_j, \beta_j];$$

- for an edge $e = (\mathrm{h}, \mathrm{h}') \in \mathrm{E}(\Gamma)$, there exists $k \in \chi(\mathrm{h})$, $k' \in \chi(\mathrm{h}')$, such that

$$kk' = \mathrm{Id} \in H.$$

For each $\Gamma \in G_{g,n}$ and $\chi \in \chi_{\Gamma,H}$, there is a component $\overline{M}_{\Gamma,\chi} \subset B_{\mathrm{node}}$ parametrizing maps with nodal domains of topological types given by Γ and orbifold structures given by χ. Let

$$\zeta_{\Gamma,\chi} : \overline{M}_{\Gamma,\chi} \longrightarrow \overline{M}_{g,n}(BH; \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$$

be the restriction of $\mathfrak{i}$ to this component. Then $c(-V_{g,n})$ is

(3)

$$\sum_{\Gamma \in G_{g,n}} \sum_{\chi \in \chi_{\Gamma,H}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \zeta_{\Gamma,\chi*} \Bigg[\prod_{v \in V(\Gamma)} \exp\left(-\sum_{m \ge 1} (-1)^{m-1} \frac{B_{m+1}}{m(m+1)} \kappa_m(v) \right) \times$$
$$\times \prod_{i=1}^{n} \exp\left(\sum_{m \ge 1} (-1)^{m-1} \frac{1}{m(m+1)} B_{m+1} \left(\frac{a_{\mathfrak{c}_i}(V)}{r(\mathfrak{c}_i)} \right) \bar{\psi}_{\mathrm{h}_i}^m \right) \times$$
$$\times \prod_{\substack{e \in \mathrm{E}(\Gamma) \\ e = (\mathrm{h}_+, \mathrm{h}_-)}} r(\chi(\mathrm{h}_+)) \frac{1}{\bar{\psi}_{\mathrm{h}_+} + \bar{\psi}_{\mathrm{h}_-}} \left(1 - \exp\left(\sum_{m \ge 1} \frac{(-1)^{m-1}}{m(m+1)} B_{m+1} \left(\frac{a_{\chi(\mathrm{h}_+)}(V)}{r(\chi(\mathrm{h}_+))} \right) \left(\bar{\psi}_{\mathrm{h}_+}^m - (-\bar{\psi}_{\mathrm{h}_-})^m \right) \right) \right) \Bigg].$$

Remark 3.3.

(i) For a half-edge h, $\bar{\psi}_h$ denotes the descendant at the marked point/node corresponding to h.

(ii) For a vertex v, let $\overline{M}_v(BH)$ be the moduli space of stable maps to BH described by v and let $\pi_v : \mathcal{C}_v \to \overline{M}_v(BH)$ be the universal curve. Write $\bar{\psi}_v \in A^1(\mathcal{C}_v)$ for the descendant corresponding to the additional marked point. Then define $\kappa_m(v) := \pi_{v*}(\bar{\psi}_v^{m+1})$.

3.2. **Cyclic extensions.** Let $r \in \mathbb{Z}_{>0}$, the r-th power map

$$\mathbb{C}^* \longrightarrow \mathbb{C}^*, \quad z \longmapsto z^r$$

gives the map

$$\mu_{ar} \longrightarrow \mu_a.$$

The kernel of the map is μ_r. Hence this gives the exact sequence

$$1 \longrightarrow \mu_r \xrightarrow{g} \mu_{ar} \xrightarrow{f} \mu_a \longrightarrow 1,$$

where

$$g\left(\exp\left(\frac{2\pi\sqrt{-1}l}{r}\right)\right) = \exp\left(\frac{2\pi\sqrt{-1}la}{ra}\right), \quad 0 \le l \le r-1,$$

and

$$f\left(\exp\left(\frac{2\pi\sqrt{-1}k}{ar}\right)\right) = \exp\left(\frac{2\pi\sqrt{-1}k}{a}\right), \quad 0 \le k \le r-1.$$

There is a unique finite group $G(r)$ which fits into the following diagram with exact rows and columns:

(4)
$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mu_r & = & \mu_r & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K & \longrightarrow & G(r) & \xrightarrow{\alpha} & \mu_{ar} & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\
1 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\varphi} & \mu_a & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Geometrically, the map $\mu_{ar} \to \mu_a$ gives a μ_r-gerbe over $B\mu_a$,

$$B\mu_{ar} \longrightarrow B\mu_a.$$

The map $\varphi : G \to \mu_a$ gives a map

$$BG \longrightarrow B\mu_a.$$

Pulling back the μ_r-gerbe to BG using this map, we obtain the gerbe

$$BG(r) \longrightarrow BG. \tag{5}$$

Moreover, when viewing the representation L as a line bundle on BG, $BG(r)$ is the gerbe of r-th roots of $L \to BG$. The homomorphism

$$G(r) \longrightarrow \mu_{ar} \subset \mathbb{C}^*$$

is a one-dimensional representation of $G(r)$ which corresponds to the universal r-th root of L on $BG(r)$. We denote this r-th root by

$$L^{1/r} \longrightarrow BG(r).$$

Let $\mathfrak{c} \in \mathrm{Conj}(G)$. Then $\varphi(\mathfrak{c}) \in \mu_a$ is a single number. The inverse image of $\varphi(\mathfrak{c})$ under the r-th power map $\mu_{ar} \to \mu_a$ has size r. The inverse image $\beta^{-1}(\mathfrak{c}) \subset G(r)$ can be partitioned into conjugacy classes of $G(r)$. Moreover, α maps these conjugacy classes to the set of inverse images of $\varphi(\mathfrak{c})$, which has size r. So there are at least r conjugacy classes in $\beta^{-1}(\mathfrak{c})$. By the counting result [7, Example 3.4], there are at most r conjugacy classes. Therefore, there are exactly r conjugacy classes of $G(r)$ that map to $\mathfrak{c}$ and they are determined by their images under $G(r) \xrightarrow{\alpha} \mu_{ar}$.

A canonical splitting of

$$1 \longrightarrow \mu_r \longrightarrow \mu_{ar} \longrightarrow \mu_a \longrightarrow 1$$

is given by

$$\mu_a \longrightarrow \mu_{ar}, \quad g \mapsto \exp\left(\frac{2\pi\sqrt{-1}\,\mathrm{age}_g(L)}{r}\right). \tag{6}$$

Using this, for $g \in \mu_a$, we may identify the inverse image of g under $\mu_{ar} \to \mu_a$ as

$$\left\{\exp\left(2\pi\sqrt{-1}\left(\frac{\mathrm{age}_g(L)}{r} + \frac{e}{r}\right)\right)\middle| 0 \le e \le r-1\right\}$$

and hence with

$$\mu_r = \left\{\exp\left(\frac{2\pi\sqrt{-1}e}{r}\right)\middle| 0 \le e \le r-1\right\}.$$

In summary, given $\mathfrak{c} \in \mathrm{Conj}(G)$, to specify the lifting $\tilde{\mathfrak{c}} \in \mathrm{Conj}(G(r))$ such that $\beta(\tilde{\mathfrak{c}}) = \mathfrak{c}$ is equivalent to specifying $e \in \{0, \dots, r-1\}$.

Moreover, given $\mathfrak{c}_1, \dots, \mathfrak{c}_n \in \mathrm{Conj}(G)$ satisfying monodromy condition in genus g, selecting $\tilde{\mathfrak{c}}_1, \dots, \tilde{\mathfrak{c}}_n \in \mathrm{Conj}(G(r))$ with $\beta(\tilde{\mathfrak{c}}_i) = \mathfrak{c}_i$ satisfying monodromy condition in genus g is equivalent to selecting $e_1, \dots, e_n \in \{0, \dots, r-1\}$ such that

$$\sum_{i=1}^n e_i \equiv -\sum_{i=1}^n \mathrm{age}_{\mathfrak{c}_i}(L) \mod r \tag{7}$$

This can be deduced from the lifting analysis in [8, Section 5]. We can also argue more directly as follows. Since $\mathfrak{c}_1, \dots, \mathfrak{c}_n$ satisfy monodromy condition in genus g, there exists a stable map $f : \mathcal{C} \to BG$ with $\mathcal{C}$ smooth of genus g and $\mathcal{C}$ has orbifold points described by $\mathfrak{c}_1, \dots, \mathfrak{c}_n$. Calculating $\chi(\mathcal{C}, f^*L)$ by Riemann-Roch, we see that $\sum_{i=1}^n \mathrm{age}_{\mathfrak{c}_i}(L) \in \mathbb{Z}$. Similarly, having the required $\tilde{\mathfrak{c}}_1, \dots, \tilde{\mathfrak{c}}_n$ implies the existence of a stable map $\tilde{f} : \tilde{\mathcal{C}} \to BG(r)$ with $\tilde{\mathcal{C}}$ smooth of genus g and $\tilde{\mathcal{C}}$ has orbifold points described by $\tilde{\mathfrak{c}}_1, \dots, \tilde{\mathfrak{c}}_n$. Calculating $\chi(\tilde{\mathcal{C}}, \tilde{f}^* L^{1/r})$ by Riemann-Roch, we see that $\sum_{i=1}^n \mathrm{age}_{\tilde{\mathfrak{c}}_i}(L^{1/r}) \in \mathbb{Z}$. Equation (7) follows because by construction $\mathrm{age}_{\tilde{\mathfrak{c}}_i}(L^{1/r}) = (\mathrm{age}_{\mathfrak{c}_i}(L) + e_i)/r$. This shows that equation (7) is necessary. That (7) is also sufficient can be seen by a direct calculation using the description of $G(r)$ as a set $G \times \mu_r$ endowed with the multiplication defined using the splitting (6), as in [6, Section 3]. We omit the details.

The above discussion allows us to split a sum over $\chi_{\Gamma,G(r)}$ as a double sum over $\chi_{\Gamma,G}$ and the set $W_{\Gamma,\chi,r}$ defined as follows.

Definition 3.4. A weighting mod r associated to a stable graph Γ and a map $\chi \in \chi_{\Gamma,G}$ is a function

$$w : \mathrm{H}(\Gamma) \longrightarrow \{0, \dots, r-1\}$$

such that

(i) For legs $\mathrm{h}_1, \ldots, \mathrm{h}_n$, $w(\mathrm{h}_i) \equiv 0 \mod r$.
(ii) For $e = (\mathrm{h}_+, \mathrm{h}_-) \in \mathrm{E}(\Gamma)$, if $\mathrm{age}_{\chi(\mathrm{h}_+)}(L) = 0$, then $w(\mathrm{h}_+) + w(\mathrm{h}_-) \equiv 0 \mod r$. If $\mathrm{age}_{\chi(\mathrm{h}_+)}(L) \neq 0$, then $w(\mathrm{h}_+) + w(\mathrm{h}_-) \equiv -1 \mod r$.
(iii) For $v \in \mathrm{V}(\Gamma)$, $\sum_{\mathrm{h}\in v} w(\mathrm{h}) \equiv A(v,\chi) \mod r$, where $A(v,\chi) := -\sum_{\mathrm{h}\in v} \mathrm{age}_{\chi(\mathrm{h})}(L)$.

We write $W_{\Gamma,\chi,r}$ for the set of weightings mod r associated to Γ and χ.

Remark 3.5.

(i) For $e = (\mathrm{h}_+, \mathrm{h}_-) \in \mathrm{E}(\Gamma)$, the conditions on $w(h_\pm)$ ensure that

$$(\mathrm{age}_{\chi(h_-)}(L) + w(h_-))/r = 1 - (\mathrm{age}_{\chi(h_+)}(L) + w(h_+))/r.$$

(ii) For $v \in \mathrm{V}(\Gamma)$, We have $A(v,\chi) \in \mathbb{Z}$ by applying Riemann-Roch to $\chi(f^*L)$, where $f : C \to BG$ is a stable map with C smooth of genus $g(v)$ and orbifold marked points described by $\{\chi(\mathrm{h}) | \mathrm{h} \in v\}$.

3.3. **Total Chern class on moduli spaces of stable maps to $BG(r)$.** We begin with the following notations.

Definition 3.6 (Liftings)**.** For $\{\mathfrak{c}_{0i}\}_i, \{\mathfrak{c}_{\infty j}\}_j \subset \mathrm{Conj}(G)$, we select liftings

$$\{\tilde{\mathfrak{c}}_{0i}\}_i, \{\tilde{\mathfrak{c}}_{\infty j}\}_j \subset \mathrm{Conj}(G(r))$$

by

$$\alpha(\tilde{\mathfrak{c}}_{0i}) = \exp\left(\frac{2\pi\sqrt{-1}\,\mathrm{age}_{\mathfrak{c}_{0i}}(L)}{r}\right) \in \mu_{ar},$$

$$\alpha(\tilde{\mathfrak{c}}_{\infty j}) = \exp\left(\frac{2\pi\sqrt{-1}\,\mathrm{age}_{\mathfrak{c}_{\infty j}}(L)}{r}\right) \in \mu_{ar},$$

The lifts of $\mathfrak{c}_1, \ldots, \mathfrak{c}_k \in \mathrm{Conj}(K) \subset \mathrm{Conj}(G)$ are chosen to be themselves, viewed via $\mathrm{Conj}(K) \subset \mathrm{Conj}(G(r))$.

Let $\overline{M}_{g,\tilde{\mu}_0+\tilde{\mu}_\infty+I}(BG(r))$ be the moduli space of stable maps to $BG(r)$ of genus g whose marked points have orbifold structures given by

$$\{\tilde{\mathfrak{c}}_{0i}\} \cup \{\tilde{\mathfrak{c}}_{\infty j}^{-1}\} \cup \{\mathfrak{c}_1, \ldots, \mathfrak{c}_k\}.$$

Let $\overline{M}_{g,\mu+\mu_\infty+I}(BG)$ be similarly defined. There is a natural map

$$\epsilon : \overline{M}_{g,\tilde{\mu}_0+\tilde{\mu}_\infty+I}(BG(r)) \longrightarrow \overline{M}_{g,\mu_0+\mu_\infty+I}(BG).$$

Strata of $\overline{M}_{g,\mu_0+\mu_\infty+I}(BG)$ are indexed by pairs $\Gamma \in G_{g,n}$ and $\chi \in \chi_{\Gamma,G}$. Let $\zeta_{\Gamma,\chi}$ be the map from this stratum to $\overline{M}_{g,\mu_0+\mu_\infty+I}(BG)$. Strata of $\overline{M}_{g,\tilde{\mu}_0+\tilde{\mu}_\infty+I}(BG(r))$ are indexed by Γ, χ, and $w \in W_{\Gamma,\chi,r}$. Let $\zeta_{\Gamma,\chi,w}$ be the natural map from the stratum to $\overline{M}_{g,\tilde{\mu}_0+\tilde{\mu}_\infty+I}(BG(r))$.

Applying the results of Section 3.1, we obtain the following formula for $c(-L_{g,n}^{1/r})$ on $\overline{M}_{g,\tilde{\mu}_0+\tilde{\mu}_\infty+I}(BG(r))$:

$$\sum_{\Gamma\in G_{g,n}}\sum_{\chi\in\chi_{\Gamma,G}}\sum_{w\in W_{\Gamma,\chi,r}}\frac{1}{|\mathrm{Aut}(\Gamma)|}\zeta_{\Gamma,\chi,w*}\Bigg[\prod_{v\in \mathrm{V}(\Gamma)}\exp\left(-\sum_{m\geq 1}(-1)^{m-1}\frac{B_{m+1}}{m(m+1)}\kappa_m(v)\right)\times$$
$$\times\prod_i\exp\left(\sum_{m\geq 1}(-1)^{m-1}\frac{1}{m(m+1)}B_{m+1}\left(\frac{\mathrm{age}_{\mathfrak{c}_{0i}}(L)}{r}\right)\bar{\psi}_i^m\right)\times$$
$$\times\prod_j\exp\left(\sum_{m\geq 1}(-1)^{m-1}\frac{1}{m(m+1)}B_{m+1}\left(1-\frac{\mathrm{age}_{\mathfrak{c}_{\infty j}}(L)}{r}\right)\bar{\psi}_j^m\right)\times$$
$$\times\prod_{l=1}^k\exp\left(\sum_{m\geq 1}(-1)^{m-1}\frac{1}{m(m+1)}B_{m+1}\bar{\psi}_l^m\right)$$
$$\times\prod_{\substack{e\in\mathrm{E}(\Gamma)\\ e=(\mathrm{h}_+,\mathrm{h}_-)}}\frac{r(\chi(\mathrm{h}_+))r}{\bar{\psi}_{\mathrm{h}_+}+\bar{\psi}_{\mathrm{h}_-}}\left(1-\exp\left(\sum_{m\geq 1}\frac{(-1)^{m-1}}{m(m+1)}B_{m+1}\left(\frac{\mathrm{age}_{\chi(\mathrm{h}_+)}(L)}{r}+\frac{w(\mathrm{h}_+)}{r}\right)\left(\bar{\psi}_{\mathrm{h}_+}^m-(-\bar{\psi}_{\mathrm{h}_-})^m\right)\right)\right)\Bigg].$$

By the calculation of [8, Section 5], the degree of ϵ on strata indexed by Γ is $r^{\sum_{v\in V(\Gamma)}(2g(v)-1)}$. This yields the following formula for $\epsilon_*c(-L_{g,n}^{1/r})$:

$$\sum_{\Gamma\in G_{g,n}}\sum_{\chi\in\chi_{\Gamma,G}}\sum_{w\in W_{\Gamma,\chi,r}}\frac{r^{2g-1-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\zeta_{\Gamma,\chi*}\Bigg[\prod_{v\in \mathrm{V}(\Gamma)}\exp\left(-\sum_{m\geq 1}(-1)^{m-1}\frac{B_{m+1}}{m(m+1)}\kappa_m(v)\right)\times$$
$$\times\prod_i\exp\left(\sum_{m\geq 1}(-1)^{m-1}\frac{1}{m(m+1)}B_{m+1}\left(\frac{\mathrm{age}_{\mathfrak{c}_{0i}}(L)}{r}\right)\bar{\psi}_i^m\right)\times$$
$$\times\prod_j\exp\left(\sum_{m\geq 1}(-1)^{m-1}\frac{1}{m(m+1)}B_{m+1}\left(1-\frac{\mathrm{age}_{\mathfrak{c}_{\infty j}}(L)}{r}\right)\bar{\psi}_j^m\right)\times$$
$$\times\prod_{l=1}^k\exp\left(\sum_{m\geq 1}(-1)^{m-1}\frac{1}{m(m+1)}B_{m+1}\bar{\psi}_l^m\right)$$
$$\times\prod_{\substack{e\in\mathrm{E}(\Gamma)\\ e=(\mathrm{h}_+,\mathrm{h}_-)}}\frac{r(\chi(\mathrm{h}_+))}{\bar{\psi}_{\mathrm{h}_+}+\bar{\psi}_{\mathrm{h}_-}}\left(1-\exp\left(\sum_{m\geq 1}\frac{(-1)^{m-1}}{m(m+1)}B_{m+1}\left(\frac{\mathrm{age}_{\chi(\mathrm{h}_+)}(L)}{r}+\frac{w(\mathrm{h}_+)}{r}\right)\left(\bar{\psi}_{\mathrm{h}_+}^m-(-\bar{\psi}_{\mathrm{h}_-})^m\right)\right)\right)\Bigg].$$

Note that we have

- $\frac{\mathrm{age}_{\chi(\mathrm{h}_+)}}{r}+\frac{w(\mathrm{h}_+)}{r}=1-\frac{\mathrm{age}_{\chi(\mathrm{h}_-)}(L)}{r}-\frac{w(\mathrm{h}_-)}{r}$ if $\mathrm{age}_{\chi(\mathrm{h}_\pm)}(L)\neq 0$.
- $\frac{w(\mathrm{h}_+)}{r}=1-\frac{w(\mathrm{h}_-)}{r}$, if $\mathrm{age}_{\chi(\mathrm{h}_\pm)}(L)=0$.

Bernoulli polynomials satisfy the following property

$$B_m(x+y)=\sum_{l=0}^m\binom{m}{k}B_k(x)y^{m-k}.$$

This implies that terms of $\epsilon_*c(-L_{g,n}^{1/r})$ depend polynomially on $\{w(\mathrm{h})|\mathrm{h}\in\mathrm{H}(\Gamma)\}$. The proof of [2, Proposition 3”] may be modified to show that the polynomiality result remains valid for sums over $W_{\Gamma,\chi,r}$. Therefore we may apply the arguments of [2, Proposition 5] to conclude the following.

Proposition 3.7. *There exists a polynomial in r which coincides with the cycle class* $r^{2d-2g+1}\epsilon_* c_d(-L_{g,n}^{1/r})$ *for* $r \gg 1$ *and prime.*

Remark 3.8. The orbifold structure at $\mathrm{h}_\pm$ has order $r(\chi(\mathrm{h}_\pm))r$ when $r \gg 1$ are primes. For our purpose this suffices.

3.4. **A formula for stacky double ramification cycle.**

Theorem 3.9. *Given a finite group G and double ramification data* $A = \{\mu_0, \mu_\infty, I\}$, *the stacky double ramification cycle* $DR_g^G(A)$ *is the constant term in r of the cycle class*

$$a^{1-l(\mu_\infty)} r \cdot \epsilon_* c_g(-L_{g,n}^{1/r}) \in A^g(\overline{M}_{g,n}(BG)),$$

for r sufficiently large. In other words,

$$DR_g^G(A) = a^{1-l(\mu_\infty)} \operatorname{Coeff}_{r^0}[r \cdot \epsilon_* c_g(-L_{g,n}^{1/r})] \in A^g(\overline{M}_{g,n}(BG))$$

We denote by $P_g^{G,d,r}(A) \in A^d(\overline{M}_{g,n}(BG))$ the degree d component of the class

$$\sum_{\Gamma \in G_{g,n}} \sum_{\chi \in \chi_{\Gamma,G}} \sum_{w \in W_{\Gamma,\chi,r}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \zeta_{\Gamma *} \Bigg[\prod_i \exp(\operatorname{age}_{\mathfrak{c}_{0i}}(L)^2 \bar{\psi}_i) \prod_j \exp(\operatorname{age}_{\mathfrak{c}_{\infty j}}(L)^2 \bar{\psi}_j) \cdot$$
$$\prod_{\substack{e \in \mathrm{E}(\Gamma) \\ e = (\mathrm{h}_+, \mathrm{h}_-)}} r(\chi(\mathrm{h}_+)) \frac{1 - \exp((\operatorname{age}_{\chi(\mathrm{h}_+)}(L) + w(\mathrm{h}_+))(\operatorname{age}_{\chi(\mathrm{h}_-)}(L) + w(\mathrm{h}_-))(\bar{\psi}_{\mathrm{h}_+} + \bar{\psi}_{\mathrm{h}_-}))}{\bar{\psi}_{\mathrm{h}_+} + \bar{\psi}_{\mathrm{h}_-}} \Bigg].$$

When the finite group G is trivial, $P_g^{G,d,r}$ reduces to Pixton's polynomial in [2]. Arguing as in the proof of [2, Proposition 5], we see that $r^{2d-2g+1}\epsilon_* c_d(-L_{g,n}^{1/r})$ and $2^{-d} P_g^{G,d,r}(A)$ have the same constant term. Then the following corollary is a result of Theorem 3.9.

Corollary 3.10. *The stacky double ramification cycle* $DR_g^G(A)$ *is the constant term in r of* $a^{1-l(\mu_\infty)} 2^{-g} P_g^{G,g,r}(A) \in A^g(\overline{M}_{g,n}(BG))$, *for r sufficiently large.*

4. Localization analysis

In this section, we give a proof of Theorem 3.9 by virtual localization on the moduli space of stable relative maps to the target obtained from $[\mathbb{P}^1/G]$ by a root construction.

4.1. **Set-up.**

Let $[\mathbb{P}^1/G]_{r,1}$ be the stack of r-th roots of $[\mathbb{P}^1/G]$ along the divisor $[0/G]$. By construction, there is a map $[\mathbb{P}^1/G]_{r,1} \to [\mathbb{P}^1/G]$. Over the divisor $[0/G] \simeq BG$, this map is the μ_r-gerbe $BG(r) \to BG$ studied in Section 3.2.

Let μ_0, μ_∞, I be as in Definition 2.2. Let $\tilde{\mu}_0 = \{(c_{0i}, f_{0i}, \tilde{\mathfrak{c}}_{0i})\}_i$, where $\tilde{\mathfrak{c}}_{0i}$ are given in Definition 3.6. Let

$$\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)$$

be the moduli space of stable relative maps to the pair $([\mathbb{P}^1/G]_{r,1}, [\infty/G])$. The moduli space parametrizes connected, semistable, twisted curves C of genus g with non-relative marked points together with a map

$$f : C \to P$$

where P is an expansion of $[\mathbb{P}^1/G]_{r,1}$ over $[\infty/G]$ such that

(i) orbifold structures at the non-relative marked points are described by $\tilde{\mu}_0$ and I;
(ii) relative conditions over $[\infty/G]$ are described by μ_∞.
(iii) The map f satisfies the ramification matching condition over the internal nodes of the destabilization P.

By [1], $\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)$ has a perfect obstruction theory and its virtual fundamental class has complex dimension

$$\mathrm{vdim}_{\mathbb{C}}[\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)]^{\mathrm{vir}} = 2g - 2 + n + \frac{|\mu_\infty|}{r} - \sum_{i=1}^{l(\mu_0)} \mathrm{age}_{\tilde{\mathfrak{c}}_{0i}}(L^{1/r})$$

where $n = l(\mu_0) + l(\mu_\infty) + \#I$ and $|\mu_\infty| := \sum_j \frac{c_{\infty j}}{f_{\infty j}}$

For $r >> 1$, we have $\mathrm{age}_{\tilde{\mathfrak{c}}_{0i}}(L^{1/r}) = c_{0i}/r f_{0i}$. In this case the virtual dimension is $2g - 2 + n$.

In what follows, we assume that r is large and is a prime number.

4.2. **Fixed loci.** The standard $\mathbb{C}^*$-action on $\mathbb{P}^1$ is given by

$$\xi \cdot [z_0, z_1] := [z_0, \xi z_1], \quad \xi \in \mathbb{C}^*, \ [z_0, z_1] \in \mathbb{P}^1.$$

This induces $\mathbb{C}^*$-actions on $[\mathbb{P}^1/G]$, $[\mathbb{P}^1/G]_{r,1}$, and $\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)$. The $\mathbb{C}^*$-fixed loci of $\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)$ are labeled by decorated graphs Γ.

Notation 4.1. A decorated graph Γ is defined as follows:

(i) (Graph data)
- $V(\Gamma)$: the set of vertices of Γ;
- $E(\Gamma)$: the set of edges of Γ;
- $F(\Gamma)$: the set of flags of Γ, defined to be

$$F(\Gamma) = \{(e, v) \in E(\Gamma) \times V(\Gamma) | v \in e\};$$

- $L(\Gamma)$: the set of legs;

(ii) (Decoration data)
- each vertex $v \in V(\Gamma)$ is assigned a genus $g(v)$, a label of either $[0/G(r)]$ or $[\infty/G]$, and a group

$$G_v := \begin{cases} G(r) & \text{if } v \text{ is over } [0/G(r)], \\ G & \text{if } v \text{ is over } [\infty/G]. \end{cases}$$

- each edge $e \in E(\Gamma)$ is labelled with a conjugacy class $(k_e) \subset G_e := K$ and a positive integer d_e, called the degree;
- each flag (e, v) is labelled with a conjugacy class $(k_{(e,v)}) \subset G_v$;
- a map $s : L(\Gamma) \to V(\Gamma)$ that assigns legs to vertices of Γ;

- legs are labelled with markings in $\mu_0 \cup I \cup \mu_\infty$. Namely $j \in L(\Gamma)$ is labelled with a conjugacy class $(k_j) \subset G_v$ where

$$\begin{cases} (k_j) \in \{\tilde{\mathfrak{c}}_{0i}\}_i \cup I & \text{if } v \text{ is over } [0/G(r)] \\ (k_j) \in \{\mathfrak{c}_{\infty j}\}_j & \text{if } v \text{ is over } [\infty/G]. \end{cases}$$

The data above satisfy certain compatibility conditions. We omit them as they do not enter our analysis.

A vertex $v \in V(\Gamma)$ over $[0/G(r)]$ corresponds to a contracted component mapping to $[0/G(r)]$ given by an element of the moduli space

$$\overline{M}_v := \overline{M}_{g(v),I(v),\mu_0(v)}(BG(r))$$

of genus $g(v)$ stable maps to $BG(r)$ such that orbifold structures at marked points are given by corresponding entries of $\tilde{\mu}_0$ and $(k_{(e,v)})^{-1}$ for flags attached to v. The dimension of $\overline{M}_{g(v),I(v),\mu_0(v)}(BG(r))$ is $3g(v) - 3 + \#I(v) + l(\mu_0(v)) + |E(v)|$.

The discussion on fixed stable maps over $[\infty/G] \in [\mathbb{P}^1/G]_{r,1}$ is similar to that in [2, Section 2.3], we omit the details.

Let $\overline{M}^{\sim}_\infty$ be the moduli space of stable maps to rubber. Its virtual class $[\overline{M}^{\sim}_\infty]^{\text{vir}}$ has complex dimension $2g(\infty) - 3 + n(\infty)$, where $g(\infty)$ is the domain genus and $n(\infty) = \#I(\infty) + l(\mu_\infty) + |E(\Gamma)|$ is the total number of markings and incidence edges.

We write $V_0^S(\Gamma)$ for the set of stable vertices of Γ over $[0/G(r)]$. If the target degenerates, define

$$\overline{M}_\Gamma = \prod_{v \in V_0^S(\Gamma)} \overline{M}_{g(v),I(v),\mu_0(v)}(BG(r)) \times \overline{M}^{\sim}_\infty,$$

If the target does not degenerate, define

$$\overline{M}_\Gamma = \prod_{v \in V_0^S(\Gamma)} \overline{M}_{g(v),I(v),\mu_0(v)}(BG(r)).$$

The fixed locus corresponding to Γ is isomorphic the quotient of $\overline{M}_\Gamma$ quotiented by the automorphism group of Γ and the product of cyclic groups associated to the Galois covers of the edges. There is a natural map $\iota : \overline{M}_\Gamma \to \overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)$.

Assuming $r >> 1$, we may argue as in [2, Lemma 6] to conclude that there are only two types of unstable vertices:

- v is mapped to $[0/G]$, $g(v) = 0$, v carries one marking and one incident edge;
- v is mapped to $[\infty/G]$, $g(v) = 0$, v carries one marking and one incident edge.

4.3. **Contributions to localization formula.** By convention, the $\mathbb{C}^*$-equivariant Chow ring of a point is identified with $\mathbb{Q}[t]$ where t is the first Chern class of the standard representation.

Let $[f : C \to [\mathbb{P}^1/G]_{r,1}] \in \overline{M}_\Gamma$. The $\mathbb{C}^*$-equivariant Euler class of the virtual normal bundle in $\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)$ to the $\mathbb{C}^*$-fixed locus indexed by Γ can be described as

$$e(N^{\text{vir}})^{-1} = \frac{e(H^1(C, f^*T_{[\mathbb{P}^1/G]_{r,1}}(-[\infty/G])))}{e(H^0(C, f^*T_{[\mathbb{P}^1/G]_{r,1}}(-[\infty/G])))} (\prod_i e(N_i))^{-1} e(N_\infty)^{-1}.$$

Let $V^S(\Gamma)$ be the set of stable vertices in $V(\Gamma)$. The set of stable flags is defined to be

$$F^S(\Gamma) = \{(e,v) \in F(\Gamma) | v \in V^S(\Gamma)\}.$$

We have

$$[\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1},\mu_\infty)]^{\text{vir}} = \sum_\Gamma \frac{1}{|Aut(\Gamma)|} \frac{1}{\prod_{e\in E(\Gamma)} d_e |G_e|} \left(\prod_{(e,v)\in F^S(\Gamma)} \frac{|G_v|}{r_{(e,v)}} \right) \iota_* \left(\frac{[\overline{M}_\Gamma]^{\text{vir}}}{e(N^{\text{vir}})} \right) \tag{8}$$

where $r_{(e,v)}$ is the order of $k_{(e,v)} \in G_v$.

The localization contributions are given as follows.

(i) Contributions to

$$\frac{e(H^1(C, f^* T_{[\mathbb{P}^1/G]_{r,1}}(-[\infty/G])))}{e(H^0(C, f^* T_{[\mathbb{P}^1/G]_{r,1}}(-[\infty/G])))}.$$

(a) For each stable vertex $v \in V(\Gamma)$ over 0, the contribution is

$$c_{\text{rk}}^{\mathbb{C}^*}((-L^{1/r}(v))) = \sum_{d\geq 0} c_d(-L_{g,n}^{1/r}) \left(\frac{t}{r(e,v)}\right)^{g(v)-1+|E(v)|-d}.$$

Here $r(e,v) = \frac{|G_v|}{|G_e|} = ar$, and the virtual rank rk is $g(v) - 1 + |E(v)|$. This follows from Riemann-Roch together with the observation that because $r >> 1$, the age terms in Riemann-Roch add up to $|E(v)|$.

(b) The two possible unstable vertices contribute to 1.

(c) The edge contribution is trivial because $r >> 1$.

(d) The contribution of a node N over $[0/G(r)]$ is trivial.

(e) Nodes over $[\infty/G]$ contribute 1.

(ii) Contributions to $\prod_i e(N_i)$.

The product $\prod_i e(N_i)$ is over all nodes over $[0/G(r)]$ formed by edges of Γ attaching to vertices. If N is such a node, then

$$e(N) = \frac{t}{r_{(e,v)} d_e} - \frac{\psi_e}{r_{(e,v)}}.$$

Hence, the contribution of this stable vertex v is:

$$\prod_{e\in E(v)} \frac{1}{\frac{t}{r_{(e,v)}d_e} - \frac{\psi_e}{r_{(e,v)}}} \sum_{d\geq 0} c_d(-L_{g,n}^{1/r}) \left(\frac{t}{r(e,v)}\right)^{g(v)-1+|E(v)|-d}.$$

(iii) Contributions to $e(N_\infty)$.

If the target degenerates, there is an additional factor

$$\frac{1}{e(N_\infty)} = \frac{\prod_{e\in E(\Gamma)} d_e r_{(e,v)}}{t + \psi_\infty}$$

4.4. **Extraction.** The virtual class of the moduli space of rubber maps has non-equivariant limit, and $\mathbb{C}^*$ acts trivially on $\overline{M}_{g,n}(BG)$. Therefore the $\mathbb{C}^*$-equivariant push-forward

$$\epsilon_*([\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1},\mu_\infty)]^{\text{vir}})$$

via the natural map

$$\epsilon : \overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1},\mu_\infty) \to \overline{M}_{g,n}(BG)$$

is a polynomial in t. Hence its coefficient of t^{-1} is equal to 0.

Set $s = tr$, we will extract the coefficient of $s^0 r^0$ in $\epsilon_*(t[\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)]^{\mathrm{vir}})$. We denote the map

$$\epsilon : \overline{M}_{g(v),I(v),\mu(v)}(BG(r)) \to \overline{M}_{g(v),n(v)}(BG)$$

We write

$$\hat{c}_d = r^{2d-2g(v)+1}\epsilon_* c_d(-L_{g,n}^{1/r}) \in A^d(\overline{M}_{g(v),n(v)}(BG)),$$

then by Proposition 3.7, $\hat{c}_d$ is a polynomial in r for r sufficiently large. So the operation of extracting the coefficient of r^0 is valid.

We have

$$\begin{aligned}&\epsilon_*(t[\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)]^{\mathrm{vir}})\\ =&\frac{s}{r}\cdot\sum_\Gamma \frac{1}{|Aut(\Gamma)|}\frac{1}{\prod_{e\in E(\Gamma)} d_e |G_e|}\prod_{(e,v)\in F^S(\Gamma)}\frac{|G_v|}{r_{(e,v)}}\epsilon_*\iota_*\left(\frac{[\overline{M}_\Gamma]^{\mathrm{vir}}}{e(N^{\mathrm{vir}})}\right),\end{aligned}$$

where $\epsilon_*\iota_*\left(\frac{[\overline{M}_\Gamma]^{\mathrm{vir}}}{e(N^{\mathrm{vir}})}\right)$ is the product of the following factors:

(i) For each stable vertex $v \in V(\Gamma)$ over 0, the factor is

$$\frac{r}{s}\prod_{e\in E(v)}\frac{r_{(e,v)}}{r}\frac{d_e}{1-\frac{rd_e}{s}\psi_e}\sum_{d\geq 0}\hat{c}_d s^{g(v)-d}\cdot a^{-g(v)+1-E(v)+d}.$$

Each edge contributes a factor $\frac{r_{(e,v)}}{r}$ which cancels with the factor $\frac{|G_v|}{r_{(e,v)}} = \frac{r|G|}{r_{(e,v)}}$ in equation (8) which comes from the contribution of the automorphism group of the node labelled by $(k_{(e,v)})^{-1}$. Therefore, we have at least one positive power of r for each stable vertex of the graph over 0.

(ii) When the target degenerates, there is a factor

$$-\frac{r}{s}\cdot\frac{\prod_{e\in E(\gamma)} d_e r_{(e,v)}}{1+\frac{r}{s}\psi_\infty}$$

we have at least one positive power of r when the target degenerates.

There are only two graphs which have exactly one r factor in the numerator:

- the graph with a stable vertex of genus g over 0 and $l(\mu_\infty)$ unstable vertices over ∞;
- the graph with a stable vertex of genus g over ∞ and $l(\mu_0)$ unstable vertices over 0.

Therefore, the r^0 coefficient is

$$\begin{aligned}&\mathrm{Coeff}_{r^0}[\epsilon_*(t[\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)]^{\mathrm{vir}})]\\ =&\frac{|G|^{l(\mu_0)}}{|G_e|^{l(\mu_0)}}\cdot\mathrm{Coeff}_{r^0}[\sum_{d\geq 0}\hat{c}_d s^{g-d}\cdot a^{-g+1-l(\mu_0)+d}] - \frac{|G|^{l(\mu_\infty)}}{|G_e|^{l(\mu_\infty)}} DR_g^G(A)\end{aligned}$$

To extract the coefficient of s^0, we take $d = g$,

$$\mathrm{Coeff}_{r^0 s^0}[\epsilon_*(t[\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)]^{\mathrm{vir}})] = \frac{|G|^{l(\mu_0)}}{|G_e|^{l(\mu_0)}}\cdot\mathrm{Coeff}_{r^0}[\hat{c}_g\cdot a^{1-l(\mu_0)}] - \frac{|G|^{l(\mu_\infty)}}{|G_e|^{l(\mu_\infty)}} DR_g^G(A)$$

By the vanishing of $\mathrm{Coeff}_{r^0s^0}[\epsilon_*(t[\overline{M}_{g,I,\mu_0}([\mathbb{P}^1/G]_{r,1},\mu_\infty)]^{\mathrm{vir}})]$, we have

$$DR_g^G(A) = a^{1-l(\mu_\infty)}\,\mathrm{Coeff}_{r^0}[r\cdot\epsilon_* c_g(-L_{g,n}^{1/r})] \in A^g(\overline{M}_{g,n}(BG)).$$

The proof is complete.

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