GROMOV–WITTEN THEORY OF FANO ORBIFOLD CURVES, GAMMA INTEGRAL STRUCTURES AND ADE-TODA HIERARCHIES

TODOR MILANOV, YEFENG SHEN, AND HSIAN-HUA TSENG

Abstract. We construct an integrable hierarchy in the form of Hirota quadratic equations (HQE) that governs the Gromov–Witten (GW) invariants of the Fano orbifold projective curve $\mathbb{P}^1_{a_1,a_2,a_3}$. The vertex operators in our construction are given in terms of the K-theory of $\mathbb{P}^1_{a_1,a_2,a_3}$ via Iritani's Γ-class modification of the Chern character map. We also identify our HQEs with an appropriate Kac–Wakimoto hierarchy of ADE type. In particular, we obtain a generalization of the famous Toda conjecture about the GW invariants of $\mathbb{P}^1$.

Contents

Date: July 30, 2014.

1. Introduction

Witten's conjecture [64], proven by Kontsevich [48] states that the GW theory of $X = \mathrm{pt}$ is governed by the KdV hierarchy. Although Witten was cautious in proposing that there should be an integrable hierarchy for every target X, several groups of physicists and mathematicians, including Witten himself [65], have tried to find a generalization of Witten's conjecture. The next important discovery was the Toda conjecture [20, 29, 66], proven by [29, 19, 51]. It states that the GW theory of $X = \mathbb{P}^1$ is governed by the extended Toda hierarchy (see [11] for the definition in terms of a Lax operator and [29, 66] for the bi-Hamiltonian definition). The Toda conjecture was further generalized by Milanov and Tseng [53] (see also [43, 12]) by allowing the target X to be a projective line with two orbifold points. The corresponding integrable hierarchy is the extended bigraded Toda hierarchy, which was introduced and studied by Carlet [10]. The relationship between topological field theories and integrable hierarchies is studied in other examples, such as [31, 34, 22, 23, 24, 50].

Motivated by GW theory, Dubrovin–Zhang [17] proposed a general construction based on the theory of semi-simple Frobenius manifolds. While their construction produces flows that are rational functions on the jet variables, it was expected that for the important classes of semi-simple Frobenius manifold, such as quantum cohomology, the flows are in fact polynomial and that the hierarchy can be used to compute uniquely the higher genus invariants. The polynomiality of the flows for a semi-simple Frobenius manifold associated with a cohomological field theory (this includes the case of GW theory) was proved recently by Buryak–Posthuma–Shadrin [7, 8] using the higher genus reconstruction of Givental. In particular, Witten's conjecture generalizes for all targets X that have semi-simple quantum cohomology. The discovery of this new class of integrable hierarchies is a major breakthrough in the theory of integrable systems. It is natural to study further their properties and to look for applications to other areas of Mathematics and even beyond.

The higher genus reconstruction of Givental which was mentioned above is one of the major achievements in GW theory. The reconstruction was discovered and proved by Givental in the equivariant settings when X is equipped with a torus action with isolated fixed points [32]. Based on his work [32], Givental conjectured a certain higher genus reconstruction formula for the total ancestor potential of X with semi-simple quantum cohomology. Givental's conjecture was proved in various cases in [33, 42, 38, 5], and in full generality by C. Teleman [61]. Givental's reconstruction inspires an approach to study the relation between GW theory, representation theory of vertex algebras, and integrable systems. In this approach one aims at constructing an integrable hierarchy in the form of *Hirota quadratic equations* (HQE)[1] and show that the generating function of GW invariants is a tau-function of the hierarchy (i.e. it satisfies the HQEs). This approach has been successfully worked out for GW theory of X when $X = \mathbb{P}^1$ [51, 52] and $X = \mathbb{P}^1_{a,b}$ [53]. See also [31, 34, 25] for instances of this approach in the setting of singularity theory.

While the construction of Dubrovin and Zhang is general, the approach with HQEs is not so easy to generalize. The main difficulty is that we have to deal with vanishing cycles and period integrals whose properties are still not very well understood. This is probably one of the main motivation to pursue the HQEs approach. It gives us a new motivation and a new view point in the theory of vanishing cycles and period integrals. Let us point out that there are no examples of targets X of dimension > 1 for which the HQEs are known to exist, although there are some indications that such examples exists (see [6]). Even for orbifolds of dimension 1 (with semi-simple quantum cohomology) the HQEs are not known in general. In this paper, we would like to solve this problem for Fano orbifolds of dimension 1, i.e., $\mathbb{P}^1$-orbifolds $\mathbb{P}^1_{a_1,a_2,a_3}$ (with 3 orbifold points), s.t., $1/a_1 + 1/a_2 + 1/a_3 > 1$. It was already noticed in [11] that the extended Toda hierarchy is equivalent to an extended Kac–Wakimoto hierarchy of type A_1. While KdV is the so called principal

[1]The word "quadratic" in HQE was used by Givental in [31]. The equations are also known as "Hirota bilinear equations."

Kac–Wakimoto hierarchy of type A_1, the extended Toda hierarchy is obtained by extending the homogeneous Kac–Wakimoto hierarchy of type A_1. Our main result is that for the remaining Fano $\mathbb{P}^1_{a_1,a_2,a_3}$-orbifolds the corresponding integrable hierarchy is an extension of a Kac–Wakimoto hierarchy as well, but this time it is neither the homogeneous, nor the principal realization, but something in between.

1.0.1. *GW theory of Fano orbifold curves.* Let

$$\mathbf{a} = \{a_1, a_2, a_3\}, \quad a_1 \le a_2 \le a_3,$$

be a triple of positive integers. Let $\mathbb{P}^1_{\mathbf{a}}$ be the orbifold projective line obtained from $\mathbb{P}^1$ by adding[2] $\mathbb{Z}_{a_1}$-, $\mathbb{Z}_{a_2}$-, and $\mathbb{Z}_{a_3}$-orbifold points. The nature of the problem of constructing HQEs depends on the orbifold Euler characteristic of $\mathbb{P}^1_{\mathbf{a}}$:

$$\chi := \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1.$$

In this paper we will study the Fano case $\chi > 0$, leaving the other two cases $\chi = 0$ (elliptic) and $\chi < 0$ (hyperbolic) for a future investigation.

We consider the Chen-Ruan orbifold cohomology of $\mathbb{P}^1_{\mathbf{a}}$,

$$H := H_{\mathrm{CR}}(\mathbb{P}^1_{\mathbf{a}}, \mathbb{C}).$$

As a vector space H is just $H^*(\mathrm{I}\mathbb{P}^1_{\mathbf{a}}, \mathbb{C})$, where $\mathrm{I}\mathbb{P}^1_{\mathbf{a}}$ is the *inertia orbifold* of $\mathbb{P}^1_{\mathbf{a}}$,

$$\mathrm{I}\mathbb{P}^1_{\mathbf{a}} = \{(x, g) \mid x \in \mathbb{P}^1_{\mathbf{a}},\ g \in \mathrm{Aut}(x)\}.$$

We can fix a homogeneous basis $\{\phi_i\}_{i\in\mathfrak{I}}$ of H, where the index set is defined by

$$\mathfrak{I} := \mathfrak{I}_{\mathrm{tw}} \cup \{(01), (02)\} := \{(k, p) \mid 1 \le k \le 3, 1 \le p \le a_k - 1\} \cup \{(01), (02)\}. \tag{1}$$

The index set $\mathfrak{I}$ reflects the structure of the forgetful map $\mathrm{I}\mathbb{P}^1_{\mathbf{a}} \to \mathbb{P}^1_{\mathbf{a}}$, $(x, g) \mapsto x$. The connected components of $\mathrm{I}\mathbb{P}^1_{\mathbf{a}}$ split into several types depending on their fate under the forgetful map. The entry k enumerates the different types, while p enumerates the cohomology classes supported on the connected components of type k. Motivated by GW theory, Chen and Ruan (see [13]) have introduced a new product, called *Chen–Ruan* or *orbifold cup product*. It is defined as the degree-0 component of the quantum cup product. It is graded homogeneous with respect to a new grading denote by $\deg_{\mathrm{CR}}$. In our notation, $\phi_{01} = \mathbf{1}$ is the unit, $\deg_{\mathrm{CR}} \phi_{02} = 1$, and $\deg_{\mathrm{CR}} \phi_{k,p} = p/a_k$.

The main objects in the orbifold GW theory of $\mathbb{P}^1_{\mathbf{a}}$ are the moduli spaces $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\mathbf{a}}, d)$ of orbifold stable maps f from a domain orbifold genus g curve Σ with n marked points, to the target orbifold $\mathbb{P}^1_{\mathbf{a}}$, such that the homology class of the image of f is d times the fundamental class of the underlying curve of $\mathbb{P}^1_{\mathbf{a}}$. The descendant GW invariants (see (7)) are intersection numbers on the moduli space of stable maps, denoted by

$$\langle \phi_1\, \psi_1^{k_1}, \dots, \phi_n \psi_n^{k_n} \rangle_{g,n,d},$$

where $\phi_j \in H$ and ψ_j is the j-th ψ-class on the moduli space of stable maps.

Our main interest is in the so-called *total descendant potential*, defined by the following generating series of GW invariants:

$$\mathcal{D}_{\mathbf{a}}(\hbar; \mathbf{t}) = \exp\Big(\sum_{g,n,d} \hbar^{g-1} \frac{Q^d}{n!} \langle \mathbf{t}(\psi_1), \dots, \mathbf{t}(\psi_n) \rangle_{g,n,d} \Big), \tag{2}$$

where Q is a non-zero complex number called the Novikov variable, $\mathbf{t}(z) := t_0 + t_1 z + t_2 z^2 + \cdots$, with $t_0, t_1, \ldots \in H$ and $\hbar$ are formal variables. Using the so called *dilaton shift* $q_m = t_m - \delta_{m,1}\mathbf{1}$ we denote $\mathcal{D}_{\mathbf{a}}(\hbar; \mathbf{t})$ by $\mathcal{D}_{\mathbf{a}}(\hbar; \mathbf{q})$ and identify it with a vector in a certain *Fock space* (see (33)).

The construction of the HQEs is given in Section 4.5. It relies on the theory of vanishing cycles and period integrals associated to a Landau-Ginzburg (LG) mirror model of $\mathbb{P}^1_{\mathbf{a}}$. An appropriate

[2] For example, by root constructions [2], [9].

mirror was constructed in [53] in the case $a_1 = 1$ and in general by P. Rossi [55], who managed to compute the quantum cohomology of $\mathbb{P}^1_{\mathbf{a}}$. We also need to know how to solve the quantum differential equations in terms of period integrals. This was achieved recently by Ishibashi–Shiraishi–Takahashi [41], where the mirror model was constructed from the miniversal deformation space M of the *affine cusp polynomial*

$$f_{\mathbf{a}}(x_1, x_2, x_3) := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - \frac{1}{Q} x_1 x_2 x_3.$$

With such a mirror model at hands we can pursue the same idea as in [31, 25, 34, 51, 53] to construct HQEs for GW theory of $\mathbb{P}^1_{\mathbf{a}}$ by studying periods of the mirror model $f_{\mathbf{a}}$.

1.0.2. *Γ-conjecture for the Milnor lattice.* Compared to the earlier works, one novelty of this paper is that we made use of Iritani's integral structure [39] (see also [47]), which allows us to express the vertex operators in our construction in terms of K-theory. This observation seems to be quite general, so we formulated a conjecture for the general case (see Conjecture 11 below), which we refer to as the *Γ-conjecture for the Milnor lattice.*

The homology space $\mathfrak{h}$ of the Milnor fiber at a reference point $(0, 1) \in M \times \mathbb{C}$ has a lattice structure on the vanishing cycles, called *the Milnor lattice*. Conjecture 11 in the case of $\mathbb{P}^1_{\mathbf{a}}$ says that for each element of the K-group $K(\mathbb{P}^1_{\mathbf{a}})$, there exists a corresponding integral cycle in the Milnor lattice of the mirror model, such that both integrals structures match. We will give a proof of this conjecture for the Fano orbifold curves $\mathbb{P}^1_{\mathbf{a}}$, based on Iritani's proof for the Γ-conjecture of toric orbifolds $\mathbb{P}^2_{\mathbf{a}}$. After inverse Laplace transformations, this allows us to get explicit formulas for the calibrated periods over the Milnor lattice, in terms of integral structures in the A-model quantum cohomology, see formula (45). Then we can embed the root system of vanishing cycles into the quantum cohomology via period maps. The vanishing cycles form an affine root system of type A, D, or E, and we can identify the classical monodromy of the Milnor lattice with an affine Coxeter transformation, see Proposition 16. The calibrated periods over the vanishing cycles are very important for the construction of the vertex operators later.

1.0.3. *The Kac–Wakimoto hierarchy.* The triplets $\mathbf{a} = \{a_1, a_2, a_3\}$ with $\chi > 0$ are classified by the Dynkin diagrams of type ADE together with a choice of a *branching node*. In the D and E cases there is a unique choice of a branching node, while in the A-case any node can be chosen. By removing the branching node we obtain 3 diagrams of type[3] A_{a_k-1}, $k = 1, 2, 3$. Let us denote by $\mathfrak{h}^{(0)}$ the Cartan subalgebra of the corresponding simple Lie algebra $\mathfrak{g}^{(0)}$ and define (cf. eqn. (50))

$$\sigma_b = \prod_{k=1}^{3} \Big(s^{(0)}_{k,a_k-1} \cdots s^{(0)}_{k,2} s^{(0)}_{k,1} \Big), \tag{3}$$

where $s^{(0)}_{k,p} : \mathfrak{h}^{(0)} \to \mathfrak{h}^{(0)}$ is the reflection through the hyperplanes orthogonal to $\gamma^{(0)}_{k,p}$, which is the p-th simple root on the k-th branch of the Dynkin diagram. The automorphism σ_b can be extended to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$. Let us denote by κ the order of σ_b as an automorphism of $\mathfrak{g}^{(0)}$. Due to a mirror symmetry phenomenon the spectrum of σ_b is given by the degrees of the cohomology classes ϕ_i. More precisely, there exists a σ_b-eigenbasis $\{H_i\}_{i\in\mathfrak{I}}$ of $\mathfrak{h}^{(0)}$, s.t., $\sigma_b(H_i) = e^{-2\pi\sqrt{-1}d_i} H_i$, where $d_i = 1 - \deg_{\mathrm{CR}}(\phi_i)$. The index set $\mathfrak{I}$ admits a natural involution $*$ compatible with the Poincaré pairing:

$$d_i + d_{i^*} = 1.$$

The σ_b-eigenbasis can be normalized so that $(H_i|H_{j^*}) = \kappa\, \delta_{i,j}$.

The Kac-Wakimoto hierarchy corresponding to the conjugacy class of σ_b in the Weyl group can be described as follows. Let $\mathbb{C}[y]$ be the algebra of polynomials on $y = (y_{i,\ell})$, $i \in \mathfrak{I}\backslash\{(01)\}$ and

[3] if $a_k = 1$ then the corresponding diagram is empty.

$\ell \geq 0$. The vector space[4] $\mathbb{C}[y]^{\mathbb{Z}}$ is equipped with the structure of a module over the algebra of differential operators in e^{ω} by setting

$$(e^{\omega}\cdot\tau)_n = \tau_{n-1}, \quad (\partial_\omega\cdot\tau)_n = n\tau_n, \quad \tau = (\tau_n)_{n\in\mathbb{Z}} \in \mathbb{C}[y]^{\mathbb{Z}}.$$

For every root $\alpha \in \Delta^{(0)}$ of $\mathfrak{g}^{(0)}$ we define *vertex operators* $E^{(0)}_\alpha(\zeta)$ (see (77)) and $E^*_\alpha(\zeta)$ (see (63)) in Section 4.2, both acting on $\mathbb{C}[y]^{\mathbb{Z}}$. Let $E_\alpha(\zeta) = E^{(0)}_\alpha(\zeta)E^*_\alpha(\zeta)$. The HQE of the σ_b-twisted Kac–Wakimoto hierarchy are given by the following bilinear equation for $\tau = (\tau_n(y))_{n\in\mathbb{Z}}$:

$$\begin{aligned}\mathrm{Res}_{\zeta=0}\frac{d\zeta}{\zeta}\Big(\sum_{\alpha\in\Delta^{(0)}} a_\alpha(\zeta)E_\alpha(\zeta)\otimes E_{-\alpha}(\zeta)\Big)\tau\otimes\tau = \Big(\frac{1}{12}\sum_{k=1}^{3}\frac{a_k^2-1}{a_k} + \frac{\chi}{2}\,(\partial_\omega\otimes 1 - 1\otimes\partial_\omega)^2 + \\ + \sum_{i\in\mathfrak{I}\setminus\{(01)\}}\sum_{\ell\geq 0}^{\infty}(d_{i^*}+\ell)(y_{i,\ell}\otimes 1 - 1\otimes y_{i,\ell})(\partial_{y_{i,\ell}}\otimes 1 - 1\otimes\partial_{y_{i,\ell}})\Big)\tau\otimes\tau,\end{aligned} \tag{4}$$

with the coefficients $a_\alpha(\zeta)$ defined by (78) in Section 4.3.

1.0.4. *The main result.* We can write the Kac–Wakimoto HQE in terms of the descendant variables $\{q_m\}_{m\geq 0}$, using the change of variables between $y_{i,\ell}$ and q^i_ℓ (see (79)–(80)). Our main result can be stated as follows.

Theorem 1. *Let $\mathcal{D}_{\mathbf{a}}(\hbar;\mathbf{q})$ (with $\mathbf{a} = \{a_1,a_2,a_3\}$) be the total descendant potential* (2) *of an orbifold projective line $\mathbb{P}^1_{\mathbf{a}}$ with a positive orbifold Euler characteristic, then the sequence $(\tau_n(\hbar;\mathbf{q}))_{n\in\mathbb{Z}}$ of formal power series defined by*

$$\tau_n(\hbar;\mathbf{q}) = (\kappa^\chi Q)^{\frac{1}{2}n^2}\mathcal{D}_{\mathbf{a}}(\hbar;\mathbf{q}+n\sqrt{\hbar}\mathbf{1}), \quad n\in\mathbb{Z}.$$

is a solution to the σ_b-twisted Kac–Wakimoto HQE (4)*, where σ_b is the element* (3) *of the Weyl group of the corresponding finite root system.*

In other words, Theorem 1 shows that the GW theory of $\mathbb{P}^1_{\mathbf{a}}$ is governed by the Kac-Wakimoto hierarchy associated to the triple $\mathbf{a}$. Let us emphasize that the variables $q^{01}_1, q^{01}_2, \ldots$ appear as parameters in the differential equations for τ. It is natural to expect that the σ_b-twisted Kac–Wakimoto HQE can be extended in order to include differential equations in $q^{01}_1, q^{01}_2, \ldots$ as well. We hope that our work will motivate the specialists in integrable systems and representation theory to investigate more systematically the possibility of extending the Kac–Wakimoto hierarchies. For example, in the case of Dynkin diagrams of type A, our hierarchy should agree with a certain reduction of the 2D Toda hierarchy and the required extension was constructed by G. Carlet [10] based on the ideas of [11]. For the type D and E cases, the extension can be constructed with the same idea as in [52] with a slight necessary modification. The details will be presented elsewhere. We suggest to call the σ_b-twisted Kac–Wakimoto hierarchy appearing in Theorem 1 the *ADE-Toda hierarchy*, and call the corresponding extension the *Extended ADE-Toda hierarchy*.

Our approach to Theorem 1 systematically explores representation theoretic properties of the Landau-Ginzburg mirror of $\mathbb{P}^1_{\mathbf{a}}$ and realizes these properties in quantum cohomology of $\mathbb{P}^1_{\mathbf{a}}$ using the period maps. A new observation is that we can also use K-theory to obtain explicit formulas for the leading terms of the period mapping. In particular, this simplifies the analysis of the monodromy representation. Such an approach should be helpful for more general target spaces as well.

To our knowledge, Theorem 1 is the first case where the problem of constructing HQEs governing GW theory of a target X is solved for a *non-toric* X.

Finally, it is very interesting also to investigate the relation between the integrable hierarchies obtained by applying Dubrovin and Zhang's construction [18] to the quantum cohomology of $\mathbb{P}^1_{\mathbf{a}}$ and the integrable hierarchies in Theorem 1. It is natural to expect that the two approaches yield the same integrable hierarchy. We hope to return to this problem in the near future.

[4]This is a direct product of copies of $\mathbb{C}[y]$ indexed by $n\in\mathbb{Z}$.

1.0.5. *Outline of the proof of Theorem 1.* First, the hierarchy (4) is shown to be equivalent (via a Laplace transform) to another hierarchy (88) defined for affine cusp polynomials, see Theorem 31. Then by Proposition 34, the descendant potential $\mathcal{D}_{\mathbf{a}}$ satisfies the hierarchy (88) if and only if the ancestor potential $\mathcal{A}_t$ (see equation (35)) satisfies another hierarchy (100). Finally, the most difficult step is to prove (Theorem 42) that $\mathcal{A}_t$ indeed satisfies (100).

Let us point out that although our proof of Theorem 42 follows closely the argument of [34], we managed to simplify one of the crucial steps in [34]. Namely, there is a certain analyticity property (c.f. Section 5.4) of the so called phase factors that was previously established via the theory of finite reflection groups and their relation to Artin groups. This is one of the main obstacles to generalize the result of [34] to other singularities. Our argument now seems to apply in much more general settings, since it relies only on the fact that the Gauss–Manin connection has regular singularities and that the vertex operators are local to each other (in the sense of the theory of vertex operator algebras).

The rest of this paper is organized as follows. In Section 2, we recall the orbifold GW theory for Fano projective curves $\mathbb{P}^1_{\mathbf{a}}$ and the corresponding LG mirror model. In Section 3 we recall Iritani's integral structure (see [39]) in the quantum cohomology of a smooth projective orbifold X. Furthermore, we prove that for $X = \mathbb{P}^1_{\mathbf{a}}$ the integral structure corresponds to the Milnor lattice under mirror symmetry. Finally, using the period mapping we identify the root system arising from the set of vanishing cycles with an affine root system in the quantum cohomology of $\mathbb{P}^1_{\mathbf{a}}$. The integral structures allows us to obtain an explicit description of the leading order terms of the period mapping in terms of finite root systems. In Section 4, using the results from Section 3, we give a Fock-space realization of the basic representations of the affine Lie algebras of ADE type. Then we recall the Kac-Wakimoto hierarchies and construct integrable hierarchies for affine cusp polynomials and show that these hierarchies are related by a Laplace transform (Theorem 31). In Section 5 we construct another hierarchy (100) and describe its relation with the hierarchies from previous sections, see Proposition 34. Then we show that the ancestor potential of $\mathbb{P}^1_{\mathbf{a}}$ satisfies the integrable hierarchy (100) and deduce Theorem 1. In Section 6 we consider the example $\mathbf{a} = \{2, 2, 2\}$. In the appendix, we give an alternative proof for the higher genus reconstruction of total ancestor potential.

Acknowledgments. We thank Yongbin Ruan for his interests in this work, and for his comments and suggestions. T. M. would like to thank Tadashi Ishibe for useful discussions on the affine Artin group, Atsushi Takahashi and Yuuki Shiraishi for useful discussion on mirror symmetry for affine cusp polynomials, and especially to S. Galkin and H. Iritani for very useful discussions on the Γ-conjecture. Y. S. would like to thank Shifra Reif for helpful discussions on affine Kac–Moody algebras. H.-H. T. thanks H. Iritani for discussions on mirror symmetry.

T. M. and Y. S. acknowledge the World Premiere International Research Center Initiative (WPI Initiative), Mext, Japan. H.-H. T. thanks the hospitality and support for his visits to IPMU.

T. M. is supported in part by JSPS Grant-in-Aid 26800003. H.-H. T. is supported in part by Simons Foundation Collaboration Grant.

2. Orbifold GW theory of Fano orbifold curves $\mathbb{P}^1_{\mathbf{a}}$ and their mirror symmetry

2.1. **Orbifold GW theory of $\mathbb{P}^1_{\mathbf{a}}$.** Fano orbifold curves are closed orbifold curves with positive orbifold Euler characteristics. They are classified by triplets of positive integers $\mathbf{a} = \{a_1, a_2, a_3\}$ where $a_1 \le a_2 \le a_3$ and $\chi := \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1 > 0$. Each Fano orbifold curve is an orbifold curve with an underlying curve $\mathbb{P}^1$ and has at most three orbifold points p_k $(k = 1, 2, 3)$ with local isotropy groups $\mathbb{Z}_{a_k}$. We denote such an Fano orbifold curve by $\mathbb{P}^1_{\mathbf{a}}$. Note that such notation also includes the smooth curve $\mathbb{P}^1$ with $a_1 = a_2 = a_3 = 1$. It is easy to see that χ is the orbifold Euler characteristic of $\mathbb{P}^1_{\mathbf{a}}$.

We use the index set $\mathfrak{I}$ (see (1)) to label a fixed basis of the Chen-Ruan orbifold cohomology $H := H_{\rm CR}(\mathbb{P}^1_{\mathbf{a}};\mathbb{C})$ as follows:

$$\phi_{01} = \mathbf{1}, \quad \phi_{02} = P$$

are the unit and the hyperplane class of the underlying $\mathbb{P}^1$ respectively and

$$\phi_i = \phi_{k,p}, \quad i := (k,p) \in \mathfrak{I}_{\rm tw}.$$

are the units of the corresponding twisted sectors of $\mathbb{P}^1_{\mathbf{a}}$. The cohomology degrees of the classes are:

$$\deg_{\rm CR}\phi_{01} = 0, \quad \deg_{\rm CR}\phi_{02} = 1, \quad \deg_{\rm CR}\phi_i = \frac{p}{a_k}, \quad i = (k,p) \in \mathfrak{I}_{\rm tw},$$

where slightly violating the standard conventions we work with complex degree, i.e., half of the usual real degrees. There is a natural involution $*$ on $\mathfrak{I}$ induced by orbifold Poincaré duality

$$(01)^* = (02), \quad (k,p)^* = (k, a_k - p). \tag{5}$$

The orbifold Poincaré pairing $(-,-)$ on H is non-zero only for the following cases:

$$(\phi_{01},\phi_{02}) = 1, \quad (\phi_i,\phi_j) = \frac{1}{a_i}\delta_{i,j^*},$$

where $i,j \in \mathfrak{I}_{\rm tw}$ correspond to twisted classes, and we set $a_i := a_k$ for $i = (k,p) \in \mathfrak{I}_{\rm tw}$.

GW theory studies integrals over moduli spaces of stable maps. In this paper, we will use both the descendant invariants and the ancestor invariants. Let us introduce their definitions for Fano orbifold curves $\mathbb{P}^1_{\mathbf{a}}$. For more details on orbifold GW theory we refer to [13] for the analytic approach and to [2] for the algebraic geometry approach. Let $d \in \mathrm{Eff}(\mathbb{P}^1_{\mathbf{a}}) \subset H_2(\mathbb{P}^1_{\mathbf{a}};\mathbb{Z}) \cong \mathbb{Z}$ be an effective curve class. By choosing the homology class $[\mathbb{P}^1_{\mathbf{a}}]$ as a $\mathbb{Z}$-basis of $H_2(\mathbb{P}^1_{\mathbf{a}};\mathbb{Z})$ we may identify d with a non-negative integer. Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\mathbf{a}},d)$ be the moduli space of stable orbifold maps f from a genus-g nodal orbifold Riemann surface Σ to $\mathbb{P}^1_{\mathbf{a}}$, such that $f_*[\Sigma] = d$. In addition, Σ is equipped with n marked points $z_1,\dots,z_n$ that are pairwise distinct and not nodal and the orbifold structure of Σ is non-trivial only at the marked points and the nodes. The moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\mathbf{a}},d)$ has a virtual fundamental cycle $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\mathbf{a}},d)]^{\rm virt}$. Its homology degree is

$$2\left((3-\dim\mathbb{P}^1_{\mathbf{a}})(g-1) + \chi\cdot d + n\right). \tag{6}$$

The moduli space is naturally equipped with line bundles $\mathcal{L}_j$ formed by the cotangent lines[5] $T^*_{\bar z_j}\bar\Sigma/\mathrm{Aut}(\Sigma,z_1,\dots,z_n;f)$ and with evaluation map

$$\mathrm{ev}: \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\mathbf{a}},d) \to \underbrace{\mathrm{I}\,\mathbb{P}^1_{\mathbf{a}}\times\cdots\times\mathrm{I}\,\mathbb{P}^1_{\mathbf{a}}}_{n},$$

obtained by evaluating f at the (orbifold) marked points $z_1,...,z_n$ and landing at the connected component of the inertia orbifold $\mathrm{I}\,\mathbb{P}^1_{\mathbf{a}}$ corresponding to the generator of the automorphism group of the orbifold point z_j (c.f. [13]).

The *descendant orbifold GW invariants* of $\mathbb{P}^1_{\mathbf{a}}$ are intersection numbers

$$\langle \phi_1\psi_1^{k_1},\dots,\phi_n\psi_n^{k_n}\rangle_{g,n,d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\mathbf{a}},d)]^{\rm virt}} \mathrm{ev}^*(\phi_1\otimes\cdots\otimes\phi_n)\,\psi_1^{k_1}\cdots\psi_n^{k_n}, \tag{7}$$

where $\phi_j \in H := H_{\rm CR}(\mathbb{P}^1_{\mathbf{a}};\mathbb{C})$, $\psi_j = c_1(\mathcal{L}_j)$. The *total descendant potential* is

$$\mathcal{D}_{\mathbf{a}}(\hbar;\mathbf{t}) = \exp\Big(\sum_{g,n,d}\hbar^{g-1}\frac{Q^d}{n!}\langle\mathbf{t}(\psi_1),\dots,\mathbf{t}(\psi_n)\rangle_{g,n,d}\Big),$$

where Q is a non-zero complex number called the *Novikov variable*, $\hbar$, $t_0,t_1,\ldots \in H$ are formal variables and $\mathbf{t}(z) := t_0 + t_1 z + t_2 z^2 + \cdots$.

[5]Here $\bar\Sigma$ is the nodal Riemann surface underlying Σ and $\bar z_j \in \bar\Sigma$ is the i-th marked point on $\bar\Sigma$.

Let $\pi : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\mathbf{a}}, d) \to \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism and

$$\Lambda_{g,n,d}\,(\phi_1, \cdots, \phi_n) := \pi_*\left([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\mathbf{a}}, d)]^{\mathrm{virt}} \cap \mathrm{ev}^*(\phi_1 \otimes \cdots \otimes \phi_n)\right).$$

The *ancestor orbifold GW invariants* of $\mathbb{P}^1_{\mathbf{a}}$ are intersections numbers over the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ $(2g-2+n>0)$:

$$\langle \phi_1\, \bar{\psi}_1^{k_1}, \ldots, \phi_n \bar{\psi}_n^{k_n} \rangle_{g,n,d} := \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n,d}\,(\phi_1, \cdots, \phi_n)\ \bar{\psi}_1^{k_1} \cdots \bar{\psi}_n^{k_n}, \tag{8}$$

where $\bar{\psi}_j$ is the j-th ψ-class over $\overline{\mathcal{M}}_{g,n}$. We define the total ancestor potential of $\mathbb{P}^1_{\mathbf{a}}$ as follows

$$\mathcal{A}_{\mathbf{a}}(\hbar; \mathbf{t}) := \exp\Big(\sum_{g,n,d} \hbar^{g-1} \frac{Q^d}{n!} \langle \mathbf{t}(\bar{\psi}_1), \ldots, \mathbf{t}(\bar{\psi}_n) \rangle_{g,n,d} \Big). \tag{9}$$

For each element $t \in H$, it is useful to introduce the double bracket notation:

$$\langle\langle \phi_1\, \bar{\psi}_1^{k_1}, \ldots, \phi_n \bar{\psi}_n^{k_n} \rangle\rangle_{g,n}(t) := \sum_{k,d} \frac{Q^d}{k!} \langle \phi_1\, \bar{\psi}_1^{k_1}, \ldots, \phi_n \bar{\psi}_n^{k_n}, t, \ldots, t \rangle_{g,n+k,d}$$

We define a total ancestor potential that depends on the choice of t,

$$\mathcal{A}_t(\hbar; \mathbf{t}) = \exp\Big(\sum_{g,n} \hbar^{g-1} \frac{1}{n!} \langle\langle \mathbf{t}(\bar{\psi}_1), \ldots, \mathbf{t}(\bar{\psi}_n) \rangle\rangle_{g,n}(t) \Big). \tag{10}$$

According to [33] the total ancestor potential $\mathcal{A}_t(\hbar; \mathbf{t})$ and the total descendant potential $\mathcal{D}_{\mathbf{a}}(\hbar; \mathbf{t})$ are related by the quantization of a calibration operator $S_t(z)$ in Section 2.3.2. We will explain the details of the quantization in Section 2.4.

The *quantum cup product* is a family of associative commutative multiplications $\bullet_t$ (or just $\bullet$ if the reference point t is mentioned) in H defined for each $t \in H$ via the correlators

$$(\phi_i \bullet_t \phi_j, \phi_k) = \langle\langle \phi_i, \phi_j, \phi_k \rangle\rangle(t).$$

The degree-0 part of $\bullet_t$ at $t=0$ is called the *Chen-Ruan cup product.* We denote it by

$$\cup_{\mathrm{CR}} = \bullet_{t=0}|_{Q=0}$$

Let t_i, $i \in \mathfrak{I}$ be the corresponding coordinates of ϕ_i. The quantum cup product induces on H a Frobenius structure of conformal dimension 1 with respect to the *Euler vector field*

$$E = \sum_{i \in \mathfrak{I}} d_i t_i \frac{\partial}{\partial t_i} + \chi \frac{\partial}{\partial t_{02}}$$

where $d_i = 1 - \deg_{\mathrm{CR}}(\phi_i)$, i.e.,

$$d_{01} = 1, \quad d_{02} = 0, \quad d_i = 1 - \frac{p}{a_k}, \quad i = (k,p) \in \mathfrak{I}_{\mathrm{tw}}.$$

2.2. **Mirror symmetry for the quantum cohomology.** The Frobenius structure on H arising from quantum cohomology can be identified with the Frobenius structure on a certain deformation space of the *affine cusp polynomial*

$$f_{\mathbf{a}}(x) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - \frac{1}{Q} x_1 x_2 x_3, \quad x = (x_1, x_2, x_3). \tag{11}$$

where $Q \in \mathbb{C}^*$ is the Novikov variable. The isomorphism in the case $a_1 = 1$ was established in [53] and the general case can be found in [55]. According to Ishibashi–Shiraishi–Takahashi (see [41]), the Frobenius structure can be described also in the general framework of K. Saito's theory of primitive forms. This is precisely the point of view suitable for our purposes.

Denote the Milnor number of $f_{\mathbf{a}}$ (i.e., the number of critical points of a Morsification of $f_{\mathbf{a}}$) by

$$N + 1 = a_1 + a_2 + a_3 - 1.$$

Denote the space of a miniversal deformation of the polynomial $f_{\mathbf{a}}$ by

$$M = \mathbb{C}^{N+1}.$$

Note that the cardinality of the set $\mathfrak{I}$ is $N+1$, so we can enumerate the coordinates on M via $s = (s_i)_{i \in \mathfrak{I}}$. Recall $\mathfrak{I}_{\rm tw} = \mathfrak{I} \setminus \{(01),(02)\}$. Given $s \in M$, we put

$$F(x,s) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - \frac{1}{Qe^{s_{02}}}\, x_1x_2x_3 + s_{01} + \sum_{i=(k,p)\in \mathfrak{I}_{\rm tw}} s_i\, x_k^p.$$

Let $C \subset M \times \mathbb{C}^3$ be the analytic subvariety with structure sheaf

$$\mathcal{O}_C = \mathcal{O}_{M\times\mathbb{C}^3}/(\partial_{x_1}F, \partial_{x_2}F, \partial_{x_3}F);$$

then the *Kodaira-Spencer map*

$$\mathcal{T}_M \to p_*\mathcal{O}_C, \quad \frac{\partial}{\partial s_i} \mapsto \frac{\partial F}{\partial s_i} \bmod (\partial_{x_1}F, \partial_{x_2}F, \partial_{x_3}F), \tag{12}$$

where $p : M \times \mathbb{C}^3 \to M$ is the projection onto the first factor, is an isomorphism which allows us to define an associative, commutative multiplication $\bullet$ on $\mathcal{T}_M$. The main result in [41] is that

$$\omega = \frac{\sqrt{-1}}{Qe^{s_{02}}}\, dx_1 \wedge dx_2 \wedge dx_3$$

is a *primitive form* in the sense of K. Saito (see [56]), which allows us to construct a Frobenius structure on M (see [57]). More precisely, the form ω gives rise to a residue pairing on $\mathcal{O}_C$

$$(\phi_1,\phi_2) = -\frac{1}{Q^2e^{2\,s_{02}}}\, \mathrm{Res}_{M\times\mathbb{C}^3/M}\, \frac{\phi_1\phi_2\, dx_1\wedge dx_2\wedge dx_3}{\partial_{x_1}F\partial_{x_2}F\partial_{x_3}F}\,,$$

which via the Kodaira–Spencer isomorphism (12) induces a non-degenerate bilinear form on $\mathcal{T}_M$. Let us form the following family of connections on $\mathcal{T}_M$

$$\nabla = \nabla^{\rm L.C.} - \frac{1}{z}\sum_{i\in\mathfrak{I}} (\partial_{s_i}\bullet)\, ds_i,$$

where $\nabla^{\rm L.C.}$ is the Levi-Cevita connection associated with the residue pairing and $\partial_{s_i}\bullet$ is the operator of multiplication by the vector field $\partial/\partial s_i$. Let us also introduce the *oscillatory integrals*

$$J_{\mathcal{A}}(s,z) = (-2\pi z)^{-3/2}\, z d_M \int_{\mathcal{A}_{s,z}} e^{F(x,s)/z}\, \omega \in T_s^*M,$$

where d_M is the de Rham differential on M, and $\mathcal{A}$ is a flat section of the bundle on $M\times\mathbb{C}^*$, whose fiber over a point (s,z) is given by the space of semi-infinite homology cycles

$$H_3(\mathbb{C}^3, \{x|\mathrm{Re}(F(x,s)/z) \ll 0\}; \mathbb{C}) \cong \mathbb{C}^{N+1}.$$

The fact that ω is primitive means that the connection ∇ is flat for all $z\neq 0$ and that after identifying $\mathcal{T}_M \cong \mathcal{T}_M^*$ via the residue pairing, the oscillatory integrals $J_{\mathcal{A}}$ give rise to flat sections of ∇. Moreover, since the oscillatory integrals are weighted-homogeneous functions if one assigns weights d_i ($i\in\mathfrak{I}$), $1/a_j$ ($1\leq j\leq 3$), and χ to s_i, x_j, and Q respectively, they satisfy an additional differential equation with respect to z. Let $E\in\mathcal{T}_M$ be the *Euler vector field*

$$E = \sum_{i\in\mathfrak{I}} d_i s_i \frac{\partial}{\partial s_i} + \chi\frac{\partial}{\partial s_{02}}.$$

Note that under the Kodaira–Spencer isomorphism E corresponds to the equivalence class of F in $p_*\mathcal{O}_C$. The oscillatory integrals satisfy the following differential equation:

$$(z\partial_z + E)\, J_{\mathcal{A}}(t,z) = \theta\, J_{\mathcal{A}}(t,z), \tag{13}$$

where $\theta : \mathcal{T}_M \to \mathcal{T}_M$ is the *Hodge grading operator* defined via

$$\theta(X) = \nabla_X^{L.C.}(E) - \frac{1}{2}X \tag{14}$$

where the constant $\frac{1}{2}$ is chosen in such a way that θ is *anti-symmetric* with respect to the residue pairing: $(\theta(X), Y) = -(X, \theta(Y))$.

The quantum cohomology computed at $t = 0$ is isomorphic as a Frobenius algebra with T_0M (see [41, 55]). The identification has the following form

$$\phi_i = x_k^p + \cdots, \quad \phi_{01} = 1, \quad \phi_{02} = \frac{1}{Q}x_1x_2x_3 + \cdots.$$

where $i = (k, p)$ is the index of a twisted class and the dots stand for some polynomials that involve higher-order powers of Q. More precisely, using the Kodaira-Spencer isomorphism we have

$$\phi_i = \partial_{s_i} + \cdots, \quad \phi_{01} = \partial_{s_{01}}, \quad \phi_{02} = \partial_{s_{02}} + \cdots,$$

where the dots stand for some vector fields depending holomorphically on Q near $Q = 0$ and vanishing at $Q = 0$. These additional terms are uniquely fixed by the requirement that the vector fields ϕ_i ($i \in \mathfrak{I}$) are flat, i.e., the residue pairing is constant independent of Q. On the other hand the flatness of ∇ implies that the residue pairing is flat, therefore we can extend uniquely the isomorphism $H \cong T_0M$ to an isomorphism

$$TH \cong TM$$

such that the residue pairing coincides with the Poincaré pairing. In other words, the linear coordinates t_i, $i \in \mathfrak{I}$ on H are functions on M such that $t_i(0) = 0$, the vector field $\partial/\partial t_i$ is flat with respect to the Levi–Civita connection, and at $s = 0$ it coincides with ϕ_i. The mirror symmetry for quantum cohomology can be stated as follows.

Theorem 2 ([41], Theorem 4.1)**.** *The isomorphism $M \cong H$, $s \mapsto t(s)$ is an isomorphism of Frobenius manifolds, i.e., $T_sM \cong T_{t(s)}H$ as Frobenius algebras.*

Remark 3. *Theorem 2 can be proved also by using the extended J-function of $\mathbb{P}^1_{\mathbf{a}}$ (see Section 3.2.3). Namely, it is not hard to derive an identification between the quantum cohomology D-module of $\mathbb{P}^1_{\mathbf{a}}$ and the D-module defined by $f_{\mathbf{a}}(x)$.*

From now on we will make use of the residue pairing to identify $T^*M \cong TM$. Also the flat Levi–Civita connection $\nabla^{\mathrm{L.C.}}$ allows us to construct a trivialization

$$TM \cong M \times T_0M,$$

and finally, the Kodaira–Spencer map (12) together with the mirror symmetry isomorphism gives $T_0M \cong H$. In other words, we have natural trivializations

$$T^*M \cong TM \cong M \times H. \tag{15}$$

2.3. **The period integrals and the calibration operator.** Givental noticed that certain period integrals (c.f. formula (16) below) in singularity theory play a crucial role in the theory of integrable systems. In this section, we recall Givental's construction as well as some of its basic properties. See [31] for more details.

Put $X = M \times \mathbb{C}^3$ and let

$$\varphi : X \to M \times \mathbb{C}, \quad (s, x) \mapsto (s, F(x, s)).$$

Let

$$X_{s,\lambda} = \varphi^{-1}(s, \lambda)$$

be the fibers of φ. The set of all $(s, \lambda) \in M \times \mathbb{C}$ such that the fiber $X_{s,\lambda}$ is singular is an analytic hypersurface, called *discriminant*. Its complement in $M \times \mathbb{C}$ will be denoted by $(M \times \mathbb{C})'$. The homology and cohomology groups $H_2(X_{s,\lambda};\mathbb{C})$ and $H^2(X_{s,\lambda};\mathbb{C})$, $(s,\lambda)\in(M\times\mathbb{C})'$ form vector bundles over the base $(M\times\mathbb{C})'$. Moreover, the integral structure in the fibers allows us to define a flat connection known as the *Gauss–Manin* connection.

Let us fix the point $(0,1)\in(M\times\mathbb{C})'$ (for $Q\ll 1$) to be our reference point. The vector space

$$\mathfrak{h}=H_2(X_{0,1};\mathbb{C})$$

has a very rich structure, which we would like to recall. Let

$$\Delta\subset\mathfrak{h}$$

be the set of *vanishing cycles*, and $(\cdot|\cdot)$ be the *negative* of the intersection pairing. The negative sign is chosen so that $(\alpha|\alpha)=2$ for all $\alpha\in\Delta$. The parallel transport with respect to the Gauss–Manin connection induces a monodromy representation

$$\pi_1((M\times\mathbb{C})')\to \mathrm{GL}(\mathfrak{h}).$$

The image

$$W\subset\mathrm{GL}(\mathfrak{h})$$

of the fundamental group under this representation is a subgroup of the group of linear transformations of $\mathfrak{h}$ that preserve the intersection form. The Picard–Lefschetz theory can be applied in our setting as well and W is in fact a reflection group generated by the reflections

$$s_\alpha(x)=x-(\alpha|x)\alpha,\quad \alpha\in\Delta.$$

The reflection s_α is the monodromy transformation along a simple loop that goes around a generic point on the discriminant over which the cycle α vanishes. Finally, recall that the *classical* monodromy $\sigma\in W$ is the monodromy transformation along a big loop around the discriminant. For more details on vanishing homology and cohomology and the Picard–Lefschetz theory we refer to the book [3]. We will see in Proposition 16 below that Δ is an affine root system.

The main objects in our construction are the following multi-valued analytic functions:

$$I^{(n)}_\alpha(t,\lambda)=-\frac{1}{2\pi}\,\partial_\lambda^{n+1}\,d_M\int_{\alpha_{t,\lambda}}d^{-1}\omega, \tag{16}$$

where the value of the RHS depends on the choice of a path avoiding the discriminant, connecting the reference point with (t,λ). The cycle $\alpha_{t,\lambda}$ is obtained from $\alpha\in\mathfrak{h}$ via a parallel transport (along the chosen path), $d^{-1}\omega$ is any holomorphic 2-form η on $\mathbb{C}^3$ such that $\omega=d\eta$, and d_M is the de Rham differential on M. The RHS in (16) defines naturally a cotangent vector in T^*_tM, which via the trivialization (15) is identified with a vector in H.

The period vectors (16) are uniquely defined for all $n\geq -1$. For $n\leq -2$ there is an ambiguity in choosing integration constants, which can be removed by means of the following differential equations:

$$\partial_{t_i}\,I^{(n)}_\alpha(t,\lambda)=-\phi_i\bullet I^{(n+1)}_\alpha(t,\lambda),\quad i\in\mathfrak{I}, \tag{17}$$

$$\partial_\lambda\,I^{(n)}_\alpha(t,\lambda)=I^{(n+1)}_\alpha(t,\lambda), \tag{18}$$

$$(\lambda-E\bullet)\partial_\lambda I^{(n)}_\alpha(t,\lambda)=\Big(\theta-n-1/2\Big)I^{(n)}_\alpha(t,\lambda). \tag{19}$$

Finally, note that the unit vector $\mathbf{1}\in H\cong M$ has coordinates $t_{01}=1$, $t_i=0$ for $i\neq(01)$ and that the period vectors have the following translation symmetry:

$$I^{(n)}_\alpha(t,\lambda)=I^{(n)}_\alpha(t-\lambda\mathbf{1},0),\quad \forall n\in\mathbb{Z},\quad\forall\alpha\in\mathfrak{h}.$$

The oscillatory integrals are related to the period integrals via a Laplace transform along an appropriately chosen path:

$$J_{\mathcal{A}}(t,z)=(-2\pi z)^{-1/2}\int_{u_j}^{\infty}e^{\lambda/z}I^{(0)}_\alpha(t,\lambda)d\lambda, \tag{20}$$

where $u_j(t)$ is such that $(t, u_j(t))$ is a point on the discriminant over which the cycle α vanishes. The differential equations (17) are the Laplace transform of $\nabla J_{\mathcal{A}} = 0$, while the equation (19) is the Laplace transform of the differential equation (13). Using equations (18) and (19) we can express $I^{(n)}$ in terms of $I^{(n+1)}$ as long as the operator $\theta - n - 1/2$ is invertible. This is the case for $n \leq -2$, which allows us to extend the definition of $I^{(n)}$ to all $n \in \mathbb{Z}$.

2.3.1. *Stationary phase asymptotic.* Let $u_j(t)$, $1 \leq j \leq N+1$ be the critical values of $F(x,t)$. The set

$$M_{\rm ss} \subset M$$

of all points $t \in M$ such that the critical values $u_j(t)$ form locally near t a coordinate system is open and dense. Let us fix some $t_0 \in M_{\rm ss}$; then in a neighborhood of t_0 the critical values give rise to a coordinate system in which the pairing and the product $\bullet$ are diagonal, i.e.,

$$\partial/\partial u_j \bullet \partial/\partial u_{j'} = \delta_{j,j'} \partial/\partial u_j, \quad (\partial/\partial u_j, \partial/\partial u_{j'}) = \delta_{j,j'}/\Delta_j,$$

where Δ_j are some multi-valued analytic functions on $M_{\rm ss}$. Following Dubrovin's terminology (see [16]), we refer to u_j as *canonical coordinates.*

Remark 4. *It is easy to see that the critical variety C of the function F is non-singular, i.e., it is a manifold. It can be proved that the projection map $p : C \subset M \times \mathbb{C}^3 \to M$ is a finite branched covering of degree $N+1$. The branching points are precisely $M \setminus M_{ss}$.*

Using the canonical coordinates we can construct a trivialization of the tangent bundle

$$\Psi : M_0 \times \mathbb{C}^{N+1} \cong TM_0, \quad (t, e_j) \mapsto (t, \sqrt{\Delta_j}\, \frac{\partial}{\partial u_j}).$$

Here $M_0 \subset M_{\rm ss}$ is an open contractible neighborhood of t_0 and $\{e_j\}$ is the standard basis of $\mathbb{C}^{N+1}$, where the j-th component of e_j is 1, while the remaining ones are 0. According to Givental (see [32]), there exists a unique formal asymptotic series $\Psi_t R_t(z) e^{U_t/z}$ that satisfies the same differential equations as the oscillatory integrals $J_{\mathcal{A}}$, where

$$R_t(z) = 1 + \sum_{\ell=1}^{\infty} R_\ell(t) z^\ell, \quad R_\ell(t) \in \mathrm{End}(\mathbb{C}^{N+1}). \tag{21}$$

We will make use of the following formal series

$$\mathbf{f}_\alpha(t, \lambda; z) = \sum_{n \in \mathbb{Z}} I^{(n)}_\alpha(t, \lambda)\, (-z)^n \,, \alpha \in \mathfrak{h}. \tag{22}$$

Example 5. Note that for A_1-singularity $F(t,x) = x^2/2 + t$ we have $u := u_1(t) = t$. Up to a sign there is a unique vanishing cycle. The series (22) will be denoted simply by $\mathbf{f}_{A_1}(t, \lambda; z)$. The corresponding period vectors can be computed explicitly:

$$I^{(n)}_{A_1}(u, \lambda) = (-1)^n \, \frac{(2n-1)!!}{2^{n-1/2}} \, (\lambda - u)^{-n-1/2}, \quad n \geq 0$$

$$I^{(-n-1)}_{A_1}(u, \lambda) = 2\, \frac{2^{n+1/2}}{(2n+1)!!} \, (\lambda - u)^{n+1/2}, \quad n \geq 0.$$

The key lemma (see [31]) is the following.

Lemma 6. *Let $t \in M_{\rm ss}$ and β be a vanishing cycle vanishing over the point $(t, u_j(t))$. Then for all λ near $u_j := u_j(t)$, we have*

$$\mathbf{f}_\beta(t, \lambda; z) = \Psi_t R_t(z)\, e_j\, \mathbf{f}_{A_1}(u_j, \lambda; z)\,.$$

An important corollary of Lemma 6 is the following remarkable formula due to K. Saito ([56]):

$$(\alpha|\beta) = (I^{(0)}_\alpha(t,\lambda), (\lambda - E\bullet) I^{(0)}_\beta(t,\lambda)). \tag{23}$$

To prove this formula, first note that the the differential equations (17)–(19) imply that the RHS is independent of t and λ. In order to compute the RHS, let us fix $t \in M_{\rm ss}$ and let λ approach one of the critical values $u_j(t)$ in such a way that the cycle β vanishes over $(t, u_j(t))$. According to Lemma 6 we have

$$I^{(0)}_\beta(t,\lambda) = 2(2(\lambda - u_j))^{-1/2} e_j + O((\lambda - u_j)^{1/2}).$$

Similarly, decomposing $\alpha = \alpha' + (\alpha|\beta)\beta/2$, where α' is invariant with respect to the local monodromy, we get

$$I^{(0)}_\alpha(t,\lambda) = (\alpha|\beta)\,(2(\lambda - u_j))^{-1/2} e_j + O((\lambda - u_j)^{1/2}).$$

It is well known (see [16]) that in canonical coordinates the Euler vector field has the form $E = \sum u_j \partial_{u_j}$. Now it is easy to see that the RHS of (23), up to higher order terms in $(\lambda - u_j)$ is $(\alpha|\beta)$ and since the latter must be independent of λ the higher-order terms must vanish.

2.3.2. *The calibration operator.* The *calibration* of the Frobenius structure on H is by definition a gauge transformation S of the form

$$S_t(z) = 1 + \sum_{\ell=1}^{\infty} S_\ell(t) z^{-\ell}, \quad S_\ell(t) \in \mathrm{End}(H), \tag{24}$$

such that $\nabla = S d S^{-1}$. In GW theory there is a canonical choice of calibration given by genus-0 descendant invariants as follows (see [33]):

$$(S_t(z)\phi_i, \phi_j) = (\phi_i, \phi_j) + \sum_{\ell=0}^{\infty} \langle\langle \phi_i \psi^\ell, \phi_j \rangle\rangle_{0,2}(t) z^{-\ell-1}. \tag{25}$$

Here

$$\langle\langle \phi_i \psi^\ell, \phi_j \rangle\rangle_{0,2}(t) = \sum_{m\geq 0}\sum_{d\geq 0} \frac{Q^d}{m!} \langle \phi_i\psi^\ell, \phi_j, t, \ldots, t\rangle_{0,2+m,d}.$$

It is a general fact in GW theory (see [33]) that

$$S_t(z)^{-1}\Big(\partial_z - z^{-1}\theta + z^{-2} E\,\bullet\Big) S_t(z) = \partial_z - z^{-1}\theta + z^{-2}\rho, \tag{26}$$

where $\rho = \chi\, P \cup_{\rm CR}$. By definition the operator ρ acts on H as follows

$$\rho(\phi_{01}) = \chi\phi_{02}, \quad \rho(\phi_i) = 0, \quad \text{for } i \in \mathfrak{I}\backslash\{(01)\}. \tag{27}$$

We define a new series

$$\widetilde{\mathbf{f}}_\alpha(\lambda; z) := S_t(z)^{-1}\, \mathbf{f}_\alpha(t,\lambda; z). \tag{28}$$

Note that the RHS is independent of t. Put

$$\widetilde{\mathbf{f}}_\alpha(\lambda; z) = \sum_{n\in\mathbb{Z}} \widetilde{I}^{(n)}_\alpha(\lambda)\,(-z)^n. \tag{29}$$

We will refer to $\widetilde{I}^{(n)}_\alpha(\lambda)$ as the *calibrated* limit of the period vector $I^{(n)}_\alpha(t,\lambda)$.

In our general set up the Novikov variable Q is a fixed non-zero constant. However, it will be useful also to allow Q to vary in a small contractible neighborhood and to study the dependence of the periods and their calibrated limits on Q. By definition $I^{(n)}_\alpha(t,\lambda)$ depend on $Qe^{t_{02}}$, so we simply have

$$Q\partial_Q\, I^{(n)}_\alpha(t,\lambda) = \partial_{t_{02}}\, I^{(n)}_\alpha(t,\lambda).$$

Using the *divisor equation* in GW theory, it is easy to prove (c.f. [33]) that the gauge transformation $S_t(z)$ satisfies the following differential equation:

$$zQ\partial_Q\, S_t(z) = z\partial_{t_{02}} S_t(z) - S_t(z)\,(P\cup_{\rm CR}).$$

Finally, the gauge identity $\nabla = SdS^{-1}$ and the differential equations (17)–(19) imply that the calibrated limit of the period vectors satisfy the following system of differential equations:

$$Q\partial_Q \widetilde{I}^{(n)}_\alpha(\lambda) = -P\cup_{\rm CR}\, \widetilde{I}^{(n+1)}_\alpha(\lambda) \tag{30}$$

$$\partial_\lambda\, \widetilde{I}^{(n)}_\alpha(\lambda) = \widetilde{I}^{(n+1)}_\alpha(\lambda), \tag{31}$$

$$(\lambda-\rho)\partial_\lambda \widetilde{I}^{(n)}_\alpha(\lambda) = \Big(\theta - n - 1/2\Big)\widetilde{I}^{(n)}_\alpha(\lambda). \tag{32}$$

Lemma 7. *a) Let $\{B_i\}_{i\in\mathfrak{I}}$ be a basis of $\mathfrak{h}^\vee := H^2(X_{0,1};\mathbb{C})$, then the following formula holds*

$$\widetilde{I}^{(-1)}_\alpha(\lambda) = \langle B_{01},\alpha\rangle \Big(\lambda\mathbf{1} + (\chi\log\lambda - \log Q)\,P\Big) + \langle B_{02},\alpha\rangle\, P + \sum_{i\in\mathfrak{I}_{\rm tw}} \langle B_i,\alpha\rangle\,\lambda^{d_i}\,\phi_i,$$

b) The analytic continuation of $\widetilde{I}^{(n)}_\alpha(\lambda)$ along a closed loop around 0 is $\widetilde{I}^{(n)}_{\sigma(\alpha)}(\lambda)$, where σ is the classical monodromy.

Proof. a) Recall ρ acts on H by (27), while the operator θ defined in (14) has the form (via (15))

$$\theta(\phi_i) = (d_i - 1/2)\,\phi_i, \quad i\in\mathfrak{I}.$$

Note that the H-valued functions that follow the pairings $\langle B_i,\alpha\rangle$ are solutions to the system (30)–(32) with $n=-1$. These solutions are linearly independent, therefore they must give a basis in the space of all solutions.

b) Now the statement follows, because it is true for $I^{(n)}_\alpha(t,\lambda)$, for $|\lambda|\gg 1$, where

$$I^{(n)}_\alpha(t,\lambda) = \widetilde{I}^{(n)}_\alpha(\lambda) + \sum_{\ell=1}^\infty (-1)^\ell\, S_\ell(t)\widetilde{I}^{(n+\ell)}_\alpha(\lambda).$$

□

2.4. **Mirror symmetry at higher genus.** A Frobenius manifold is called *semi-simple* if the multiplication has a semi-simple basis. The Frobenius manifold $(H, (\ ,\ ), \bullet_t, \phi_{01}, E)$ is isomorphic to the Frobenius manifold constructed from the mirror model of $\mathbb{P}^1_{\mathbf{a}}$ [53, 55, 41], see Theorem 2. Using the mirror model, it is easy to see that $\bullet_t$ is semi-simple for generic t.

For any semi-simple Frobenius manifold, Givental introduced a higher genus reconstruction formula [32] using the symplectic loop space formalism [33]. Furthermore, he conjectured that the higher genus GW ancestor invariants are uniquely determined from its semi-simple quantum cohomology. Teleman [61] has proved this conjecture. Let us recall the construction.

2.4.1. *Canonical quantization.* Equip the space

$$\mathcal{H} := H(\!(z^{-1})\!)$$

of formal Laurent series in z^{-1} with coefficients in H with the following *symplectic form*:

$$\Omega(\phi_1(z),\phi_2(z)) := \mathrm{Res}_z\,(\phi_1(-z),\phi_2(z))\;, \qquad \phi_1(z),\phi_2(z)\in\mathcal{H}\,,$$

where, as before, $(,)$ denotes the residue pairing on H and the formal residue Res_z gives the coefficient in front of z^{-1}.

Let $\{\phi_i\}_{i\in\mathfrak{I}}$ and $\{\phi^i\}_{i\in\mathfrak{I}}$ be dual bases of H with respect to the residue pairing. Then

$$\Omega(\phi^i(-z)^{-\ell-1},\phi_j z^m) = \delta_{ij}\delta_{\ell m}\,.$$

Hence, a Darboux coordinate system is provided by the linear functions q^i_ℓ, $p_{\ell,i}$ on $\mathcal{H}$ given by:

$$q^i_\ell = \Omega(\phi^i(-z)^{-\ell-1},\cdot)\,, \qquad p_{\ell,i} = \Omega(\cdot,\phi_i z^\ell)\,.$$

In other words,

$$\phi(z) = \sum_{\ell=0}^{\infty}\sum_{i\in\mathfrak{I}} q_\ell^i(\phi(z))\phi_i z^\ell + \sum_{\ell=0}^{\infty}\sum_{i\in\mathfrak{I}} p_{\ell,i}(\phi(z))\phi^i(-z)^{-\ell-1}\,, \qquad \phi(z)\in\mathcal{H}\,.$$

The first of the above sums will be denoted by $\phi(z)_+$ and the second by $\phi(z)_-$.

The *quantization* of linear functions on $\mathcal{H}$ is given by the rules

$$\widehat{q}_\ell^i = \hbar^{-1/2}q_\ell^i\,, \qquad \widehat{p}_{\ell,i} = \hbar^{1/2}\frac{\partial}{\partial q_\ell^i}\,,$$

where the RHSs of the above definitions are operators acting on the Fock space

$$\mathbb{C}_\hbar[\![\mathbf{q}]\!] := \mathbb{C}_\hbar[\![q_0, q_1+\mathbf{1}, q_2, \cdots]\!], \quad \text{where} \quad \mathbb{C}_\hbar := \mathbb{C}(\!(\hbar)\!) \quad q_\ell := (q_\ell^i)_{i\in\mathfrak{I}}. \tag{33}$$

Every $\phi(z)\in\mathcal{H}$ gives rise to the linear function $\Omega(\phi(z),\cdot)$ on $\mathcal{H}$, so we can define the quantization $\widehat{\phi(z)}$. Explicitly,

$$(\phi_i z^\ell)\widehat{\ } = -\hbar^{1/2}\frac{\partial}{\partial q_\ell^i}, \quad (\phi^i(-z)^{-\ell-1})\widehat{\ } = \hbar^{-1/2}\,q_\ell^i. \tag{34}$$

The quantization also makes sense for $\phi(z)\in H[[z,z^{-1}]]$ if we interpret $\widehat{\phi(z)}$ as a formal differential operator in the variables q_ℓ^i with coefficients in $\mathbb{C}_\hbar$.

Lemma 8. *For all $\phi_1(z),\phi_2(z)\in\mathcal{H}$, we have $[\widehat{\phi_1(z)},\widehat{\phi_2(z)}] = \Omega(\phi_1(z),\phi_2(z))$.*

Proof. It is enough to check this for the basis vectors $\phi^i(-z)^{-\ell-1}$, $\phi_i z^\ell$, in which case it is true by definition. □

2.4.2. *Quantization of symplectic transformations.* It is known that both series $S_t(z)$ and $R_t(z)$ described in Sections 2.3.1 and 2.3.2 are symplectic transformations on $(\mathcal{H},\Omega)$. Moreover, they both have the form $e^{A(z)}$, where $A(z)$ is an infinitesimal symplectic transformation.

A linear operator $A(z)$ on $\mathcal{H} := H(\!(z^{-1})\!)$ is infinitesimal symplectic if and only if the map $\mathcal{H}\ni\phi(z)\mapsto A(\phi(z))\in\mathcal{H}$ is a Hamiltonian vector field with a Hamiltonian given by the quadratic function

$$h_A(\phi(z)) = \frac{1}{2}\Omega(A(\phi(z)),\phi(z)).$$

By definition, the *quantization* of $e^{A(z)}$ is given by the differential operator $e^{\widehat{h}_A}$, where the quadratic Hamiltonians are quantized according to the following rules:

$$(p_{\ell,i}p_{m,j})\widehat{\ } = \hbar\frac{\partial^2}{\partial q_\ell^i\partial q_m^j}\,, \quad (p_{\ell,i}q_m^j)\widehat{\ } = (q_m^j p_{\ell,i})\widehat{\ } = q_m^j\frac{\partial}{\partial q_\ell^i}\,, \quad (q_\ell^i q_m^j)\widehat{\ } = \frac{1}{\hbar}q_\ell^i q_m^j\,.$$

In the case of the orbifold $\mathbb{P}^1_{\mathbf{a}}$, the Frobenius manifold is semi-simple at a generic point $t\in H$. Teleman's higher genus reconstruction theorem [61] implies that the total ancestor potential defined in (10) can be identified with Givental's higher genus reconstruction formula [33]

$$\mathcal{A}_t(\hbar;\mathbf{q}(z)) = \widehat{\Psi_t R_t \Psi_t^{-1}}\prod_{j=1}^{N+1}\mathcal{D}_{\rm pt}(\hbar\Delta_j;{}^j\mathbf{q}(z)\sqrt{\Delta_j})\in\mathbb{C}_{\hbar,Q}[[q_0,q_1+\mathbf{1},q_2\ldots]] \tag{35}$$

and the total descendant potentials defined in (2) can be identified with

$$\mathcal{D}_{\mathbf{a}}(\hbar;\mathbf{q}(z)) = e^{F^{(1)}(t)}\widehat{S}_t^{-1}\mathcal{A}_t(\hbar;\mathbf{q}(z))\,, \tag{36}$$

where ${}^j\mathbf{q}(z) := \sum_{\ell=0}^{\infty}{}^jq_\ell z^\ell$ and the coefficents ${}^jq_\ell$ are defined by

$$\sum_{j=1}^{N+1}{}^jq_\ell\Psi(e_j) = \sum_{i\in\mathfrak{I}}q_\ell^i\phi_i\,.$$

Recall that $\mathcal{D}_{\mathrm{pt}}$ is the total descendant potential of a point and the factor

$$F^{(1)}(t) = \sum_{d,n=0}^{\infty} \frac{Q^d}{n!} \langle t, \ldots, t \rangle_{1,n,d}$$

is the genus-1 primary (i.e. no descendants) potential. Let us examine more carefully the quantized action of the operators in formula (35) and (36).

2.4.3. *The action of the asymptotical operator.* The operator $\widehat{U_t/z}$ is known to annihilate the Witten–Kontsevich tau-function. Therefore, $e^{\widehat{U_t/z}}$ is redundant and it can be dropped from the formula. The action of the operator $\widehat{R}_t$ on formal functions, whenever it makes sense, is given as follows.

Lemma 9 (Givental [32]). *We have*

$$\widehat{R}_t^{-1} F(\mathbf{q}) = \Big(e^{\frac{\hbar}{2} V_t(\partial,\partial)} F(\mathbf{q}) \Big)\Big|_{\mathbf{q} \mapsto R_t \mathbf{q}} ,$$

where $V_t(\partial,\partial)$ is the quadratic differential operator

$$V_t(\partial,\partial) = \sum_{\ell,m=0}^{\infty} \sum_{i,j\in\mathfrak{I}} (\phi^i, V_{\ell m}(t)\phi^j)\, \frac{\partial^2}{\partial q_\ell^i \partial q_m^j}$$

whose coefficients $V_{\ell m}(t)$ are given by

$$\sum_{\ell,m=0}^{\infty} V_{\ell m}(t) z^\ell w^m = \frac{1 - R_t(z)({}^T R_t(w))}{z+w}$$

and ${}^T R_t(w)$ denotes the transpose of $R_t(w)$ with respect to the Poincare pairing.

The substitution $\mathbf{q} \mapsto R_t \mathbf{q}$ can be written more explicitly as follows:

$$q_0 \mapsto q_0, \quad q_1 \mapsto R_1(t) q_0 + q_1, \quad q_2 \mapsto R_2(t) q_0 + R_1(t) q_1 + q_2\,, \ldots .$$

The above substitution is not a well-defined operation on the space of formal functions. This complication, however, is offset by a certain property of the Witten–Kontsevich tau-function, which we now explain. By definition, an *asymptotical function* is a formal function of the type:

$$\mathcal{A}(\mathbf{q}) = \exp\Big(\sum_{g=0}^{\infty} \hbar^{g-1} F^{(g)}(\mathbf{q}) \Big) .$$

Such a function is called *tame* if the following $(3g-3+n)$-*jet constraints* are satisfied:

$$\left. \frac{\partial^n F^{(g)}}{\partial q_{k_1}^{i_1} \cdots \partial q_{k_n}^{i_n}} \right|_{\mathbf{q}=0} = 0 \quad \text{if} \quad k_1 + \cdots + k_n > 3g - 3 + n\,.$$

The Witten–Kontsevich tau-function (up to the shift $q_1 \mapsto q_1 + 1$) is tame for dimensional reasons: $\dim \overline{\mathcal{M}}_{g,n} = 3g-3+n$. The total ancestor potential $\mathcal{A}_t$ is also tame, as it can be seen from its geometric definition (cf. [33]) or by using the fact that the action of the operator $\widehat{R}_t$ on tame functions is well defined and it preserves the tameness property ([31]).

2.4.4. *The action of the calibration.* The quantized symplectic transformation $\widehat{S}_t^{-1}$ acts on formal functions as follows.

Lemma 10 (Givental [32])**.** *We have*

$$\widehat{S}_t^{-1} F(\mathbf{q}) = e^{\frac{1}{2\hbar} W_t(\mathbf{q},\mathbf{q})} F\big((S_t \mathbf{q})_+\big)\,, \tag{37}$$

where $W_t(\mathbf{q},\mathbf{q})$ *is the quadratic form*

$$W_t(\mathbf{q},\mathbf{q}) = \sum_{\ell,m=0}^{\infty} (W_{\ell m}(t) q_m, q_\ell)$$

whose coefficients are defined by

$$\sum_{\ell,m=0}^{\infty} W_{\ell m}(t) z^{-\ell} w^{-m} = \frac{{}^T S_t(z) S_t(w) - 1}{z^{-1} + w^{-1}}\,.$$

The subscript $+$ in (37) means truncation of all negative powers of z, i.e., in $F(\mathbf{q})$ we have to substitute (cf. (24)):

$$q_\ell \mapsto q_\ell + S_1(t) q_{\ell+1} + S_2(t) q_{\ell+2} + \cdots\,, \quad \ell = 0, 1, 2, \dots\,.$$

This operation is well-defined on the space of formal power series.

3. Γ-integral structures and the root system

If X is a compact complex orbifold, then using the K-ring $K(X)$ of orbifold vector bundles on X and a certain Γ-modification of the Chern character map, Iritani has introduced an integral lattice in the Chen-Ruan cohomology group $H_{\mathrm{CR}}(X;\mathbb{C})$ (see [39] and also [47]). If X has semi-simple quantum cohomology, then it is expected that X has a LG mirror model and it is natural to conjecture that Iritani's embedding of the K-theoretic lattice coincide with the image of the Milnor lattice via an appropriate period map. In our case, when $X = \mathbb{P}^1_{\mathbf{a}}$, we prove the above conjecture by using the same argument as in [39], where the toric case was proved. Moreover, we obtain an explicit identification of the set of vanishing cycles with a certain K-theoretic affine root system.

3.1. Iritani's integral structure and mirror symmetry.

Let us recall Iritani's construction in the most general case when X is a compact complex orbifold. Let IX be the *inertia orbifold* of X, i.e., as a groupoid the points of IX are

$$(IX)_0 = \{(x,g) \mid x \in X_0,\ g \in \mathrm{Aut}(x)\}$$

while the arrows from (x',g') to (x'',g'') consists of all arrows $g \in X_1$ from x' to x'', s.t., $g''\circ g = g\circ g'$. It is known that IX is an orbifold consisting of several connected components X_v, $v \in T := \pi_0(|IX|)$. Following Iritani, we define a linear map

$$\Psi : K(X) \to H^*(IX;\mathbb{C}) = \bigoplus_{v\in T} H^*(X_v;\mathbb{C})$$

via

$$\Psi(V) = (2\pi)^{-\dim_{\mathbb{C}} X/2}\, \widehat{\Gamma}(X) \cup (2\pi\sqrt{-1})^{\deg}\, \mathrm{inv}^*\, \widetilde{\mathrm{ch}}(V). \tag{38}$$

Here $\cup$ is the usual cup product in $H^*(IX;\mathbb{C})$. Let us recall the notation. The linear operator

$$\deg : H^*(IX;\mathbb{C}) \to H^*(IX;\mathbb{C})$$

is defined by $\deg(\phi) = r\phi$ if $\phi \in H^{2r}(IX;\mathbb{C})$. The involution inv : $IX \to IX$ inverts all arrows while on the points it acts as $(x,g) \mapsto (x,g^{-1})$. If V is an orbifold vector bundle, then we have an eigenbasis decomposition

$$\mathrm{pr}^*(V) = \bigoplus_{v\in T} V_v = \bigoplus_{v\in T}\bigoplus_{0\le f<1} V_{v,f},$$

where pr : $IX \to X$ is the forgetful map $(x,g)\mapsto x$ and $V_{v,f}$ is the subbundle of $V_v := \mathrm{pr}^*(V)|_{X_v}$ whose fiber over a point $(x,g) \in (IX)_0$ is the eigenspace of g corresponding to the eigenvalue $e^{2\pi\sqrt{-1}f}$. Let us denote by $\delta_{v,f,j}$ $(1 \le j \le l_{v,f} := \mathrm{rk}(V_{v,f}))$ the Chern roots of $V_{v,f}$, then the Chern character and the Γ-class of V are defined by

$$\widetilde{\mathrm{ch}}(V) = \sum_{v\in T}\sum_{0\le f<1} e^{2\pi\sqrt{-1}f}\,\mathrm{ch}(V_{v,f}),$$

$$\widehat{\Gamma}(V) = \sum_{v\in T}\prod_{0\le f<1}\prod_{j=1}^{l_{v,f}} \Gamma(1-f+\delta_{v,f,j}),$$

where the value of the Γ-function $\Gamma(1-f+y)$ at $y=\delta_{v,f,j}$ is obtained by first expanding in Taylor's series at $y=0$ and then formally substituting $y=\delta_{v,f,j}$. By definition $\widehat{\Gamma}(X) := \widehat{\Gamma}(TX)$.

3.1.1. *The Γ-conjecture for the Milnor lattice.* We denote by $H_{\mathrm{CR}}(X;\mathbb{C})$ the vector space $H^*(IX;\mathbb{C})$ equipped with the Chen–Ruan cup product $\cup_{\mathrm{CR}}$. We define a shift function $\iota : T \to \mathbb{Q}$ by

$$\iota(v) = \sum_{0\le f<1} f\,\dim_{\mathbb{C}}(TX)_{v,f}.$$

The Chen–Ruan product is graded homogeneous with respect to the following grading

$$\deg_{\mathrm{CR}}(\phi) = (r+\iota(v))\phi, \quad \phi\in H^{2r}(X_v;\mathbb{C}).$$

The vector space $H^*(IX;\mathbb{C})$ is equipped with a Poincaré pairing, i.e.

$$(\phi_1,\phi_2) = \int_{IX} \phi_1 \cup \mathrm{inv}^*(\phi_2).$$

This pairing turns both algebras $H^*(IX;\mathbb{C})$ and $H_{\mathrm{CR}}(X;\mathbb{C})$ into Frobenius algebras. Let us point out also that by using the Kawasaki Riemann–Roch formula we can also prove that the map Ψ is compatible (up to a sign) with the natural pairing on $K(X)$ and the Poincaré pairing

$$\chi(V_1\otimes V_2^\vee) = (e^{\pi\sqrt{-1}\theta_X} e^{\pi\sqrt{-1}\rho_X}\Psi(V_1), \Psi(V_2)), \tag{39}$$

where $\rho_X = c_1(TX)\cup_{\mathrm{CR}}$ and θ_X is the *Hodge grading operator* of X,

$$\theta_X = \frac{1}{2}\dim_{\mathbb{C}} X - \deg_{\mathrm{CR}}.$$

On the other hand, if X has a LG-mirror model, then we can define the calibrated periods $\widetilde{I}_\alpha^{(-\ell)}(\lambda)$ in the same way as in formulas (16) and (28). The main motivation for the above construction is the following conjecture, which is motivated by Iritani's mirror symmetry theorem in [39]. To simplify the formulation we set all Novikov variables to be 1. Using the divisor equation one can recover easily the Novikov variables.

Conjecture 11 (Γ-Conjecture for the Milnor lattice)**.** *Given an integral cycle α there exists a class $V_\alpha \in K(X)$ in the K-theory of vector bundles, s.t. for all $\ell \gg 0$,*

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\lambda s}\widetilde{I}_\alpha^{(-\ell)}(\lambda)d\lambda = s^{-\theta_X-\ell-1/2}s^{-\rho_X}\Psi(V_\alpha).$$

The conjecture can be refined even further, by saying that if α is a vanishing cycle then V_α can be represented by an exceptional object in the derived category $\mathcal{D}^b(X)$ and that the monodromy transformations of α correspond to certain mutation operations in $\mathcal{D}^b(X)$. See [27] for more discussions. Now we describe Conjecture 11 in the case of $\mathbb{P}^1_{\mathbf{a}}$.

3.1.2. *The K-ring of* $\mathbb{P}^1_{\mathbf{a}}$. Let $\mathbf{a} = (a_1, a_2, a_3)$ be a triple of non-negative integers and put $X = \mathbb{P}^1_{\mathbf{a}}$. The orbifold $\mathbb{P}^1_{\mathbf{a}}$ can be constructed as follows. Put

$$G = \{t = (t_1, t_2, t_3) \in (\mathbb{C}^*)^3 \mid t_1^{a_1} = t_2^{a_2} = t_3^{a_3}\}.$$

We have

$$\mathbb{P}^1_{\mathbf{a}} = [Y_{\mathbf{a}}/G], \quad Y_{\mathbf{a}} = \{y = (y_1, y_2, y_3) \in \mathbb{C}^3 \setminus \{0\} \mid y_1^{a_1} + y_2^{a_2} + y_3^{a_3} = 0\},$$

where the quotient is taken in the category of orbifolds, i.e., it should be viewed as an orbifold groupoid. The K-ring of orbifold vector bundles on $\mathbb{P}^1_{\mathbf{a}}$ can be presented as a quotient of the polynomial ring $\mathbb{C}[L_1, L_2, L_3]$ by the following relations

$$L = L_1^{a_1} = L_2^{a_2} = L_3^{a_3}, \quad (1 - L_k)(1 - L_{k'}) = 0 \ (1 \le k < k' \le 3).$$

Here L is the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ under the natural map $\mathbb{P}^1_{\mathbf{a}} \to \mathbb{P}^1$, and the product is given by tensor product of vector bundles. The orbifold vector bundle L_k is the trivial line bundle $Y_{\mathbf{a}} \times \mathbb{C}$ equipped with the following G-action

$$G \times L_k \to L_k, \quad (t, y, v) \mapsto (ty, t_k v).$$

It is easy to see that the K-ring is generated by L_1, L_2, L_3, L. The first set of relations follows from the definition of G. To see the remaining ones, note that the coordinate function y_k on $Y_{\mathbf{a}}$ gives rise to a section of L_k. The Koszul complex associated with the sections $(y_k, y_{k'})$ is G-equivariaint and it gives rise to the exact sequence

$$0 \to L_k^\vee \otimes L_{k'}^\vee \to L_k^\vee \oplus L_{k'}^\vee \to \mathcal{O}_{\mathbb{P}^1_{\mathbf{a}}} \to 0.$$

This proves that $(1 - L_k)(1 - L_{k'}) = 0$.

3.1.3. *The image of* $K(X)$. The connected components of $\mathbb{P}^1_{\mathbf{a}}$ are indexed by $\{(0,0)\} \cup \mathfrak{I}_{\rm tw}$. Let us denote by $P = c_1(L)$, then $c_1(L_k) = P/a_k$. By the adjunction formula $TX = L_1 L_2 L_3 L^{-1}$, we get

$$c_1(TX) = \chi\, P, \quad \chi = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1.$$

Furthermore, note that

$$(L_k)_{k',p,f} = \begin{cases} 0 & \text{if } k \neq k' \text{ and } f \neq 0 \\ 0 & \text{if } k = k' \text{ and } f \neq p/a_{k'} \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

From here we get that the eigenspace decomposition of TX is

$$(TX)_{k,p,f} = \begin{cases} TX & \text{if } k = 0, \quad f = 0, \\ \mathbb{C} & \text{if } k \neq 0, \quad f = p/a_k, \\ 0 & \text{otherwise .} \end{cases}$$

Recall that for $i = (k,p) \in \mathfrak{I}_{\rm tw}$, $d_i = d_{k,p} = 1 - p/a_k$, we get the following formulas

$$\begin{aligned} \widehat{\Gamma}(X) &= \Gamma(1 + \chi P) + \sum_{i \in \mathfrak{I}_{\rm tw}} \Gamma(d_i)\phi_i, \\ \widetilde{\rm ch}(L_k^m) &= \mathbf{1} + \frac{m}{a_k}\, P + \sum_{(j,p) \in \mathfrak{I}_{\rm tw}} \zeta_j^{mp\delta_{k,j}}\, \phi_{j,p}, \quad \zeta_j := e^{2\pi\sqrt{-1}/a_j}. \end{aligned}$$

Let us point out that in the above formulas $\mathbf{1}, P \in H^*(X_{0,0})$, while $\phi_{k,p} \in H^0(X_{k,p})$ is the standard generator for the twisted sector. Note that the unit of the algebra $(H^*(IX;\mathbb{C}),\cup)$ is

$$\widetilde{\mathrm{ch}}(\mathcal{O}) = \mathbf{1} + \sum_{i\in\mathfrak{I}_{\mathrm{tw}}} \phi_i.$$

Finally, since

$$(2\pi\sqrt{-1})^{\deg}\mathrm{inv}^*\widetilde{\mathrm{ch}}(L_k^m) = \mathbf{1} + \frac{2\pi\sqrt{-1}m}{a_k}\, P + \sum_{(j,p)\in\mathfrak{I}_{\mathrm{tw}}} \zeta_j^{-mp\delta_{k,j}}\phi_{j,p},$$

we get the following formula

$$(2\pi)^{1/2}\Psi(L_k^m) = \mathbf{1} + \left(-\gamma\chi + \frac{2\pi\sqrt{-1}m}{a_k}\right) P + \sum_{(j,p)\in\mathfrak{I}_{\mathrm{tw}}} \frac{\Gamma(d_{j,p})}{\zeta_j^{mp\delta_{k,j}}}\phi_{j,p}. \tag{40}$$

3.2. **Γ-conjecture for Fano orbifold curves.** Now we give a proof[6] of the Γ-conjecture for the Milnor lattice for $\mathbb{P}^1_{\mathbf{a}}$. The proof is obtained by applying Iritani's argument of the proof of [40, Theorem 4.11] and [40, Theorem 5.7] and relies on the Γ-conjecture for the Milnor lattice for the Fano toric orbifold (proven in [39])

$$Y := \mathbb{P}^2_{\mathbf{a}} = [(\mathbb{C}^3 \setminus \{0\})/G]$$

and the explicit formulas for the J-functions of $X := \mathbb{P}^1_{\mathbf{a}}$ and Y. Note that X is a suborbifold of Y.

Remark 12. *There is a natural map $p : \mathbb{P}^2_{\mathbf{a}} \to \mathbb{P}^2$. The above description of $X = \mathbb{P}^1_{\mathbf{a}}$ realizes X as the locus of zero of a section of the line bundle $p^*\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2_{\mathbf{a}}$. Applying the recipe of constructing mirrors of complete intersections in* [30]*, we obtain $f_{\mathbf{a}}$ as the mirror of X.*

Notice that the line bundles L_k are restrictions of line bundles on Y and the K-ring of Y is the quotient of the polynomial ring $\mathbb{C}[L_1, L_2, L_3]$ by the following relations

$$L = L_1^{a_1} = L_2^{a_2} = L_3^{a_3}, \quad (1 - L_1)(1 - L_2)(1 - L_3) = 0.$$

Put $L = p^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $P = c_1(L)$. We have isomorphisms

$$\mathbb{Q} \cong H_2(X;\mathbb{Q}) \cong H_2(Y;\mathbb{Q}), \quad d \mapsto d[\mathbb{P}^1_a]$$

and

$$H^2(Y;\mathbb{Q}) \cong H^2(X;\mathbb{Q}) \cong \mathbb{Q}, \quad \alpha \mapsto \langle \alpha, [\mathbb{P}^1_a]\rangle.$$

The J-function of an orbifold X used by Iritani is

$$J_X(\tau, z) = L(\tau, z)^{-1}\mathbf{1},$$

where $\tau \in H_{\mathrm{CR}}(X)$,

$$L(\tau, z) := S_\tau(-z)e^{-P\log Q/z}$$

and S is the calibration operator (25). Note that this definition differs from Givental's one by a sign and by the exponential factor.

[6]Note that $\mathbb{P}^1_{\mathbf{a}}$ is not covered by results in [39, 40].

3.2.1. *Combinatorics of the inertia orbifolds.* The orbifold Y is toric. We describe its stacky fan as follows. Put

$$b_1 = (a_1, 0), \quad b_2 = (0, a_2), \quad b_3 = (-a_3, -a_3) \in \mathbb{Z}^2.$$

The fan of Y is

$$\Sigma \cong \{\emptyset, \{k\}, \{k, k'\} \mid 1 \le k, k' \le 3\}$$

where each set I on the RHS determines a cone in $\mathbb{R}^2$ spanned by b_k, $k \in I$. Note that Σ is the fan for $\mathbb{P}^2$. The fan map for Y sends the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{Z}^3$ to $\mathbb{Z}^2$ by

$$\mathbb{Z}^3 \to \mathbb{Z}^2, \quad e_k \mapsto b_k.$$

The connected components of IY are parametrized by

$$\mathrm{Box}(\Sigma) = \{(c_1, c_2, c_3) \mid 0 \le c_k < 1, \quad \sum_k c_k b_k \in \mathbb{Z}^2 \cap \sigma \text{ for some cones } \sigma \in \Sigma\},$$

where $c \in \mathrm{Box}(\Sigma)$ determines the twisted sector

$$Y_c = [\{y \in \mathbb{C}^3 \mid y_k = 0 \text{ if } c_k \ne 0\}/G]$$

which has a generic stabilizer given by the cyclic subgroup of G generated by

$$(e^{2\pi\sqrt{-1}c_1}, e^{2\pi\sqrt{-1}c_2}, e^{2\pi\sqrt{-1}c_3}) \in G.$$

The inertia orbifold IX is a suborbifold of IY and the twisted sectors of IX are parametrized by those $c \in \mathrm{Box}(\Sigma)$ for which $\dim(Y_c) > 0$, i.e., at most one component of c is non-zero.

3.2.2. *The J-function of Y.* Let $\mathbf{1}_c \in H^0(Y_c)$ be the dual of the fundamental class for $c \in \mathrm{Box}(\Sigma)$ and

$$\tau = \tau_1 \mathbf{1}_{(1/a_1,0,0)} + \tau_2 \mathbf{1}_{(0,1/a_2,0)} + \tau_3 \mathbf{1}_{(0,0,1/a_3)}.$$

According to the mirror theorem of [15], the J-function $J_Y(\tau, z)$ depending on τ is equal to the S-extended I-function [15, Definition 28] with $S = \{(1,0), (0,1), (-1,-1)\}$. This gives

$$J_Y(\tau, z) = e^{P \log Q/z} \Big(\sum_{d=0}^{\infty} \sum_{n_1,n_2,n_3=0}^{\infty} \frac{Q^d}{z^{\deg_Y(Q^d)}} \frac{t^n}{n! z^{\deg_Y(t^n)}} J^Y_{d,n}(\tau, z) \Big),$$

where we introduced homogeneous parameters $t = (t_1, t_2, t_3)$, whose dependance on τ and Q can be determined from the expansion $J_Y = 1 + \tau/z + \cdots$, the degrees of Q and t are

$$\deg_Y(Q^d) := \int_d c_1(TY) = d\Big(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\Big), \quad \deg_Y(t_k) := \deg_Y(\tau_k) = 1 - 1/a_k.$$

Finally, we denoted $n = (n_1, n_2, n_3)$ and we used the standard multi-index notations

$$t^n = t_1^{n_1} t_2^{n_2} t_3^{n_3}, \quad n! = n_1! n_2! n_3!.$$

In order to define the component $J^Y_{d,n}$ let us define $m_k \in \mathbb{Z}$ and $c_k \in \mathbb{Q}$ by

$$\frac{n_k - d}{a_k} = -m_k + c_k, \quad 0 \le c_k < 1.$$

Then we have

$$J^Y_{d,n}(\tau, z) = \frac{\mathbf{1}_c}{z^{\deg_Y(\mathbf{1}_c)}} \prod_{k=1}^{3} \frac{\Gamma(1 - c_k + (P/a_k) z^{-1})}{\Gamma(1 - c_k + m_k + (P/a_k) z^{-1})},$$

where if $c \notin \mathrm{Box}(\Sigma)$ then we set $\mathbf{1}_c = 0$. In other words we sum over all (d, n), s.t., at least one of the numbers c_k is 0.

3.2.3. *The J-function of X.* Since $p^*\mathcal{O}_{\mathbb{P}^2}(1)$ is a convex line bundle in the sense of [14, Example B], the J-function of $\mathbb{P}^1_{\mathbf{a}}$ can be computed from that of $\mathbb{P}^2_{\mathbf{a}}$ using the quantum Lefschetz theorem of [62] and [14].

Using the embedding $j: IX \to IY$ we restrict τ and $\mathbf{1}_c$ to $H^*(IX)$. Slightly abusing the notation we use the same notation for the restrictions. Note that now $\mathbf{1}_c = 0$ if c has more than one non-zero component. The formula for J_X has the same form

$$J_X(\tau, z) = e^{P\log Q/z}\Big(\sum_{d=0}^{\infty}\sum_{n_1,n_2,n_3=0}^{\infty} \frac{Q^d}{z^{\deg_X(Q^d)}}\,\frac{t^n}{n!z^{\deg_X(t^n)}}\,J^X_{d,n}(\tau,z)\Big),$$

where

$$J^X_{d,n}(\tau,z) = \frac{\mathbf{1}_c}{z^{\deg_X(\mathbf{1}_c)}}\,\frac{\Gamma(1+d+Pz^{-1})}{\Gamma(1+Pz^{-1})}\prod_{k=1}^{3}\frac{\Gamma(1-c_k+(P/a_k)z^{-1})}{\Gamma(1-c_k+m_k+(P/a_k)z^{-1})}$$

Note that the grading takes the form

$$\deg_X(Q^d) := \int_d c_1(TX) = d\Big(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}-1\Big)$$

while the degrees of t and $\mathbf{1}_c$ do not change, because the restriction map preserves the grading.

3.2.4. *The Galois action.* The Picard group $\mathrm{Pic}(X)$ of isomorphism classes of (topological) orbifold line bundles on X can be presented as a quotient

$$\mathbb{Z}^3 \to \mathrm{Pic}(X), \quad (r_1,r_2,r_3)\mapsto L_1^{r_1}L_2^{r_2}L_3^{r_3}$$

with kernel given by the relations

$$a_1e_1 = a_2e_2 = a_3e_3,$$

where $\{e_1,e_2,e_3\}$ is the standard basis of $\mathbb{Z}^3$. The group $\mathrm{Pic}(X)$ acts naturally on the Milnor fibration via

$$\nu\cdot(x,t) = (\nu\cdot x, \nu\cdot t),\quad \nu=(r_1,r_2,r_3)\in\mathrm{Pic}(X),$$

where

$$(\nu\cdot x)_k \;=\; e^{2\pi\sqrt{-1}r_k/a_k}x_k,$$

and the action on the remaining components is defined in such a way that

$$F(\nu\cdot x,\nu\cdot t) = F(x,t),$$

i.e.,

$$\begin{aligned}
(\nu\cdot t)_{k,p} &= e^{-2\pi\sqrt{-1}r_kp/a_k}\,t_{k,p},\quad 1\le k\le 3,\quad 1\le p\le a_k-1,\\
(\nu\cdot t)_{02} &= t_{02}+2\pi\sqrt{-1}\sum_{k=1}^{3}\frac{r_k}{a_k},\\
(\nu\cdot t)_{01} &= t_{01}.
\end{aligned}$$

Let us fix some $(t,\lambda)\in M\times\mathbb{C}$ with λ sufficiently large, then for every $\nu=(r_1,r_2,r_3)$ we can construct a path from (t,λ) to $(\nu\cdot t,\lambda)$ as follows. Using the above formulas we let $c\in\mathbb{R}^3$ act on M. As c varies along the straight segment from 0 to $(r_1,r_2,r_3)\in\mathbb{Z}^3\subset\mathbb{R}^3$ we get a path in M connecting t and $\nu\cdot t$. The parallel transport along this path with respect to the Gauss-Manin connection gives an identification $H_2(X_{\nu\cdot t,\lambda};\mathbb{Z})\cong H_2(X_{t,\lambda};\mathbb{Z})$. Combined with the $\mathrm{Pic}(X)$-action on $\mathbb{C}^3$ we get an action

$$\mathrm{Pic}(X)\times H_2(X_{t,\lambda};\mathbb{Z})\to H_2(X_{t,\lambda};\mathbb{Z}),\quad (\nu,\alpha)\mapsto\nu(\alpha).$$

Following Iritani, we refer to the above action as Galois action of $\mathrm{Pic}(X)$ on the Milnor lattice.

Lemma 13. *If the* Γ*-conjecture for the Milnor lattice is true for some cycle* α *and* $V_\alpha \in K(X)$ *is the corresponding* K*-theoretic vector bundle, then the conjecture is true for all* $\nu(\alpha)$*,* $\nu = (r_1, r_2, r_3) \in \mathrm{Pic}(X)$*. Moreover,*

$$V_{\nu(\alpha)} = V_\alpha \otimes L_\nu, \quad L_\nu = L_1^{r_1} L_2^{r_2} L_3^{r_3}.$$

Proof. Using the vector space decomposition

$$H_{\mathrm{CR}}(X) = H^*(X) \bigoplus \left(\bigoplus_{(k,p)\in \mathfrak{I}_{\mathrm{tw}}} H_{\mathrm{CR}}^{p/a_k}(X) \right),$$

we define a linear operator

$$\theta_\nu : H_{\mathrm{CR}}(X) \to H_{\mathrm{CR}}(X), \quad \theta_\nu = \sum_{k=1}^{3} \sum_{p=1}^{a_k-1} \frac{r_k p}{a_k} \mathrm{pr}_{k,p},$$

where $\mathrm{pr}_{k,p}$ is the projection onto the subspace $H_{\mathrm{CR}}^{p/a_k}(X)$. By changing the variables $y = \nu \cdot x$ in the period integrals we get

$$I^{(\ell)}_{\nu(\alpha)}(t,\lambda) = e^{-2\pi\sqrt{-1}\theta_\nu} I^{(\ell)}_\alpha(\nu^{-1}\cdot t, \lambda), \quad \forall \ell \in \mathbb{Z}.$$

On the other hand the calibration operator satisfies

$$S_{\nu^{-1}(t)}(z) = e^{2\pi\sqrt{-1}\theta_\nu} S_t(z) e^{-2\pi\sqrt{-1}\theta_\nu} e^{-2\pi\sqrt{-1}c_1(L_\nu)/z},$$

which can be seen easily by using that if the correlator

$$\langle \alpha_1\psi_1^{k_1}, \dots, \alpha_n\psi_n^{k_n}\rangle_{0,n,d}$$

is non-zero then, since we have at least one stable map $f : C \to X$, we have

$$\chi(f^*L_\nu) = \int_d c_1(L_\nu) - \sum_{j=1}^{n} \theta_\nu(\alpha_j) \in \mathbb{Z}.$$

Since by definition

$$I^{(\ell)}_\alpha(t,\lambda) = S_t(-\partial_\lambda^{-1}) \widetilde{I}^{\ell}_\alpha(\lambda),$$

the above formulas imply that

$$\widetilde{I}^{(\ell)}_{\nu(\alpha)}(\lambda) = e^{-2\pi\sqrt{-1}\theta_\nu} e^{2\pi\sqrt{-1}c_1(L_\nu)\partial_\lambda} \widetilde{I}^{\ell}_\alpha(\lambda).$$

In particular, after taking a Laplace transform, we get

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\lambda s} \widetilde{I}^{(-\ell)}_{\nu(\alpha)}(\lambda) = e^{-2\pi\sqrt{-1}\theta_\nu} e^{2\pi\sqrt{-1}c_1(L_\nu)s} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\lambda s} \widetilde{I}^{(-\ell)}_\alpha(\lambda).$$

On the other hand, using the definition of Ψ we get

$$\Psi(V\otimes L_\nu) = e^{-2\pi\sqrt{-1}\theta_\nu} e^{2\pi\sqrt{-1}c_1(L_\nu)} \Psi(V).$$

It remains only to notice that $s^{-\theta}P = (Ps)s^{-\theta}$ and that θ_ν commutes with both θ and the Chen-Ruan product multiplication operators. □

The Milnor lattice is known to be unimodular with respect to the K-theoretic bilinear form

$$(\ ,\ ) : K(X)\otimes_{\mathbb{Z}} K(X) \to \mathbb{Z}, \quad (L_1, L_2) = \chi(L_1 \otimes L_2^\vee)$$

(see [40], Section 2). The above Lemma implies that it is enough to prove that the Γ-conjecture holds for the structure sheaf. Indeed, if this is true, then since $K(X)$ is generated by $\mathrm{Pic}(X)$, the Γ conjecture correspondence will embed $K(X)$ into a sublattice of the Milnor lattice. Since both lattices are unimodular, they must coincide.

3.2.5. *The central charge.* Iritani's Γ-conjecture for the Milnor lattice looks different since he works with Lefschetz thimbles. Nevertheless, our formulation is completely equivalent. The reason is the following. Let us take a Lefschetz thimble $\mathcal{A}$ corresponding to a vanishing cycle α, i.e., for fixed $(t,z)\in M\times\mathbb{C}^*$ we fix a path C in $\mathbb{C}$ from u_j to ∞, s.t., $\mathrm{Re}(\lambda/z)>0$ for all $\lambda\in C$ and the cycle $\alpha_{t,\lambda}$ vanishes when λ approaches u_j. In this way we can identify the Milnor lattice with a lattice of Lefschetz thimbles.

We claim that

$$L(t,z)^{-1}\int_{u_j}^{\infty} e^{-\lambda/z} I^{(-\ell)}_\alpha(t,\lambda)d\lambda = e^{z^{-1}P\log Q}\int_0^\infty e^{-\lambda/z}\widetilde{I}^{(-\ell)}_\alpha(\lambda)d\lambda,$$

where $L(t,z)=S_t(-z)e^{-z^{-1}P\log Q}$. Indeed, one can check easily using the quantum differential equations that the LHS is independent of t and Q. On the other hand we have

$$L(t,z)=1-z^{-1}P\log Q+\cdots,\quad u_j=0+\cdots\quad\text{and}\quad I^{(-\ell)}_\alpha(t,\lambda)=\widetilde{I}^{(-\ell)}_\alpha(\lambda)+\cdots,$$

where the dots stand for terms that vanish at $t=Q=0$. So modulo terms that vanish at $t=Q=0$ the LHS coincides with the RHS. Our claim follows.

We define the *central charge* of $V_\alpha\in K(X)$ by

$$Z^{(0)}_X(V_\alpha)(t,z):=\Big(L(t,z)z^\theta z^\rho\Psi(V_\alpha),\mathbf{1}\Big).$$

Since we will use the result of Iritani, let us clarify the relation between our notations. In Iritani's notation, the central charge is defined to be

$$Z^{(n)}_X(V)(t,z):=(2\pi z)^{n/2}(2\pi\sqrt{-1})^{-n}\Big(L(t,z)z^\theta z^\rho\Psi(V_\alpha),\mathbf{1}\Big),\quad n=\dim_{\mathbb{C}}(X).$$

For the LG models studied in [39] the Γ-conjecture for the central charge is stated as

$$(2\pi\sqrt{-1})^{-n}\int_{\mathcal{A}} e^{-F(x,t)/z}\omega = Z^{(n)}_X(V_\alpha)(t,z).$$

As we see from the LG model that we use in general one should choose n to be the number of variables in the LG potentials. For the LG models in [39] the number of variables coincides with the dimension of the orbifold, so this difference does not matter.

The identity in the Γ-conjecture for the Milnor lattice is equivalent to

$$\frac{1}{\sqrt{2\pi}}\int_{u_j}^\infty e^{-\lambda/z}I^{(-\ell)}_\alpha(t,\lambda)d\lambda = L(t,z)z^{\theta+\ell+1/2}z^\rho\Psi(V_\alpha). \tag{41}$$

The number ℓ must be chosen sufficiently large. We will see that in our case $\ell=1$ works. In general ℓ can be chosen, s.t., the number of variables in the LG potential is $2\ell+1$. Recalling the definition of the period integrals, we transform the LHS into

$$(-zd_M)(2\pi)^{-3/2}\int_{\mathcal{A}} e^{-F(x,t)/z}\omega,$$

where d_M is the de Rham differential on M. In particular, since the Poincaré pairing of the RHS with $\mathbf{1}$ corresponds to contracting the LHS with ∂_{01} we get

$$(2\pi z)^{-3/2}\int_{\mathcal{A}} e^{-F(x,t)/z}\omega = Z^{(0)}_X(V_\alpha)(t,z). \tag{42}$$

In order to prove the Γ-conjecture for the Milnor lattice it is enough to prove that if $V=\mathcal{O}_X$, then we can find an integral cycle $\mathcal{A}$, s.t., the identity (42) holds for all parameters t of the form

$$t=t_{1,1}\mathbf{1}_{1,1}+t_{2,1}\mathbf{1}_{2,1}+t_{3,1}\mathbf{1}_{3,1}.$$

One can check that the partial derivatives of the LHS and the RHS of (42) with respect to any other parameter $t_{k,p}$ can be expressed in terms the same differential operator involving only $t_{k,1}$, $1\le k\le 3$. Therefore if (42) holds for all t of the above form, then (41) holds also for all such t.

As it was explained above if the identity (41) holds for a single point $t = t_0$ then it holds for all t and it is equivalent to the identity in our Γ-conjecture. In other words, the Γ-conjecture holds for the structure sheaf. Recalling Lemma 13 we get that the Γ-conjecture holds for the entire Milnor lattice.

3.2.6. *The central charge as an oscillatory integral.* It remains only to prove (42). Following Iritani, it is convenient to rewrite the RHS of (42) in terms of the so-called H-function

$$H_X^{(0)}(t,z) = \widetilde{\mathrm{ch}}(H_K^{(0)}(t,z)),$$

where the $K(X)$-valued function

$$H_K^{(0)} : M \times \mathbb{C}^* \to K(X)$$

is defined by the equation

$$\mathbf{1} = L(t,z) z^\theta z^\rho \Psi(H_K^{(0)}(t,z)).$$

For the central charge $Z_X^{(0)}(V)$ we have

$$(L(t,z)z^\theta z^\rho \Psi(V), L(t,-z)(-z)^\theta(-z)^\rho \Psi(H_K^{(0)}(t,-z))) = (\Psi(V), e^{\pi\sqrt{-1}\theta} e^{\pi\sqrt{-1}\rho}\Psi(H_K^{(0)})),$$

where we define $(-1)^R := (e^{\pi\sqrt{-1}R})$ for all linear operators R. Recalling (39) and the Kawasaki Riemann–Roch formula we get

$$Z_X^{(0)}(V) = \chi(H_K^{(0)} \otimes V^\vee) = \int_{IX} H_X^{(0)}(t,-z) \cup \widetilde{\mathrm{ch}}(V^\vee) \cup \widetilde{\mathrm{Td}}(TX),$$

where in the notation of Section 3.1 the Todd class of an orbifold vector bundle is a multiplicative characteristic class defined by

$$\widetilde{\mathrm{Td}}(V) = \sum_{v\in T} \prod_{j=1}^{l_{v,0}} \frac{\delta_{v,0,j}}{1-e^{-\delta_{v,0,j}}} \prod_{0<f<1} \prod_{j=1}^{l_{v,f}} \frac{1}{1-e^{-2\pi\sqrt{-1}f} e^{-\delta_{v,f,j}}}.$$

The proof of formula (42) requires a simple lemma. The main ingredient is a slight modification of the usual Laplace transform defined as follows. Let $f(t,Q;z)$ be any function, then we define

$$\mathfrak{L}(f)(t,Q;z) = \int_0^\infty e^{-\eta} f(t,-\eta z Q;z) d\eta.$$

The integral is convergent if for example f depends polynomially on Q and $\log Q$, which is the case that we have. Note that this Laplace transform does not commute with the involution $z \mapsto -z$.

Lemma 14. *Let $j : IX \to IY$ be the natural embedding, then*

$$j_* H_X^{(0)}(t,Q;-z) = (-z/2\pi)^{1/2}\, \widetilde{e}(L) \cup \mathfrak{L}(H_Y^{(0)})(t,Q;-z),$$

where $\widetilde{e}(L) = \sum_{v\in T} e(L_v)$ is the orbifold Euler class of L.

Proof. Since $j_*(j^*\alpha) = \widetilde{e}(L) \cup \alpha$ for every $\alpha \in H^*(IY)$, it is enough to prove that

$$\mathfrak{L}\left(j^* H_Y^{(0)}\right)(t,Q;-z) = (-z/2\pi)^{-1/2}\, H_X^{(0)}(t,Q;-z). \tag{43}$$

We have

$$(2\pi)^{-1/2}\widehat{\Gamma}(X) \cup (2\pi\sqrt{-1})^{\deg} \mathrm{inv}^* H_X^{(0)}(t,Q,z) = z^{-\rho_X} z^{-\theta_X} J_X(t,Q;z)$$

and

$$(2\pi)^{-1}\widehat{\Gamma}(Y) \cup (2\pi\sqrt{-1})^{\deg} \mathrm{inv}^* H_Y^{(0)}(t,Q,z) = z^{-\rho_Y} z^{-\theta_Y} J_Y(t,Q;z)$$

On the other hand, using the explicit formulas for the J-functions it is easy to check that

$$\mathfrak{L}(j^* J_Y)(t,Q;-z) = (-z)^{-P/z}\Gamma(1-P/z) \cup J_X(t,Q;-z).$$

In order to prove formula (43), it is enough only to recall the following identities

$$j^*(-z)^{-\rho_Y}(-z)^{-\theta_Y} = (-z)^{-P-1/2}(-z)^{-\rho_X}(-z)^{-\theta_X} j^*,$$

$$(-z)^{-\rho_X}(-z)^{-\theta_X}(-z)^{-P/z}\Gamma(1-P/z) = (-z)^P\Gamma(1+P)(-z)^{-\rho_X}(-z)^{-\theta_X},$$

and

$$j^*\widehat{\Gamma}(Y) = \widehat{\Gamma}(L)\widehat{\Gamma}(X).$$

□

Lemma 14 yields the following relation between the central charges of sheaves on X and Y. Let $V \in K(Y)$, then

$$Z_X^{(0)}(j^*V) = (-z/2\pi)^{1/2}\mathfrak{L}\left(Z_Y^{(0)}(V - V\otimes L)\right).$$

In particular

$$Z_X^{(0)}(1) = (-z/2\pi)^{1/2}\mathfrak{L}\left(Z_Y^{(0)}(1-L)\right). \tag{44}$$

Theorem 15. *For a Fano orbifold curve $X = \mathbb{P}^1_{\mathbf{a}}$, given a class $L_k^m \in K(X)$ in the K-theory of vector bundles, there exists an integral cycle $\alpha_{k,m} \in \mathfrak{h}$, s.t. for all $\ell \gg 0$,*

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\lambda s}\widetilde{I}^{(-\ell)}_{\alpha_{k,m}}(\lambda)d\lambda = s^{-\theta_X-\ell-1/2}s^{-\rho_X}\Psi(L_k^m).$$

Proof. It is enough to prove (42). Let us look at the corresponding oscillatory integrals. Recall that the LG model of Y is given by the restriction of

$$F_{\mathbb{P}^2}(x,t) = \sum_{k=1}^3 (x_k^{a_k} + t_{k,1}x_k),$$

to the complex torus $x_1x_2x_3 = Q$, while the corresponding primitive form is

$$\omega_{\mathbb{P}^2} = \frac{dx_1dx_2dx_3}{d(x_1x_2x_3)}.$$

Let us assume now that z and Q are real numbers, s.t., $z>0$ and $Q<0$. Let $\mathcal{C}\subset\mathbb{C}^3$ be the chain

$$\mathcal{C} = \{x\in\mathbb{R}^3 \mid x_k \geq 0, k=1,2,3\}.$$

The oscillatory integral

$$(2\pi z)^{-3/2}\int_{\mathcal{C}} e^{-F(x,t)/z}\omega = (2\pi z)^{-3/2}(-1)^{1/2}\int_0^\infty e^{-\eta}\int_{\Gamma_{-zQ\eta}} e^{-F_{\mathbb{P}^2}(x,t)/z}\frac{dx_1dx_2dx_3}{d(x_1x_2x_3)}(-z)d\eta,$$

where we presented the chain $\mathcal{C}$ as a family of cycles

$$\Gamma_{-z\eta Q} = \{x\in\mathcal{C} \mid x_1x_2x_3 = -z\eta Q\}.$$

and used the Fubini theorem. The Γ-conjecture for Y was proved by Iritani [39]. Moreover, the real cycle $\Gamma_{-z\eta Q}$ corresponds to the structure sheaf $\mathcal{O}_Y$, so the above integral coincides with

$$(-1)^{3/2}z(2\pi z)^{-3/2}\int_0^\infty e^{-\eta}(2\pi z)Z_Y^{(0)}(1)(t,-z\eta Q;z)d\eta = (-1)^{3/2}(z/2\pi)^{1/2}\mathfrak{L}\left(Z_Y^{(0)}(1)\right).$$

Recalling the argument in Lemma 13 it is easy to see that the analytic continuation around $Q=0$ in clockwise direction of $\mathfrak{L}\left(Z_Y^{(0)}(1)\right)$ is $\mathfrak{L}\left(Z_Y^{(0)}(L)\right)$, therefore the cycle that we are looking for is $\widetilde{\mathcal{C}}-\mathcal{C}$, where $\widetilde{\mathcal{C}}$ is the chain obtained from $\mathcal{C}$ via the monodromy transformation around $Q=0$ in the clockwise direction. More precisely, $\widetilde{\mathcal{C}}$ is the family of cycles $\widetilde{\Gamma}_{-z\eta Q}$ obtained from $\Gamma_{-z\eta Q}$ by the monodromy transformation around $Q=0$. It remains only to notice that the boundaries of $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ are the same. Together with (44), this proves (42). □

3.3. **Affine root systems and vanishing cycles.** According to Theorem 15 (recall that we have to put $Q=1$) and formula (40), we have

$$\int_0^\infty e^{-\lambda s}\widetilde{I}^{(-\ell)}_{\alpha_{k,m}}(\lambda)d\lambda \;=\; \frac{\mathbf{1}}{s^{\ell+1}}+\Big(\frac{2\pi\sqrt{-1}m}{a_k}-\gamma\chi-\chi\log s\Big)\frac{P}{s^\ell}+\sum_{(j,p)\in\mathfrak{I}_{\rm tw}}\frac{\Gamma(d_j)}{e^{2\pi\sqrt{-1}mp\delta_{k,j}/a_j}}\frac{\phi_{j,p}}{s^{\ell+d_{j,p}}}.$$

where $d_{j,p}=1-p/a_j$, γ is the Euler's gamma constant defined by

$$\gamma=\lim_{m\to\infty}H_m-\ln m,\quad H_m:=1+\frac{1}{2}+\cdots+\frac{1}{m},$$

If $\ell\geq 1$, then we can recall the inverse Laplace transform and also the divisor equation (30) to get

$$\begin{aligned}\widetilde{I}^{(-\ell)}_{\alpha_{k,m}}(\lambda)=&\frac{\lambda^\ell}{\ell!}\mathbf{1}+\frac{\lambda^{\ell-1}}{(\ell-1)!}\left(\frac{2\pi\sqrt{-1}m}{a_k}+\chi(\log\lambda-C_{\ell-1})\right)P+\\ &+\sum_{(j,p)\in\mathfrak{I}_{\rm tw}}\frac{\lambda^{d_{j,p}+\ell-1}e^{2\pi\sqrt{-1}m\delta_{k,j}d_{j,p}}}{(d_{j,p}+\ell-1)\cdots(d_{j,p})}\phi_{j,p},\end{aligned}\tag{45}$$

where if $\ell=1$ we set $C_0:=\frac{1}{\chi}\log Q$ and if $\ell>1$ then $C_\ell=C_{\ell-1}+\frac{1}{\ell}$.

Proposition 16.

(1) *The set of vanishing cycles* $\Delta\subset\mathfrak{h}=H_2(X_{0,1};\mathbb{C})$ *is an affine root system of type* $X_N^{(1)}$, *where* $N=a_1+a_2+a_3-2$ *and*

$$X=\begin{cases}A & \textit{if } a_1=1,\\ D & \textit{if } a_1=a_2=2,\\ E & \textit{otherwise.}\end{cases}$$

(2) *There exists a basis of simple roots such that the classical monodromy* σ *is an affine Coxeter transformation.*

Part (1) of Proposition 16 is due to A. Takahashi (see [60]). The proof is based on a standard method developed by Gusein-Zade and A'Campo. We give a proof of Proposition 16 based on Iritani's integral structure.

We will be interested in the two maps from the sequence

$$\widetilde{I}^{(n)}(1):\mathfrak{h}\to H,\quad \alpha\mapsto\widetilde{I}^{(n)}_\alpha(1)\tag{46}$$

corresponding to $n=-1$ and $n=0$. According to Lemma 7 we have

$$\widetilde{I}^{(-1)}(1)=B_{01}\,(\mathbf{1}-(\log Q)P)+B_{02}\,P+\sum_{i\in\mathfrak{I}_{\rm tw}}B_i\,\phi_i,$$

which proves that the map for $n=-1$ is an isomorphism. Using $\widetilde{I}^{(-1)}(1)$ we equip H with an intersection pairing $(\cdot|\cdot)$, i.e.,

$$(\phi'|\phi''):=(\alpha'|\alpha''),\quad\text{for}\quad \phi'=\widetilde{I}^{(-1)}_{\alpha'}(1),\quad \phi''=\widetilde{I}^{(-1)}_{\alpha''}(1).$$

The period map (46) with $n=0$ has a 1-dimensional kernel because (using (32))

$$\widetilde{I}^{(0)}(1)=(1-\rho)^{-1}(\theta+1/2)\widetilde{I}^{(-1)}(1)=(1+\rho)(1-\deg_{\rm CR})\widetilde{I}^{(-1)}(1),$$

so the kernel is $\mathbb{C}\,P$. We denote the image of $\widetilde{I}^{(0)}(1)$ by $H^{(0)}$. Let us denote by $r:H\to H^{(0)}$ the map defined by $\widetilde{I}^{(0)}(1)=r\circ\widetilde{I}^{(-1)}(1)$, i.e.,

$$r(b)=(1+\rho)(1-\deg_{\rm CR})(b).$$

According to Saito's formula (23) the intersection pairing on H takes the form

$$(\phi'|\phi'')=(r(\phi'),(1-\rho)r(\phi'')),\quad \phi',\phi''\in H.\tag{47}$$

It follows that we can pushforward the intersection form to a non-degenerate bilinear pairing on $H^{(0)}$, which we denote again by $(\cdot|\cdot)$. More precisely we define

$$(\phi'|\phi'') = (\phi', (1-\rho)\phi''), \quad \phi', \phi'' \in H^{(0)}.$$

Let us denote by $\Delta^{(-1)} \subset H$ and $\Delta^{(0)} \subset H^{(0)}$ the images of the set of vanishing cycles, i.e.,

$$\Delta^{(-1)} = \{\widetilde{I}^{(-1)}_\alpha(1) \mid \alpha \in \Delta\}, \quad \Delta^{(0)} = \{\widetilde{I}^{(0)}_\alpha(1) \mid \alpha \in \Delta\}.$$

A straightforward computation with formula (47) implies

Lemma 17. *Consider $\alpha_{k,m}$ as in (45). Then the cycles $\alpha_{k,m}$ ($1 \le k \le 3$, $m \in \mathbb{Z}$) satisfy*

$$(\alpha_{k,m}|\alpha_{k,n}) = \begin{cases} 2 & \text{if } m = n (\text{mod } a_k), \\ 1 & \text{if } m \neq n (\text{mod } a_k), \end{cases}$$

and for $k \neq k'$

$$(\alpha_{k,m}|\alpha_{k',n}) = \begin{cases} 2 & \text{if } m = 0(\text{mod } a_k) \text{ and } n = 0(\text{mod } a_{k'}), \\ 0 & \text{if } m \neq 0(\text{mod } a_k) \text{ and } n \neq 0(\text{mod } a_{k'}), \\ 1 & \text{otherwise.} \end{cases}$$

3.3.1. *The toroidal cycle.* Let $\Gamma_\varepsilon \subset \mathbb{C}^3$ be the torus

$$\Gamma_\epsilon := \{|x_1| = |x_2| = 1, |x_3| = \varepsilon\}.$$

If ε is sufficiently large, Γ_ε does not intersect the Milnor fiber $X_{0,1}$. Hence we have a well-defined cycle

$$[\Gamma_\varepsilon] \in H_3(\mathbb{C}^3 \setminus X_{0,1}; \mathbb{Z}) \cong H_2(X_{0,1}; \mathbb{Z}),$$

where the isomorphism is given by the so called *tube mapping* (for more details see [35]). Let us denote by φ the image of $[\Gamma_\varepsilon]$ under the above isomorphism.

Proposition 18. *We have $I^{(-1)}_\varphi(t,\lambda) = 2\pi\sqrt{-1}\, P$.*

Proof. Increasing ε does not change the homology class $[\Gamma_\varepsilon]$, therefore by choosing $\varepsilon \gg 0$ we may arrange that Γ_ε does not intersect the Milnor fiber $X_{t,\lambda}$ for all (t,λ) sufficiently close to $(0,1)$. In particular, the cycle $\varphi_{t,\lambda}$ obtained from φ via a parallel transport with respect to the Gauss–Manin connection coincides with the image of $[\Gamma_\varepsilon]$ via the tube mapping. We have (c.f. [35])

$$I(t,\lambda,Q) := \int_{[\Gamma_\varepsilon]} \frac{\omega}{F(t,x)-\lambda} = 2\pi\sqrt{-1} \int_{\varphi_{t,\lambda}} \frac{\omega}{dF} = 2\pi\sqrt{-1}\,\partial_\lambda \int d^{-1}\omega. \tag{48}$$

Comparing with the definition (16) we get

$$I(t,\lambda,Q) = -(2\pi)^2\sqrt{-1}\,(I^{(-1)}_\varphi(t,\lambda), \mathbf{1}).$$

Using the differential equation (19), we get

$$(\lambda\partial_\lambda + E) I(t,\lambda,Q) = 0. \tag{49}$$

The integral $I(t,\lambda,Q)$ is analytic at $(t,\lambda,Q) = (0,0,0)$ because it has the form

$$\sqrt{-1} \int_{[\Gamma_\varepsilon]} \frac{dx_1 dx_2 dx_3}{Q e^{t_{02}}\,(G(t,x)-\lambda) - x_1x_2x_3}\,,$$

where $G(t,x)$ is a holomorphic function in t and x. However, equation (49) means that $I(t,\lambda,Q)$ is homogeneous of degree 0 and since the weights of all variables are positive, the integral must be a constant. In particular, we may set $t = Q = \lambda = 0$, which gives

$$I(t,\lambda,Q) = -\sqrt{-1} \int_{[\Gamma_\varepsilon]} \frac{dx_1dx_2dx_3}{x_1x_2x_3} = (2\pi)^3.$$

Note that equation (48) implies that $I^{(0)}_{\varphi}(t,\lambda)=0$. Recalling again the differential equation (19), we get

$$I^{(-1)}_{\varphi}(t,\lambda)=(I^{(-1)}_{\varphi}(t,\lambda),\mathbf{1})\,P=(2\pi)^{-2}\sqrt{-1}\,I(t,\lambda,Q)\,P=2\pi\sqrt{-1}\,P.$$

□

We immidiately have a corollary

Corollary 19. *The cycle φ corresponds to the skyscraper sheaf $\mathcal{O}_{\rm pt}:=L-\mathcal{O}$, i.e.,*

$$\delta:=\widetilde{I}^{(-1)}_{\alpha_{k,a_k}}(1)-\widetilde{I}^{(-1)}_{\alpha_{k,0}}(1)=2\pi\sqrt{-1}P.$$

Proof of Proposition 16 (1). The image of the Milnor lattice in H has a $\mathbb{Z}$-basis given by

$$\delta,\quad \gamma_b^{(-1)},\quad \gamma_i^{(-1)},\quad i\in\mathfrak{I}_{\rm tw},$$

where for $n=0$ or -1, and $i=(k,p)$, we get

$$\begin{aligned}\gamma_b^{(n)} &:= \widetilde{I}^{(n)}_{\alpha_{k,0}}(1)\quad (\text{corresponds to } \mathcal{O})\\ \gamma_{k,p}^{(n)} &:= \widetilde{I}^{(n)}_{\alpha_{k,-p}}(1)-\widetilde{I}^{(n)}_{\alpha_{k,-p+1}}(1).\end{aligned}$$

It is easy to check that the intersection diagram of the set of cycles $\gamma_b^{(-1)},\gamma_i^{(-1)}, i\in\mathfrak{I}_{\rm tw}$ is given by the Dynkin diagram on Figure 1. As usual, each node has self-intersection 2, each edge means that the intersection of the cycles corresponding to the nodes of the edge is -1, and no edge means that the intersection is 0. It follows that the intersection form of the Milnor lattice is semi-positive definite with 1 dimensional kernel. This is possible only if $\Delta^{(-1)}$ is an affine root system. □

In particular we get also that δ is a $\mathbb{Z}$-basis for the imaginary roots and that $\Delta^{(0)}=r(\Delta^{(-1)})$ is a finite root system.

3.3.2. *Splitting of the affine root system.* It is convenient to enumerate the roots $\gamma_b^{(n)},\gamma_i^{(n)}, i\in\mathfrak{I}_{\rm tw}$ also by $\gamma_j^{(n)} (1\le j\le N)$. The Dynkin diagram on Figure 1 is of type X_N, $X=ADE$. Let us denote by $\gamma_0^{(-1)}$ the *affine vertex*, i.e., the extra node that we have to attach to X_N in order to obtain the corresponding affine Dynkin diagram $X_N^{(1)}$.

Vectors $\gamma_j^{(0)}$, $1\le j\le N$, form a basis of simple roots of $\Delta^{(0)}$. Let $W^{(0)}$ be the reflection group generated by $\gamma_j^{(0)}$. It is well known that there exists a group embedding $W^{(0)}\to W$ which is induced by the map

$$s_j^{(0)}:=s_{\gamma_j^{(0)}}\mapsto s_j^{(-1)}:=s_{\gamma_j^{(-1)}},\quad 1\le j\le N$$

Given $\alpha\in\Delta^{(0)}$, let us define a *lift* $\tilde\alpha\in\Delta^{(-1)}$ as follows

$$\alpha=\sum_{j=1}^N b_j\gamma_j^{(0)}\quad\mapsto\quad \tilde\alpha:=\sum_{j=1}^N b_j\gamma_j^{(-1)}.$$

Then the root system $\Delta^{(-1)}$ coincides with the set

$$\left\{\tilde\alpha+n\,\delta\mid \alpha\in\Delta^{(0)},\ n\in\mathbb{Z}\right\},$$

where $\delta=\gamma_0^{(-1)}+\theta^{(-1)}$ and $\theta\in\Delta^{(0)}$ is the highest root with respect to the basis $\{\gamma_j^{(0)}\}_{j=1}^N$ (see [44]). Following Kac, we will refer to $n\,\delta$ ($n\in\mathbb{Z}$) as *imaginary roots*. Finally, let us denote by

$$\Lambda^{(-1)}:=H_2(X_{0,1};\mathbb{Z})$$

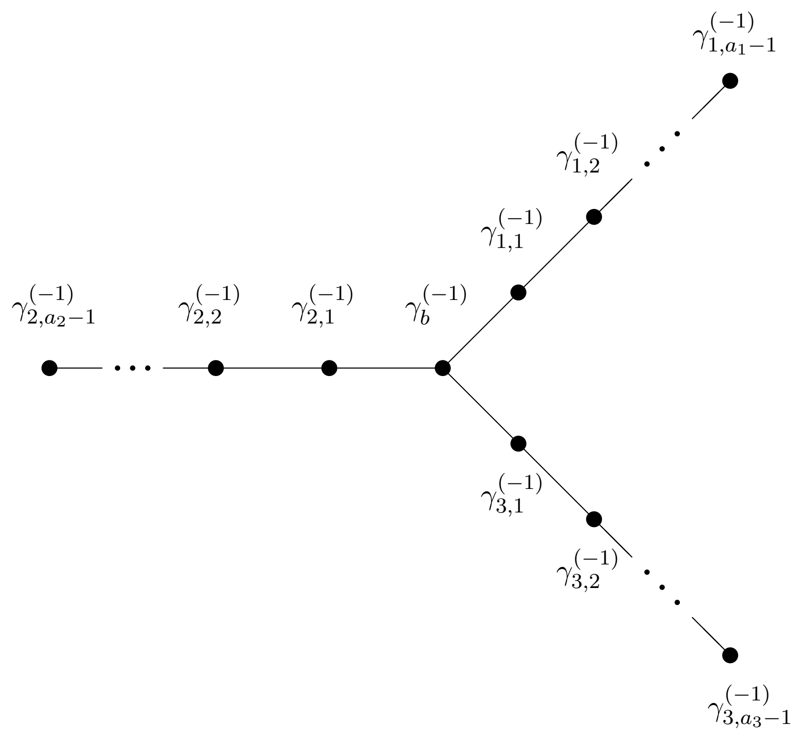


FIGURE 1. The branching node

the root lattice of $\Delta^{(-1)}$. Given $\alpha \in \Lambda^{(-1)}$ such that $|\alpha|^2 := (\alpha|\alpha) \neq 0$, recall that the reflection with respect to α is defined by

$$s_\alpha(x) = x - 2\frac{(\alpha|x)}{(\alpha|\alpha)}\,\alpha.$$

We also define the following translation:

$$T_\alpha(x) := s_{\alpha+\delta} s_\alpha(x) = x + 2\frac{(\alpha|x)}{(\alpha|\alpha)}\,\delta.$$

This definition induces a group embedding $T : \Lambda^{(0)} \to W$. Recall that $w\, s_\alpha\, w^{-1} = s_{w(\alpha)}$ for all $w \in W$ and $\alpha \in \Lambda^{(-1)}$ such that $|\alpha|^2 \neq 0$. Therefore, $\Lambda^{(0)}$ is a normal subgroup of W and we have an isomorphism

$$W \cong \Lambda^{(0)} \rtimes W^{(0)}.$$

Let us emphasize that the above isomorphism is not canonical – it depends on the choice of a basis of simple roots of $\Delta^{(-1)}$.

3.3.3. *The Coxeter transformation.* Put $\sigma_b := \sigma_b^{(0)}$, where

$$\sigma_b^{(\ell)} = \prod_{k=1}^{3} \left(s^{(\ell)}_{k,a_k-1} \cdots s^{(\ell)}_{k,2} s^{(\ell)}_{k,1} \right) \quad \in \quad \mathrm{Aut}(\Delta^{(\ell)}), \quad \ell = -1, 0. \tag{50}$$

Note that while the order of the reflections that enter each factor of the above product is important, the order in which the 3 factors are arranged is irrelevant since they pairwise commute.

Proposition 20. *The automorphism of $\Delta^{(0)}$ induced by the action of the classical monodromy σ coincides with σ_b.*

Proof. The analytic continuation in λ around $\lambda = 0$ of the period $\widetilde{I}^{(-1)}_{\alpha_{k,m}}(\lambda)$ is equivalent to tensoring the line bundle L_k^m by $TX = L_1 L_2 L_3 L^{-1}$ and then taking the corresponding periods. Using

$$(L_k - 1)L_{k'} = L_k - 1, \quad \text{for} \quad k \neq k'$$

it is easy to check that

$$\begin{aligned} (L_k^{-m} - L_k^{-m+1})TX &= L_k^{-m+1} - L_k^{-m+2} \quad \forall m \in \mathbb{Z}, \\ TX^{-1} &= 1 + L_1^{-1} - 1 + L_2^{-1} - 1 + L_3^{-1} - 1 + L - 1, \\ (L_k^{-1} - 1)TX &= 1 - L_k^{-(a_k-1)} + 1 - L. \end{aligned}$$

According to the above remark, the classical monodromy acts as follows

$$\begin{aligned} \sigma(\gamma_{k,p}) &= \gamma_{k,p-1}, \quad (k,p) \in \mathfrak{I}_{\rm tw}, \\ \sigma^{-1}(\gamma_b) &= \gamma_b + \gamma_{1,1} + \gamma_{2,1} + \gamma_{3,1} + \delta, \\ \sigma(\gamma_{k,1}) &= -\gamma_{1,1} - \cdots - \gamma_{k,a_k-1} - \delta. \end{aligned}$$

It remains only to check that the action of $\sigma_b^{(-1)}$ is given by the same formulas modulo the imaginary root δ. □

It is known that up to a translation the affine Coxeter transformation coincides with σ_b (see [58, 59]), so part (2) of Proposition 16 follows from Proposition 20.

3.4. **Calibrated periods in terms of the finite root system.** Let $\omega_j^{(-1)} \in H^\vee$ $(0 \le j \le N)$ be the fundamental weights of $\Delta^{(-1)}$, i.e.,

$$\langle \omega_j^{(-1)}, \gamma_m^{(-1)} \rangle = \delta_{j,m}.$$

Using the intersection form we identify $H^{(0)}$ and its dual. Let $\omega_j^{(0)} \in H^{(0)}$ $(1 \le j \le N)$ be the fundamental weights of $\Delta^{(0)}$, i.e.,

$$(\omega_j^{(0)} | \gamma_m^{(0)}) = \delta_{j,m}, \quad 1 \le j, m \le N.$$

We have the following relation

$$\langle \omega_j^{(-1)}, \tilde{\alpha} \rangle = (\omega_j^{(0)} | r(\tilde{\alpha})) - k_j \langle \omega_0^{(-1)}, \tilde{\alpha} \rangle, \quad \tilde{\alpha} \in \Delta,$$

where k_j $(1 \le j \le N)$ are the Kac labels defined by

$$\delta = \gamma_0^{(-1)} + \sum_{j=1}^N k_j \gamma_j^{(-1)}.$$

In terms of the fundamental weights, the splitting of the affine root system from the previous section can be stated also as the following isomorphism

$$\Delta^{(-1)} \cong \Delta^{(0)} \times \mathbb{Z}, \quad \tilde{\alpha} \mapsto (\alpha, n), \quad \alpha = r(\tilde{\alpha}), \quad n = \langle \omega_0^{(-1)}, \tilde{\alpha} \rangle.$$

Lemma 21. *The following identity holds*

$$\omega_b^{(0)} + \sum_{m=1}^{a_k-1} (\zeta_k^{mp} - \zeta_k^{(m-1)p}) \omega_{k,m}^{(0)} = a_k \phi_{k,p^*}, \quad 1 \le k \le 3.$$

Proof. We have explicit formulas for the simple roots

$$\gamma_b^{(0)} = \mathbf{1} + \chi P + \sum_{k=1}^{3}\sum_{p=1}^{a_k-1}\phi_{k,p}$$

$$\gamma_{k,m}^{(0)} = \sum_{p=1}^{a_k-1}(\zeta_k^{mp}-\zeta_k^{(m-1)p})\phi_{k,p}.$$

It remains only to check that the LHS and the RHS have the same intersection pairing with the above set of simple roots of $\Delta^{(0)}$. □

Let κ be a positive constant whose value will be specified later on. Put

$$H_0 := H_{01} := H_{02} := (\kappa\chi)^{\frac{1}{2}}\,\omega_b^{(0)},\quad H_i := (\kappa\, a_i)^{\frac{1}{2}}\,\phi_i,\quad i\in\mathfrak{I}_{\rm tw}. \tag{51}$$

Note that according to Lemma 7 $\{H_i\}_{i\in\mathfrak{I}}$ is a σ_b-eigenbasis of $H^{(0)}$ with $\sigma_b(H_i)=e^{-2\pi\sqrt{-1}d_i}H_i$ in which the intersection form takes the form

$$(H_i|H_j) = \kappa\,\delta_{i,j^*},\quad i,j\in\mathfrak{I}, \tag{52}$$

where for $i=j=01\in\mathfrak{I}$ we used that $\omega_b^{(0)} = \chi^{-1}\mathbf{1}+P$. Finally, put

$$\rho_b = -\sum_{(k,p)\in\mathfrak{I}_{\rm tw}}\frac{1}{a_k}\omega_{k,p}^{(0)}.$$

Proposition 22. *Let $\tilde\alpha=(\alpha,n)\in\Delta^{(0)}\times\mathbb{Z}\cong\Delta$, be a vanishing cycle, then the corresponding calibrated periods are given by the following formulas:*

$$\widetilde{I}^{(\ell)}_{\tilde\alpha}(\lambda) = (-1)^\ell\ell!(\alpha|\omega_b^{(0)})\chi\lambda^{-\ell-1}P + \sum_{i\in\mathfrak{I}_{\rm tw}}(\alpha|H_{i^*})(d_i-1)\cdots(d_i-\ell)\lambda^{d_i-\ell-1}\sqrt{a_i/\kappa}\phi_i,$$

$$\widetilde{I}^{(0)}_{\tilde\alpha}(\lambda) = (\alpha|\omega_b^{(0)})\mathbf{1} + (\alpha|\omega_b^{(0)})\chi\lambda^{-1}P + \sum_{i\in\mathfrak{I}_{\rm tw}}(\alpha|H_{i^*})\lambda^{d_i-1}\sqrt{a_i/\kappa}\phi_i,$$

$$\widetilde{I}^{(-1-\ell)}_{\tilde\alpha}(\lambda) = (\alpha|\omega_b^{(0)})\frac{\lambda^{\ell+1}}{(\ell+1)!}\mathbf{1} + \Big((\alpha|\omega_b^{(0)})\chi\frac{\lambda^\ell}{\ell!}(\log\lambda - C_\ell) + 2\pi\sqrt{-1}(n+(\rho_b|\alpha))\,\frac{\lambda^\ell}{\ell!}\Big)P + \sum_{i\in\mathfrak{I}_{\rm tw}}(\alpha|H_{i^*})\sqrt{a_i/\kappa}\,\frac{\lambda^{d_i+\ell}}{d_i(d_i+1)\cdots(d_i+\ell)}\,\phi_i,$$

where $\ell\geq 1$ and C_ℓ $(\ell\geq 1)$ are constants defined recursively by

$$C_\ell = C_{\ell-1}+\frac{1}{\ell},\quad C_0=\frac{1}{\chi}\log Q.$$

Proof. It is enough to check the statement for the following basis of the Milnor lattice

$$\gamma_b^{(-1)},\quad \delta,\quad \gamma_i^{(-1)},\quad i=(k,p)\in\mathfrak{I}_{\rm tw}.$$

Let us check the last identity for $\tilde\alpha=\gamma_{k,p}^{(-1)}$, i.e., $n=0$ and $\alpha=\gamma_{k,p}^{(0)}$. Recalling the explicit formulas for $\widetilde{I}^{(-1-\ell)}_{\alpha_{k,p}}(\lambda)$ and $\gamma_{k,p}=\alpha_{k,-p}-\alpha_{k,-p+1}$, we get (recall that $d_{k,m}=1-m/a_k$)

$$\widetilde{I}^{(-1-\ell)}_{\gamma_{k,p}}(\lambda) = -\frac{2\pi\sqrt{-1}}{a_k}\frac{\lambda^\ell}{\ell!}P + \sum_{m=1}^{a_k-1}\frac{\zeta_k^{mp}-\zeta_i^{m(p-1)}}{(\ell+d_{k,m})\cdots(1+d_{k,m})}\lambda^{\ell+d_{k,m}}\phi_{k,m}.$$

On the other hand, by definition $H_{i*}\sqrt{a_i/\kappa}=a_i\phi_{i*}$, so the identity follows from Lemma 21. The remaining two cases are proved in the same way. □

4. ADE-Toda hierarchies

4.1. **Twisted realization of the affine Lie algebra.** Let $\mathfrak{g}^{(0)}$ be a simple Lie algebra of type ADE with an invariant bilinear form $(\;|\;)$, normalized in such a way that all roots have length $\sqrt{2}$. By definition, the affine Kac–Moody algebra corresponding to $\mathfrak{g}$ is the vector space

$$\mathfrak{g} := \mathfrak{g}^{(0)}[t,t^{-1}] \bigoplus \mathbb{C}\,K \bigoplus \mathbb{C}\,d$$

equipped with a Lie bracket defined by the following relations:

$$[X\;t^n, Y\;t^m] := [X,Y]\;t^{n+m} + n\delta_{n,-m}(X\,|\,Y)K,$$
$$[d, X\;t^n] := n(X\;t^n), \quad [K,\mathfrak{g}^{(0)}] := 0,$$

where $X, Y \in \mathfrak{g}^{(0)}$.

We fix a Cartan subalgebra $\mathfrak{h}^{(0)} \subset \mathfrak{g}^{(0)}$ and a basis γ_b, γ_i $(i \in \mathfrak{I})$ of simple roots, s.t., the corresponding Dynkin diagram has the standard shape with γ_b corresponding to the branching node. If the root system is of type A; then we choose any of the nodes to be a branching node and we have (at most) 2 instead of 3 branches. Let us define $\sigma_b := \sigma_b^{(0)}$ by formula (50).

Let $\Delta^{(0)} \subset \mathfrak{h}^{(0)}$ be the root system of $\mathfrak{g}^{(0)}$, i.e.,

$$\mathfrak{g}^{(0)} = \bigoplus_{\alpha\in\Delta^{(0)}} \mathfrak{g}^{(0)}_\alpha.$$

The Lie algebra $\mathfrak{g}^{(0)}$ can be constructed in terms of the root system via the so-called *Frenkel–Kac* construction [26]. Let $\Lambda^{(0)} \subset \mathfrak{h}^{(0)}$ be the root lattice. There exists a bimultiplicative function

$$\epsilon : \Lambda^{(0)} \times \Lambda^{(0)} \to \{\pm 1\}$$

satisfying

$$\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)}, \quad \epsilon(\alpha,\alpha) = (-1)^{|\alpha|^2/2},$$

where $|\alpha|^2 := (\alpha|\alpha)$. The map $(\alpha,\beta) \mapsto \varepsilon(\sigma_b(\alpha),\sigma_b(\beta))$ is another bimultiplicative function satisfying the above properties. It is known that all bi-multiplicative functions of the above form are equivalent (see [45], Corollary 5.5). Hence there exists a function $\upsilon : \Lambda^{(0)} \to \{\pm 1\}$ such that

$$\upsilon(\alpha)\upsilon(\beta)\varepsilon(\alpha,\beta) = \upsilon(\alpha+\beta)\varepsilon(\sigma_b(\alpha),\sigma_b(\beta)). \tag{53}$$

There exists a set of root vectors

$$A_\alpha \in \mathfrak{g}^{(0)}_\alpha \tag{54}$$

such that

$$[A_\alpha, A_{-\alpha}] = \epsilon(\alpha,-\alpha)\alpha$$
$$[A_\alpha, A_\beta] = \epsilon(\alpha,\beta)A_{\alpha+\beta}, \quad \text{if} \quad (\alpha|\beta) = -1$$
$$[A_\alpha, A_\beta] = 0, \quad \text{if} \quad (\alpha|\beta) \geq 0.$$

We can extend σ_b to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$ as follows

$$\sigma_b(A_\alpha) = \upsilon(\alpha)^{-1}\,A_{\sigma_b(\alpha)}, \quad \alpha \in \Delta^{(0)}.$$

Let us denote by κ the order of the extended automorphism $\sigma_b : \mathfrak{g}^{(0)} \to \mathfrak{g}^{(0)}$. Clearly we have $\kappa = |\sigma_b|$ or $2|\sigma_b|$. Since $(\;|\;)$ is both $\mathfrak{g}^{(0)}$-invariant (with respect to the adjoint representation) and $W^{(0)}$-invariant, we have

$$(A_\alpha|A_{-\alpha}) := \epsilon(\alpha,-\alpha), \quad (A_\alpha|A_\beta) := (A_\alpha|H) = 0, \quad \forall\beta \neq -\alpha,\; H \in \mathfrak{h}^{(0)}.$$

Put $\eta = e^{2\pi\sqrt{-1}/\kappa}$. We extend the action of σ_b to the affine Lie algebra $\mathfrak{g}$ by

$$\sigma_b \cdot (X \otimes t^n) = \sigma_b(X) \otimes (\eta^{-1}t)^n, \quad \sigma_b \cdot K = K, \quad \sigma_b \cdot d = d.$$

Let

$$\mathfrak{g}^{\sigma_b} \subset \mathfrak{g}$$

be the Lie subalgebra of σ_b-fixed points. According to Kac (see [44], Theorem 8.6.), $\mathfrak{g}^{\sigma_b} \cong \mathfrak{g}$. Let us recall the isomorphism. The fixed points subspace $(\mathfrak{g}^{(0)})^{\sigma_b}$ contains a Cartan subalgebra $\widetilde{\mathfrak{h}}^{(0)}$. We have a corresponding decomposition into root subspaces

$$\mathfrak{g}^{(0)} = \widetilde{\mathfrak{h}}^{(0)} \bigoplus \left(\bigoplus_{\widetilde{\alpha} \in \widetilde{\Delta}^{(0)}} \mathfrak{g}^{(0)}_{\widetilde{\alpha}} \right),$$

where $\widetilde{\Delta}^{(0)} \subset \widetilde{\mathfrak{h}}^{(0)}$ are the corresponding roots. Note that since the root subspaces are 1 dimensional, they must be eigen-subspaces of σ_b. Therefore, by choosing a set of simple roots $\widetilde{\alpha}_j$, $j = 1, 2, \ldots, N$ in $\widetilde{\Delta}^{(0)}$ we can uniquely define an integral vector $s = (s_1, \ldots, s_N)$, $0 \leq s_j < \kappa$ such that the eigenvalue of the eigensubspace $\mathfrak{g}^{(0)}_{\widetilde{\alpha}_j}$ is η^{s_j}. Put

$$\rho_s : \widetilde{\mathfrak{h}}^{(0)} \to \widetilde{\mathfrak{h}}^{(0)}, \quad \rho_s = \sum_{j=1}^{N} s_j \widetilde{\omega}_j,$$

where $\widetilde{\omega}_j \in \widetilde{\mathfrak{h}}^{(0)}$ $(1 \leq j \leq N)$ are the fundamental weights corresponding to the simple roots $\widetilde{\alpha}_j$ $(1 \leq j \leq N)$, i.e., $(\widetilde{\omega}_j | \widetilde{\alpha}_{j'}) = \delta_{j,j'}$. The isomorphism

$$\Phi : \mathfrak{g} \longrightarrow \mathfrak{g}^{\sigma_b}$$

is defined as follows

$$\Phi(Xt^n) = t^{n\kappa + \mathrm{ad}_{\rho_s}} X + \delta_{n,0}\, (\rho_s | X) K \tag{55}$$

$$\Phi(K) = \kappa\, K$$

$$\Phi(d) = \kappa^{-1} \Big(d - \rho_s - \frac{1}{2} (\rho_s | \rho_s)\, K \Big), \tag{56}$$

where

$$t^{\mathrm{ad}_{\rho_s}} X = \exp\Big(\log t\ \mathrm{ad}_{\rho_s} \Big) X.$$

Note that the RHS is single-valued in t and σ_b-invariant in X, because

$$\exp\Big(2\pi\sqrt{-1} \mathrm{ad}_{\rho_s/\kappa} \Big) = \sigma_b.$$

Finally, we make a remark on κ. There is no a canonical way to extend σ_b to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$. Therefore, the value of κ depends on our choice of the cocycle $\epsilon(\alpha, \beta)$ and the corresponding sign-function $\upsilon(\alpha)$. We will see however that replacing κ by $m\kappa$, where m is a positive integer, does not change the HQEs, so we may assume that κ is a sufficiently large integer, s.t., $\sigma_b^{\kappa} = 1$. For the sake of completeness, let us fix an extension that seems natural for our purposes. Put $\omega_{k,0} = \omega_b$ and $\omega_{k,a_k} = 0$ and define

$$\mathrm{SF}(\alpha, \beta) = \sum_{k=1}^{3} \sum_{p=0}^{a_k - 1} (\omega_{k,p} | \alpha)(\omega_{k,p} - \omega_{k,p+1} | \beta). \tag{57}$$

Since $\mathrm{SF}(\alpha, \beta) + \mathrm{SF}(\beta, \alpha) = (\alpha|\beta)$, the bi-multiplicative function $\epsilon(\ ,\ ) = (-1)^{\mathrm{SF}(\ ,\ )}$ is an acceptable choice for the Frenkel–Kac construction. Note that

$$\upsilon(\alpha) = (-1)^{\sum_{k=1}^{3} (\omega_b|\alpha)(\omega_{k,1}|\alpha)} \tag{58}$$

satisfies formula (53), so we get an explicit formula for an extension of σ_b to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$. Moreover, since

$$\prod_{m=1}^{|\sigma_b|} \upsilon(\sigma_b^m(\alpha)) = (-1)^{\chi|\sigma_b|},$$

we get that $\kappa = |\sigma_b|$ if $\chi|\sigma_b|$ is even and $\kappa = 2|\sigma_b|$ if $\chi|\sigma_b|$ is odd. Notice that $|\sigma_b| = \mathrm{lcm}(a_1, a_2, a_3)$, the least common multiple of a_1, a_2, a_3.

Remark 23. *The notation* SF *is motivated from the notion of a Seifert form in singularity theory (cf.* [3, 4]*). We do not claim that* (57) *is a Seifert form, although it would be interesting to investigate whether definition* (57) *can be interpreted as a linking number between* α *and* β*.*

4.2. **The Kac–Peterson construction.** Following [46], we would like to recall the realization of the basic level 1 representation of the affine Lie algebra $\mathfrak{g}$ corresponding to the automorphism σ_b. The idea is to construct a representation of the Lie algebra $\mathfrak{g}^{\sigma_b}$ on a Fock space, which induces via the isomorphism Φ the basic level-1 representation.

Fix a σ_b-eigenbasis $\{H_i\}_{i\in\{0\}\sqcup\mathfrak{I}_{\rm tw}}$ of $\mathfrak{h}^{(0)}$. It is convenient to define $H_{01} := H_{02} := H_0$ and to assume that the basis is normalized so that $(H_i|H_{j^*}) = \kappa\, \delta_{i,j}$ (compare with (52)). Put

$$m_{01} := 0, \quad m_{02} := \kappa, \quad m_i := d_{i^*}\, \kappa, \quad i \in \mathfrak{I}_{\rm tw},$$

so that $e^{-2\pi\sqrt{-1}\, d_i} = \eta^{m_i}$ is the eigenvalue corresponding to the eigen vector H_i. The elements

$$H_{i,\ell} := H_i t^{m_i+\ell\kappa}\, (i \in \mathfrak{I}, \ell \in \mathbb{Z})$$

generate a *Heisenberg* Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}^{\sigma_b}$, i.e., the following commutation relations hold

$$[H_{i,\ell}, H_{j,m}] = (m_i + \ell\kappa)\, \delta_{i,j^*}\, \delta_{\ell+m,-1}\, \kappa\, K.$$

Let us also fix a $\mathbb{C}$-linear basis of $\mathfrak{s}$

$$H_0 := H_{01}, \quad H_{i,\ell}, \quad H_{i^*,-\ell-1}, \quad K, \quad (i,\ell) \in I_+, \tag{59}$$

where the index set is defined by

$$I_+ = \{(i,\ell) \mid i \in \mathfrak{I} \setminus \{(01)\}, \ell \in \mathbb{Z}_{\geq 0}\}. \tag{60}$$

Let $\mathfrak{S}$ be the subgroup of the affine Kac–Moody Lie group generated by the lifts of the following loops:

$$h_{\alpha,\beta} = \exp\Big(\alpha \log t^\kappa + 2\pi\sqrt{-1}\ \beta\Big), \tag{61}$$

where $\alpha, \beta \in \mathfrak{h}^{(0)}$ are such that

$$\sigma_b(\alpha) = \alpha, \quad \sigma_b(\beta) - \beta + \alpha \in \Lambda^{(0)}.$$

Let us point out that under the analytical continuation around $t = 0$, the loop $h_{\alpha,\beta}$ gains the factor $e^{2\pi\sqrt{-1}\kappa\alpha}$. The latter must be 1 because

$$\kappa\alpha = (\alpha + \sigma_b(\beta) - \beta) + \sigma_b(\alpha + (\sigma_b(\beta) - \beta)) + \cdots + \sigma_b^{\kappa-1}(\alpha + (\sigma_b(\beta) - \beta)) \in \Lambda^{(0)}.$$

It follows that $h_{\alpha,\beta}$ is single-valued and σ_b-invariant, i.e., it defines an element of the affine Kac–Moody loop group acting on $\mathfrak{g}^{\sigma_b}$ by conjugation. The main result of Kac and Peterson [46] is the following: *the basic representation of* $\mathfrak{g}^{\sigma_b}$ *remains irreducible when restricted to the pair* $(\mathfrak{s}, \mathfrak{S})$.

Let us recall the construction of the representation. Let us denote by

$$\pi_0 : \mathfrak{h}^{(0)} \to \mathfrak{h}_0^{(0)} \text{ and } \pi_* : \mathfrak{h}^{(0)} \to (\mathfrak{h}_0^{(0)})^\perp$$

the orthogonal projections of $\mathfrak{h}^{(0)}$ onto $\mathfrak{h}_0^{(0)} := \mathbb{C}\, H_0$ and $(\mathfrak{h}_0^{(0)})^\perp$ respectively. Given $x \in \mathfrak{h}^{(0)}$, let

$$x_0 := \pi_0(x), \quad x_* := \pi_*(x).$$

Let $\mathfrak{s}_- \subset \mathfrak{s}$ be the Lie subalgebra of $\mathfrak{s}$ spanned by the vectors $H_{i^*,-\ell-1}, (i,\ell) \in I_+$. The basic representation can be realized on the following vector space:

$$V_x = S^*(\mathfrak{s}_-) \otimes \mathbb{C}[e^{\omega}] e^{x\omega}, \tag{62}$$

where x is a complex number and $\omega := \pi_0(\gamma_b)$. The first factor of the tensor product in (62) is the symmetric algebra on $\mathfrak{s}_-$, and the second one is isomorphic to the group algebra of the lattice $\pi_0(\Lambda^{(0)}) = \mathbb{Z}\,\pi_0(\gamma_b)$. We will refer to $|0\rangle := 1 \otimes e^{x\omega}$ as the *vacuum vector*. Slightly abusing the notation we define the operator

$$\partial_\omega := \frac{\partial}{\partial \omega} - x,$$

acting on V_x, so that $\partial_\omega \,|0\rangle = 0$.

Put

$$X_\alpha(\zeta) = \sum_{n\in\mathbb{Z}} A_{\alpha,n}\,\zeta^{-n} = \frac{1}{\kappa}\sum_{m=1}^{\kappa}\sum_{n\in\mathbb{Z}} \eta^{-nm}(\sigma_b^m(A_\alpha)t^n)\zeta^{-n}, \quad \alpha\in\Delta^{(0)},$$

where A_α appears in (54). Let $E^*_\alpha(\zeta)$ be the *vertex operator*

$$E^*_\alpha(\zeta) = \exp\Big(\sum_{(i,\ell)\in I_+} (\alpha|H_i) H_{i^*,-\ell-1}\frac{\zeta^{m_i+\ell\kappa}}{m_i+\ell\kappa}\Big)\exp\Big(\sum_{(i,\ell)\in I_+} (\alpha|H_{i^*}) H_{i,\ell}\frac{\zeta^{-m_i-\ell\kappa}}{-m_i-\ell\kappa}\Big). \tag{63}$$

Lemma 24. *There are operators C_α, $\alpha\in\Delta^{(0)}$, independent of ζ, that commute with all basis vectors* (59) *of $\mathfrak{s}$ different from H_0, such that*

$$X_\alpha(\zeta) = X^0_\alpha(\zeta)E^*_\alpha(\zeta),$$

where

$$X^0_\alpha(\zeta) = \zeta^{\kappa\,|\alpha_0|^2/2} C_\alpha \zeta^{\kappa\,\alpha_0}, \quad \alpha_0 := \pi_0(\alpha). \tag{64}$$

Proof. After a direct computation we get

$$[H_{i,\ell}, X_\alpha(\zeta)] = (\alpha|H_i)\zeta^{m_i+\ell\kappa}X_\alpha(\zeta).$$

It follows that $X_\alpha(\zeta) = X^0_\alpha(\zeta)E^*_\alpha(\zeta)$, where $X^0_\alpha(\zeta)$ is an operator commuting with all $H_{i,\ell}\neq H_0$.

After a direct computation we get the following commutation relations:

$$\begin{aligned} h_{\alpha,\beta}\ (-d)\ h^{-1}_{\alpha,\beta} &= -d + \kappa\,\alpha + \frac{1}{2}|\alpha|^2\,\kappa^2 K, \\ h_{\alpha,\beta}\ A_{\gamma,n}\ h^{-1}_{\alpha,\beta} &= e^{2\pi\sqrt{-1}\,(\beta|\gamma)}A_{\gamma,n+\kappa(\alpha|\gamma)} + \delta_{n,0}(\alpha|A_\gamma)\kappa\,K, \end{aligned}$$

and $h_{\alpha,\beta}$ commute with the Heisenberg algebra $\mathfrak{s}$ except for:

$$h_{\alpha,\beta}\ H_0\ h^{-1}_{\alpha,\beta} = H_0 + (\alpha|H_0)\kappa K.$$

Here $h_{\alpha,\beta}$ are given in (61). In order to determine the dependence on ζ of $X^0_\alpha(\zeta)$ we first have to notice that

$$-\,d = \frac{1}{2}|\rho_s|^2 K + \frac{1}{2}H_0^2 + \sum_{(i,\ell)\in I_+} H_{i^*,-l-1}H_{i,\ell}, \tag{65}$$

where $H_0 = H_{01} = H_{02}$. Indeed, if we decompose the basic representation into a direct sum of weight subspaces of $\mathfrak{s}$, then using the above commutation relations, we get that the LHS of (65) is an operator that preserves these weight subspaces while the difference of the LHS and the RHS commutes with $\mathfrak{s}$ and $\mathfrak{S}$. The formula follows up to the constant term $\frac{1}{2}|\rho_s|^2K$, which is fixed by examining the action of the operator $d\in\mathfrak{g}$ on the vacuum vector. Using formula (55) for $Xt^n = \rho_s$

we get that ρ_s (viewed as an element of $\mathfrak{g}^{\sigma_b}$) acts on the vacuum by the scalar $-|\rho_s|^2/\kappa$; then since the RHS of formula (56) acts by 0 on the vacuum, we get that $d\in\mathfrak{g}^{\sigma_b}$ acts by the scalar

$$-|\rho_s|^2/\kappa+\frac{1}{2}|\rho_s|^2(1/\kappa)=-\frac{1}{2\kappa}|\rho_s|^2.$$

Since we have

$$[d,X_\alpha(\zeta)]=-\zeta\partial_\zeta X_\alpha(\zeta),\quad [d,E^*_\alpha(\zeta)]=-\zeta\partial_\zeta E^*_\alpha(z),$$

we easily get $-\zeta\partial_\zeta X^0_\alpha=[d,X^0_\alpha]$. On the other hand, $X^0_\alpha(\zeta)$ commutes with $H_{i,\ell}$ for all i,ℓ, except

$$[H_0,X^0_\alpha(\zeta)]=(\alpha|H_0)X^0_\alpha. \tag{66}$$

It follows that

$$\zeta\partial_\zeta X^0_\alpha=\kappa\Big(X^0_\alpha\alpha_0+\frac{|\alpha_0|^2}{2}X^0_\alpha\Big).$$

Solving the above equation we get formula (64). □

Lemma 25. *The operators C_α in (64) satisfy the following commutation relation*

$$C_\alpha C_\beta=\epsilon(\alpha,\beta)B^{-1}_{\alpha,\beta}C_{\alpha+\beta}, \tag{67}$$

where

$$B_{\alpha,\beta}=\kappa^{-(\alpha|\beta)}\prod_{m=1}^{\kappa-1}(1-\eta^m)^{(\sigma_b^m(\alpha)|\beta)}.$$

Proof. Let us assume first that $\alpha\neq-\beta$ are two roots. After a direct computation we get that the commutator $[X_\alpha(\zeta),X_\beta(w)]$ is given by the following formula:

$$\frac{1}{\kappa}\sum_{m=1}^{\kappa}\left(\prod_{j=1}^{m-1}\upsilon^{-1}(\sigma_b^j(\beta))\right)\epsilon(\alpha,\sigma_b^m(\beta))\,\delta(\eta^{-m}\zeta,w)\,w\,X_{\alpha+\sigma_b^m(\beta)}(\zeta),$$

where $\delta(x,y):=\sum_{n\in\mathbb{Z}}x^ny^{-n-1}$ is the formal delta function. On the other hand,

$$E^*_\alpha(\zeta)E^*_\beta(w)=\prod_{m=1}^{\kappa}\Big(1-\eta^m\frac{w}{\zeta}\Big)^{(\sigma_b^m(\alpha)|\beta)}:E^*_\alpha(\zeta)E^*_\beta(w):,$$

where : : is the standard normal ordering in the Heisenberg group - all annihilation operators $H_{i,\ell}$ must be moved to the right. Substituting in the above commutator $X_\gamma(\zeta)=X^0_\gamma(\zeta)E^*_\gamma(\zeta)$ we get that the following two expressions are equal:

$$\prod_{m=1}^{\kappa}\Big(1-\eta^m\frac{w}{\zeta}\Big)^{(\sigma_b^m(\alpha)|\beta)}X^0_\alpha(\zeta)X^0_\beta(w)-\prod_{m=1}^{\kappa}\Big(1-\eta^m\frac{\zeta}{w}\Big)^{(\sigma_b^m(\beta)|\alpha)}X^0_\beta(w)X^0_\alpha(\zeta) \tag{68}$$

and

$$\frac{1}{\kappa}\sum_{m=1}^{\kappa}\left(\prod_{j=1}^{m-1}\upsilon^{-1}(\sigma_b^j(\beta))\right)\epsilon(\alpha,\sigma_b^m(\beta))\delta(\eta^{-m}\zeta,w)\,w\,X^0_{\alpha+\sigma_b^m(\beta)}(\zeta). \tag{69}$$

Both formulas have the form $i_{\zeta,w}P_1(\zeta,w)-i_{\zeta,w}P_2(\zeta,w)$, where P_1 and P_2 are some rational functions and $i_{\zeta,w}$ (resp. $i_{w,\zeta}$) means the Laurent series expansion in the region $|\zeta|>|w|$ (resp. $|w|<|\zeta|$). Since $P_1=P_2$ for the second expression, the same must be true for the first one, i.e.,

$$\prod_{m=1}^{\kappa}\Big(1-\eta^m\frac{w}{\zeta}\Big)^{(\sigma_b^m(\alpha)|\beta)}X^0_\alpha(\zeta)X^0_\beta(w)=\prod_{m=1}^{\kappa}\Big(1-\eta^m\frac{\zeta}{w}\Big)^{(\sigma_b^m(\beta)|\alpha)}X^0_\beta(w)X^0_\alpha(\zeta).$$

Recalling formula (64) and (66), the above equality implies:

$$C_\alpha C_\beta = \prod_{m=1}^{\kappa} (-\eta^m)^{(\alpha|\sigma_b^m(\beta))}\, C_\beta C_\alpha. \tag{70}$$

Using this equality we can easily write (68) as a sum of formal delta functions. Comparing with (69) we get (67). □

Lemma 26. *Let ω_b, ω_i, $i=(k,p)\in\mathfrak{I}_{\rm tw}$, be the fundamental weights corresponding to the basis of simple roots γ_b, γ_i, $i\in\mathfrak{I}_{\rm tw}$, then*

$$(\omega_i|\chi\omega_b)=d_i,\quad \pi_0(\gamma_b)=\chi\omega_b,\quad \pi_*(\gamma_b)=-\sum_{i\in\mathfrak{I}_{\rm tw}} d_i\,\gamma_i.$$

Proof. Let $\{\varepsilon_{k,p}\}_{p=1}^{a_k}$ be the standard basis of $\mathbb{C}^{a_k}$ for any fixed $k=1,2,3$. The root system of type A_{a_k-1} is given by $\{\varepsilon_{k,p}-\varepsilon_{k,q}\}$ and the standard choice of simple roots is $\gamma_{k,p}=\varepsilon_{k,p}-\varepsilon_{k,p+1}$, $1\le p\le a_k-1$. Note that the fundamental weights corresponding to the basis of simple roots are

$$\widetilde{\omega}_{k,p}=\Big(1-\frac{p}{a_k}\Big)(\varepsilon_{k,1}+\cdots+\varepsilon_{k,p})-\frac{p}{a_k}(\varepsilon_{k,p+1}+\cdots+\varepsilon_{k,a_k}).$$

It follows that the pairing between the fundamental weights is

$$(\widetilde{\omega}_{k,p}|\widetilde{\omega}_{k,q})=\min(p,q)-pq/a_k.$$

In particular, we have

$$\widetilde{\omega}_{k,p}=\Big(1-\frac{p}{a_k}\Big)\gamma_1+\cdots, \tag{71}$$

where the remaining terms involve only $\gamma_2,\dots,\gamma_{a_k-1}$.

In our settings, the roots $\{\gamma_{k,p}\}_{p=1}^{a_k-1}$ give rise to a subroot system of type A_{a_k-1}. Let us denote by $\widetilde{\omega}_{k,p}$ the corresponding fundamental weights. Note that

$$\omega_{k,p}=\widetilde{\omega}_{k,p}-(\widetilde{\omega}_{k,p}|\gamma_b)\,\omega_b,$$

so the first formula of the Lemma follows from (71) and

$$(\gamma_b|\gamma_{k,p})=-\delta_{p,1},\quad (\widetilde{\omega}_{k,p}|\omega_b)=0.$$

The other two identities follow easily from the first one. □

Using formula (67) we define C_α for all α in the root lattice $\Lambda^{(0)}$; then formula (70) still holds. Finally, a similar argument gives us that

$$C_\alpha C_{-\alpha}=\epsilon(\alpha,-\alpha)\,B_{\alpha,-\alpha}^{-1}, \text{i.e., } C_0=1. \tag{72}$$

Lemma 27. *Let c_α ($\alpha\in\Lambda^{(0)}$) be operators defined by*

$$C_\alpha=c_\alpha\exp\Big((\omega_b|\alpha)\omega\Big)\exp\Big(2\pi\sqrt{-1}\,(\rho_b|\alpha)\partial_\omega\Big). \tag{73}$$

Then $[c_\alpha,c_\beta]=0$.

Proof. To begin with, note that by definition, the commutator $C_\alpha C_\beta C_\alpha^{-1}C_\beta^{-1}$ is given by the following formula:

$$\prod_{m=1}^{\kappa}(-\eta^m)^{(\alpha|\sigma_b^m(\beta))}=e^{\pi\sqrt{-1}\,(\alpha_0|\beta)}e^{2\pi\sqrt{-1}\,((1-\sigma_b)^{-1}\alpha_*|\beta)}.$$

On the other hand, using (73), the commutator becomes

$$c_\alpha c_\beta c_\alpha^{-1}c_\beta^{-1}\exp\,2\pi\sqrt{-1}\,\Big((\rho_b|\alpha)(\omega_b|\beta)-(\rho_b|\beta)(\omega_b|\alpha)\Big). \tag{74}$$

Recall that σ_b is a composition of 3 matrices $\sigma_k^{(0)}, k = 1, 2, 3$ whose action on the subspace with basis $\{\gamma_{k,1}, \ldots, \gamma_{k,a_k-1}\}$ is represented by the matrix

$$\sigma_k^{(0)} = \begin{bmatrix} -1 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 1 \\ -1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

It is easy to check that the (p, q)-th entry

$$\left[(1 - \sigma_k^{(0)})^{-1}\right]_{pq} = \frac{p}{a_k} - \varepsilon_{pq}, \quad \varepsilon_{pq} = \begin{cases} 0 & \text{if } p \le q, \\ 1 & \text{if } p > q. \end{cases} \tag{75}$$

A straightforward computation using formula (75) and Lemma 26 yields

$$\begin{aligned} ((1 - \sigma_k^{(0)})^{-1}\gamma_{k,p}|\gamma_b) &= -\frac{1}{a_k}, \\ ((1 - \sigma_k^{(0)})^{-1}\gamma_{k,p}|\gamma_{k,q}) &= \delta_{p,q} - \delta_{p+1,q}, \\ ((1 - \sigma_b)^{-1}(\gamma_b)_*|\gamma_{k,q}) &= \frac{1}{a_k} \pmod{\mathbb{Z}}, \\ ((1 - \sigma_b)^{-1}(\gamma_b)_*|\gamma_b) &= 1 - \frac{1}{2}\chi\,. \end{aligned}$$

Using the above formulas we get

$$((1 - \sigma_b)^{-1}\pi_*(\alpha)|\beta) = (\rho_b|\alpha)\,(\omega_b|\beta) - (\rho_b|\beta)\,(\omega_b|\alpha) - \frac{1}{2}(\alpha_0|\beta_0) \pmod{\mathbb{Z}}.$$

For the commutator we get

$$C_\alpha C_\beta C_\alpha^{-1} C_\beta^{-1} = \exp\Big(2\pi\sqrt{-1}\Big((\rho_b|\alpha)\,(\omega_b|\beta) - (\rho_b|\beta)\,(\omega_b|\alpha)\Big)\Big).$$

Comparing with (74) we get $c_\alpha c_\beta c_\alpha^{-1} c_\beta^{-1} = 1$. □

Lemma 27 implies that the operators c_α can be represented by scalars, i.e., we can find complex numbers c_α, $\alpha \in \Lambda^{(0)}$ such that

$$c_\alpha c_\beta = \epsilon(\alpha, \beta)\, B_{\alpha,\beta}^{-1}\, e^{-2\pi\sqrt{-1}(\rho_b|\beta)(\omega_b|\alpha)}\ c_{\alpha+\beta}. \tag{76}$$

For example we can choose c_{α_i} arbitrarily for the simple roots α_i and then use formula (76) to define the remaining constants.

The level 1 basic representation can be realized on V_x as follows. Let us represent the Heisenberg algebra $\mathfrak{s}$ on $\mathbb{C}[e^\omega]e^{x\omega}$ by letting all generators act trivially, except for $H_0 \mapsto (H_0|\gamma_b)\,\partial_\omega$. The latter is forced by the commutation relation

$$[H_0, C_\alpha] = (\alpha|H_0)\, C_\alpha = (\omega_b|\alpha)\,(H_0|\gamma_b)\, C_\alpha.$$

In this way V_x naturally becomes an $\mathfrak{s}$-module. Furthermore, put

$$E_\alpha^0(\zeta) = \exp\Big((\omega_b|\alpha)\omega\Big) \exp\Big(\Big((\omega_b|\alpha)\,\chi\,\log\zeta^\kappa\ + 2\pi\sqrt{-1}\ (\rho_b|\alpha)\Big)\partial_\omega\Big) \tag{77}$$

and $E_\alpha(\zeta) = E_\alpha^0(\zeta)E_\alpha^*(\zeta)$, where $E_\alpha^*(\zeta)$ is defined by formula (63). Thus the representation of the Heisenberg algebra $\mathfrak{s}$ on V_x can be lifted to a representation of the affine Lie algebra $\mathfrak{g}^{\sigma_b}$ as follows:

$$\begin{aligned} X_\alpha(\zeta) &\mapsto c_\alpha \zeta^{\kappa\,|\alpha_0|^2/2} E_\alpha(\zeta), \quad \alpha \in \Delta^{(0)} \\ K &\mapsto 1/\kappa, \end{aligned}$$

$$d \quad \mapsto \quad -\frac{1}{2\kappa}|\rho_s|^2 - \frac{1}{2}H_0^2 - \sum_{(i,\ell)\in I_+} H_{i^*,-\ell-1}H_{i,\ell}.$$

4.3. **The Kac–Wakimoto hierarchy.** Following Kac–Wakimoto (see [44]), we can define an integrable hierarchy in the Hirota form whose solutions are parametrized by the orbit of the vacuum vector $|0\rangle$ of the affine Kac–Moody group. A vector $\tau \in V_x$ belongs to the orbit if and only if $\Omega_x(\tau\otimes\tau) = 0$, where Ω_x is the operator representing the following bi-linear Casimir operator:

$$\sum_{\alpha\in\Delta^{(0)}}\sum_n \frac{A_{\alpha,n}\otimes A_{-\alpha,-n}}{(A_\alpha|A_{-\alpha})} + K\otimes d + d\otimes K + \frac{H_0\otimes H_0}{\kappa} + \sum_{(i,\ell)\in I_+}\Big(\frac{H_{i,\ell}\otimes H_{i^*,-\ell-1} + H_{i^*,-\ell-1}\otimes H_{i,\ell}}{\kappa}\Big),$$

On the other hand, we have

$$\sum_n \frac{A_{\alpha,n}\otimes A_{-\alpha,-n}}{(A_\alpha|A_{-\alpha})} = \mathrm{Res}_{\zeta=0}\,\frac{d\zeta}{\zeta}a_\alpha(\zeta)E_\alpha(\zeta)\otimes E_{-\alpha}(\zeta),$$

where the coefficients a_α can be computed explicitly thanks to formula (76), i.e.,

$$a_\alpha(\zeta) = B_{\alpha,\alpha}\,\zeta^{\kappa\,|\alpha_0|^2}\,e^{2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}. \tag{78}$$

We identify the symmetric algebra $S^*(\mathfrak{s}_-)$ with the Fock space $\mathbb{C}[y]$, where $y = (y_{i,\ell})$ is a sequence of formal variables indexed by $(i,\ell)\in I_+$ as defined in (60), by identifying $H_{i^*,-\ell-1} = (m_i+\ell\kappa)y_{i,\ell}$. Then (note that $(H_0|\gamma_b) = (\kappa\chi)^{1/2}$)

$$H_{i,\ell} = \frac{\partial}{\partial y_{i,\ell}},\quad H_0 = (\kappa\chi)^{1/2}\,\partial_\omega,\quad K = 1/\kappa,$$

and

$$d = -\frac{|\rho_s|^2}{2\kappa} - \frac{\kappa\chi}{2}\partial_\omega^2 - \sum_{(i,\ell)\in I_+}(m_i+\ell\kappa)y_{i,\ell}\partial_{y_{i,\ell}}.$$

The elements in V_x can also be thought as sequences of polynomials in the following way:

$$V_x \cong \mathbb{C}[y]^{\mathbb{Z}},\quad \sum_{n\in\mathbb{Z}}\tau_n(y)e^{(n+x)\omega} \mapsto \tau := (\tau_n(y))_{n\in\mathbb{Z}}.$$

The above isomorphism turns $\mathbb{C}[y]^{\mathbb{Z}}$ into a module over the algebra of differential operators in e^ω:

$$(e^\omega\cdot\tau)_n = \tau_{n-1},\quad (\partial_\omega\cdot\tau)_n = n\tau_n.$$

The HQEs of the σ_b-twisted Kac–Wakimoto hierarchy will assume the form (4) stated in Section 1.0.3 provided we prove the following identity.

Lemma 28. *The following identities hold*

$$|\rho_s|^2/\kappa^2 = \frac{1}{12}\sum_{k=1}^3\frac{a_k^2-1}{a_k} = \frac{1}{2}\,\mathrm{tr}\Big(\frac{1}{4}+\theta\,\theta^{\mathrm{T}}\Big),$$

where θ is the Hodge grading operator (14).

Proof. Since $\tau = |0\rangle$ is a solution to the hierarchy, we must have

$$|\rho_s|^2/\kappa^2 = \sum_{\alpha:\ (\omega_b|\alpha)=0} a_\alpha(\zeta).$$

Let $\alpha\in\Delta^{(0)}$ be such that $(\omega_b|\alpha) = 0$, then formula (78) reduces simply to

$$a_\alpha(\zeta) = B_{\alpha,\alpha} = \kappa^{-2}\prod_{m=1}^{\kappa-1}(1-\eta^m)^{(\sigma_b^m(\alpha)|\alpha)}.$$

Recall the notation in the proof of Lemma 26. We claim that α must belong to one of the root subsystems $\Delta_k^{(0)}$ of type A_{a_k-1} corresponding to the legs of the Dynkin diagram for some k. Indeed, let us write α as a linear combination $\sum_{k,p} c_{k,p}\gamma_{k,p}$ for some integers $c_{k,p}$. If this linear combination involves a simple root $\gamma_{k,p}$ for some k, then using reflections $s_{k,p}$ with $p>1$ we can transform α to a cycle α' such that the decomposition of α' as a sum of simple roots will involve $\gamma_{k,1}$. Moreover, we still have $(\omega_b|\alpha')=0$. In other words, we may assume that $c_{k,1}\neq 0$ as long as $c_{k,p}\neq 0$ for some p. However, since $(\alpha|\gamma_b) = -\sum_k c_{k,1}$ and the coefficients $c_{k,p}$ have the same sign (depending on whether α is a positive or a negative root) we get that there is precisely one k for which $c_{k,1}\neq 0$.

Assume that $\alpha\in\Delta_k^{(0)}$, then since σ_b is a product of the Coxeter transformations $\sigma_{k'} = \cdots s_{k',2}s_{k',1}$, in the above formula for a_α only σ_k contributes and since the order of σ_k is a_k, after a short computation we get

$$a_\alpha(\zeta) = a_k^{-2}\prod_{m=1}^{a_k-1}(1-\eta_k^m)^{(\sigma_k^m(\alpha)|\alpha)},\quad \eta_k = e^{2\pi\sqrt{-1}/a_k}.$$

These are precisely the coefficients of the principal Kac-Wakimoto hierarchy of type A_{a_k-1}. Let ρ_k be the sum of the fundamental weights of $\Delta_k^{(0)}$. It is well known that $|\rho_k|^2 = (a_k-1)a_k(a_k+1)/12$. According to [25] the sum

$$|\rho_s|^2/\kappa^2 = \sum_{\alpha\in\Delta_k^{(0)}} a_\alpha(\zeta) = |\rho_k|^2/a_k^2 = \frac{1}{12}\sum_{k=1}^{3}\Big(a_k-\frac{1}{a_k}\Big).$$

It remains only to notice (using $\theta^{\mathrm{T}}=-\theta$) that

$$\frac12\operatorname{tr}\Big(\frac14+\theta\,\theta^{\mathrm{T}}\Big) = \frac12\operatorname{tr}\Big(\frac12+\theta\Big)\Big(\frac12-\theta\Big) = \frac12\sum_{i\in\mathfrak{I}_{\mathrm{tw}}} d_i(1-d_i) = \frac{1}{12}\sum_{k=1}^{3}\Big(a_k-\frac{1}{a_k}\Big).$$

□

4.4. **Formal discrete Laplace transform.** Let $\alpha\in\Delta^{(0)}$ and $\tilde\alpha\in\Delta$ be as in Section 3.4. We would like to compare the vertex operators $E_\alpha(\zeta)$ and $\widetilde{\Gamma}^{\tilde\alpha}(\lambda) := e^{(\widetilde{\mathbf{f}}_{\tilde\alpha}(\lambda;z))^\wedge}$, where $(-)^\wedge$ is the quantization operation defined in Section 2.4.1 and

$$\widetilde{\mathbf{f}}_{\tilde\alpha}(\lambda;z) = \sum_{n\in\mathbb{Z}} \widetilde{I}^{(n)}_{\tilde\alpha}(\lambda)\,(-z)^n,$$

see (29). Using the formulas for the calibrated periods from Section 3.4 we get

$$\widetilde{\Gamma}^{\tilde\alpha}(\lambda) = U_{\tilde\alpha}(\lambda)\,\widetilde{\Gamma}^{\tilde\alpha}_0(\lambda)\,\widetilde{\Gamma}^{\tilde\alpha}_*(\lambda),$$

where (we dropped the superscript and set $\omega_b := \omega_b^{(0)}$)

$$U_{\tilde\alpha}(\lambda) = \exp\Big(\sum_{\ell=1}^{\infty}\Big((\omega_b|\alpha)\chi(\log\lambda - C_\ell) + 2\pi\sqrt{-1}(n+(\rho_b|\alpha))\Big)\frac{\lambda^\ell}{\ell!}\,q_\ell^{01}/\sqrt{\hbar}\Big),$$

$$\widetilde{\Gamma}^{\tilde\alpha}_0(\lambda) = \exp\Big(\Big((\omega_b|\alpha)\,\chi\,(\log\lambda - C_0) + 2\pi\sqrt{-1}(n+(\rho_b|\alpha))\Big)\frac{q_0^{01}}{\sqrt{\hbar}}\Big)\times\exp\Big(-(\omega_b|\alpha)\sqrt{\hbar}\frac{\partial}{\partial q_0^{01}}\Big),$$

$$\widetilde{\Gamma}^{\tilde\alpha}_*(\lambda) = \exp\Big(\sum_{(i,\ell)\in I_+}(\alpha|H_i)\,\zeta^{m_i+\ell\kappa}\,y_{i,\ell}\Big)\,\exp\Big(\sum_{(i,\ell)\in I_+}(\alpha|H_{i^*})\,\frac{\zeta^{-m_i-\ell\kappa}}{-m_i-\ell\kappa}\frac{\partial}{\partial y_{i,\ell}}\Big),$$

where $\lambda=\zeta^\kappa/\kappa$, and we use the change of variables

$$y_{02,\ell} = \frac{1}{\sqrt{\hbar}}\frac{\kappa^{d_{02}}}{\sqrt{\kappa\chi}}\frac{q_\ell^{02}}{m_{02}(m_{02}+\kappa)\cdots(m_{02}+\ell\kappa)}, \tag{79}$$

$$y_{i,\ell} \quad = \quad \frac{1}{\sqrt{\hbar}}\, \frac{\kappa^{d_i}}{\sqrt{\kappa a_i}}\, \frac{q_\ell^i}{m_i(m_i+\kappa)\cdots(m_i+\ell\kappa)}, \qquad (i,\ell)\in \mathfrak{I}_{\rm tw}\times \mathbb{Z}_{\geq 0}. \tag{80}$$

Comparing with (63) and (77) we get that $\widetilde{\Gamma}^{\tilde{\alpha}}_*(\lambda) = E^*_\alpha(\zeta)$ and that $\widetilde{\Gamma}^{\tilde{\alpha}}_0(\lambda)$ is a Laplace transform of $E^0_\alpha(\zeta)$. We make the last statement precise as follows. Put

$$\widehat{V} := \mathbb{C}_\hbar[\![y,x,q_1^{01}+1,q_2^{01},\dots]\!]^{\mathbb{Z}}.$$

The space $\widehat{V}$ contains a completion of the basic representation V_x. It has also some additional variables q_ℓ^{01}, $\ell\geq 1$ which will be treated as parameters. Just like before, we identify the elements of $\widehat{V}$ with formal Fourier series

$$f=(f_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} f_n\, e^{(n+x)\omega}.$$

Given $f(\hbar;\mathbf{q})\in \mathbb{C}_\hbar[\![\mathbf{q}]\!]$ satisfying the condition

$$f(\hbar;\mathbf{q})\Big|_{q_0^{01}=x\sqrt{\hbar}} \in \mathbb{C}_\hbar[\![\mathbf{q}]\!] \quad \forall x\in\mathbb{C}, \tag{81}$$

define the formal Laplace transform of f depending on a parameter C ($C\neq 0$)

$$\mathcal{F}_C(f(q_0^{01},\dots)) := \sum_{n\in\mathbb{Z}} f((x+n)\sqrt{\hbar},\dots)\, e^{(n+x)\omega}\, C^{\frac{1}{2}n^2} \qquad \in \widehat{V},$$

where the dots stand for the remaining $\mathbf{q}$-variables on which f depends. It is easy to check that

$$\mathcal{F}_C\ \circ\ q_0^{01}/\sqrt{\hbar} = \frac{\partial}{\partial\omega}\ \circ\ \mathcal{F}_C \tag{82}$$

and

$$\mathcal{F}_C\ \circ\ e^{-m\sqrt{\hbar}\partial/\partial q_0^{01}} = e^{m\omega}\, C^{\frac{1}{2}m^2+m\partial_\omega}\ \circ\ \mathcal{F}_C, \tag{83}$$

where recall that $\partial_\omega = \frac{\partial}{\partial\omega}-x$.

Lemma 29. *Let $C=\kappa^\chi\, e^{\chi C_0}$, then*

$$E^0_\alpha(\zeta)\ \mathcal{F}_C = \mathcal{F}_C\ e^{-AB-\frac{1}{2}B^2\log C}\, e^{Ax}\, \widetilde{\Gamma}^{\tilde{\alpha}}_0,$$

where

$$A=(\omega_b|\alpha)\,\chi\,(\log\,\lambda - C_0)+2\pi\sqrt{-1}(n+(\rho_b|\alpha)), \qquad B=(\omega_b|\alpha).$$

Proof. Using (82) and (83) we get that the vertex operators in q_0^{01} transform as follows:

$$\mathcal{F}_C\ e^{A\,q_0^{01}/\sqrt{\hbar}}\, e^{-B\sqrt{\hbar}\partial/\partial q_0^{01}} = e^{AB+\frac{1}{2}B^2\log C}\, e^{Ax}\, e^{B\omega} e^{(A+B\log C)\partial_\omega}\ \mathcal{F}_C.$$

On the other hand, after a straightforward computation, we get

$$e^{AB+\frac{1}{2}B^2\log C} = \zeta^{\kappa|\alpha_0|^2} e^{-\frac{|\alpha_0|^2}{2\chi}\,(2\chi(C_0+\log\kappa)-\log C)} e^{2\pi\sqrt{-1}(\omega_b|\alpha)(\rho_b|\alpha)}$$

and

$$A+B\log C = (\omega_b|\alpha)\Big(\chi\log\zeta^\kappa+\log C-\chi(C_0+\log\kappa)\Big)+2\pi\sqrt{-1}(n+(\rho_b|\alpha)). \tag{84}$$

Furthermore, note that since the operator $e^{2\pi\sqrt{-1}\partial_\omega}$ acts as the identity on $\widehat{V}$, the integer n in (84) may be set to 0. Finally, it remains only to compare with (77) and to recall our assumption

$$\log C = \chi(C_0+\log\kappa). \tag{85}$$

□

4.5. **Integrable hierarchies for the affine cusp polynomials.** For every root $\alpha \in \Delta^{(0)} \subset H^{(0)}$ we fix an arbitrary lift $\tilde{\alpha} \in \Delta \subset \mathfrak{h}$ (cf. Section 3.4). The subset of affine roots obtained in this way will be denoted by Δ'. Following the construction of Givental and Milanov in [34] we introduce the following Casimir-like operator

$$\begin{aligned}\widetilde{\Omega}_{\Delta'}(\lambda) \quad = \quad & -\frac{\lambda^2}{2}\left(\sum_{m=1}^{N} :(\widetilde{\phi}_m(\lambda)\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\widetilde{\phi}_j(\lambda))(\widetilde{\phi}^m(\lambda)\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\widetilde{\phi}^m(\lambda)):\right) + \\ & + \sum_{\tilde{\alpha}\in\Delta'} \tilde{b}_{\tilde{\alpha}}(\lambda)\widetilde{\Gamma}^{\tilde{\alpha}}(\lambda)\otimes_{\mathfrak{a}}\widetilde{\Gamma}^{-\tilde{\alpha}}(\lambda) - \frac{1}{2}\operatorname{tr}\left(\frac{1}{4}+\theta\,\theta^{\mathrm{T}}\right),\end{aligned}$$

where the notation is as follows. Let $\{\tilde{\alpha}_m\}_{m=1}^N$ and $\{\tilde{\alpha}^m\}_{m=1}^N$ be two sets of vectors in $\mathfrak{h}$ such that under the projection $\widetilde{I}^{(0)}(1) : \mathfrak{h} \to H^{(0)}$ they project to bases dual with respect to the intersection form $(\cdot|\cdot)$, i.e., $(\tilde{\alpha}_j|\tilde{\alpha}^m) = \delta_{j,m}$. Then

$$\widetilde{\phi}_m(\lambda) = (\partial_\lambda \widetilde{\mathbf{f}}_{\tilde{\alpha}_m}(\lambda;z))\widehat{\ }, \quad \widetilde{\phi}^m(\lambda) = (\partial_\lambda \widetilde{\mathbf{f}}_{\tilde{\alpha}^m}(\lambda;z))\widehat{\ }, \quad 1\le m\le N.$$

The tensor product is over the polynomial algebra $\mathfrak{a} := \mathbb{C}_\hbar[q_1^{01}, q_2^{01}, \ldots]$, which in particular means that almost all terms that involve $\log\lambda$ cancel.

The first sum in the definition of $\widetilde{\Omega}_{\Delta'}$ is monodromy invariant around $\lambda=\infty$ and hence it expands in only integral powers of λ. In fact one can check that the corresponding coefficients give rise to a representation of the Virasoro algebra, which can be identified with an instance of the so called *coset Virasoro construction*[7]. After a straightforward computation using the formulas for the periods from Section 3.4, we get the following formula for the coefficient in front of λ^{-2} (i.e., the L_0-Virasoro operator)

$$\frac{\chi}{2\hbar}(q_0^{01}\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}} q_0^{01})^2 + \sum_{(i,\ell)\in I_+}\left(\frac{m_i}{\kappa}+\ell\right)(q_\ell^i\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}} q_\ell^i)(\partial_{q_\ell^i}\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\partial_{q_\ell^i}).$$

The coefficient $\tilde{b}_{\tilde{\alpha}}$ are defined in terms of the vertex operators $\widetilde{\Gamma}^{\tilde{\alpha}}(\lambda)$ as follows

$$\tilde{b}_{\tilde{\alpha}}^{-1}(\lambda) = \lim_{\mu\to\lambda}\left(1-\frac{\mu}{\lambda}\right)^2 \widetilde{B}_{\tilde{\alpha},-\tilde{\alpha}}(\lambda,\mu), \quad \tilde{\alpha},\tilde{\beta}\in\Delta, \tag{86}$$

where $\widetilde{B}_{\tilde{\alpha},\tilde{\beta}}(\lambda,\mu)$ is the phase factor from the composition of the following two vertex operators:

$$\widetilde{\Gamma}^{\tilde{\alpha}}(\lambda)\widetilde{\Gamma}^{\tilde{\beta}}(\mu) = \widetilde{B}_{\tilde{\alpha},\tilde{\beta}}(\lambda,\mu)\ :\widetilde{\Gamma}^{\tilde{\alpha}}(\lambda)\widetilde{\Gamma}^{\tilde{\beta}}(\mu):\ .$$

After a straightforward computation as in Section 4.2, we get

$$\widetilde{B}_{\tilde{\alpha},\tilde{\beta}}(\lambda,\mu) = \mu^{-(\alpha_0|\beta_0)} e^{C_0(\alpha_0|\beta_0) - 2\pi\sqrt{-1}(\omega_b|\alpha)\,(\rho_b|\beta)}\prod_{m=1}^{\kappa}\left(1-\eta^m(\mu/\lambda)^{1/\kappa}\right)^{(\sigma_b^m(\alpha)|\beta)}. \tag{87}$$

We are interested in the following system of Hirota quadratic equations: for every integer $n\in\mathbb{Z}$

$$\operatorname{Res}_{\lambda=\infty}\frac{d\lambda}{\lambda}\left(\widetilde{\Omega}_{\Delta'}(\lambda)\,(\tau\otimes_{\mathfrak{a}}\tau)\right)\Big|_{q_0^{01}\otimes 1 - 1\otimes q_0^{01} = n\sqrt{\hbar}} = 0 \tag{88}$$

where $\tau\in\mathbb{C}_\hbar[\![q_0, q_1+1, q_2\ldots]\!]$. The operator $\widetilde{\Omega}_{\Delta'}(\lambda)$ is multivalued near $\lambda=\infty$: the analytic continuation around $\lambda=\infty$ corresponds to a monodromy transformation of each cycles $\tilde{\alpha}\in\Delta'$ of the type $\tilde{\alpha}\mapsto\sigma_b(\tilde{\alpha}) + n_{\tilde{\alpha}}\varphi$, where $n_{\tilde{\alpha}}\in\mathbb{Z}$. Using Proposition 18 we get that the analytic continuation transforms $\widetilde{\Omega}_{\Delta'}(\lambda)$ by permuting the cycles $\tilde{\alpha}$ and multiplying each vertex operator term by $e^{2\pi\sqrt{-1}n_{\tilde{\alpha}}(q_0^{01}\otimes 1 - 1\otimes q_0^{01})}$. Therefore the 1-form in (88) is invariant with respect to the analytic continuation near $\lambda=\infty$. Moreover, for the same reason the equations (88) are independent of the choice of a lift Δ' of $\Delta^{(0)}$.

[7] We are thankful to B. Bakalov for this remark.

Remark 30. *The Hirota quadratic equations* (88) *are a straightforward generalization of the construction of Givental and Milanov* [34] *(see also* [25]*, where the coefficients* $\widetilde{b}_{\tilde{\alpha}}$ *were interpreted in terms of the vertex operators) of integrable hierarchies for simple singularities.*

The following is the main result of this section.

Theorem 31. *If* τ *is a solution to the Hirota quadratic equations* (88)*, then* $\mathcal{F}_C(\tau)$ *with* $C = \kappa^\chi Q$ *is a tau-function of the* σ_b*-twisted Kac–Wakimoto hierarchy.*

Proof. We just have to find the Laplace transform of the Hirota quadratic equations (4) of the Kac–Wakimoto hierarchy. Let $\alpha \in \Delta^{(0)}$ and $\tilde{\alpha} \in \Delta$ be as in Section 3.4. Using Lemma 29 we get

$$\Big(a_\alpha(\zeta)\, E_\alpha(\zeta) \otimes E_{-\alpha}(\zeta)\Big)\, \Big(\mathcal{F}_C \otimes \mathcal{F}_C\Big) = \Big(\mathcal{F}_C \otimes \mathcal{F}_C\Big)\, \Big(b_{\tilde{\alpha}}(\lambda)\, \widetilde{\Gamma}^{\tilde{\alpha}}(\lambda) \otimes_{\mathfrak{a}} \widetilde{\Gamma}^{-\tilde{\alpha}}(\lambda)\Big),$$

where the coefficient $b_{\tilde{\alpha}}$ is given by

$$a_\alpha(\zeta)\, \zeta^{-2\kappa|\alpha_0|^2} e^{\frac{|\alpha_0|^2}{\chi}\log C} e^{-4\pi\sqrt{-1}(\omega_b|\alpha)(\rho_b|\alpha)}.$$

Recalling formula (78) and $\lambda = \zeta^\kappa/\kappa$ we get

$$b_{\tilde{\alpha}}(\lambda) = B_{\alpha,\alpha}\, \lambda^{-|\alpha_0|^2}\, e^{|\alpha_0|^2 C_0}\, e^{-2\pi\sqrt{-1}(\omega_b|\alpha)(\rho_b|\alpha)}. \tag{89}$$

Using (86) and (87), it is not hard to verify that $b_{\tilde{\alpha}}(\lambda) = \widetilde{b}_{\tilde{\alpha}}(\lambda)$.

In other words, $\mathcal{F}_C(\tau)$ is a solution to the Kac–Wakimoto hierarchy if τ satisfies the following equations:

$$\mathrm{Res}_{\lambda=\infty} \frac{d\lambda}{\lambda}\Big((\mathcal{F}_C \otimes \mathcal{F}_C)\widetilde{\Omega}_{\Delta'}(\lambda)\,(\tau \otimes_{\mathfrak{a}} \tau)\Big) = 0.$$

Comparing the coefficients in front of $e^{(n'+x)\omega} \otimes e^{(n''+x)\omega}$ we get (88) with $n = n' - n''$. □

5. The main Theorem

5.1. **Vertex operators.** The symplectic loop space formalism in GW theory was introduced by Givental [33]. We apply this natural framework to describe and investigate further the Hirota quadratic equations (88). In this section, we again adopt the notation that α, β are in the affine root system Δ.

Recall the series (22). We are interested in the vertex operators

$$\Gamma^\alpha(t,\lambda) =: e^{\widehat{\mathbf{f}}^\alpha(t,\lambda)} :, \quad \alpha \in \Delta, \tag{90}$$

and their *phase factors* $B_{\alpha,\beta}(t,\lambda,\mu)$ defined by

$$\Gamma^\alpha(t,\lambda)\Gamma^\beta(t,\mu) = B_{\alpha,\beta}(t,\lambda,\mu) : \Gamma^\alpha(t,\lambda)\Gamma^\beta(t,\mu) : \quad \alpha,\beta \in \Delta,$$

where $:\cdot:$ is the usual normal ordering – move all differentiation operators to the right of the multiplication operators. Note that

$$B_{\alpha,\beta}(t,\lambda,\mu) := e^{\Omega(\mathbf{f}_\alpha(t,\lambda;z)_+,\mathbf{f}_\beta(t,\mu;z)_-)}. \tag{91}$$

The action of the vertex operators on the Fock space is not well defined in general. We would like to recall the conjugation laws from [31] and to make sense of the vertex operator action on the Fock space.

5.1.1. *Vertex operators at infinity.* Let us fix $t \in M$ and expand the vertex operators $\Gamma^\alpha(t,\lambda)$ in a neighborhood of $\lambda = \infty$. By definition (see (28)) we have $f_\alpha(t,\lambda;z) = S_t \widetilde{\mathbf{f}}_\alpha(\lambda;z)$. Using formula (37), it is easy to prove that

$$\widetilde{\Gamma}^\alpha(\lambda)\, \widehat{S}_t^{-1} = e^{\frac{1}{2} W(\widetilde{\mathbf{f}}_\alpha(\lambda)_+, \widetilde{\mathbf{f}}_\alpha(\lambda)_+)} \widehat{S}_t^{-1}\, \Gamma^\alpha(t,\lambda). \tag{92}$$

In particular, using the formal λ^{-1}-adic topology we get that the vertex operator $\Gamma^\alpha(t,\lambda)$ defines a linear map $\mathbb{C}_\hbar[\![\mathbf{q}]\!] \to K_\hbar[\![\mathbf{q}]\!]$, where K is an appropriate field extension of the field $\mathbb{C}(\!(\lambda^{-1})\!)$.

Let us explain the relation between the phase factors. Recall formula (87), the RHS is interpreted as an element in $\mathbb{C}(\!(\lambda^{-1/\kappa})\!)(\!(\mu^{-1/\kappa})\!)$ by taking the Laurent series expansion with respect to λ at $\lambda = \infty$.

Proposition 32. *The following formula holds:*

$$B_{\alpha,\beta}(t,\lambda,\mu) = \widetilde{B}_{\alpha,\beta}(\mu,\lambda) e^{W_t(\widetilde{\mathbf{f}}_\alpha(\mu)_+, \widetilde{\mathbf{f}}_\beta(\lambda)_+)}.$$

Proof. Conjugating the identity $\widetilde{\Gamma}^\alpha(\lambda)\widetilde{\Gamma}^\beta(\mu) = \widetilde{B}_{\alpha,\beta}(\lambda,\mu)\ :\widetilde{\Gamma}^\alpha(\lambda)\widetilde{\Gamma}^\beta(\mu):$ by $\widehat{S}_t$ and using formula (92) we get that

$$e^{\frac{1}{2}\left(W_t(\widetilde{\mathbf{f}}_\alpha(\lambda)_+, \widetilde{\mathbf{f}}_\alpha(\lambda)_+) + W_t(\widetilde{\mathbf{f}}_\beta(\mu)_+, \widetilde{\mathbf{f}}_\beta(\mu)_+)\right)} B_{\alpha,\beta}(t,\lambda,\mu)$$

coincides with

$$e^{\frac{1}{2} W_t(\widetilde{\mathbf{f}}_\alpha(\lambda)_+ + \widetilde{\mathbf{f}}_\beta(\mu)_+, \widetilde{\mathbf{f}}_\alpha(\lambda)_+ + \widetilde{\mathbf{f}}_\beta(\mu)_+)} \widetilde{B}_{\alpha,\beta}(\lambda,\mu).$$

The quadratic form W is symmetric, so comparing the above identities yields the desired formula. □

5.1.2. *Vertex operators at a critical value.* Let us assume now that λ is near one of the critical values $u_j(t)$ and that β is a cycle vanishing over $\lambda = u_j(t), 1 \le j \le N+1$. According to Lemma 6 we have $\mathbf{f}_\beta(t,\lambda;z) = \Psi_t R_t(z) \mathbf{f}_{A_1}(u_j,\lambda;z)$. Using Lemma 9 it is easy to prove (see [31], Section 7) that

$$\Gamma^\beta(t,\lambda)\, \widehat{\Psi}_t \widehat{R}_t = e^{\frac{1}{2} V_t(\mathbf{f}_\beta(t,\lambda)_-, \mathbf{f}_\beta(t,\lambda)_-)} \widehat{\Psi}_t \widehat{R}_t\, \Gamma^\pm_{A_1}(u_j,\lambda), \tag{93}$$

where $\Gamma^\pm_{A_1}(u_j,\lambda) =: e^{\pm \widehat{\mathbf{f}}_{A_1}(u_j,\lambda)}:$ is the vertex operator of the A_1-singularity, V_t is the second order differential operator defined in Lemma 9, and

$$V_t(\mathbf{f}_\beta(t,\lambda)_-, \mathbf{f}_\beta(t,\lambda)_-) = \sum_{\ell,m=0}^\infty (I^{(-\ell)}_\beta(t,\lambda), V_{\ell m}\, I^{(-m)}_\beta(t,\lambda)).$$

In this case, the action of the vertex operators is well-defined on the subspace spanned by the tame asymptotical functions and it yields a linear map

$$\Gamma^\beta(t,\lambda) : \mathbb{C}_\hbar[\![\mathbf{q}]\!]_{\text{tame}} \to K_\hbar[\![\mathbf{q}]\!],$$

where $K = \mathbb{C}(\!((\lambda - u_j)^{1/2})\!)$. Furthermore, the phase factor $B_{\alpha,\beta}(t,\lambda,\mu)$ is well defined if β is a vanishing cycle, since it can be interpreted as an element in $\mathbb{C}(\!((\mu - u_j)^{1/2})\!)(\!((\lambda - u_j)^{1/2})\!)$. Finally, similarly to Proposition 32, we have

$$B_{\beta,\beta}(t,\lambda,\mu) = B_{A_1}(u_j,\lambda,\mu) e^{-V_t(\mathbf{f}_\beta(t,\lambda)_-, \mathbf{f}_\beta(t,\mu)_-)}, \tag{94}$$

where $B_{A_1}(u_j,\lambda,\mu)$ is the phase factor of the product $\Gamma^\pm_{A_1}(u_j,\lambda)\Gamma^\pm_{A_1}(u_j,\mu)$. A straightforward computation gives

$$B_{A_1}(u_j,\lambda,\mu) = \left(\frac{\sqrt{\lambda - u_j} - \sqrt{\mu - u_j}}{\sqrt{\lambda - u_j} + \sqrt{\mu - u_j}} \right)^2, \tag{95}$$

where the RHS should be expanded into a Laurent series with respect to μ at $\mu = u_j$.

5.2. **From descendants to ancestors.** Following our construction of the HQEs from Section 4.5 we would like to introduce an integrable hierarchy for the ancestor potential $\mathcal{A}_t$. Let us introduce the Heisenberg fields

$$\phi_\beta(t,\lambda) = \partial_\lambda \widehat{\mathbf{f}}^\beta(t,\lambda), \quad \beta\in\Delta',$$

and the corresponding Casimir operator

$$\begin{aligned}\Omega_{\Delta'}(t,\lambda) &= -\frac{\lambda^2}{2}\Big(\sum_{m=1}^N :(\phi_{\beta_m}(t,\lambda)\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\phi_{\beta_m}(t,\lambda))(\phi^{\beta_m}(t,\lambda)\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\phi^{\beta_m}(t,\lambda)):\Big) + \\ &\quad + \sum_{\beta\in\Delta'} b_\beta(t,\lambda)\Gamma^\beta(t,\lambda)\otimes_{\mathfrak{a}}\Gamma^{-\beta}(t,\lambda) - \frac{1}{2}\operatorname{tr}\Big(\frac14 + \theta\,\theta^{\mathrm{T}}\Big),\end{aligned}$$

where $\{\beta_m\}$ and $\{\beta^m\}$ are chosen as $\{\tilde\alpha_m\}$ and $\{\tilde\alpha^m\}$ as in Section 4.5, and the coefficients $b_\beta(t,\lambda)$ are defined by

$$b_\beta(t,\lambda)^{-1} = \lim_{\mu\to\lambda}\ \Big(1-\frac{\mu}{\lambda}\Big)^2 B_{\beta,-\beta}(t,\lambda,\mu). \tag{96}$$

Finally, we need also to discretize the HQEs corresponding to the above Casimir operator so that we offset the problem of multivaluedness. Note that, for the toroidal cycle φ in Section 3.3.1, according to Proposition 18 the vector $\mathbf{f}^\varphi(t,\lambda;z)$ has only negative powers of z, so the quantization $\widehat{\mathbf{f}}^\varphi(t,\lambda)$ is a linear function in $\mathbf{q}$.

Lemma 33. *Let φ be the toroidal cycle. Then the equation*

$$\widehat{\mathbf{f}}^\varphi(t,\lambda)\otimes 1 - 1\otimes\widehat{\mathbf{f}}^\varphi(t,\lambda) = 2\pi\sqrt{-1}\,n \tag{97}$$

is equivalent to

$$[S_t^{-1}\mathbf{q}(z)]_{0,01}\otimes 1 - 1\otimes[S_t^{-1}\mathbf{q}(z)]_{0,01} = n\sqrt{\hbar} \tag{98}$$

$$[S_t^{-1}\mathbf{q}(z)]_{\ell,01}\otimes 1 - 1\otimes[S_t^{-1}\mathbf{q}(z)]_{\ell,01} = 0,\quad \forall\ell\geq 1, \tag{99}$$

where $[S_t^{-1}\mathbf{q}(z)]_{\ell,i}$ denotes the coefficient of $S_t^{-1}\mathbf{q}(z)$ in front of $\phi_i z^\ell$.

Proof. Note that

$$\widetilde{\mathbf{f}}^\varphi(\lambda;z) = 2\pi\sqrt{-1}\ \sum_{\ell=0}^\infty \frac{\lambda^\ell}{\ell!}\,\phi_{02}\,(-z)^{-\ell-1}.$$

The equations (98)–(99) can be written equivalently as

$$\Omega(\widetilde{\mathbf{f}}^\varphi(\lambda;z), S_t^{-1}\mathbf{q}(z)) = 2\pi\sqrt{-1}\,n\sqrt{\hbar}.$$

It remains only to recall that S_t is a symplectic transformation and that

$$\mathbf{f}^\varphi(t,\lambda;z) = S_t\widetilde{\mathbf{f}}^\varphi(\lambda;z).$$

□

We will be interested in the following HQEs: for every integer $n\in\mathbb{Z}$

$$\operatorname{Res}_{\lambda=\infty}\frac{d\lambda}{\lambda}\Big(\Omega_{\Delta'}(t,\lambda)\,(\tau\otimes\tau)\Big)\Big|_{\widehat{\mathbf{f}}^\varphi(t,\lambda)\otimes 1 - 1\otimes\widehat{\mathbf{f}}^\varphi(t,\lambda) = 2\pi\sqrt{-1}\,n} = 0, \tag{100}$$

where τ belongs to an appropriate Fock space and we have to require also that the discretization is well defined. For our purposes the HQEs (100) will be on the Fock space $\mathbb{C}_\hbar[\![q_0+t,q_1+1,q_2,\dots]\!]$. On the other hand the operator $\widehat{S}_t^{-1}$ gives rise to an isomorphism

$$\widehat{S}_t^{-1}:\mathbb{C}_\hbar[\![q_0+t,q_1+\mathbf{1},q_2,\dots]\!]\to\mathbb{C}_\hbar[\![q_0,q_1+1,q_2,\dots]\!].$$

which allows us to identify the HQEs (88) and (100).

Proposition 34. *A function τ is a solution to the HQEs* (100) *iff $\widehat{S}_t^{-1}\tau$ is a solution to the HQEs* (88)*.*

Proof. Using Proposition 32 we get that

$$\widetilde{\Omega}_{\Delta'}(\lambda)\,(\widehat{S}_t^{-1}\otimes\widehat{S}_t^{-1}) = (\widehat{S}_t^{-1}\otimes\widehat{S}_t^{-1})\,\Omega_{\Delta'}(t,\lambda).$$

It remains only to notice that the discretization in both HQEs are compatible with the action of $\widehat{S}_t$, which follows easily from Lemma 10 and Lemma 33. □

5.3. **The integrable hierarchy for A_1-singularity.** It was conjectured by Witten [64] and first proved by Kontsevich [48] that the total descendant potential of a point is a tau-function of the KdV hierarchy. The latter can be written in two different ways: via the Kac-Wakimoto construction and as a reduction of the KP hierarchy. We will need both realizations, so let us recall them.

5.3.1. *The Kac–Wakimoto construction of KdV.* The Casimir operator (cf. Section 5.2) for the A_1-singularity $f(x) = x^2/2 + u$ takes the form

$$\begin{aligned}\Omega_{A_1}(u,\lambda) \;&=\; -\frac{\lambda^2}{4}\,:\phi_\beta^{V\otimes V}(u,\lambda)\phi_\beta^{V\otimes V}(u,\lambda):+\\ &\quad + b_\beta(u,\lambda)\Big(\Gamma^{\beta}_{A_1}(u,\lambda)\otimes\Gamma^{-\beta}_{A_1}(u,\lambda)+\Gamma^{-\beta}_{A_1}(u,\lambda)\otimes\Gamma^{\beta}_{A_1}(u,\lambda)\Big)-\frac{1}{8},\end{aligned}$$

where the coefficient

$$b_\beta(u,\lambda) = \lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^{-2} B_{\beta,\beta}(u,\mu,\lambda) = \frac{\lambda^2}{16(\lambda-u)^2}.$$

We denoted by V the Fock space $\mathbb{C}_\hbar[\![\mathbf{q}]\!]$, and

$$\phi_\beta^{V\otimes V}(u,\lambda) := \phi_\beta(u,\lambda)\otimes 1 - 1\otimes\phi_\beta(u,\lambda).$$

Witten's conjecture (Kontsevich's theorem) can be stated as follows:

$$\mathrm{Res}_{\lambda=\infty}\,\Omega_{A_1}(0,\lambda)\,(\mathcal{D}_{\rm pt}\otimes\mathcal{D}_{\rm pt}) = 0. \tag{101}$$

To compare the above equation with the principal Kac-Wakimoto hierarchy of type A_1, note that

$$\Gamma^{\beta}_{A_1}(u,\lambda) = \exp\Big(2\sum_{n=0}^{\infty}\frac{(2(\lambda-u))^{n+1/2}}{(2n+1)!!}\,\frac{q_n}{\sqrt{\hbar}}\Big)\exp\Big(-2\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2(\lambda-u))^{n+1/2}}\sqrt{\hbar}\partial_n\Big),$$

and that the coefficient in front of λ^{-2} in $\frac{1}{4}:\phi_\beta^{V\otimes V}(0,\lambda)\phi_\beta^{V\otimes V}(0,\lambda):$ is precisely

$$\sum_{n=0}^{\infty}\Big(n+\frac{1}{2}\Big)(q_n\otimes 1 - 1\otimes q_n)(\partial_n\otimes 1 - 1\otimes\partial_n),$$

where $\partial_n := \partial/\partial q_n$. It follows that the above equations coincide with the Kac–Wakimoto form of the KdV hierarchy up to the rescaling $q_n = t_{2n+1}(2n+1)!!$.

On the other hand, the total descendant potential $\mathcal{D}_{\rm pt}$ satisfies the string equation, which can be stated as follows (see [33]): $e^{(u/z)\widehat{}}\mathcal{D}_{\rm pt} = \mathcal{D}_{\rm pt}$. Using that

$$\Omega_{A_1}(0,\lambda)\,\Big(e^{(u/z)\widehat{}}\otimes e^{(u/z)\widehat{}}\Big) = \Big(e^{(u/z)\widehat{}}\otimes e^{(u/z)\widehat{}}\Big)\,\Omega_{A_1}(u,\lambda)$$

we get that $\mathcal{D}_{\rm pt}$ satisfies also the following HQEs:

$$\mathrm{Res}_{\lambda=\infty}\,\Omega_{A_1}(u,\lambda)\,(\mathcal{D}_{\rm pt}\otimes\mathcal{D}_{\rm pt}) = 0. \tag{102}$$

5.3.2. *The KdV hierarchy as a reduction of KP.* According to Givental [31] the KdV hierarchy (101) can be written also as

$$\mathrm{Res}_{\lambda=0}\Big(\sum_{\pm}\frac{d\lambda}{\pm\sqrt{\lambda}}\,\Gamma^{\pm\beta/2}_{A_1}(0,\lambda)\otimes\Gamma^{\mp\beta/2}_{A_1}(0,\lambda)\Big)\,(\mathcal{D}_{\rm pt}\otimes\mathcal{D}_{\rm pt})=0.$$

Using again the string equation and Proposition 32 we get that $\mathcal{D}_{\rm pt}$ satisfies also the following HQEs:

$$\mathrm{Res}_{\lambda=u}\Big(\sum_{\pm}\frac{d\lambda}{\pm\sqrt{\lambda-u}}\,\Gamma^{\pm\beta/2}_{A_1}(u,\lambda)\otimes\Gamma^{\mp\beta/2}_{A_1}(u,\lambda)\Big)\,(\mathcal{D}_{\rm pt}\otimes\mathcal{D}_{\rm pt})=0. \tag{103}$$

5.4. **The phase factors.** In this section we will prove Proposition 41, that the phase factors $B_{\alpha,\beta}(t,\lambda,\mu)$ (see (91)) are multivalued analytic function and that the analytic continuation is compatible with the monodromy action on the cycles α and β. To begin with put

$$B^{\infty}_{\alpha,\beta}(t,\lambda,\mu)=\exp\Omega^{\infty}_{\alpha,\beta}(t,\lambda,\mu),$$

where

$$\Omega^{\infty}_{\alpha,\beta}(t,\lambda,\mu):=\iota_{\lambda^{-1}}\iota_{\mu^{-1}}\sum_{n=0}^{\infty}(-1)^{n+1}(I^{(n)}_{\alpha}(t,\lambda),I^{(-n-1)}_{\beta}(t,\mu)), \tag{104}$$

where $\iota_{\lambda^{-1}}$ (resp. $\iota_{\mu^{-1}}$) is the Laurent series expansion at $\lambda=\infty$ (resp. $\mu=\infty$). The differential of (104) with respect to t is

$$\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu):=I^{(0)}_{\alpha}(t,\lambda)\bullet I^{(0)}_{\beta}(t,\mu)=\sum_{i\in\mathfrak{I}}(I^{(0)}_{\alpha}(t,\lambda),\partial_i\bullet I^{(0)}_{\beta}(t,\mu))\,dt_i,$$

which will be interpreted as a 1-form on M depending on the parameters λ and μ. Furthermore, for each $t\in M$, put $r(t)=\max_j|u_j(t)|$, where $\{u_j(t)\}_{j=1}^{N+1}$ is the set of all critical values of $F(x,t)$. In other words, $r(t)$ is the radius of the smallest disk (with center at 0) that contains all critical values of $F(x,t)$. Let

$$D^{+}_{\infty}=\{(t,\lambda,\mu)\in M\times\mathbb{C}^2\ :\ |\lambda-\mu|<|\mu|-r(t)<|\lambda|-r(t)\}.$$

Note that since $|\lambda-\mu|\geq 0$ we have $|\lambda|>r(t)$ and $|\mu|>r(t)$ for all $(t,\lambda,\mu)\in D^{+}_{\infty}$, which implies that the Laurent series expansions of $I^{(0)}_{\alpha}(t,\lambda)$ and $I^{(0)}_{\beta}(t,\mu)$ at respectively $\lambda=\infty$ and $\mu=\infty$ are convergent. The first inequality in the definition of D^{+}_{∞} guarantees that the line segment $[\lambda,\mu]$ is outside the disk $|x|\leq r(t)$. In particular, in order to specify a branch of $\mathcal{W}_{\alpha,\beta}(\lambda,\mu)$ it is enough to specify the branches of the period vectors only at the point (t,λ), the branch of the periods at (t,μ) is determined via the line segment $[\lambda,\mu]$.

Proposition 35. *The series* (104) *is convergent for all* $(t,\lambda,\mu)\in D^{+}_{\infty}$.

Proof. Using Proposition 32 we can write (104) as a sum of two formal series

$$\Omega^{\infty}_{\alpha,\beta}(t,\lambda,\mu)=\widetilde{\Omega}^{\infty}_{\alpha,\beta}(\lambda,\mu)+W_t(\widetilde{\mathbf{f}}_{\alpha}(\lambda)_+,\widetilde{\mathbf{f}}_{\beta}(\mu)_+), \tag{105}$$

where $\widetilde{\Omega}^{\infty}_{\alpha,\beta}$ is the Laurent series expansion of $\log\widetilde{B}_{\alpha,\beta}$ in the domain $|\lambda|>|\mu|$. Since the series $\widetilde{\Omega}^{\infty}_{\alpha,\beta}$ is convergent for $|\lambda|>|\mu|>|\lambda-\mu|$, it is enough to prove the proposition for the second series on the RHS of (105). Recalling the definition of W_t and using the fact that modulo Q the series $S_t(z)=e^{\frac{1}{z}t\cup}$, where $t\cup$ means the classical orbifold cup product multiplication by t, we get that

$$\lim_{\mathrm{Re}(t_{02})\to-\infty}\ \lim_{t\to(0,\cdots,0,t_{02})}(W_t-t_{02}\,P)=0,$$

On the other hand, since

$$dW_t(\widetilde{\mathbf{f}}_{\alpha}(\lambda)_+,\widetilde{\mathbf{f}}_{\beta}(\mu)_+)=d\Omega_{\alpha,\beta}(t,\lambda,\mu)=I^{(0)}_{\alpha}(t,\lambda)\bullet I^{(0)}_{\beta}(t,\mu),$$

the series

$$W_t(\widetilde{\mathbf{f}}_\alpha(\lambda)_+, \widetilde{\mathbf{f}}_\beta(\mu)_+) - t_{02}(\alpha_0|\beta_0)/\chi, \tag{106}$$

viewed as a formal Laurent series in λ^{-1} and μ^{-1} can be identified with the improper integral

$$\lim_{\varepsilon\to\infty} \int_\varepsilon^t \Big(I^{(0)}_\alpha(t',\lambda) \bullet I^{(0)}_\beta(t',\mu) - dt'_{02}(\alpha_0|\beta_0)/\chi \Big), \tag{107}$$

where $\varepsilon \in M$ and the limit is taken along a straight segment, s.t., $\varepsilon_i \to 0$ for $i \neq 02$ and $\mathrm{Re}(\varepsilon_{02}) \to -\infty$. More precisely, if we take the Laurent series expansion of the integrand at $\lambda = \infty$ and $\mu = \infty$ and integrate termwise, we get (106). It remains only to notice that the integrand extends holomorphically at the limiting point $\varepsilon = \infty$ (because we removed the singularity), so the termwise integration preserves the convergence. □

The proof of Proposition 35 yields slightly more. Namely, we proved that the 2nd summand on the RHS of (105) is a convergent Laurent series in λ^{-1} and μ^{-1} and that the corresponding limit is a multi-valued analytic function on

$$D_\infty := \{(t,\lambda,\mu) \in M \times \mathbb{C}^2 \ : \ |\lambda-\mu| < \min(|\lambda| - r(t), |\mu| - r(t))\}.$$

On the other hand, the phase factor $\widetilde{B}_{\alpha,\beta}(\lambda,\mu)$ is also a multivalued analytic function on D_∞ except for a possible pole along $\lambda = \mu$. Hence we have the following corollary (of the proof).

Corollary 36. *The series $B^\infty_{\alpha,\beta}(t,\lambda,\mu)$ extends analytically to a multivalued analytic function on D_∞ except for a possible pole along the diagonal $\lambda = \mu$.*

Using the analytic extension of $B^\infty_{\alpha,\beta}(t,\lambda,\mu)$ we define a multi-valued function with values in the space $\mathbb{C}\{\{\xi\}\}$ of convergent Laurent series at $\xi = 0$ in the following way:

$$B_{\alpha,\beta} : (M\times\mathbb{C})_\infty \to \mathbb{C}\{\{\xi\}\}, \quad (t,\lambda) \mapsto \iota_{\mu-\lambda} B^\infty_{\alpha,\beta}(t,\lambda,\mu),$$

where $\xi = \mu - \lambda$, $\iota_{\mu-\lambda}$ is the Laurent series expansion at $\mu = \lambda$, and

$$(M\times\mathbb{C})_\infty := \{(t,\lambda) \in M\times\mathbb{C} \ : \ |\lambda| > r(t)\}.$$

It is convenient to introduce the 1-form $\mathcal{W}_{\alpha,\beta}(\xi) := \widetilde{\mathcal{W}}_{\alpha,\beta}(0,\xi)$. Following [31] we call $\mathcal{W}_{\alpha,\beta}(\xi)$ the *phase form.* Note that if $C \subset (M\times\mathbb{C})_\infty$ is a path from (t,λ) to (t',λ'), then

$$B_{\alpha,\beta}(t',\lambda') = B_{\alpha,\beta}(t,\lambda)\, e^{\int_C \mathcal{W}_{\alpha,\beta}(\xi)}. \tag{108}$$

Therefore we can uniquely extend the function $B_{\alpha,\beta}$ to a function on $(M\times\mathbb{C})'$, so that formula (108) holds for all paths $C \subset (M\times\mathbb{C})'$. Finally, for every $(t,\lambda) \in (M\times\mathbb{C})'$ and μ sufficiently close to λ we define

$$B_{\alpha,\beta}(t,\lambda,\mu) = B_{\alpha,\beta}(t,\lambda)|_{\xi=\mu-\lambda}, \quad \Omega_{\alpha,\beta}(t,\lambda,\mu) := \log B_{\alpha,\beta}(t,\lambda,\mu).$$

Note that $B_{\alpha,\beta}(t,\lambda,\mu) = B^\infty_{\alpha,\beta}(t,\lambda,\mu)$ if $(t,\lambda,\mu) \in D^+_\infty$.

Let $t_0 \in M$ be a generic point, so that all critical points of $F(x,t_0)$ are of type A_1 and the absolute values of the corresponding critical values are pairwise distinct. Let $u_j(t_0)$ be a critical value of $F(x,t_0)$ with a maximal absolute value, i.e., $|u_j(t_0)| = r(t_0)$. There exists a real number $\epsilon_0 > 0$, s.t., if $|x| < \epsilon_0$, then $r(t_0 + x\mathbf{1}) = |u_j(t_0) + x|$. We fix $t = t_0 + x_0\mathbf{1}$, λ, and μ, s.t., the line segment $[\mu - u_j(t_0), x_0]$ is contained inside the disk $\{|x| < \epsilon_0\} \subset \mathbb{C}$ and the line segment $[t_0, t] \times \{(\lambda,\mu)\} \subset D^+_\infty$. For example, fix μ, s.t., $|\mu| > u_j(t_0)$ and $|\mu - u_j(t_0)| < \varepsilon_0$ and put $x_0 = \frac{1}{2}(\mu - u_j(t_0))$, then we can find λ such that all requirements are fulfilled.

Lemma 37. *If $\beta \in H_2(X_{t,\mu};\mathbb{Z})$ is a cycle vanishing over $t = t_0 + (\mu - u_j(t_0))\mathbf{1}$, then*

$$\Omega_{\alpha,\beta}(t,\lambda,\mu) = \lim_{\varepsilon\to 0} \int_{t_0 + (\varepsilon + \mu - u_j(t_0))\mathbf{1}}^t \widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu), \tag{109}$$

where the integration is along a straight segment.

Proof. By definition $\Omega_{\alpha,\beta}(t,\lambda,\mu)$ is the Laurent series expansion near $\lambda=\infty$ of the series

$$\sum_{n=0}^{\infty}(-1)^{n+1}(I^{(n)}_\alpha(t,\lambda),I^{(-n-1)}_\beta(t,\mu)), \tag{110}$$

while the RHS of (109) is

$$\lim_{\varepsilon\to 0}\int_{\varepsilon+\mu-u_j}^{x_0}(I^{(0)}_\alpha(t_0,\lambda-x),I^{(0)}_\beta(t_0,\mu-x))dx. \tag{111}$$

Using integration by parts $(n+1)$ times and the fact that the periods $I^{(-p-1)}_\beta(t_0,\mu-x)$ vanish at $x=\mu-u_j$, we get that the integral (111) coincides with

$$\sum_{p=0}^{n}(-1)^{p+1}(I^{(p)}_\alpha(t_0',\lambda),I^{(-p-1)}_\beta(t_0',\mu))+$$
$$\lim_{\varepsilon\to 0}(-1)^{n+1}\int_{\varepsilon+\mu-u_j}^{x_0}(I^{(n+1)}_\alpha(t_0,\lambda-x),I^{(-n-1)}_\beta(t_0,\mu-x))dx. \tag{112}$$

The Laurent series expansion of $I^{(n+1)}_\alpha(t_0,\lambda-x)=I^{(n+1)}_\alpha(t_0+x\mathbf{1},\lambda)$ in λ^{-1} has radius of convergence $r(t_0+x\mathbf{1})$. Hence, it is uniformly convergent for all x that vary along a compact subset of the open subset in $\mathbb{C}$ defined by the inequality

$$\{x\in\mathbb{C}\ :\ |\lambda|>r(t_0+x\mathbf{1})\}.$$

On the other hand, according to our choice of x_0,λ, and μ, the point $(t_0+x\mathbf{1},\lambda,\mu)\in D^+_\infty$ for all x on the integration path. In particular, $|\lambda|>|\mu|>r(t_0+x\mathbf{1})$, which means that the integration path is entirely contained in the above open subset. Hence the integral (112) has a convergent Laurent series in λ^{-1}. Moreover, the leading order term of the expansion is λ^{-e} for some rational number $e>n$. This proves that the Laurent series expansions (in λ^{-1}) of the integral (111) and of the series (110) coincide. □

Our next goal is to prove that the analytic continuation of the phase factor $B_{\alpha,\beta}(t,\lambda,\mu)$ is compatible with the monodromy representation in the following sense. Recall the monodromy representation (cf. Section 2.3)

$$\rho:\pi_1((M\times\mathbb{C})')\to \mathrm{GL}(\mathfrak{h}).$$

Let $U\subset(M\times\mathbb{C})'$ be an open subdomain and $f_{\alpha,\beta}(t,\lambda)$ be a (vector-valued) function depending bi-linearly on $(\alpha,\beta)\in\mathfrak{h}\times\mathfrak{h}$ and analytic in a neighborhood of some point $(t_0,\lambda_0)\in U$. We say that $f_{\alpha,\beta}$ is *multi-valued analytic* on U if it can be extended analytically along any path in U. Furthermore, we say that $f_{\alpha,\beta}$ is *compatible* with the monodromy representation ρ, if for every closed loop C in U, the analytic continuation of $f_{\alpha,\beta}(t,\lambda)$ along C coincides with $f_{w(\alpha),w(\beta)}(t,\lambda)$, where $w=\rho(C)$ is the corresponding monodromy transformation.

Recall that (see Corollary 36) the Laurent series $\Omega^\infty_{\beta,\alpha}(t,\lambda,\mu)$ extends analytically to a multi-valued analytic function $\Omega_{\beta,\alpha}(t,\lambda,\mu)$ defined for all $(t,\lambda,\mu)\in D_\infty$, s.t., $\lambda\neq\mu$.

Lemma 38. *Let α and β be cycles in the vanishing cohomology, s.t., $(\alpha|\beta)=0$ then*

$$\Omega_{\alpha,\beta}(t,\lambda,\mu)-\Omega_{\beta,\alpha}(t,\mu,\lambda)=2\pi\sqrt{-1}\,\mathrm{SF}(\alpha,\beta)\quad\forall(t,\lambda,\mu)\in D^+_\infty,$$

where SF *is the bi-linear form* (57).

Proof. Since the difference

$$\Omega_{\beta,\alpha}(t,\lambda,\mu)-\widetilde{\Omega}_{\beta,\alpha}(\lambda,\mu),\quad\text{where}\quad \widetilde{\Omega}_{\beta,\alpha}(\lambda,\mu):=\log\widetilde{B}_{\beta,\alpha}(\lambda,\mu),$$

has a convergent Laurent series expansion in D_∞ and it is invariant under switching $(\beta,\lambda)\leftrightarrow(\alpha,\mu)$, it is enough to prove the statement for $\widetilde{\Omega}_{\alpha,\beta}(\lambda,\mu)$ where (λ,μ) is a point in the open subset

$$\{|\lambda-\mu|<\min(|\lambda|,|\mu|)\}\subset\mathbb{C}^2.$$

Recalling formula (87), the rest of the proof is a straightforward computation (see also the proof of Lemma 27, where some of the computations were already done). □

Remark 39. *If we omit the condition $(\alpha|\beta)=0$ in Lemma 38, then the identity is true only up to an integer multiple of $2\pi\sqrt{-1}(\alpha|\beta)$. The ambiguity comes from the fact that the phase factor $\widetilde{\Omega}_{\alpha,\beta}(\lambda,\mu)$ has a logarithmic singularity along $\lambda=\mu$ of the type $(\alpha|\beta)\log(\lambda-\mu)$.*

Proposition 40. *The phase factor $B_{\alpha,\beta}(t,\lambda)$ is compatible with the monodromy representation in the domain $(M\times\mathbb{C})'$.*

Proof. By definition we have to prove that if $C'\subset(M\times\mathbb{C})'$ is an arbitrary loop based at (t,λ) and μ is sufficiently close to λ, then

$$B_{w(\alpha),w(\beta)}(t,\lambda,\mu)=B_{\alpha,\beta}(t,\lambda,\mu)\,e^{\int_C\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)},$$

where $w=\rho(C')$ and $C\subset M$ is the path parametrized by

$$t'+(\lambda-\lambda')\mathbf{1},\quad (t',\lambda')\in C'.$$

We may assume that $(t,\lambda,\mu)\in D_\infty^+$, because by definition the value of $B_{\alpha,\beta}$ at any other point differs by an integral along the path of the phase form $\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)$. Under this assumption the above equality is equivalent to

$$\Omega^\infty_{w(\alpha),w(\beta)}(t,\lambda,\mu)=\Omega^\infty_{\alpha,\beta}(t,\lambda,\mu)+\int_C\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)\pmod{2\pi\sqrt{-1}\,\mathbb{Z}}. \tag{113}$$

We first prove a special case of the above formula. Namely, let us choose a generic point $t_0\in M$, s.t., the absolute values of the critical values of $F(x,t_0)$ are pairwise distinct and let $u_j(t_0)$ be the critical value with maximal absolute value (here the notation is the same as in Lemma 37). We will assume that $t=t_0+x_0\mathbf{1}$ is sufficiently close to $t_0+(\mu-u_j(t_0))\mathbf{1}$ and that C is a closed loop of the type $t_0+x\mathbf{1}$, where the parameter x varies along a small closed loop based at $x_0\in\mathbb{C}$ going around $\mu-u_j(t_0)$, so that the line segment $[\lambda-x,\mu-x]$ moves around u_j. Let us denote by $\gamma\in H_2(X_{t,\lambda};\mathbb{Z})$ the vanishing cycle vanishing over $(t_0,u_j(t_0))$, then we have the following decompositions:

$$\alpha=\alpha'+\frac{(\alpha|\gamma)}{2}\gamma,\quad \beta=\beta'+\frac{(\beta|\gamma)}{2}\gamma,$$

where α' and β' are cycles invariant w.r.t. the local monodromy around the point $(t_0,u_j(t_0))$. After a straightforward computation we get

$$\Omega_{w(\alpha),w(\beta)}(t,\lambda,\mu)-\Omega_{\alpha,\beta}(t,\lambda,\mu)=-(\alpha|\gamma)\Omega_{\gamma,\beta'}(t,\lambda,\mu)-(\beta|\gamma)\Omega_{\alpha',\gamma}(t,\lambda,\mu),$$

while $\int_C\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)$ is

$$\frac{1}{2}(\beta|\gamma)\int_C\widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu)+\frac{1}{2}(\alpha|\gamma)\int_C\widetilde{\mathcal{W}}_{\gamma,\beta'}(\lambda,\mu)+\frac{1}{4}\,(\alpha|\gamma)\,(\beta|\gamma)\int_C\widetilde{\mathcal{W}}_{\gamma,\gamma}(\lambda,\mu), \tag{114}$$

where we used that $\int_C\widetilde{\mathcal{W}}_{\alpha',\beta'}(\lambda,\mu)=0$, because the periods $I^{(0)}_{\alpha'}(t_0,\lambda-x)$ and $I^{(0)}_{\beta'}(t_0,\mu-x)$ are holomorphic respectively at $x=\lambda-u_j$ and $x=\mu-u_j$, which means that the phase form is holomorphic inside the loop C. The last integral in the above formula is easy to compute because only the singular terms of $I^{(0)}_\gamma(t_0,\lambda-x)$ and $I^{(0)}_\gamma(t_0,\mu-x)$ contribute, i.e.,

$$\int_C\widetilde{\mathcal{W}}_{\gamma,\gamma}(\lambda,\mu)=2\oint\frac{dx}{\sqrt{(\lambda-u_j(t_0)-x)(\mu-u_j(t_0)-x)}}=4\pi\sqrt{-1}.$$

According to Lemma 37

$$\Omega_{\alpha',\gamma}(t,\lambda,\mu) = \int_{t_0+(\mu-u_j(t_0))\mathbf{1}}^{t} \widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu)$$

and the integral on the RHS has a convergent Laurent series expansion in $\lambda - u_j(t)$ and $(\mu - u_j(t))^{1/2}$, which allows us to evaluate the integral

$$\int_C \widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu) = -2\int_{t_0+(\mu-u_j(t_0))\mathbf{1}}^{t} \widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu) = -2\Omega^{\infty}_{\alpha',\gamma}(t,\lambda,\mu) = -2\Omega_{\alpha',\gamma}(t,\lambda,\mu).$$

It remains only to evaluate the 2nd integral in (114). We have

$$\int_C \widetilde{\mathcal{W}}_{\gamma,\beta'}(\lambda,\mu) = \int_C \widetilde{\mathcal{W}}_{\beta',\gamma}(\mu,\lambda) = -2\Omega_{\beta',\gamma}(t,\mu,\lambda),$$

where the 2nd identity is derived just like above when $|\mu| > |\lambda|$, and then we use analytic continuation to extend the formula for $|\mu| < |\lambda|$ as well. Recalling Lemma 38, we get

$$\Omega_{\beta',\gamma}(t,\mu,\lambda) = \Omega_{\gamma,\beta'}(t,\lambda,\mu) + 2\pi\sqrt{-1}\,\mathrm{SF}(\beta',\gamma).$$

Using that $\beta' = \beta - (\beta|\gamma)\gamma/2$ and that $\mathrm{SF}(\gamma,\gamma) = 1$, we finally get

$$\int_C \widetilde{\mathcal{W}}_{\gamma,\beta'}(\lambda,\mu) = -2\Omega_{\gamma,\beta'}(t,\lambda,\mu) - 4\pi\sqrt{-1}\,\mathrm{SF}(\beta,\gamma) + 2\pi\sqrt{-1}(\beta|\gamma).$$

Since $\mathrm{SF}(\beta,\gamma) \in \mathbb{Z}$, the proof of formula (113) in the special case is complete.

The general case follows easily, because the fundamental group $\pi_1((M\times\mathbb{C})')$ is generated by loops like the above one. Indeed, we already know that the affine cusp polynomial $f(x)$ has a real Morsification $F(x,t_0')$, i.e., all critical points of $F(x,t_0')$ are real and the corresponding critical values are real as well. In particular, we can find a small deformation $F(x,t_0)$ of the real Morsification, s.t., the critical values u_j are vertices of a convex polygon. The fundamental group $\pi_1((M\times\mathbb{C})')$ is generated by simple loops in $\{t_0\}\times\mathbb{C}$ that go around the vertices of the polygon. Let us pick one of these loops and let $(t_0, u_j(t_0))$ be the corresponding vertex of the polygon. Since the translations of the type $t_0 \mapsto t_0 + c\mathbf{1}$, $c\in\mathbb{C}$, do not change the homotopy class of the loop, we can find a representative (namely, pick c, s.t., the $|u_j(t_0)+c| > |u_j(t_0)+c|$ for all other vertices $(t_0,u_j(t_0))$) of the homotopy class, which has the special form from above. □

Proposition 41. *There exists a generic point $t_0\in M$ (i.e. $F(x,t_0)$ is a Morse function) and a critical value $u_j(t_0)$, s.t.,*

$$B_{\alpha,\beta}(t,\lambda,\mu) = \lim_{\varepsilon\to 0} \exp\Big(-\int_{t}^{t_0+(\varepsilon+\mu-u_j(t_0))\mathbf{1}} \widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)\Big), \tag{115}$$

where the integration is along any path avoiding the poles of the 1-form $\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)$, s.t., the cycle $\beta\in H_2(X_{t,\mu},\mathbb{Z})$ vanishes along it.

Proof. Let us assume that t_0 is a generic point and that $u_j(t_0)$ is the critical value with maximal absolute value. It is enough to prove the statement for an arbitrary point $(t,\lambda,\mu)\in D^+_\infty$, because by definition the value of $B_{\alpha,\beta}(t',\lambda,\mu)$ at any other point (t',λ,μ) differs by an integral of $\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)$ along a path connecting t and t', while the RHS of (115) clearly has the same property. Let $(t,\lambda,\mu)\in D^+_\infty$ be a point such that Lemma 37 holds and let C''_ε be the straight segment $[t, t_0+(\varepsilon+\mu-u_j(t_0))\mathbf{1}]$. Put $C' = (C''_\varepsilon)^{-1}\circ C_\varepsilon$ and $w = \rho(C')$, where C_ε is the integration path (from t to $t_0+(\varepsilon+\mu-u_j(t_0))\mathbf{1}$), then by definition the cycle $w(\beta)\in H_2(X_{t,\mu};\mathbb{Z})$ is the vanishing cycle along the line segment $[t, t_0+(\mu-u_j(t_0))\mathbf{1}]$. According to Lemma 37, formula (115) holds for C'' and $B_{w(\alpha),w(\beta)}$. Therefore, we need to prove that

$$-\int_{C'} \widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu) = \Omega_{\alpha,\beta}(t,\lambda,\mu) - \Omega_{w(\alpha),w(\beta)}(t,\lambda,\mu) \pmod{2\pi\sqrt{-1}\mathbb{Z}}, \tag{116}$$

which follows from Proposition 40. □

5.5. **The ancestor solution.** Now we are in a position to prove

Theorem 42. *The total ancestor potential* $\mathcal{A}_t(\hbar;\mathbf{q})$ *is a solution to the HQEs* (100).

To begin with, put $\mathbf{q}'=\mathbf{q}\otimes 1$, $\mathbf{q}''=1\otimes\mathbf{q}$, and let us assume that the discretization condition (97) is satisfied for some integer n. The tameness of $\mathcal{A}(\hbar;\mathbf{q})$ implies that the LHS of (100) (for $\tau=\mathcal{A}(\hbar;\mathbf{q})$) is a formal series in $\mathbf{q}'$ and $\mathbf{q}''$ with coefficients formal Laurent series in $\sqrt{\hbar}$, whose coefficients are polynomial expressions of the period vectors $I^{(n)}_\alpha(t,\lambda)$. In particular, the residue in (100) can be computed via the residue theorem, i.e., we have to compute the residues at all critical points and at $\lambda=0$ and prove that their sum is 0.

Let $u_j(t)$ be one of the critical points of F, where $t\in M$ is a generic point such that all critical values are pairwise different. Furthermore, we assume that λ is near $u_j(t)$ and that a path in $(M\times\mathbb{C})'$ from the reference point $(0,1)$ to (t,λ) is fixed in such a way that the vanishing cycle β, vanishing over $\lambda=u_j(t)$, belongs to the subset Δ' of affine roots defined in Section 4.5.

5.5.1. *The Virasoro term.* Let us compute

$$-\operatorname{Res}_{\lambda=u_j(t)}\frac{\lambda}{2}d\lambda\sum_{m=1}^N :\phi^{V\otimes V}_{\beta_m}(t,\lambda)\phi^{V\otimes V}_{\beta^m}(t,\lambda):\ \mathcal{A}_t^{\otimes 2}, \tag{117}$$

where $\phi^{V\otimes V}_\alpha:=\phi_\alpha\otimes 1-1\otimes\phi_\alpha$. Put $\beta_m=\alpha_m+(\beta_m|\beta)\beta/2$ and $\beta^m=\alpha^m+(\beta^m|\beta)\beta/2$, where $(\alpha_m|\beta)=(\alpha^m|\beta)=0$. The above operator can be written as the sum of

$$\sum_{m=1}^N :\phi^{V\otimes V}_{\alpha_m}(t,\lambda)\phi^{V\otimes V}_{\alpha^m}(t,\lambda):+\Big(\sum_{m=1}^N(\beta_m|\beta)(\beta^m|\beta)\Big)\frac14 :\phi^{V\otimes V}_{\beta}(t,\lambda)\phi^{V\otimes V}_{\beta}(t,\lambda):$$

and

$$\sum_{m=1}^N\frac12\Big((\beta_m|\beta):\phi^{V\otimes V}_{\beta}(t,\lambda)\phi^{V\otimes V}_{\alpha^m}(t,\lambda):+(\beta^m|\beta):\phi^{V\otimes V}_{\beta}(t,\lambda)\phi^{V\otimes V}_{\alpha_m}(t,\lambda):\Big) \tag{118}$$

The Picard-Lefschetz formula implies that the periods $I^{(n)}_{\alpha_m}(t,\lambda)$ and $I^{(n)}_{\alpha^m}(t,\lambda)$ are invariant with respect to the local monodromy around $\lambda=u_j(t)$, so they must be holomorphic in a neighborhood of $\lambda=u_j(t)$. The operator $\phi^{V\otimes V}_\varphi(t,\lambda)$, where φ is the toroidal cycle, vanishes after we impose the discretization condition (97). On the other hand, since $\sum_m(\beta_m|\beta)(\beta^m|\alpha)=(\beta|\alpha)$, the cycles

$$-\beta+\sum_{m=1}^N(\beta_m|\beta)\beta^m\quad\text{and}\quad-\beta+\sum_{m=1}^N(\beta^m|\beta)\beta_m$$

are in the kernel of the intersection form, so they must be proportional to φ. Hence the operator (118) vanishes after the discretization condition (97) is imposed. The residue (117) turns into

$$-\operatorname{Res}_{\lambda=u_j(t)}\frac{\lambda}{4}d\lambda\ :\phi^{V\otimes V}_{\beta}(t,\lambda)\phi^{V\otimes V}_{\beta}(t,\lambda):\ \mathcal{A}_t(\hbar;\mathbf{q}')\mathcal{A}_t(\hbar;\mathbf{q}'').$$

To compute the above residue, note that the expression

$$:\phi^{V\otimes V}_{\beta}(t,\lambda)\phi^{V\otimes V}_{\beta}(t,\lambda):(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}$$

can be written as

$$(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}:\phi^{V\otimes V}_{A_1}(u_j,\lambda)\phi^{V\otimes V}_{A_1}(u_j,\lambda):+2V_t(\phi_\beta(t,\lambda)_-,\phi_\beta(t,\lambda)_-).$$

Let us compute

$$-\operatorname{Res}_{\lambda=u_j(t)}\frac{\lambda}{4}d\lambda\,2V_t(\phi_\beta(t,\lambda)_-,\phi_\beta(t,\lambda)_-)=-\operatorname{Res}_{\lambda=u_j(t)}\frac{\lambda}{2}d\lambda\,(V_{00}(t)I^{(0)}_\beta(t,\lambda),I^{(0)}_\beta(t,\lambda)),$$

where we used the fact that only the leading term (w.r.t. z) of $\phi_\beta(t,\lambda;z)_- = -I^{(0)}_\beta(t,\lambda)z^{-1}+\cdots$ will contribute because the remaining ones have a zero at $\lambda = u_j(t)$ of order at least $\frac{1}{2}$. Furthermore, the Laurent series expansion of $I^{(0)}_\beta$ at $\lambda = u_j(t)$ has the form

$$I^{(0)}_\beta(t,\lambda) = 2(2(\lambda-u_j))^{-1/2}e_j + \cdots, \quad e_j = du_j/\sqrt{\Delta_j},$$

where the dots stand for terms that have at $\lambda = u_j$ a zero of order at least $\frac{1}{2}$. These terms do not contribute to the residue, so we get

$$-\operatorname{Res}_{\lambda=u_j(t)} \frac{\lambda}{2} d\lambda\, (V_{00}(t)e_j, e_j) \frac{2}{\lambda - u_j(t)} = u_j(t)\,(R_1(t)e_j, e_j).$$

We get the following formula for the residue (117):

$$(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}\Big(u_j R_1^{jj} - \operatorname{Res}_{\lambda=u_j}\frac{\lambda}{4}d\lambda\, :\phi^{V\otimes V}_{A_1}(u_j,\lambda)\phi^{V\otimes V}_{A_1}(u_j,\lambda):\Big)\prod_{m=1}^{N+1}\mathcal{D}_{\rm pt}(\hbar\Delta_m;{}^m\mathbf{q})^{\otimes 2},$$

where $R_1^{jj} = (R_1e_j, e_j)$ is the j-th diagonal entry of R_1.

5.5.2. *The A_1-subroot system.* The vanishing cycles $\{-\beta,\beta\}$ form a subroot system of type A_1. Let us compute the residue of the corresponding vertex operator terms, i.e.,

$$\operatorname{Res}_{\lambda=u_j(t)}\frac{d\lambda}{\lambda}\Big(\sum_{\pm} b_{\pm\beta}(t,\lambda)\Gamma^{\pm\beta}(t,\lambda)\otimes\Gamma^{\mp\beta}(t,\lambda)\Big)\mathcal{A}_t^{\otimes 2}. \tag{119}$$

We have $b_\beta(t,\lambda) = b_{-\beta}(t,\lambda)$ and

$$b_\beta(t,\lambda)\Gamma^{\pm\beta}(t,\lambda)\otimes\Gamma^{\mp\beta}(t,\lambda)(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2} = (\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2} b_{A_1}(u_j,\lambda)\Gamma^{\pm\beta}_{A_1}(u_j,\lambda)\otimes\Gamma^{\mp\beta}_{A_1}(u_j,\lambda),$$

where we used formula (93) together with the identity

$$b_\beta(t,\lambda)e^{V_t(\mathbf{f}_\beta(t,\lambda)_-,\mathbf{f}_\beta(t,\lambda)_-)} = b_{A_1}(u_j,\lambda),$$

which follows immediately from (94). Using that $\mathcal{A}_t = \widehat{\Psi}_t\widehat{R}_t\prod_j\mathcal{D}^{(j)}_{\rm pt}$, where the factors $\mathcal{D}^{(j)}_{\rm pt} = \mathcal{D}_{\rm pt}(\hbar\Delta_j;{}^j\mathbf{q})$ are solutions to KdV, we can compute the residue (119) via the Kac-Wakimoto form of the KdV hierarchy (102). After a short computation we get that the residue (119) is

$$(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}\Big(\frac{1}{8} + \operatorname{Res}_{\lambda=u_j}\frac{\lambda}{4}d\lambda\, :\phi^{V\otimes V}_{A_1}(u_j,\lambda)\phi^{V\otimes V}_{A_1}(u_j,\lambda):\Big)\prod_{m=1}^{N+1}\mathcal{D}_{\rm pt}(\hbar\Delta_m;{}^m\mathbf{q})^{\otimes 2}.$$

5.5.3. *The A_2-subroot subsystem.* Let $\alpha\in\Delta'$ be a cycle such that $(\alpha|\beta) = 1$. We claim that the expression

$$\Big(b_\alpha(t,\lambda)\Gamma^{\alpha}(t,\lambda)\otimes\Gamma^{-\alpha}(t,\lambda) + b_{\alpha-\beta}(t,\lambda)\Gamma^{\alpha-\beta}(t,\lambda)\otimes\Gamma^{-\alpha+\beta}(t,\lambda)\Big)\mathcal{A}_t^{\otimes 2} \tag{120}$$

is analytic near $\lambda = u_j$. Using the decompositions

$$\alpha = \alpha' + \beta/2, \quad \alpha-\beta = \alpha' - \beta/2,$$

where $(\alpha'|\beta) = 0$, the above expression can be written as

$$\Gamma^{\alpha'}\otimes\Gamma^{-\alpha'}\Big(a'\Gamma^{\beta/2}\otimes\Gamma^{-\beta/2} + a''\Gamma^{-\beta/2}\otimes\Gamma^{\beta/2}\Big)\,\mathcal{A}_t^{\otimes 2},$$

where the coefficients a' and a'' are given by

$$\begin{aligned} a'(t,\lambda) &= \lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^{-2} B_{\alpha,\alpha}(t,\lambda,\mu)B^{u_j}_{\alpha',-\beta}(t,\lambda,\mu), \\ a''(t,\lambda) &= \lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^{-2} B_{\alpha-\beta,\alpha-\beta}(t,\lambda,\mu)B^{u_j}_{\alpha',\beta}(t,\lambda,\mu), \end{aligned}$$

where the phase factor $B^{u_j}_{\alpha',\beta} = \exp \Omega^{u_j}_{\alpha',\beta}$ with

$$\Omega^{u_j}_{\alpha',\beta}(t,\lambda,\mu) = \iota_{\lambda-u_j}\iota_{\mu-u_j}\sum_{n=0}^{\infty}(-1)^{n+1}(I^{(n)}_{\alpha'}(t,\lambda), I^{(-n-1)}_{\beta}(t,\mu)),$$

where $\iota_{\lambda-u_j}$ (resp. $\iota_{\mu-u_j}$) is the Laurent series expansion at $\lambda = u_j$ (resp. $\mu = u_j$). Since the Laurent series expansions are convergent for λ and μ sufficiently close to u_j, integration by parts yields

$$\Omega^{u_j}_{\alpha',\beta}(t,\lambda,\mu) = \lim_{\varepsilon\to 0}\int_{L_\varepsilon} \mathcal{W}_{\alpha',\beta}(\mu-\lambda),$$

where L_ε is the straight segment $[t + (\varepsilon+\mu-\lambda-u_j)\mathbf{1}, t-\lambda\mathbf{1}]$. On the other hand we have

$$\Gamma^{\pm\beta/2}\otimes\Gamma^{\mp\beta/2}(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2} = (\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}\, e^{V_t(\mathbf{f}_{\beta/2}(t,\lambda)_-,\mathbf{f}_{\beta/2}(t,\lambda)_-)}\Gamma^{\pm\beta/2}_{A_1}\otimes\Gamma^{\mp\beta/2}_{A_1}.$$

The exponential factor can be expressed in terms of the phase factors as follows (cf. Section 5.1.2):

$$e^{V_t(\mathbf{f}_{\beta/2}(t,\lambda)_-,\mathbf{f}_{\beta/2}(t,\lambda)_-)} = \frac{1}{2\sqrt{\lambda-u_j}}\lim_{\mu\to\lambda}(\lambda-\mu)^{1/2}B^{u_j}_{\beta/2,-\beta/2}(t,\lambda,\mu),$$

where the limit is taken in the region $|\lambda| > |\mu|$. Recalling the KP-reduction HQEs of KdV (103) we get that if the coefficients

$$c'(t,\lambda) = \lambda^2 \lim_{\mu\to\lambda}(\lambda-\mu)^{-3/2}B_{\alpha,\alpha}(t,\lambda,\mu)B^{u_j}_{\alpha',-\beta}(t,\lambda,\mu)B^{u_j}_{\beta/2,-\beta/2}(t,\lambda,\mu)$$

and

$$c''(t,\lambda) = \lambda^2 \lim_{\mu\to\lambda}(\lambda-\mu)^{-3/2}B_{\alpha-\beta,\alpha-\beta}(t,\lambda,\mu)B^{u_j}_{\alpha',\beta}(t,\lambda,\mu)B^{u_j}_{\beta/2,-\beta/2}(t,\lambda,\mu)$$

are analytic near $\lambda = u_j$, and $c'/c'' = -1$, then the expression (120) is analytic near $\lambda = u_j$.

Let us prove the analyticity of c'. The argument for c'' is similar. Let us choose a small $\varepsilon\in\mathbb{C}$ and a generic point $t_0 \in M$ on the discriminant, so that Proposition 41 holds. Furthermore, we fix 2 paths C'_ε, and C''_ε in $M' = M\setminus\{\mathrm{discr}\}$ from $t_0 + (\mu-\lambda+\varepsilon)\mathbf{1}$ to $t-\lambda\mathbf{1}$ such that the parallel transport transforms the cycle φ vanishing over t_0 respectively into α, and $\alpha-\beta$. The phase factors in the definition of c' can be written in terms of integrals along the path as follows

$$\begin{aligned} B_{\alpha,\alpha}(t,\lambda,\mu) &= \lim_{\varepsilon\to 0}\exp\Big(\int_{C'_\varepsilon}\mathcal{W}_{\alpha,\alpha}(\mu-\lambda)\Big),\\ B_{\alpha',-\beta}(t,\lambda,\mu) &= \lim_{\varepsilon\to 0}\exp\Big(\int_{L_\varepsilon}\mathcal{W}_{\alpha',-\beta}(\mu-\lambda)\Big),\\ B_{\beta/2,-\beta/2}(t,\lambda,\mu) &= \lim_{\varepsilon\to 0}\exp\Big(\int_{L_\varepsilon}\mathcal{W}_{\beta/2,-\beta/2}(\mu-\lambda)\Big). \end{aligned}$$

Using these formulas, we can express the coefficient $c'(t,\lambda)$ as the limit $\varepsilon\to 0$ of the following expression:

$$\lambda^2\lim_{\mu\to\lambda}(\lambda-\mu)^{-3/2}\exp\Big(\int_{C'_\varepsilon}\mathcal{W}_{\alpha,\alpha}(\mu-\lambda) - \int_{L_\varepsilon}\mathcal{W}_{\alpha,\alpha}(\mu-\lambda) + \int_{L_\varepsilon}\mathcal{W}_{\alpha',\alpha'}(\mu-\lambda)\Big).$$

Let us examine the dependence on the parameters t,λ, and $\xi := \mu-\lambda$. The difference of the first two integrals in the above formula does not depend on λ, because the paths C'_ε and L_ε have the same ending point, while the starting points are independent of λ. After passing to the limit the difference contributes a constant independent of λ, and μ. The last integral is analytic near $\lambda = u_j$, because the cycle α' is invariant with respect to the local monodromy, which means that the period vector $I^{(0)}_{\alpha'}(t',\xi)$ and respectively the phase form $\mathcal{W}_{\alpha',\alpha'}(\xi)$ are analytic for t' sufficiently close to $t-u_j\mathbf{1}$ and $|\xi|\ll 1$. This proves the analyticity of c'.

It remains only to prove that $c'/c''=-1$. Using the above path integrals, we can write $\log(c'/c'')$ in the following way:

$$\int_{C'_\varepsilon}\mathcal{W}_{\alpha,\alpha}-\int_{L_\varepsilon}\mathcal{W}_{\alpha,\alpha}-\int_{C''_\varepsilon}\mathcal{W}_{\alpha-\beta,\alpha-\beta}+\int_{L_\varepsilon}\mathcal{W}_{\alpha-\beta,\alpha-\beta}+\int_{\gamma_\varepsilon}\mathcal{W}_{\alpha,\alpha}-\int_{\gamma_\varepsilon}\mathcal{W}_{\alpha,\alpha},$$

where γ_ε is a small loop in M' based at the starting point of L_ε (i.e. $t+(\varepsilon+\mu-\lambda-u_j)\mathbf{1}$) that goes counterclockwise around the discriminant and the branch of the phase form is determined by its value at the point $t-\lambda\mathbf{1}$ (which belongs to the integration paths of the first 4 integrals and it is connected via the line segment L_ε to the contour of the last 2 ones). The above expression coincides with

$$\oint_{(C''_\varepsilon)^{-1}\circ L_\varepsilon\circ\gamma_\varepsilon\circ L_\varepsilon^{-1}\circ C'_\varepsilon}\mathcal{W}_{\alpha,\alpha}-\oint_{\gamma_\varepsilon}\mathcal{W}_{\alpha,\alpha}.$$

By definition the cycle α is invariant along the integration contour of the first integral, so the first integral is an integer multiple of $2\pi\sqrt{-1}$. We get

$$c'/c''=\lim_{\xi\to 0}\lim_{\varepsilon\to 0}\exp\Big(-\oint_{\gamma_\varepsilon}\mathcal{W}_{\alpha,\alpha}(\xi)\Big).$$

The limit here is easy to compute because the integral involves only local information. Using again the decomposition $\alpha=\alpha'+\beta/2$ and Lemma 6 we get

$$I^{(0)}_\beta(t',\xi)=2(2(\xi-u))^{-1/2}\,\frac{du}{\sqrt{\Delta}}+\cdots,$$

where the dots stand for higher order terms. On the other hand, the period vector $I^{(0)}_{\alpha'}(t',\xi)$ is analytic for (t',ξ) sufficiently close (t,u_j). Expanding the phase form into a Laurent series about $\xi=u$ we get

$$\lim_{\varepsilon\to 0}\oint_{\gamma_\varepsilon}\mathcal{W}_{\alpha,\alpha}(\xi)=\frac14\oint_{\gamma_\varepsilon}\mathcal{W}_{\beta,\beta}(\xi)=\frac14\oint\frac{2du}{\sqrt{(-u)(\xi-u)}}=\pi\sqrt{-1},$$

i.e., $c'/c''=-1$.

5.5.4. *Proof of Theorem 42.* The 1-form

$$\frac{d\lambda}{\lambda}\Omega_{\Delta'}(t,\lambda)\,\mathcal{A}_t(\hbar;\mathbf{q}')\mathcal{A}_t(\hbar;\mathbf{q}'')$$

has poles only at $\lambda=0,\infty$, and the critical values u_j, $1\le j\le N+1$. Let u_j be one of the critical values and β be the cycle vanishing over $\lambda=u_j$. Note that non-trivial contributions to the residue at $\lambda=u_j$ come only from vertex operator terms corresponding to vanishing cycles that have non-zero intersection with β. Recalling our computations in Sections 5.5.1, 5.5.2, and 5.5.3, we get that the residue at $\lambda=u_j$ is $(1/8+u_jR_1^{jj})\mathcal{A}_t^{\otimes 2}$, while the residue at $\lambda=0$ is $-\frac12\operatorname{tr}\left(\frac14+\theta\theta^T\right)\mathcal{A}_t^{\otimes 2}$. In order to prove that the residue at $\lambda=\infty$ is 0, we just need to check that

$$\sum_{j=1}^{N+1}u_jR_1^{jj}=\frac12\operatorname{tr}\left(\theta\theta^T\right).$$

The above identity is well-known from the theory of Frobenius manifolds (see [34, 36]). Hence the ancestor potential $\mathcal{A}_t(\hbar;\mathbf{q})$ is a solution to the HQEs (100). Theorem 42 is thus proved. □

Proof of Theorem 1. Given Theorem 42, Proposition 34 implies that the total descendant potential $\mathcal{D}_{\mathbf{a}}(\hbar;\mathbf{q})$ is a solution to the HQEs (88). Theorem 1 then follows from Theorem 31. □

6. An example: $\mathbb{P}^1_{2,2,2}$

In this section we consider the example $\mathbf{a} = \{2,2,2\}$, namely $\mathbb{P}^1_{\mathbf{a}} = \mathbb{P}^1_{2,2,2}$. In this case $\Delta^{(0)}$ is the root system of type D_4. It is convenient to denote the indexes in the index set $\mathfrak{I}_{\rm tw} = \{(1,1),(2,1),(3,1)\}$ simply by $1,2,3$. There are 12 positive roots

$$\gamma_i\ (1\le i\le 3),\quad \gamma_b,\quad \gamma_b+\gamma_i\ (1\le i\le 3),\quad \gamma_b+\gamma_i+\gamma_j\ (1\le i<j\le 3),$$
$$\gamma_b+\gamma_1+\gamma_2+\gamma_3,\quad 2\gamma_b+\gamma_1+\gamma_2+\gamma_3,$$

where γ_b is the simple root corresponding to the branching node of the Dynkin diagram and $\gamma_i\,(1\le i\le 3)$ are the remaining simple roots. The fundamental weight is $\omega_b = 2\gamma_b+\gamma_1+\gamma_2+\gamma_3$. The eigenbasis for σ_b used in our construction is

$$H_i := -(\kappa/2)^{1/2}\gamma_i \quad (1\le i\le 3),\quad H_0 := (\kappa/2)^{1/2}\omega_b,$$

and we have $m_i = \frac{\kappa}{2}$, $d_i = \frac{1}{2}$, $1\le i\le 3$, where $\kappa = 4$.

Let us write the HQEs for $\tau = (\tau_n(y))_{n\in\mathbb{Z}}$. We have

$$a_\alpha(\zeta) = \frac{1}{4}2^{(\sigma_b(\alpha)|\alpha)}\zeta^{\kappa|\alpha_0|^2}e^{2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}$$

and

$$\Big(E_\alpha(\zeta)\tau\Big)_0 = \zeta^{-\kappa|\alpha_0|^2}e^{-2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}E^*_\alpha(\zeta)\tau_{-(\omega_b|\alpha)},$$

where the subscript 0 on the LHS means the 0-th component of the corresponding vector in our Fock space. Recall that the HQEs give rise to a system of PDEs in the following way. First we make a substitution

$$\mathbf{y}' := \mathbf{y}\otimes 1 = \mathbf{x}+\mathbf{t},\quad \mathbf{y}'' := 1\otimes\mathbf{y} = \mathbf{x}-\mathbf{t},$$

which implies that

$$\mathbf{y}'-\mathbf{y}'' = 2\mathbf{t},\quad \frac{\partial}{\partial\mathbf{y}'}-\frac{\partial}{\partial\mathbf{y}''} = \frac{\partial}{\partial\mathbf{t}},$$

and that

$$\mathrm{Res}_{\zeta=0}\Big(a_\alpha(\zeta)E_\alpha(\zeta)\tau\otimes E_{-\alpha}(\zeta)\tau\Big)_{0,0}$$

is the coefficient in front of ζ^0 in the following expression

$$2^{(\sigma_b(\alpha)|\alpha)-2}e^{-2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}\Big(\zeta^{-\kappa|\alpha_0|^2}e^{\sum_{i,\ell}2(\alpha|H_i)\zeta^{m_i+\ell\kappa}t_{i,\ell}}\Big)$$
$$\Big(e^{-\sum_{i,\ell}(\alpha|H_{i^*})\frac{\zeta^{-m_i-\ell\kappa}}{m_i+\ell\kappa}\partial_{x_{i,\ell}}}\tau_{-(\omega_b|\alpha)}(\mathbf{x}+\mathbf{t})\Big)\Big(e^{\sum_{i,\ell}(\alpha|H_{i^*})\frac{\zeta^{-m_i-\ell\kappa}}{m_i+\ell\kappa}\partial_{x_{i,\ell}}}\tau_{(\omega_b|\alpha)}(\mathbf{x}-\mathbf{t})\Big).$$

By definition the HQEs are

$$\mathrm{Res}_{\zeta=0}\sum_{\alpha\in\Delta^{(0)}}\Big(a_\alpha(\zeta)E_\alpha(\zeta)\tau\otimes E_{-\alpha}(\zeta)\tau\Big)_{m,n} =$$
$$\Big(\frac{3}{8}+\frac{1}{4}(m-n)^2+2\sum_{i,\ell}(d_{i^*}+\ell)t_{i,\ell}\partial_{t_{i,\ell}}\Big)\tau_m(\mathbf{x}+\mathbf{t})\tau_n(\mathbf{x}-\mathbf{t}).$$

Comparing the coefficients in front of the various monomials in $\mathbf{t}$ we obtain a system of PDEs whose equations are some quadratic polynomials in the partial derivatives of τ. Let us specialize to the case $m=n=0$. In order to get non-trivial equations we have to compare coefficients in front of monomials that are invariant under the involution $\mathbf{t}\mapsto -\mathbf{t}$. The simplest case is $\mathbf{t}^0$, which corresponds to the identity

$$\sum_{\alpha\in\Delta^{(0)}:(\omega_b|\alpha)=0} 2^{(\sigma_b(\alpha)|\alpha)-2}=\frac{3}{8}.$$

Comparing the coefficients in front of the monomial $t_{02,0}^2$, we get

$$4\frac{\partial^2}{\partial x_{02,0}^2}\log\tau(\mathbf{x}) = 8\kappa\frac{\tau_{-2}(\mathbf{x})\tau_2(\mathbf{x})}{\tau^2(\mathbf{x})} - 4(2/\kappa)^{1/2}\frac{\partial^3}{\partial t_{1,0}\partial t_{2,0}\partial t_{3,0}}\Big(\frac{\tau_{-1}(\mathbf{x}+\mathbf{t})\tau_1(\mathbf{x}-\mathbf{t})}{\tau^2(\mathbf{x})}\Big)\Big|_{\mathbf{t}=0}.$$

Recalling the substitution (79)–(80), which in this case is

$$\begin{aligned} y_{02,0} &= \frac{1}{\sqrt{\hbar}}\,\frac{\sqrt{2}}{\kappa\sqrt{\kappa}}q_0^{02},\\ y_{i,0} &= \frac{1}{\sqrt{\hbar}}\,\frac{\sqrt{2}}{\kappa}\,q_0^i,\quad 1\le i\le 3, \end{aligned}$$

we get

$$\hbar\frac{\partial^2}{\partial(q_0^{02})^2}\log\tau(\mathbf{q}) = \frac{4}{\kappa^2}\frac{\tau_{-2}(\mathbf{q})\tau_2(\mathbf{q})}{\tau^2(\mathbf{q})} - \frac{\hbar^{3/2}}{\kappa^{1/2}}\,\partial_1\partial_2\partial_3\Big(\frac{\tau_{-1}(\mathbf{q}+\mathbf{t})\tau_1(\mathbf{q}-\mathbf{t})}{\tau^2(\mathbf{q})}\Big)\Big|_{\mathbf{t}=0},$$

where for brevity we put $\partial_i:=\partial/\partial t_0^i$. To get a differential equation for the total descendant potential we just have to substitute

$$\tau_{\pm 2}(\mathbf{q}) = C^2\mathcal{D}(\hbar;\mathbf{q}\pm 2\sqrt{\hbar}),\quad \tau_{\pm 1}(\mathbf{q}) = C^{1/2}\mathcal{D}(\hbar;\mathbf{q}\pm\sqrt{\hbar})$$

where $C=\kappa^{1/2}Q$.

Let us use the above equation to compute the genus-0 primary potential F. Put $q_k^i=0$, $\forall k>0$, and compare the leading terms of the genus expansion. We get the following PDE for F:

$$F_{02,02} = 4Q^4e^{4F_{01,01}} + Qe^{F_{01,01}}\Big(8F_{01,1}F_{01,2}F_{01,3} + 4(F_{01,1}F_{2,3}+F_{01,2}F_{1,3}+F_{01,3}F_{1,2})\Big),$$

where $F_{i,j}:=\partial^2F/\partial q_0^i\partial q_0^j$. To simplify the notation, let us put $t_i:=q_0^i$. String equation gives

$$F_{01,01}=t_{02},\quad F_{01,i}=\frac{1}{2}t_i,$$

so from the above equation we get the following relation

$$F_{02,02} = 4Q^4e^{4t_{02}} + Qe^{t_{02}}\Big(t_1t_2t_3 + 2(t_1F_{2,3}+t_2F_{1,3}+t_3F_{1,2})\Big). \tag{121}$$

Equation (121) allows us to compute the potential F recursively, by the degree of the Novikov variable Q. Indeed, it is easy to see that up to degree-1 terms, F is given by

$$\frac{1}{2}t_{01}^2t_{02} + \frac{1}{4}t_{01}(t_1^2+t_2^2+t_3^2) + \frac{1}{96}(t_1^4+t_2^4+t_3^4) + Qe^{t_{02}}t_1t_2t_3.$$

Comparing the degree-2 terms in (121) we get that the degree-2 term of F must be $\frac{1}{2}(t_1^2+t_2^2+t_3^2)Q^2e^{2t_{02}}$. Arguing in the same way we get that F does not have degree-3 terms, while the degree 4 term must be $\frac{1}{4}Q^4e^{4t_{02}}$. The potential F takes the form

$$\begin{aligned} F(t) &= \frac{1}{2}t_{01}^2t_{02} + \frac{1}{4}t_{01}(t_1^2+t_2^2+t_3^2) + \frac{1}{96}(t_1^4+t_2^4+t_3^4) + Qe^{t_{02}}t_1t_2t_3 + \\ &\quad + \frac{1}{2}Q^2e^{2t_{02}}(t_1^2+t_2^2+t_3^2) + \frac{1}{4}Q^4e^{4t_{02}}. \end{aligned}$$

The above formula agrees with the computation of P. Rossi [55, Example 3.2] based on Symplectic Field Theory.

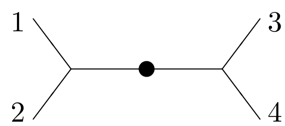
Appendix A. An alternative proof of higher genus reconstruction

In this subsection, we use the degree of virtual fundamental cycle and tautological relations to give a simple proof for Teleman's higher genus reconstruction theorem for the target $\mathbb{P}^1_{\mathbf{a}}$, see Proposition 44 below. This proof does not require the semi-simple assumption.

We first recall the g-reduction property introduced in [22], which is a consequence of results by Ionel [37], and by Faber and Pandharipande [21]:

Lemma 43 ([37, 21])**.** *If $M(\psi,\kappa)$ is a polynomial of ψ-classes and κ-classes with $\deg M \geq g$ for $g \geq 1$ or $\deg M \geq 1$ for $g = 0$, then $M(\psi,\kappa)$ can be presented as a linear combination of dual graphs on the boundary of $\overline{\mathcal{M}}_{g,n}$.*

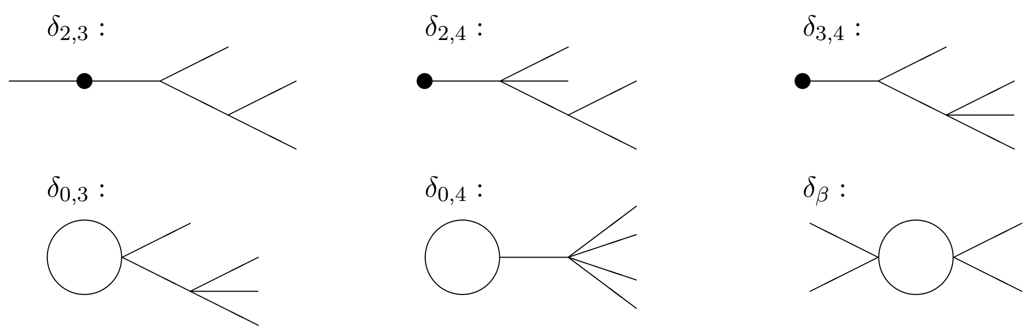
Our second tool is the Getzler's relation in [28]. It is a linear relation between codimension two cycles in $H_*(\overline{\mathcal{M}}_{1,4},\mathbb{Q})$. Here we briefly introduce this relation for our purpose. The dual graph

$\Delta_{12,34} =$ 

represents a codimension-two stratum in $\overline{\mathcal{M}}_{1,4}$: A filled circle represents a genus-1 component, other vertices represent genus-0 components. An edge connecting two vertices represents a node, a tail (or half-edge) represents a marked point on the component of the corresponding vertex. $\Delta_{2,2}$ is defined to be the S_4-invariant of the codimension-two stratum in $\overline{\mathcal{M}}_{1,4}$,

$$\Delta_{2,2} = \Delta_{12,34} + \Delta_{13,24} + \Delta_{14,23}.$$

We denote $\delta_{2,2} = [\Delta_{2,2}]$ the corresponding cycle in $H_4(\overline{\mathcal{M}}_{1,4},\mathbb{Q})$. We list the corresponding unordered dual graph for other strata below, see [28] for more details.



In [28], Getzler found the following identity:

$$12\delta_{2,2} + 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} + \delta_{0,3} + \delta_{0,4} - 2\delta_{\beta} = 0 \in H_4(\overline{\mathcal{M}}_{1,4},\mathbb{Q}). \tag{122}$$

Now we prove the following higher genus reconstruction result.

Proposition 44. *The total ancestor potential $\mathcal{A}_{\mathbf{a}}(\hbar;\mathbf{t})$ is uniquely determined by the quantum cohomology of $\mathbb{P}^1_{\mathbf{a}}$ when $\mathbf{a} \neq \{1,1,1\}$ and $\chi > 0$.*

Proof. We consider the ancestor correlator $\langle \phi_1\bar{\psi}_1^{k_1},\ldots,\phi_n\bar{\psi}_n^{k_n}\rangle_{g,n,d}$ in (8). According to the degree formula (6), if the correlator is nonzero, then

$$\frac{1}{2}\sum_{j=1}^{n}\deg\phi_j + \sum_{j=1}^{n}k_j = (3-\frac{1}{2}\dim\mathbb{P}^1_{\mathbf{a}})(g-1) + \chi\cdot d + n. \tag{123}$$

Now if $\sum_{j=1}^n k_j \geq g$ for $g \geq 1$ or $\sum_{j=1}^n k_j \geq 1$ for $g = 0$, then we can apply Lemma 43 of g-reduction to rewrite the ancestor correlator as a linear combination of intersection numbers over the corresponding homology cycles of some dual graphs, each of the dual graph lives on the boundary of $\overline{\mathcal{M}}_{g,n}$. The splitting axiom in GW theory allows us to reconstruct the ancestor correlator in (8) using intersection numbers over each component of the boundaries. We can keep doing this process until on each component, the g-reduction property does not hold. In other words, all the ancestor correlators are determined completely by those (8) which satisfies $\sum_{j=1}^n k_j \leq g-1$ for $g \geq 1$ or $\sum_{j=1}^n k_j = 0$ for $g = 0$. On the other hand, since $\deg \phi_j \leq 2$, $\chi > 0$ and $\dim \mathbb{P}^1_{\mathbf{a}} = 1$, the formula (123) implies such intersection numbers must vanish unless $g = 0$ and all $k_j = 0$, or $g = 1, d = 0$, all $k_j = 0$ and all $\deg \phi_j = 2$.

In order to finish the proof, it only remains to consider genus 1 correlator $\langle P\rangle_{1,1,0}$. If $\mathbf{a} \neq \{1,1,1\}$, then according to Rossi's computation [55], we can always find a twisted sector $\phi_i \in H$, such that

$$\langle \phi_i, \phi_i, \phi_{i^*}, \phi_{i^*}\rangle_{0,4,0} \neq 0. \tag{124}$$

Consider the integration of the cohomology cycle $\Lambda_{1,4,0}(\phi_i, \phi_i, \phi_{i^*}, \phi_{i^*})$ over the Getzler's relation (122), with four fixed insertions $\phi_i, \phi_i, \phi_{i^*}, \phi_{i^*}$. By the splitting axiom in GW theory, it is not hard to see that the integration vanishes on those homology classes with a genus-1 component except that

$$\int_{\delta_{3,4}} \Lambda_{1,4,0}(\phi_i, \phi_i, \phi_{i^*}, \phi_{i^*})$$

is a multiplication by a nonzero scalar and $\langle P\rangle_{1,1,0}$, because of (124). Thus the equality (122) implies $\langle P\rangle_{1,1,0}$ is reconstructed from genus-0 correlators. ☐

Remark 45. *The technique above only uses properties of cohomology field theories and tautological relations over the moduli space of stable curves. So it also works for the reconstruction of the ancestor potential in* (35)*. It also works for elliptic orbifold projective curves* $\mathbb{P}^1_{\mathbf{a}}$*, where* $\chi = 0$*, see* [49]*. The genus-1 correlator* $\langle P\rangle_{1,1,0}$ *in GW theory can be calculated directly using virtual cycle or virtual localization, see* [63]*.*

References

[1] M. J. Ablowitz, J. F. Ladik. *Nonlinear differential-difference equations.* J. Math. Phys. 16 (1975), 598–603.
[2] D. Abramovich, T. Graber, A. Vistoli. *Gromov-Witten theory of Deligne-Mumford stacks.* Amer. J. Math. 130 (2008), no. 5, 1337–1398.
[3] V.I. Arnol'd, S.M. Gusein-Zade, A.N. Varchenko. *Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals.* Monographs in Mathematics, 83. Birkhäuser Boston, Inc., Boston, MA, 1988
[4] B. Bakalov, T. Milanov. *$\mathcal{W}$-constraints for the total descendant potential of a simple singularity.* Compositio Math. 149 (2013), no. 5, 840–888.
[5] A. Bertram, I. Ciocan-Fontanine, B. Kim. *Two proofs of a conjecture of Hori and Vafa.* Duke Math. J. 126 (2005), no. 1, 101–136.
[6] A. Brini. *The local Gromov-Witten theory of $\mathbb{CP}^1$ and integrable hierarchies.* Comm. Math. Phys. 313 (2012), no. 3, 571–605.
[7] A. Buryak, H. Posthuma, S. Shadrin. *On deformations of quasi-Miura transformations and the Dubrovin–Zhang bracket.* J. Geom. Phys. 62 (2012), no. 7, 1639–1651.
[8] A. Buryak, H. Posthuma, S. Shadrin. *A polynomial bracket for the Dubrovin-Zhang hierarchies.* J. Differential Geom. 92 (2012), no. 1, 153–185.
[9] C. Cadman. *Using stacks to impose tangency conditions on curves.* Amer. J. Math. 129 (2007), no. 2, 405–427.
[10] G. Carlet. *The extended bigraded Toda hierarchy.* J. Phys. A 39 (2006), no. 30, 9411–9435.
[11] G. Carlet, B. Dubrovin, Y. Zhang. *The extended Toda hierarchy.* Mosc. Math. J. 4 (2004), no. 2, 313–332, 534.
[12] G. Carlet, J. van de Leur. *Hirota equations for the extended bigraded Toda hierarchy and the total descendent potential of $\mathbb{CP}^1$ orbifolds.* J. Phys. A: Math. Theor. 46 (2013), 405205.
[13] W. Chen, Y. Ruan. *Orbifold Gromov-Witten theory.* In: "Orbifolds in mathematics and physics". 25–85. Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

[14] T. Coates, A. Corti, H. Iritani, H.-H. Tseng. *Computing Genus-Zero Twisted Gromov-Witten Invariants.* Duke Math. J. 147 (2009), no. 3, 377–438.
[15] T. Coates, A. Corti, H. Iritani, H.-H. Tseng. *A Mirror Theorem for Toric Stacks.* arXiv:1310.4163.
[16] B. Dubrovin. *Geometry of 2D topological field theories.* In: "Integrable Systems and Quantum Groups". 120–348. Lecture Notes in Math. 1620, Springer-Verlag, New York, 1996.
[17] B. Dubrovin, Y. Zhang. *Extended Affine Weyl groups and Frobenius manifolds.* Compos. Math. 111 (1998), no. 2, 167–219.
[18] B. Dubrovin, Y. Zhang. *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants.* a new 2005 version of arXiv:math/0108160v1, 295 pp.
[19] B. Dubrovin, Y. Zhang. *Virasoro Symmetries of the Extended Toda Hierarchy.* Comm. Math. Phys. 250 (2004), no. 1, 161193.
[20] T. Eguchi, S.-K. Yang. *The topological $\mathbb{C}P^1$-model and the large-N matrix integral.* Mod. Phys. Lett. A9 (1994), 2893–2902.
[21] C. Faber, R. Pandharipande. *Relative maps and tautological classes.* J. Eur. Math. Soc. 7 (2005), 13–49.
[22] C. Faber, S. Shadrin, D. Zvonkine. *Tautological relations and the r-spin Witten conjecture.* Ann. Sci. c. Norm. Supr. (4) 43 (2010), no. 4, 621–658.
[23] H. Fan, T. Jarvis, Y. Ruan. *Quantum singularity theory for $A_{(r-1)}$ and r-spin theory.* Ann. Inst. Fourier (Grenoble) 61 (2011), no. 7, 2781–2802.
[24] H. Fan, T. Jarvis, Y. Ruan. *The Witten equation, mirror symmetry, and quantum singularity theory.* Ann. of Math. (2) 178 (2013), no. 1, 1–106.
[25] E. Frenkel, A. Givental, T. Milanov. *Soliton equations, vertex operators, and simple singularities.* Funct. Anal. Other Math. 3 (2010), no. 1, 47–63.
[26] I. Frenkel, V. Kac. *Basic representations of affine Lie algebras and the dual resonance models.* Invent. Math. 62 (1980/81), no. 1, 23–66.
[27] S. Galkin, V. Golyshev, H. Iritani. *Gamma classes and quantum cohomology of Fano manifolds: Gamma conjectures.* arXiv:1404.6407.
[28] E. Getzler. *Intersection theory on $\overline{\mathcal{M}}_{1,4}$ and elliptic Gromov-Witten invariants.* J. Amer. Math. Soc. 10 (1997), no. 4, 973–998.
[29] E. Getzler. *The Toda conjecture.* In: "Symplectic geometry and mirror symmetry (Seoul, 2000)", 51–79, World Sci. Publ., River Edge, NJ, 2001.
[30] A. Givental. *A mirror theorem for toric complete intersections.* In: "Topological field theory, primitive forms and related topics (Kyoto, 1996)", 141–175, Progr. Math., 160, Birkhäuser Boston, Boston, MA, 1998.
[31] A. Givental. *A_{n-1} singularities and nKdV Hierarchies.* Mosc. Math. J. 3 (2003), no. 2, 475–505.
[32] A. Givental. *Semi-simple Frobenius structures at higher genus.* Internat. Math. Res. Notices 2001, no. 23, 1265–1286.
[33] A. Givental. *Gromov–Witten invariants and quantization of quadratic Hamiltonians.* Mosc. Math. J. 1 (2001), no. 4, 551–568.
[34] A. Givental, T. Milanov. *Simple singularities and integrable hierarchies.* In: "The breadth of symplectic and Poisson geometry". 173–201. Progr. Math., 232, Birkhäuser Boston, Boston, MA, 2005
[35] P. Griffiths. *On the periods of certain rational integrals: I.* Ann. of Math. (2) 90 (1969), no. 3, 460–495.
[36] C. Hertling. *Frobenius manifolds and moduli spaces for singularities.* Cambridge Tracts in Math., 151, Cambridge Univ. Press, Cambridge, 2002
[37] E. Ionel. *Topological Recursive Relations in $H^{2g}(\mathcal{M}_{g,n})$.* Invent. Math. 148 (2002), no. 3, 627–658.
[38] H. Iritani. *Convergence of quantum cohomology by quantum Lefschetz.* J. Reine Angew. Math. 610 (2007), 29–69.
[39] H. Iritani. *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds.* Adv. Math. 222 (2009), no. 3, 1016–1079.
[40] H. Iritani. *Quantum cohomology and periods.* Ann. Inst. Fourier 61 (2011), 7, 2909–2958.
[41] Y. Ishibashi, Y. Shiraishi, A. Takahashi. *Primitive forms for affine cusp polynomials.* arXiv:1211.1128.
[42] D. Joe, B. Kim. *Equivariant mirrors and the Virasoro conjecture for flag manifolds.* Int. Math. Res. Not. 2003, no. 15, 859–882.
[43] P. Johnson. *Equivariant Gromov-Witten theory of one dimensional stacks.* arXiv:0903.1068.
[44] V. Kac. *Infinite dimensional Lie algebras.* Third edition. Cambridge University Press, 1990.
[45] V. Kac. *Vertex algebras for beginners.* Second edition. University Lecture Series, 10. American Mathematical Society, Providence, RI, 1998.
[46] V. Kac, D. Peterson. 112 *constructions of the basic representation of the loop group of E_8.* In: "Symposium on anomalies, geometry, topology (Chicago, Ill., 1985)". 276–298. World Sci. Publishing, Singapore, 1985.
[47] L. Katzarkov, M. Kontsevich, T. Pantev. *Hodge theoretic aspects of mirror symmetry.* arXiv:0806.0107.

[48] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*. Comm. Math. Phys. 147 (1992), no. 1, 1–23.

[49] M. Krawitz, Y. Shen. *Landau-Ginzburg/Calabi-Yau correspondence of all genera for elliptic orbifold* $\mathbb{P}^1$. arXiv:1106.6270 [math.AG]

[50] S.-Q. Liu, Y. Ruan, Y. Zhang. *BCFG Drinfeld-Sokolov Hierarchies and FJRW-Theory*. arXiv:1312.7227.

[51] T. Milanov. *Gromov–Witten theory of* $\mathbb{C}P^1$ *and integrable hierarchies*. arXiv:math-ph/0605001.

[52] T. Milanov. *Hirota Quadratic Equations for the Extended Toda Hierarchy*. Duke Math. J. 138 (2007), no. 1, 161–178.

[53] T. Milanov, H.-H. Tseng. *The spaces of Laurent polynomials, Gromov–Witten theory of* $\mathbb{P}^1$*-orbifolds, and integrable hierarchies*. J. Reine Angew. Math. 622 (2008), 189–235.

[54] A. Okounkov, R. Pandharipande. *The equivariant Gromov-Witten theory of* $\mathbf{P}^1$. Ann. of Math. (2) 163 (2006), no. 2, 561–605.

[55] P. Rossi. *Gromov–Witten theory of orbi-curves, the space of tri-polynomials, and symplectic field theory of Seifert fibrations*. Math. Ann. 348 (2010), no. 2, 265–287.

[56] K. Saito. *On Periods of Primitive Integrals, I.* Preprint RIMS (1982).

[57] K. Saito, A. Takahashi. *From primitive forms to Frobenius manifolds.* In: "From Hodge theory to integrability and TQFT tt*-geometry". 31–48, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008

[58] R. Stekolshchik. *Notes on Coxeter transformations and the McKay correspondence.* Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg, 2008.

[59] R. Steinberg. *Finite subgroups of* SU_2*, Dynkin diagrams and affine Coxeter elements.* Pacific J. Math. 118 (1985), no. 2, 2587–598.

[60] A. Takahashi. *Mirror symmetry between orbifold projective lines and cusp singularities.* to appear in Adv. Stud. Pure Math.

[61] C. Teleman. *The structure of 2D semi-simple field theories.* Invent. Math. 188 (2012), no. 3, 525–588.

[62] H.-H. Tseng. *Orbifold quantum Riemann-Roch, Lefschetz and Serre.* Geom. Topol. 14 (2010), no. 1, 1–81.

[63] H.-H. Tseng. *On degree-0 elliptic orbifold Gromov-Witten invariants.* Internat. Math. Res. Notices 2011, no. 11, 2444–2468.

[64] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*. In: "Surveys in differential geometry," 243–310, Lehigh Univ., Bethlehem, PA, 1991.

[65] E. Witten, *Algebraic geometry associated with matrix models of two-dimensional gravity.* Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, 235–269.

[66] Y. Zhang. *On the* $\mathbb{C}P^1$ *topological sigma model and the Toda lattice hierarchy.* J. Geom. Phys. 40 (2002), no. 3-4, 215–232.

Kavli IPMU, University of Tokyo (WPI), Japan
E-mail address: `todor.milanov@ipmu.jp`

Kavli IPMU, University of Tokyo (WPI), Japan
E-mail address: `yefeng.shen@ipmu.jp`

Department of Mathematics, Ohio State University, USA
E-mail address: `hhtseng@math.ohio-state.edu`