HIGHER GENUS RELATIVE AND ORBIFOLD GROMOV-WITTEN INVARIANTS

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ABSTRACT. Given a smooth projective variety X and a smooth divisor $D \subset X$. We study relative Gromov-Witten invariants of (X, D) and the corresponding orbifold Gromov-Witten invariants of the r-th root stack $X_{D,r}$. For sufficiently large r, we prove that orbifold Gromov-Witten invariants of $X_{D,r}$ are polynomials in r. Moreover, higher genus relative Gromov-Witten invariants of (X, D) are exactly the constant terms of the corresponding higher genus orbifold Gromov-Witten invariants of $X_{D,r}$. We also provide a new proof for the equality between genus zero relative and orbifold Gromov-Witten invariants, originally proved by Abramovich-Cadman-Wise [2]. When r is sufficiently large and $X = C$ is a curve, we prove that stationary relative invariants of C are equal to the stationary orbifold invariants in all genera.

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1. Introduction

Gromov-Witten theory associated to a smooth projective variety X is an enumerative theory about counting curves in X with prescribed conditions. Gromov-Witten invariants are defined as intersection numbers on the moduli space $\overline{M}_{g,n,d}(X)$ of n-pointed, genus g, degree $d \in H_2(X,\mathbb{Z})$, stable maps to X.

Given a smooth divisor D in X, one can study the enumerative geometry of counting curves with prescribed tangency conditions along the divisor D. There are at least two ways to impose tangency conditions.

1.1. Relative Gromov-Witten Invariants.

The first way to impose tangency conditions is to consider relative stable maps to (X,D) developed in [15], [20], [21].

For a degree $d \in H_2(X,\mathbb{Z})$, we consider a partition $\vec{k} = (k_1,\ldots,k_m) \in (\mathbb{Z}_{>0})^m$ of $\int_d [D]$. That is,

$$\sum_{i=1}^m k_i = \int_d [D].$$

A *cohomology weighted partition* $\mathbf{k}$ of $\int_d[D]$ is a partition $\vec{k}$ whose parts are weighted by cohomology classes of $H^*(D,\mathbb{Q})$. More precisely,

$$\mathbf{k} = \{(k_1,\delta_1),\ldots,(k_m,\delta_m)\}$$

such that

- $\sum_{i=1}^m k_i = \int_d [D]$;
- $\delta_i \in H^*(D,\mathbb{Q}), \quad 1 \le i \le m.$

Cohomology weighted partitions will appear in the degeneration formula for Gromov-Witten invariants.

Convention 1.1. When X is a curve and D is a point, the cohomology weights are just the identity class of $H^*(pt,\mathbb{Q})$. In this case, we will not distinguish $\mathbf{k}$ and $\vec{k}$.

We consider the moduli space $\overline{M}_{g,\vec{k},n,d}(X,D)$ of $(m+n)$-pointed, genus g, degree $d \in H_2(X,\mathbb{Z})$, relative stable maps to (X,D) such that the relative conditions are given by the partition $\vec{k}$. We assume the first m marked points are relative marked points and the last n marked points are non-relative marked points. Let ev_i be the i-th evaluation map, where

$$\begin{aligned} \mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X,D) &\to D, \quad \text{for } 1 \le i \le m; \\ \mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X,D) &\to X, \quad \text{for } m+1 \le i \le m+n. \end{aligned}$$

There is a stabilization map

$$s : \overline{M}_{g,\vec{k},n,d}(X,D) \to \overline{M}_{g,m+n,d}(X).$$

Write $\bar{\psi}_i = s^*\psi_i$ which is the class pullback from the corresponding descendant class on the moduli space $\overline{M}_{g,m+n,d}(X)$ of stable maps to X. Consider

- $\delta_i \in H^*(D,\mathbb{Q})$, for $1 \le i \le m$.
- $\gamma_{m+i} \in H^*(X,\mathbb{Q})$, for $1 \le i \le n$.
- $a_i \in \mathbb{Z}_{\ge 0}$, for $1 \le i \le m+n$.

Relative Gromov-Witten invariants of (X,D) are defined as

(1)
$$\left\langle \prod_{i=1}^{m} \tau_{a_i}(\delta_i) \middle| \prod_{i=1}^{n} \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{(X,D)} :=$$
$$\int_{[\overline{M}_{g,\vec{k},n,d}(X,D)]^{vir}} \bar{\psi}_1^{a_1} \mathrm{ev}_1^*(\delta_1)\cdots\bar{\psi}_m^{a_m} \mathrm{ev}_m^*(\delta_m)\bar{\psi}_{m+1}^{a_{m+1}} \mathrm{ev}_{m+1}^*(\gamma_{m+1})\cdots\bar{\psi}_{m+n}^{a_{m+n}} \mathrm{ev}_{m+n}^*(\gamma_{m+n}).$$

We refer to [15], [20], [21] for more details about the construction of relative Gromov-Witten theory.

1.2. **Orbifold Gromov-Witten Invariants.** Another way to impose tangency conditions is to consider orbifold Gromov-Witten invariants of the r-th root stack $X_{D,r}$ of X for a positive integer r [7]. By [13], root construction is essentially the only way to construct stack structures in codimension one. The construction of root stacks can be found in [5, Appendix B] and [7].

Example 1.2. For a positive integer r, the r-th root stack of $\mathbb{P}^1$ over the point $0 \in \mathbb{P}^1$ is denoted by $\mathbb{P}^1[r]$. The root stack $\mathbb{P}^1[r]$ is the weighted projective line with a single stack point of order r at 0. We will be dealing with this stack when we study stationary Gromov-Witten theory of curves in Section 5.

The evaluation maps for orbifold Gromov-Witten invariants land on the inertia stack of the target orbifold. The coarse moduli space $\underline{I}X_{D,r}$ of the inertia stack of the root stack $X_{D,r}$ can be decomposed into disjoint union of r components

$$\underline{I}X_{D,r} = X \sqcup \coprod_{i=1}^{r-1} D,$$

where there are $r-1$ components isomorphic to D. These components are called twisted sectors.

The partition $\vec{k}$ can be used to impose orbifold data of orbifold stable maps as follows. We assume that $r > k_i$, for all $1 \leq i \leq m$. For orbifold invariants of the root stack $X_{D,r}$, we consider the moduli space $\overline{M}_{g,\vec{k},n,d}(X_{D,r})$ of $(m+n)$-pointed, genus g, degree d, orbifold stable maps to $X_{D,r}$ whose orbifold data is given by the partition $\vec{k}$, such that

- for $1 \leq i \leq m$, the coarse evaluation map ev_i at the i-th marked point lands on the twisted sector D with age k_i/r. These marked points are orbifold marked points.
- the coarse evaluation maps ev_i at the last n marked points all land on the identity component X of the coarse moduli space of the inertia stack $IX_{D,r}$. These marked points are non-orbifold marked points.

Orbifold Gromov-Witten invariants of $X_{D,r}$ are defined as

(2)
$$\left\langle \prod_{i=1}^{m} \tau_{a_i}(\delta_i) \prod_{i=1}^{n} \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{X_{D,r}} :=$$
$$\int_{[\overline{M}_{g,\vec{k},n,d}(X_{D,r})]^{vir}} \bar{\psi}_1^{a_1} \mathrm{ev}_1^*(\delta_1) \cdots \bar{\psi}_m^{a_m} \mathrm{ev}_m^*(\delta_m) \bar{\psi}_{m+1}^{a_{m+1}} \mathrm{ev}_{m+1}^*(\gamma_{m+1}) \cdots \bar{\psi}_{m+n}^{a_{m+n}} \mathrm{ev}_{m+n}^*(\gamma_{m+n}),$$

where the descendant class $\bar{\psi}_i$ is the class pullback from the corresponding descendant class on the moduli space $\overline{M}_{g,m+n,d}(X)$ of stable maps to X.

The basic constructions and fundamental properties of orbifold Gromov-Witten theory can be found in [1], [4], [5], [10] and [30].

1.3. **Relations and Questions.** By [24, Theorem 2], relative Gromov-Witten invariants of a smooth pair (X,D) can be uniquely and effectively reconstructed from the Gromov-Witten theory of X, the Gromov-Witten theory of D, and the restriction map $H^*(X,\mathbb{Q}) \to H^*(D,\mathbb{Q})$. On the other hand, for the smooth pair (X,D), we conjectured[1] and proved that the Gromov-Witten theory of root stack $X_{D,r}$ is also determined by the Gromov-Witten theory of X, the Gromov-Witten theory of D, and the restriction map

[1]For smooth Deligne-Mumford stacks Y and a smooth divisor D, we proved the conjecture when D is disjoint from the locus of stack structures of X [31]. The more general version of our conjecture is recently proved by [9].

$H^*(X,\mathbb{Q}) \to H^*(D,\mathbb{Q})$ [31]. This provides another evidence that these two theories may be related.

The relationship between relative and orbifold Gromov-Witten invariants in genus zero has been established by Abramovich-Cadman-Wise [2] when the target is a smooth pair (X,D). The relationship was first observed in [8] for genus zero maps to $X = \mathbb{P}^2$ with tangency conditions along a smooth plane cubic D. It was observed that, for large and divisible r, orbifold Gromov-Witten invariants of the root stack $X_{D,r}$ stabilize and coincide with relative Gromov-Witten invariants of (X,D). It was proved in [2] that genus zero orbifold Gromov-Witten invariants of $X_{D,r}$ for large and divisible r agree with genus zero relative Gromov-Witten invariants of (X,D) for any X and any D. The proof used comparison of virtual fundamental classes of different moduli spaces.

The goal of this paper is to study the relationship between these relative and orbifold Gromov-Witten invariants in all genera. In general the result of [2] does not hold for higher genus invariants, as shown by a counterexample (due to D. Maulik) for genus 1 invariants in [2, Section 1.7]. Naturally, we ask the following questions.

Question 1.3. What is the precise relationship between relative and orbifold Gromov-Witten invariants in higher genus?

Question 1.4. Will the equality between higher genus relative and orbifold Gromov-Witten invariants hold under some assumptions?

In this paper, we answer the first question for invariants of smooth projective varieties and answer the second question for invariants of target curves.

1.4. **Higher Genus Invariants of General Targets.** For a smooth pair (X,D), the orbifold invariants of $X_{D,r}$ in general depend on r. On the other hand, the relative invariants of (X,D) do not depend on r. Hence, it is not expected that the exact equality between invariants of $X_{D,r}$ and (X,D) holds in general. The precise relationship is the following:

Theorem 1.5. *Given a smooth projective variety X, its smooth divisor $D \subset X$, and a sufficiently large integer r, the orbifold Gromov-Witten invariant*

$$\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{X_{D,r}}$$

of $X_{D,r}$ is a polynomial in r. Moreover, relative Gromov-Witten invariants of (X,D) are the r^0-coefficients of orbifold Gromov-Witten invariants of $X_{D,r}$. More precisely,

$$\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \middle| \prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{(X,D)} = \left[\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{X_{D,r}} \right]_{r^0}, \tag{3}$$

where the notation $[]_{r^0}$ stands for taking the coefficient of r^0-term of a polynomial in r.

Remark 1.6. Theorem 1.5 can also be formulated on the cycle level. This is because the techniques that we are using in this paper are the degeneration formula and the virtual localization formula. Both formulas are on the level of virtual cycles. The virtual class version of Theorem 1.5 can be proved by straightforward adaptations of the arguments in this paper. In particular, a virtual class version of Theorem 1.5 is stated in [11] for genus zero invariants and will appear in [12] for higher genus invariants. Note that, the results in [11] and [12] extend the result of this paper to include relative invariants with negative contact orders.

Theorem 1.5 directly implies the following result.

Corollary 1.7. *Relative Gromov-Witten invariants of (X, D) are completely determined by orbifold Gromov-Witten invariants of the root stack $X_{D,r}$ for r sufficiently large.*

Example 1.8. In genus zero, relative invariants of (X, D) are equal to orbifold invariants of $X_{D,r}$, for r sufficiently large [2]. There is a counterexample in genus one given by D. Maulik in [2, Section 1.7]. It is worth to point out that Maulik's counterexample does fit into our result. The example is as follows. Let $X = E \times \mathbb{P}^1$, where E is an elliptic curve. Consider the divisor $D = X_0 \cup X_\infty$, the union of 0 and ∞ fibers of X over $\mathbb{P}^1$. One can consider the root stack $X_{D,r,s}$ obtained from taking r-th root along X_0 and s-th root along X_∞. One can compare relative invariants of (X, D) and orbifold invariants of the root stack $X_{D,r,s}$. Taking a fiber class $f \in H_2(X)$ of the fibration $X \to \mathbb{P}^1$, the genus one relative and orbifold invariants with no insertions are computed in [2, Section 1.7]:

$$\begin{aligned}\langle\rangle_{1,f}^{(X,D)} &= 0;\\ \langle\rangle_{1,f}^{X_{D,r,s}} &= r+s.\end{aligned}$$

Hence, we have

$$\langle\rangle_{1,f}^{(X,D)} = \left[\langle\rangle_{1,f}^{X_{D,r,s}}\right]_{r^0 s^0}.$$

The proof of Theorem 1.5 follows from degeneration formula and virtual localization computation.

By degeneration formula, we can reduce Theorem 1.5 to the comparison between the following invariants of (relative) local models. We can consider the degeneration of X (resp. $X_{D,r}$) to the normal cone of D (resp. $\mathcal{D}_r$). Indeed, let $Y := \mathbb{P}(\mathcal{O}_D \oplus N)$ where N is the normal bundle of $D \subset X$, we will consider relative invariants of $(Y, D_0 \cup D_\infty)$, where D_0 and D_∞ are zero and infinity sections respectively. On the other hand, we will consider orbifold-relative invariants of $(Y_{D_0,r}, D_\infty)$, where $Y_{D_0,r}$ is the r-th root stack of the zero section D_0 of Y. Theorem 1.5 reduces to the comparison between relative invariants of $(Y, D_0 \cup D_\infty)$ and orbifold-relative invariants of $(Y_{D_0,r}, D_\infty)$.

The relationship between invariants of $(Y, D_0 \cup D_\infty)$ and of (Y_{D_0}, D_∞) can be found by $\mathbb{C}^*$-virtual localization. Localization computation relates both relative invariants of $(Y, D_0 \cup D_\infty)$ and orbifold-relative invariants of $(Y_{D_0,r}, D_\infty)$ to rubber integrals with the base variety D.

A key point for the localization computation is the polynomiality of certain cohomology classes on the moduli space $\overline{M}_{g,n,d}(D)$ of stable maps to D which is proved in [17, Corollary 11], see Section 3.2.3. For the relationship between relative and orbifold Gromov-Witten theory of curves, the corresponding result is the polynomiality of certain tautological classes on the moduli space $\overline{M}_{g,n}$ of stable curves proved in [16, Proposition 5].

We can use the localization computation in the proof of Theorem 1.5, without the need of polynomiality, to provide a new proof of the main theorem of [2] in Section 4. The different behavior between genus zero invariants and higher genus invariants can be seen directly from the difference of their localization computations.

We restrict our discussions to the case when X is a smooth projective variety, but Theorem 1.5 can be extended to the case when X is an orbifold. The key ingredient is the generalization of the polynomiality in [17] to orbifolds. When X is a one dimensional orbifold, we only need the orbifold version of the polynomiality in [16], which has been proved in our previous work [32] on double ramification cycles on the moduli spaces of admissible covers.

1.5. **Stationary Invariants of Target Curves.** We answer Question 1.4 for stationary Gromov-Witten invariants of target curves.

Gromov-Witten theory of target curves has been completely determined in the trilogy [27], [28] and [29] by Okounkov-Pandharipande. Gromov-Witten theory of target curves C is closely related to Hurwitz theory of enumerations of ramified covers of C. The GW/H correspondence proved in [27] showed a correspondence between stationary Gromov-Witten invariants of C and Hurwitz numbers of C. The main result of [28] showed that equivariant Gromov-Witten theory of $\mathbb{P}^1$ is governed by the 2-Toda hierarchy. The Virasoro constraints for target curves were proven in [29], the third part of the trilogy.

Moreover, Gromov-Witten theory of $\mathbb{P}^1$ can be considered as a more fundamental object than Gromov-Witten theory of a point [28]. The stationary Gromov-Witten invariants of $\mathbb{P}^1$ arise as Eynard-Orantin invariants [25], [6].

As an application, Gromov-Witten theory of a point arises in the asymptotics of large degree Gromov-Witten invariants of $\mathbb{P}^1$ [25], [26].

Now we consider stationary invariants of curves. Let $X = C$ be a smooth projective curve and q be a point in C, we consider the following stationary relative invariants of (C,q):

$$\langle \prod_{i=1}^n \tau_{a_{m+i}}(\omega)|k\rangle^{(C,q)}_{g,n,\vec{k},d} := \int_{[\overline{M}_{g,n,\vec{k},d}(C,q)]^{\mathrm{vir}}} \prod_{i=1}^n \psi_{m+i}^{a_{m+i}} \mathrm{ev}^*_{m+i}\,\omega, \tag{4}$$

where $\omega \in H^2(C,\mathbb{Q})$ denote the class that is Poincaré dual to a point.

We consider the root stack $C[r]$ of C by taking r-th root along q. The stationary orbifold invariants of $C[r]$ are defined as

$$\langle \prod_{i=1}^n \tau_{a_i}(\omega)\rangle^{C[r]}_{g,n,\vec{k},d} := \int_{[\overline{M}_{g,n,\vec{k},d}(C[r])]^{\mathrm{vir}}} \prod_{i=1}^m \mathrm{ev}^*_i(\mathbf{1}_{k_i/r}) \prod_{i=1}^n \bar{\psi}_{m+i}^{a_{m+i}} \mathrm{ev}^*_{m+i}\,\omega, \tag{5}$$

where $\mathbf{1}_{k_i/r}$ is the identity class in twisted sector of age k_i/r.

Theorem 1.9. *Let C be a smooth target curve in any genus. When r is sufficiently large, the stationary Gromov-Witten invariants of (C,q) are equal to the stationary Gromov-Witten invariants of the root stack $C[r]$. That is,*

$$(4) = (5).$$

Remark 1.10. Theorem 1.9 can be extended slightly by string equations and dilaton equations for Gromov-Witten theory of (C,q) and $C[r]$ with insertions $\tau_0(\mathbf{1})$ and $\tau_1(\mathbf{1})$.

The proof is based on the degeneration of the target and the equality in genus zero.

As an application for the equality between stationary invariants. We obtain the GW/H correspondence for orbifold Gromov-Witten invariants of the root stack $C[r_1,\ldots,r_l]$ obtained by taking sufficiently large r_i-th root at the point $q_i \in C$ for $1 \le i \le l$.

1.6. **Further Discussions.** The exact equality between stationary relative invariants of curves and stationary orbifold invariants of curves is in fact a unique feature for Gromov-Witten theory of curves. The higher dimensional analogy of the equality between stationary invariants of curves is not correct [2]. It can already be seen from the counterexample given by Maulik in [2, Section 1.7]. The counterexample is about invariants of $X := E \times \mathbb{P}^1$, where E is an elliptic curve, with no insertions. These invariants can be viewed as stationary invariants without any insertions. Moreover, the proof

[2]In this context, based on the degeneration and localization analysis, a reasonable analogy of stationary invariants for higher dimensional target is to require the restrictions of all cohomological insertions to D vanish.

for the equality of stationary invariants of curves in Section 5.1 used degeneration formula to reduce the equality to the case of invariants with no insertions. For Gromov-Witten theory of curves, the equality reduces to the trivial case. It does not reduce to the trivial case beyond Gromov-Witten theory of curves. Indeed, Maulik's counterexample shows that the equality is not true in general. In [2, Section 1.7], this counterexample is interpreted as a result of the nontriviality of the Picard group of the elliptic curve E.

1.7. **Plan of the Paper.** The paper is organized as follows.

In Section 2, we reduce the comparison between relative and orbifold invariants to (relative) local models by applying degeneration formulas to relative and orbifold invariants. In Section 3, we prove Theorem 1.5 for local models by virtual localization. Our localization computation is also used in Section 4 to provide a new proof for the equality between genus zero relative and orbifold invariants. In Section 5, we present the proof of Theorem 1.9. As an easy consequence of Theorem 1.9, we extend the GW/H correspondence to stationary orbifold invariants of curves when the root constructions on the curve are taken to be sufficiently large.

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2. Degeneration

In this section, we show that Theorem 1.5 and Theorem 1.9 can be reduced to the case of $\mathbb{P}^1$-bundles by degeneration formula. It can be understood by observing that the comparison between relative and orbifold invariants is "local over the divisor D", hence it is sufficient to compare invariants of local models. The degeneration formula gives the precise statement for this observation.

Following [31], we consider the degeneration of $X_{D,r}$ to the normal cone of $\mathcal{D}_r$, the divisor of $X_{D,r}$ lying over $D \subset X$. The degeneration formula shows that orbifold Gromov-Witten invariants of $X_{D,r}$ are expressed in terms of relative Gromov-Witten invariants of $(X_{D,r}, \mathcal{D}_r)$ and of $(\mathcal{Y}, \mathcal{D}_\infty)$, where

$\mathcal{Y} := \mathbb{P}(\mathcal{O} \oplus \mathcal{N})$ is obtained from the normal bundle $\mathcal{N}$ of $\mathcal{D}_r \subset X_{D,r}$; the infinity section $\mathcal{D}_\infty$ of $\mathcal{Y} \to \mathcal{D}_r$ is identified with $\mathcal{D}_r \subset X_{D,r}$ under the gluing.

By [3, Proposition 4.5.1], relative Gromov-Witten invariants of $(X_{D,r}, \mathcal{D}_r)$ are equal to relative Gromov-Witten invariants of (X, D) and relative Gromov-Witten invariants of $(\mathcal{Y}, \mathcal{D}_\infty)$ are equal to relative Gromov-Witten invariants of $(Y_{D_0,r}, D_\infty)$, where $Y := \mathbb{P}(\mathcal{O} \oplus N)$ is obtained from the normal bundle N of $D \subset X$ and $Y_{D_0,r}$ is the root stack of Y constructed by taking r-th root along the zero section D_0 of $Y \to D$.

Then, the degeneration formula for the orbifold Gromov-Witten invariants of $X_{D,r}$ is indeed written as

(6)

$$\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{X_{D,r}} =$$

$$\sum \frac{\prod_i \eta_i}{|\mathrm{Aut}(\eta)|} \left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \prod_{i\in S} \tau_{a_{m+i}}(\gamma_{m+i}) \middle| \eta \right\rangle_{g,\vec{k},|S|,\vec{\eta},d_1}^{\bullet,(Y_{D_0,r},D_\infty)} \left\langle \eta^\vee \middle| \prod_{i\notin S} \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{g,\vec{\eta},n-|S|,d_2}^{\bullet,(X,D)},$$

where $\eta^\vee$ is defined by taking the Poincaré duals of the cohomology weights of the cohomology weighted partition η; $|\mathrm{Aut}(\eta)|$ is the order of the automorphism group $\mathrm{Aut}(\eta)$ preserving equal parts of the cohomology weighted partition η. The sum is over all splittings of g and d, all choices of $S \subset \{1, \dots, n\}$, and all intermediate cohomology weighted partitions η. The superscript $\bullet$ stands for possibly disconnected Gromov-Witten invariants.

Remark 2.1. The degeneration of $X_{D,r}$ can also be constructed as follows. One can first consider the degeneration of X to the normal cone of D. The total space of the degeneration admits a divisor B whose restriction to the general fiber is D and restriction to the special fiber is D_0, the zero section of $Y = \mathbb{P}(\mathcal{O}_D \oplus N)$. Taking the r-th root stack along B, we have a flat degeneration of $X_{D,r}$ to X glued together with $Y_{D_0,r}$ along the infinity section $D_\infty \subset Y_{D_0,r}$. It yields the same degeneration formula as in (6).

For relative Gromov-Witten invariants of (X, D), we consider the degeneration of X to the normal cone of D. It yields the following degeneration formula:

(7)

$$\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \middle| \prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{(X,D)} =$$

$$\sum \frac{\prod_i \eta_i}{|\mathrm{Aut}(\eta)|} \left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \middle| \prod_{i\in S} \tau_{a_{m+i}}(\gamma_{m+i}) \middle| \eta \right\rangle_{g,\vec{k},|S|,\vec{\eta},d_1}^{\bullet,(Y,D_0\cup D_\infty)} \left\langle \eta^\vee \middle| \prod_{i\notin S} \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{g,\vec{\eta},n-|S|,d_2}^{\bullet,(X,D)}.$$

The sum is also over all intermediate cohomology weighted partitions η and all splitting of g, d and n.

The degeneration formulae (6) and (7) take the same form. Hence, the comparison between orbifold invariants of $X_{D,r}$ and relative invariants of (X,D) reduces to the comparison between invariants of $(Y_{D_0,r},D_\infty)$ and invariants of $(Y,D_0\cup D_\infty)$. More precisely, it is sufficient to compare the relative invariant

$$\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i)\left|\prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i})\right|\mu\right\rangle^{(Y,D_0\cup D_\infty)}_{g,\vec{k},n,\vec{\mu},d} \tag{8}$$

of $(Y,D_0\cup D_\infty)$ and the orbifold-relative invariant

$$\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i)\prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i})\right|\mu\Big\rangle^{(Y_{D_0,r},D_\infty)}_{g,\vec{k},n,\vec{\mu},d} \tag{9}$$

of $(Y_{D_0,r},D_\infty)$, where μ is a cohomology weighted partition of $\int_d[D_\infty]$.

Remark 2.2. By the degeneration formula, we should compare disconnected invariants instead of connected invariants. However, the relationship between disconnected invariants follows from the relationship between connected invariants. Hence, it is sufficient to compare connected invariants.

As a result, the comparison can be considered as *local* over the relative/orbifold divisor D. The pairs $(Y_{D_0,r},D_\infty)$ and $(Y,D_0\cup D_\infty)$ can be viewed as (relative) local models of $X_{D,r}$ and (X,D) . Therefore, Theorem 1.5 follows from the following theorem for local models.

Theorem 2.3. *For r sufficiently large, the orbifold-relative invariant*

$$\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i)\prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i})\right|\mu\Big\rangle^{(Y_{D_0,r},D_\infty)}_{g,\vec{k},n,\vec{\mu},d}$$

is a polynomial in r and,

(10)

$$\left[\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i)\prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i})\right|\mu\Big\rangle^{(Y_{D_0,r},D_\infty)}_{g,\vec{k},n,\vec{\mu},d}\right]_{r^0} = \left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i)\left|\prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i})\right|\mu\right\rangle^{(Y,D_0\cup D_\infty)}_{g,\vec{k},n,\vec{\mu},d}.$$

Similarly, Theorem 1.9 follows from the following theorem for $(\mathbb{P}^1[r],\infty)$ and $(\mathbb{P}^1,0,\infty)$.

Theorem 2.4. *For r sufficiently large, the stationary orbifold-relative invariants of $(\mathbb{P}^1[r],\infty)$ are equal to the stationary relative invariants of $(\mathbb{P}^1,0,\infty)$:*

$$\langle \mathbf{k}|\prod_{i=1}^n \tau_{a_{m+i}}(\omega)|\mu\rangle^{(\mathbb{P}^1,0,\infty)}_{g,\vec{k},n,\vec{\mu},d} = \langle \prod_{i=1}^n \tau_{a_{m+i}}(\omega)|\mu\rangle^{(\mathbb{P}^1[r],\infty)}_{g,\vec{k},n,\vec{\mu},d}. \tag{11}$$

Remark 2.5. Theorem 2.3 and Theorem 2.4 can also be stated for disconnected invariants, since the proofs of Theorem 2.3 and Theorem 2.4 also work for disconnected invariants.

3. Local Model

In this section, we prove Theorem 2.3 by using virtual localization to obtain identities of cycle classes on moduli spaces.

Let D be a smooth projective variety equipped with a line bundle L, and let Y be the total space of the $\mathbb{P}^1$-bundle

$$\pi : \mathbb{P}(\mathcal{O}_D \oplus L) \to D.$$

Following [24], let $\delta_{b_1}, \ldots, \delta_{b_s}$ be a basis of $H^*(D, \mathbb{Q})$. We view δ_{b_i} as an element of $H^*(Y, \mathbb{Q})$ via pull-back by π. Let $[D_0], [D_\infty] \in H^2(Y, \mathbb{Q})$ denote the cohomology classes associated to the zero and infinity divisors. The cohomological insertions of the invariants will be taken from the following classes in $H^*(Y, \mathbb{Q})$:

$$\delta_{b_1}, \ldots, \delta_{b_s}, [D_0] \cdot \delta_{b_1}, \ldots, [D_0] \cdot \delta_{b_s}, [D_\infty] \cdot \delta_{b_1}, \ldots, [D_\infty] \cdot \delta_{b_s}.$$

We write $Y_{D_0,r}$ for the root stack of Y constructed by taking rth root along the zero section D_0. The r-th root of D_0 is denoted by $\mathcal{D}_r$.

3.1. Relative Invariants.

Consider the moduli space $\overline{M}_{g,\vec{k},n,\vec{\mu}}(Y, D_0 \cup D_\infty)$ of relative stable maps to $(Y, D_0 \cup D_\infty)$ with tangency conditions at relative divisor D_0 (resp. D_∞) given by the partitions $\vec{k}$ (resp. $\vec{\mu}$) of $\int_d [D_0]$ (resp. $\int_d [D_\infty]$). The length of $\vec{\mu}$ is denoted by $l(\mu)$. Recall that the length of $\vec{k}$ is still denoted by m. The following relation between moduli space $\overline{M}_{g,\vec{k},n,\vec{\mu}}(Y, D_0 \cup D_\infty)$ of relative stable maps to rigid target and moduli space $\overline{M}_{g,\vec{k},n,\vec{\mu}}(Y, D_0 \cup D_\infty)^\sim$ of relative stable maps to non-rigid target is proven in [24].

Lemma 3.1 ([24], Lemma 2). *Let p be a non-relative marking with evaluation map*

$$\operatorname{ev}_p : \overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y, D_0 \cup D_\infty) \to Y.$$

Then, the following identities hold.

$$\begin{aligned}
[\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y, D_0 \cup D_\infty)^\sim]^{\mathrm{vir}} =& \epsilon_* \left(\operatorname{ev}_p^*([D_0]) \cap [\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y, D_0 \cup D_\infty)]^{\mathrm{vir}} \right) \\
=& \epsilon_* \left(\operatorname{ev}_p^*([D_\infty]) \cap [\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y, D_0 \cup D_\infty)]^{\mathrm{vir}} \right),
\end{aligned} \tag{12}$$

where

$$\epsilon : \overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y, D_0 \cup D_\infty) \to \overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y, D_0 \cup D_\infty)^\sim$$

is the canonical forgetful map.

The proof of Lemma 3.1 is through $\mathbb{C}^*$-localization on the moduli space $\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y, D_0 \cup D_\infty)$. The following identity directly follows from Lemma 3.1.

Lemma 3.2. *For $n > 0$,*

$$\left\langle \prod_{i=1}^{m} \tau_{a_i}(\delta_i) \middle| \tau_{a_{m+1}}([D_\infty]\cdot\delta_{m+1}) \prod_{i=m+2}^{m+n} \tau_{a_i}(\delta_i) \middle| \mu \right\rangle^{(Y,D_0\cup D_\infty)}_{g,\vec{k},n,\vec{\mu},d} = \left\langle \prod_{i=1}^{m} \tau_{a_i}(\delta_i) \middle| \prod_{i=m+1}^{m+n} \tau_{a_i}(\delta_i) \middle| \mu \right\rangle^{\sim,(Y,D_0\cup D_\infty)}_{g,\vec{k},n,\vec{\mu},d}, \tag{13}$$

where $\delta_i \in \pi^*(H^*(D,\mathbb{Q}))$, for $m+1 \le i \le m+n$, are cohomology classes pulled back from $H^*(D,\mathbb{Q})$.

3.2. **Orbifold-Relative Invariants.** We study $\mathbb{C}^*$-localization over the moduli space $\overline{M}_{g,\vec{k},n,\vec{\mu}}(Y_{D_0,r}, D_\infty)$ with prescribed orbifold and relative conditions given by $\vec{k}$ and $\vec{\mu}$ respectively. Our goal is to find an identity that is similar to identity (12), then relates orbifold-relative invariants of $(Y_{D_0,r}, D_\infty)$ to rubber integrals as well.

3.2.1. *Fixed Loci.* The fiberwise $\mathbb{C}^*$-action on

$$\pi : \mathbb{P}(\mathcal{O}_D \oplus L) \to D.$$

induces a $\mathbb{C}^*$-action on $Y_{D_0,r}$ and, hence, a $\mathbb{C}^*$-action on the moduli space $\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y_{D_0,r}, D_\infty)$. The $\mathbb{C}^*$-fixed loci of $Y_{D_0,r}$ are the zero divisor $\mathcal{D}_r$ and the infinity divisor D_∞. For a $\mathbb{C}^*$-invariant stable map, a component of the domain curve is called contracted if it lands on the zero section $\mathcal{D}_r$ or the infinity section D_∞; a component is called non-contracted if its image connects zero section $\mathcal{D}_r$ and the infinity section D_∞. For a $\mathbb{C}^*$-invariant stable map, the images of all marked points, nodes and contracted components are $\mathbb{C}^*$-fixed points. In other words, they land on the zero divisor $\mathcal{D}_r$ or the infinity divisor D_∞. They are connected by non-contracted components which land on the fiber of the projection from $Y_{D_0,r}$ to D.

The $\mathbb{C}^*$-fixed loci of $\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y_{D_0,r}, D_\infty)$ are labeled by decorated graphs Γ. We follow [22] for the notation of decorated graphs. A decorated graph Γ contains the following data.

- $V(\Gamma)$ is the set of vertices of Γ.
- $E(\Gamma)$ is the set of edges of Γ. We write $E(v)$ for the set of edges attached to the vertex $v \in V(\Gamma)$ and write $|E(v)|$ for the number of edges attached to the vertex $v \in V(\Gamma)$.
- The set of flags of Γ is defined to be

$$F(\Gamma) = \{(e,v) \in E(\Gamma) \times V(\Gamma) | v \in e\}.$$

- Each vertex v is decorated by the genus $g(v)$ and the degree $d(v) \in H_2(D,\mathbb{Z})$. The degree $d(v)$ must be an effective curve class. The genus and degree conditions are required

$$g = \sum_{v \in V(\Gamma)} g(v) + h^1\Gamma \quad \text{and} \quad d = \sum_{v \in V(\Gamma)} d(v).$$

- Each vertex v is labeled by 0 or ∞. The labeling map is denoted by
$$i : V(\Gamma) \to \{0, \infty\}.$$
- Each edge e is decorated by the degree $d_e \in \mathbb{Z}_{>0}$.
- The set of legs is in bijective correspondence with the set of markings. For $1 \le j \le m$, the legs are labeled by $k_j \in \mathbb{Z}_{>0}$ and are incident to vertices labeled 0. For $m+1 \le j \le m+n$, the legs are labeled by 0. For $m+n+1 \le j \le m+n+l(\mu)$, the legs are labeled by $\mu_{j-m-n} \in \mathbb{Z}_{>0}$ and are incident to vertices labeled ∞. We write $S(v)$ to denote the set of markings assigned to the vertex v.
- If the flag is at 0, then it is labeled by an element $k_{(e,v)} \in \mathbb{Z}_r$. In fact, in our example,
$$k_{(e,v)} = d_e,$$
by compatibility along the edge. See, for example, [18], [22] and [19].
- Γ is a connected graph, and Γ is bipartite with respect to labeling i. Each edge is incident to a vertex labeled by 0 and a vertex labeled by ∞.
- A vertex $v \in V(\Gamma)$ is stable if $2g(v)-2+\mathrm{val}(v) > 0$, where $\mathrm{val}(v)$ is the total numbers of marked points and incident edges associated to the vertex $v \in V(\Gamma)$. Otherwise, $v \in V(\Gamma)$ is called unstable. We write $V^S(\Gamma)$ for the set of stable vertices of Γ. We use $F^S(\Gamma)$ to denote the set of stable flags, that is, the set of flags whose associated vertices are stable.
- The compatibility condition at a vertex v over 0:
$$\sum_{j \in S(v)} k_j - \sum_{e \in E(v)} k_{(e,v)} = \int_{d(v)} c_1(L) \mod r. \tag{14}$$
The compatibility condition at a vertex is being used in the proof of [17, Lemma 12], which will be used later in this section.
- The compatibility condition at a vertex v over ∞:
$$\sum_{e \in E(v)} k_{(e,v)} - \sum_{j \in S(v)} \mu_{j-m-n} = \int_{d(v)} c_1(L).$$

The $\mathbb{C}^*$-fixed loci are described by decorated graphs as follows.

- There is a one-to-one correspondence between contracted components and the set of vertices $V(\Gamma)$.
- There is a one-to-one correspondence between non-contracted components and the set of edges $E(\Gamma)$. In other words, edges correspond to components that lie in the fibers of the projection from $Y_{D_0,r}$ to D.

- The degree $d(v) \in H_2(D,\mathbb{Z})$ associated to a vertex is the degree of the stable map to the contracted component. A vertex labeled by 0 or ∞ represents zero or infinity section of $Y_{D_0,r}$. In other words, $i(v) = 0$ if the contracted component maps to the zero section; $i(v) = \infty$ if the contracted component maps to the infinity section.
- For $1 \le j \le m$, the element k_j associated to a leg representing the monodromy at the orbifold marked point. Recall that k_j/r is the age at the orbifold marked point. For $m+1 \le j \le m+n$, the element 0 associated to a leg representing the trivial monodromy. For $m+n+1 \le j \le m+n+l(\mu)$, the element μ_{j-m-n} associated to a leg representing the contact order with the infinity divisor.

Recall that a vertex $v \in V(\Gamma)$ is unstable if $2g(v) - 2 + \mathrm{val}(v) \le 0$. The following statement about unstable vertices is proved in [17, Lemma 12].

Lemma 3.3 ([17], Lemma 12). *For r sufficiently large, the following two types of unstable vertices can not occur:*

- *v is labeled by 0, $g(v) = 0$, v carries no marking and one incident edge;*
- *v is labeled by 0, $g(v) = 0$, v carries no marking and two incident edges;*

Hence, for r sufficiently large, there are only two types of unstable vertices

- v is labeled by 0, $g(v) = 0$, v carries one marking and one incident edge;
- v is labeled by ∞, $g(v) = 0$, v carries one marking and one incident edge.

A vertex $v \in V(\Gamma)$ over the zero section $\mathcal{D}_r$ corresponds to a stable map contracted to $\mathcal{D}_r$ given by an element of $\overline{M}_{g(v),\mathrm{val}(v),d(v)}(\mathcal{D}_r)$. The stable maps over the infinity section D_∞ have two different forms.

- If the target expands, then the stable map is a possibly disconnected rubber map to $(Y, D_0 \cup D_\infty)$. The relative data is given by incident edges over D_0 of the rubber target $(Y, D_0 \cup D_\infty)$ and by the partition $\vec{\mu}$ over D_∞ of the rubber.
- If the target does not expand over D_∞, then the stable map has $l(\mu)$ preimages of $D_\infty \subset Y$. Each preimage is described by an unstable vertex of Γ over D_∞.

Therefore, if the target expands at D_∞, the $\mathbb{C}^*$-fixed locus corresponding to the decorated graph Γ is isomorphic to

$$\overline{M}_\Gamma = \prod_{v \in V^S(\Gamma), i(v)=0} \overline{M}_{g(v),\mathrm{val}(v),d(v)}(\mathcal{D}_r) \times \prod_{v \in V^S(\Gamma), i(v)=\infty} \overline{M}_{g(v),\mathrm{val}(v),d(v)}(Y, D_0 \cup D_\infty)^\sim$$

quotiented by the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ and the product $\prod_{e\in E(\Gamma)}\mathbb{Z}_{d_e}$ of cyclic groups associated to the edges.

If the target does not expand, then the moduli spaces of rubber maps do not appear and the invariant locus is a product of moduli space of stable maps to $\mathcal{D}_r$. That is,

$$\overline{M}_\Gamma = \prod_{v\in V^S(\Gamma), i(v)=0} \overline{M}_{g(v),\mathrm{val}(v),d(v)}(\mathcal{D}_r).$$

The natural morphism

$$\iota : \overline{M}_\Gamma \to \overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y_{D_0,r}, D_\infty)$$

is of degree $|\mathrm{Aut}(\Gamma)|\prod_{e\in E(\Gamma)} d_e$. The virtual localization formula is written as

$$(15)\quad [\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y_{D_0,r}, D_\infty)]^{\mathrm{vir}} = \sum_\Gamma \frac{1}{|\mathrm{Aut}(\Gamma)|\prod_{e\in E(\Gamma)} d_e}\cdot \iota_*\left(\frac{[\overline{M}_\Gamma]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}\right),$$

where $e(\mathrm{Norm}^{\mathrm{vir}})$ is the Euler class of the virtual normal bundle.

3.2.2. *Virtual Normal Bundle.* Denote by $T \to Y_{D_0,r}$ the tangent line bundle to the fiber of $Y_{D_0,r} \to D$. The inverse of the Euler class of virtual normal bundle can be written as

$$\frac{1}{e(\mathrm{Norm}^{\mathrm{vir}})} = \frac{e_{\mathbb{C}^*}(H^1(\mathcal{C}, f^*T(-\log D_\infty)))}{e_{\mathbb{C}^*}(H^0(\mathcal{C}, f^*T(-\log D_\infty)))} \frac{1}{\prod_{(e,v)\in F(\Gamma), i(v)=0} e(N_{(e,v)})} \frac{1}{e(N_\infty)}$$

The factors are explained as follows.

- The factor $\frac{e_{\mathbb{C}^*}(H^1(\mathcal{C}, f^*T(-\log D_\infty)))}{e_{\mathbb{C}^*}(H^0(\mathcal{C}, f^*T(-\log D_\infty)))}$ can be computed using the normalization exact sequence for domain $\mathcal{C}$ tensored with the line bundle $f^*T(-\log D_\infty)$, where $e_{\mathbb{C}^*}(\cdots)$ is the $\mathbb{C}^*$-equivariant Euler class. So it can be written as a product of vertex, edge, and node contributions using the associated long exact sequence in cohomology.

  Following [17, Section 3.4] and [19, Section 2.2], the edge and node contributions are trivial when r is sufficiently large. Indeed, the edge contribution is trivial since the degree d_e/r of $f^*T(-\log D_\infty)$ is less than 1. The contribution of a node N over $\mathcal{D}_r$ is trivial because the space of sections $H^0(N, f^*T(-\log D_\infty)) = 0$ (there is no invariant sections) and $H^1(N, f^*T(-\log D_\infty))$ vanishes for dimension reasons. Nodes over D_∞ contribute 1.

  The contribution from a vertex v over the zero section $\mathcal{D}_r$ is given by the class

$$c_{g(v)-1+|E(v)|}((-R^*\pi_*\mathcal{L})\otimes\mathcal{O}^{(1/r)}) \in A^*(\overline{M}_{g(v),\mathrm{val}(v),d(v)}(\mathcal{D}_r))\otimes\mathbb{Q}[t,1/t],$$

  where

$$\pi : \mathcal{C}_{g(v),\mathrm{val}(v),d(v)}(\mathcal{D}_r) \to \overline{M}_{g(v),\mathrm{val}(v),d(v)}(\mathcal{D}_r)$$

is the universal curve,

$$\mathcal{L} \to \mathcal{C}_{g(v),\mathrm{val}(v),d(v)}(\mathcal{D}_r)$$

is the universal r-th root which is the pull-back of T to the universal curve and $\mathcal{O}^{(1/r)}$ is a trivial line bundle with a $\mathbb{C}^*$-action of weight $1/r$. By Riemann-Roch theorem for twisted curves, the virtual rank of $-R^*\pi_*\mathcal{L}$ is

$$\begin{aligned} & g(v) - 1 + \frac{\sum_{j\in S(v)} k_j}{r} + \left(|E(v)| - \frac{\sum_{e\in E(v)} k_{(e,v)}}{r}\right) - \frac{\int_{d(v)} c_1(L)}{r} \\ =& g(v) - 1 + |E(v)|, \end{aligned}$$

where $\frac{\sum_{j\in S(v)} k_j}{r}$ and $\left(|E(v)| - \frac{\sum_{e\in E(v)} k_{(e,v)}}{r}\right)$ are the sum of ages. The equation follows from the compatibility condition of (14).

- For each node connecting an edge e and a stable vertex v over 0, the factor $e(N_{(e,v)})$ is the first Chern class of the normal bundle of the divisors of source nodal curves corresponding to smoothing the node. We have
$$e(N_{(e,v)}) = \frac{t + \mathrm{ev}_e^* c_1(L)}{rd_e} - \frac{\bar{\psi}_{(e,v)}}{r},$$
where $c_1(L)/r$ is the first Chern class of the normal bundle of $\mathcal{D}_r$ in $Y_{D_0,r}$; t/r is the weight of the $\mathbb{C}^*$-action on the normal bundle; $\bar{\psi}_{(e,v)}$ is the first Chern class of the cotangent line of the domain orbi-curve to the branch of the curve at the node corresponding to the vertex v.
- The factor $1/e(N_\infty)$ appears if the target expands at D_∞. We have
$$e(N_\infty) = \frac{-t - \psi_\infty}{\prod_{e\in E(\Gamma)} d_e},$$
where ψ_∞ is the first Chern class of the tautological cotangent line bundle determined by the relative divisor D_∞. See, for example, [24, Section 1.5.2] for the precise definition of ψ_∞.

3.2.3. *Identity on cycle classes.*

Lemma 3.4. *For r sufficiently large,*

(16)
$$\left[\epsilon^{\mathrm{orb}}_*\left(\mathrm{ev}_p^*([D_\infty]) \cap [\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y_{D_0,r}, D_\infty)]^{\mathrm{vir}}\right)\right]_{r^0} = \epsilon^{\mathrm{rel}}_*\left([\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y, D_0 \cup D_\infty)^\sim]^{\mathrm{vir}}\right),$$

where ϵ^{orb} and ϵ^{rel} are forgetful maps

$$\epsilon^{\mathrm{orb}} : \overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y_{D_0,r}, D_\infty) \to \overline{M}_{g,m+n+l(\mu),d}(Y);$$

$$\epsilon^{\mathrm{rel}} : \overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y, D_0 \cup D_\infty)^\sim \to \overline{M}_{g,m+n+l(\mu),d}(Y).$$

Proof. The localization formula (15) gives

$$\operatorname{ev}_p^*([D_\infty])\cap[\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y_{D_0,r},D_\infty)]^{\mathrm{vir}}= \sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|\prod_{e\in E(\Gamma)}d_e}\cdot\iota_*\left(\left(-\operatorname{ev}_p^*(c_1(L))-t\right)\cdot\frac{[\overline{M}_\Gamma]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}\right), \tag{17}$$

where $-\operatorname{ev}_p^*(c_1(L))-t$ is the restriction of the class $[D_\infty]$ to the infinity section D_∞. Following Section 3.2.2, the inverse of the virtual normal bundle $\frac{1}{e(\mathrm{Norm}^{\mathrm{vir}})}$ is the product of the following factors

- for each stable vertex v over the zero section, there is a factor

$$\begin{aligned}
&\left(\prod_{e\in E(v)}\frac{rd_e}{t+\operatorname{ev}_e^*c_1(L)-d_e\bar\psi_{(e,v)}}\right)\cdot\left(\sum_{i=0}^\infty(t/r)^{g(v)-1+|E(v)|-i}c_i(-R^*\pi_*\mathcal{L})\right)\\
=&t^{-1}\left(\prod_{e\in E(v)}\frac{d_e}{1+(\operatorname{ev}_e^*c_1(L)-d_e\bar\psi_{(e,v)})/t}\right)\cdot\left(\sum_{i=0}^\infty t^{g(v)-i}(r)^{i-g(v)+1}c_i(-R^*\pi_*\mathcal{L})\right)\\
=&t^{-1}\left(\prod_{e\in E(v)}\frac{d_e}{1+(\operatorname{ev}_e^*c_1(L)-d_e\bar\psi_{(e,v)})/t}\right)\cdot\left(\sum_{i=0}^\infty (tr)^{g(v)-i}(r)^{2i-2g(v)+1}c_i(-R^*\pi_*\mathcal{L})\right);
\end{aligned}\tag{18}$$

- if the target expands over the infinity section, there is a factor

$$\frac{\prod_{e\in E(\Gamma)}d_e}{-t-\psi_\infty}. \tag{19}$$

We consider the pushforward to the moduli space $\overline{M}_{g,m+n+l(\mu),d}(D)$ by forgetful maps. Following [16], we want to extract the coefficient of t^0r^0 from the contributions. We set $s:=tr$ and extract r^0s^0-coefficient instead. Let

$$\hat{c}_i=r^{2i-2g+1}\epsilon_*^{\mathrm{orb}}c_i(-R^*\pi_*\mathcal{L}).$$

The inverse of the virtual normal bundle can be rewritten as the product of the factors

$$\frac{r}{s}\prod_{e\in E(v)}\frac{d_e}{1+\frac{r}{s}(\operatorname{ev}_e^*c_1(L)-d_e\psi_{(e,v)})}\left(\sum_{i=0}^\infty\hat{c}_is^{g(v)-i}\right),\quad \text{for } v\in V^S(\Gamma)\cap i^{-1}(0); \tag{20}$$

and

$$-\frac{r}{s}\epsilon_*^{\mathrm{rel}}\left(\frac{\prod_{e\in E(\Gamma)}d_e}{1+\frac{r}{s}\psi_\infty}\right),\quad \text{if target expands.} \tag{21}$$

[17, Corollary 11] states that, for each $i\geq 0$, the class $\hat{c}_i$ is a polynomial in r when r is sufficiently large.

In addition, we have

$$-\operatorname{ev}_p^*(c_1(L))-t=-\operatorname{ev}_p^*(c_1(L))-\frac{s}{r}.$$

Since the irreducible component containing the non-relative and non-orbifold marked point p maps to D_∞, the target always expands at D_∞. Therefore, there is exactly one factor of (21) from contributions at D_∞.

Each factor of (20) and (21) is of positive power in r and contributes at least one r. Therefore, to extract the coefficient of r^0, there can be only one such factor, which, of course, has to be the factor (21) from the only stable vertex over the infinity divisor (there is only one stable vertex over the infinity because there are only unstable vertices over 0 and each unstable vertex only has one edge). Note that the term $\mathrm{ev}_p^*(c_1(L))$ also disappears, because its product with (20) and (21) only produces positive powers of r. Therefore, the fixed locus is described by the decorated graph with one stable vertex of full genus g over the infinity section D_∞ and m unstable vertices over the zero section $\mathcal{D}_r$.

The appearance of higher powers of the target descendant class ψ_∞ in the expansion of (21) will also contribute positive power of r, hence the terms involving ψ_∞ are not allowed either.

Then we extract the coefficient of s^0, the result is exactly the right-hand side of (16).

□

We consider the invariant

$$\left\langle\left(\prod_{i=1}^m \tau_{a_i}(\delta_i)\right)\tau_{a_{m+1}}([D_\infty]\cdot\delta_{m+1})\prod_{i=m+2}^{m+n}\tau_{a_i}(\delta_i)\middle|\,\mu\right\rangle^{(Y_{D_0,r},D_\infty)}_{g,\vec{k},n,\vec{\mu},d}, \tag{22}$$

where $\delta_i\in\pi^*(H^*(D,\mathbb{Q}))$, for $m+1\le i\le m+n$, are cohomology classes pulled back from $H^*(D,\mathbb{Q})$. We have the following relation between orbifold-relative invariants of $(Y_{D_0,r},D_\infty)$ and rubber integrals.

Lemma 3.5. *For r sufficiently large and $n>0$, the orbifold-relative Gromov-Witten invariant (22) of $(Y_{D_0,r},D_\infty)$ is a polynomial in r. Moreover,*

$$\begin{aligned}&\left[\left\langle\left(\prod_{i=1}^m \tau_{a_i}(\delta_i)\right)\tau_{a_{m+1}}([D_\infty]\cdot\delta_{m+1})\prod_{i=m+2}^{m+n}\tau_{a_i}(\delta_i)\middle|\,\mu\right\rangle^{(Y_{D_0,r},D_\infty)}_{g,\vec{k},n,\vec{\mu},d}\right]_{r^0}\\&=\left\langle\prod_{i=1}^m \tau_{a_i}(\delta_i)\middle|\prod_{i=m+1}^{m+n}\tau_{a_i}(\delta_i)\middle|\,\mu\right\rangle^{\sim,(Y,D_0\cup D_\infty)}_{g,\vec{k},n,\vec{\mu},d}.\end{aligned} \tag{23}$$

Proof. Identity (23) follows from Identity (16) in Lemma (3.4).

Polynomiality of the invariant (22) follows from the localization analysis and the polynomiality of the class $\hat{c}_i$. Indeed, it is sufficient to consider the factor (20):

$$\frac{1}{t}\prod_{e\in E(v)}\frac{d_e}{1+\frac{1}{t}(\mathrm{ev}_e^* c_1(L)-d_e\psi_{(e,v)})}\left(\sum_{i=0}^{\infty}\hat{c}_i(tr)^{g(v)-i}\right). \tag{24}$$

Negative power of r appears only when $i > g(v)$, but the appearance of negative power of r also results in the same negative power of t in the factor. Hence, negative powers of r do not contribute to the coefficient of t^0. ☐

Combining Lemma 3.2 and Lemma 3.5, we obtain the identity between relative invariants of $(Y, D_0\cup D_\infty)$ and orbifold-relative invariants of $(Y_{D_0,r}, D_\infty)$ with exactly one class of the form $\tau_a([D_\infty]\cdot\delta)$.

Proposition 3.6. *For r sufficiently large,*

$$\left[\left\langle\left(\prod_{i=1}^{m}\tau_{a_i}(\delta_i)\right)\tau_{a_{m+1}}([D_\infty]\cdot\delta_{m+1})\prod_{i=m+2}^{m+n}\tau_{a_i}(\delta_i)\middle|\,\mu\right\rangle_{g,\vec{k},n,\vec{\mu},d}^{(Y_{D_0,r},D_\infty)}\right]_{r^0} = \tag{25}$$
$$\left\langle\prod_{i=1}^{m}\tau_{a_i}(\delta_i)\middle|\tau_{a_{m+1}}([D_\infty]\cdot\delta_{m+1})\prod_{i=m+2}^{m+n}\tau_{a_i}(\delta_i)\middle|\,\mu\right\rangle_{g,\vec{k},n,\vec{\mu},d}^{(Y,D_0\cup D_\infty)}.$$

3.3. **Proof of Theorem 2.3.** In this section, we complete the proof of Theorem 2.3, hence also complete the proof of Theorem 1.5. A special case of Theorem 2.3 is already given in Proposition 3.6. Indeed, the general case of Theorem 2.3 can be derived from Proposition 3.6.

Lemma 3.7. *Theorem 2.3 follows from Proposition 3.6.*

To prove Lemma 3.7, we need to prove the identity for the following three types of invariants.

Type I: No descendant insertions of the form $\tau_a([D_0]\cdot\delta)$ or $\tau_a([D_\infty]\cdot\delta)$.

Suppose $\int_d[D_\infty]=0$ and there is at least one non-relative marked point, we may rewrite the relative invariant (8) of $(Y, D_0\cup D_\infty)$ as

$$\left\langle\prod_{i=1}^{m}\tau_{a_i}(\delta_i)\middle|\prod_{i=1}^{n}\tau_{a_{m+i}}(\delta_{m+i})\middle|\,\mu\right\rangle_{g,\vec{k},n,\vec{\mu},d}^{(Y,D_0\cup D_\infty)}, \tag{26}$$

where $\delta_{m+i}\in\pi^*H^*(D,\mathbb{Q})$ for $1\le i\le n$. In this case, the decorated graphs in localization computation do not have edges, hence there is only one vertex. The invariant (26) is zero because the virtual dimension of the $\mathbb{C}^*$-fixed locus is 1 less than the virtual dimension of $\overline{M}_{g,\vec{k},n,\vec{\mu},d}(Y, D_0\cup D_\infty)$. Consider the corresponding orbifold invariant of $(Y_{D_0,r}, D_\infty)$,

$$\left\langle\prod_{i=1}^{m}\tau_{a_i}(\delta_i)\prod_{i=1}^{n}\tau_{a_{m+i}}(\delta_{m+i})\middle|\,\mu\right\rangle_{g,\vec{k},n,\vec{\mu},d}^{(Y_{D_0,r},D_\infty)}. \tag{27}$$

Again, the decorated graphs has no edge. By virtual dimension constraint and the localization analysis in Lemma 3.4, the coefficient of t^0r^0 of the invariant (27) is zero.

Suppose $\int_d [D_\infty] = 0$ and there is no non-relative marked point. Choose a class $H \in \pi^* H^2(D,\mathbb{Q})$, such that $\int_d H \neq 0$. By divisor equation, this type of invariants can be reduced to the **Type I** invariants with at least one insertion.

Suppose $\int_d [D_\infty] \neq 0$, by the divisor equation, we have

$$\begin{aligned}
&\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \middle| \tau_0([D_\infty]) \prod_{i=1}^n \tau_{a_{m+i}}(\delta_{m+i}) \middle| \mu \right\rangle^{(Y,D_0\cup D_\infty)}_{g,\vec{k},n+1,\vec{\mu},d} \\
= &\int_d [D_\infty] \left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \middle| \prod_{i=1}^n \tau_{a_{m+i}}(\delta_{m+i}) \middle| \mu \right\rangle^{(Y,D_0\cup D_\infty)}_{g,\vec{k},n,\vec{\mu},d} \\
&+ \sum_{j=1}^n \left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \middle| \tau_{a_{m+j}-1}([D_\infty]\cdot\delta_{m+j}) \prod_{i\in\{1,\ldots,n\}\setminus\{j\}} \tau_{a_{m+i}}(\delta_{m+i}) \middle| \mu \right\rangle^{(Y,D_0\cup D_\infty)}_{g,\vec{k},n+1,\vec{\mu},d}.
\end{aligned}$$

Applying the divisor equation to the corresponding orbifold-relative invariant of $(Y_{D_0,r}, D_\infty)$ yields

$$\begin{aligned}
&\left\langle \left(\prod_{i=1}^m \tau_{a_i}(\delta_i) \right) \tau_0([D_\infty]) \prod_{i=1}^n \tau_{a_{m+i}}(\delta_{m+i}) \middle| \mu \right\rangle^{(Y_{D_0,r},D_\infty)}_{g,\vec{k},n+1,\vec{\mu},d} \\
= &\int_d [D_\infty] \left\langle \left(\prod_{i=1}^m \tau_{a_i}(\delta_i) \right) \prod_{i=1}^n \tau_{a_{m+i}}(\delta_{m+i}) \middle| \mu \right\rangle^{(Y_{D_0,r},D_\infty)}_{g,\vec{k},n,\vec{\mu},d} \\
&+ \sum_{j=1}^n \left\langle \left(\prod_{i=1}^m \tau_{a_i}(\delta_i) \right) \tau_{a_{m+j}-1}([D_\infty]\cdot\delta_{m+j}) \prod_{i\in\{1,\ldots,n\}\setminus\{j\}} \tau_{a_{m+i}}(\delta_{m+i}) \middle| \mu \right\rangle^{(Y_{D_0,r},D_\infty)}_{g,\vec{k},n+1,\vec{\mu},d}.
\end{aligned}$$

Therefore, the divisor equations for invariants of $(Y, D_0 \cup D_\infty)$ and invariants of $(Y_{D_0,r}, D_\infty)$ take the same form. Hence Theorem 2.3 for invariants of **Type I** follows from Proposition 3.6 by divisor equations when $\int_d [D_\infty] \neq 0$.

Hence we have completed the proof for **Type I** invariants.

Type II: At least one descendant insertions of the form $\tau_a([D_\infty]\cdot\delta)$ and no descendant insertions of the form $\tau_a([D_0]\cdot\delta)$.

Lemma 3.8. *Theorem 2.3 for invariants of* **Type II** *follows from the result for* **Type I** *invariants.*

Proof. We may rewrite the invariant (8) of $(Y_{D_0,r}, D_\infty)$ as

$$\left\langle \left(\prod_{i=1}^m \tau_{a_i}(\delta_i) \right) \prod_{i=1}^{n_0} \tau_{a_{m+i}}(\delta_{m+i}) \prod_{i=1}^{n_\infty} \tau_{a_{m+n_0+i}}([D_\infty]\cdot\delta_{m+n_0+i}) \middle| \mu \right\rangle^{(Y_{D_0,r},D_\infty)}_{g,\vec{k},n_0+n_\infty,\vec{\mu},d}. \tag{28}$$

We can apply degeneration formula to $(Y_{D_0,r}, D_\infty)$ over the infinity divisor D_∞. Hence the invariant (28) equals to

(29)

$$\sum \frac{\prod_i \eta_i}{|\mathrm{Aut}(\eta)|}\left\langle \left(\prod_{i=1}^m \tau_{a_i}(\delta_i)\right)\prod_{i\in S}\tau_{a_{m+i}}(\delta_{m+i})\middle|\eta\right\rangle^{\bullet,(Y_{D_0,r},D_\infty)}_{g,\vec{k},|S|,\vec{\eta},d_1}\cdot$$
$$\left\langle \eta^\vee\middle|\prod_{i\in\{1,\ldots,n_0\}\setminus S}\tau_{a_{m+i}}(\delta_{m+i})\prod_{i=1}^{n_\infty}\tau_{a_{m+n_0+i}}([D_\infty]\cdot\delta_{m+n_0+i})\middle|\mu\right\rangle^{\bullet,(Y,D_0\cup D_\infty)}_{g,\vec{\eta},n_0-|S|+n_\infty,\vec{\mu},d_2}.$$

The relative invariant of $(Y, D_0\cup D_\infty)$ corresponding to the invariant (28) is

$$\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i)\middle|\prod_{i=1}^{n_0}\tau_{a_{m+i}}(\delta_{m+i})\prod_{i=1}^{n_\infty}\tau_{a_{m+n_0+i}}([D_\infty]\cdot\delta_{m+n_0+i})\middle|\mu\right\rangle^{(Y,D_0\cup D_\infty)}_{g,\vec{k},n_0+n_\infty,\vec{\mu},d}. \tag{30}$$

Applying the degeneration formula, the invariant (30) equals to

(31)

$$\sum \frac{\prod_i \eta_i}{|\mathrm{Aut}(\eta)|}\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i)\middle|\prod_{i\in S}\tau_{a_{m+i}}(\delta_{m+i})\middle|\eta\right\rangle^{\bullet,(Y,D_0\cup D_\infty)}_{g,\vec{k},|S|,\vec{\eta},d_1}\cdot$$
$$\left\langle \eta^\vee\middle|\prod_{i\in\{1,\ldots,n_0\}\setminus S}\tau_{a_{m+i}}(\delta_{m+i})\prod_{i=1}^{n_\infty}\tau_{a_{m+n_0+i}}([D_\infty]\cdot\delta_{m+n_0+i})\middle|\mu\right\rangle^{\bullet,(Y,D_0\cup D_\infty)}_{g,\vec{\eta},n_0-|S|+n_\infty,\vec{\mu},d_2}.$$

The **Type II** orbifold-relative invariants of $(Y_{D_0,r}, D_\infty)$ and relative invariants of $(Y, D_0\cup D_\infty)$ satisfy the same form of degeneration formula. Note that the invariants on the first line of (29) and the invariants on the first line of (31) are of **Type I**. Hence Theorem 2.3 for invariants of **Type II** follows from the result for **Type I** invariants. □

Type III: At least one descendant insertions of the form $\tau_a([D_0]\cdot\delta)$.

The basic divisor relation in $H^2(Y,\mathbb{Q})$ gives

$$[D_\infty] = [D_0] - c_1(L).$$

Using this formula, invariants of **Type III** can be written as sum of invariants of **Type I** and **Type II**. Hence Theorem 2.3 for invariants of **Type III** follows from Theorem 2.3 for **Type I** and **Type II** invariants.

It is straightforward to see that the polynomiality of the orbifold-relative invariant (9) of $(Y_{D_0,r}, D_\infty)$ follows from the above discussion.

The proof of Theorem 2.3 is completed.

4. Genus Zero Relative and Orbifold Invariants

It is proved in [2] that relative invariants of (X, D) and orbifold invariants of $X_{D,r}$ are equal in genus zero, provided that r is sufficiently large. The proof in [2] is through comparison between virtual fundamental classes on different moduli spaces. In this section we give a new proof for the exact equality between genus zero relative invariants of (X, D) and genus zero orbifold invariants of the root stack $X_{D,r}$ for sufficiently large r. Our new proof is through degeneration formula and virtual localization. The reason why the equality fails to hold for higher genus invariants can be seen directly from the localization computation.

We consider the following genus zero relative and orbifold invariants.

(32)

$$\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \middle| \prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{0,\vec{k},n,d}^{(X,D)} :=$$

$$\int_{[\overline{M}_{0,\vec{k},n,d}(X,D)]^{vir}} \psi_1^{a_1} \operatorname{ev}_1^*(\delta_1) \cdots \psi_m^{a_m} \operatorname{ev}_m^*(\delta_m) \cdot \psi_{m+1}^{a_{m+1}} \operatorname{ev}_{m+1}^*(\gamma_{m+1}) \cdots \psi_{m+n}^{a_{m+n}} \operatorname{ev}_{m+n}^*(\gamma_{m+n}),$$

and

(33)

$$\left\langle \prod_{i=1}^m \tau_{a_i}(\delta_i) \prod_{i=1}^n \tau_{a_{m+i}}(\gamma_{m+i}) \right\rangle_{0,\vec{k},n,d}^{X_{D,r}} :=$$

$$\int_{[\overline{M}_{0,\vec{k},n,d}(X_{D,r})]^{vir}} \bar{\psi}_1^{a_1} \operatorname{ev}_1^*(\delta_1) \cdots \bar{\psi}_m^{a_m} \operatorname{ev}_m^*(\delta_m) \cdot \bar{\psi}_{m+1}^{a_{m+1}} \operatorname{ev}_{m+1}^*(\gamma_{m+1}) \cdots \bar{\psi}_{m+n}^{a_{m+n}} \operatorname{ev}_{m+n}^*(\gamma_{m+n}).$$

Theorem 4.1 ([2], Theorem 1.2.1)**.** *For r sufficiently large, genus zero relative and orbifold invariants coincide:*

$$(32) = (33).$$

The degeneration formula in Section 2 shows that it is sufficient to prove equality between genus zero invariants of $(Y_{D_0,r}, D_\infty)$ and genus zero invariants of $(Y, D_0 \cup D_\infty)$. Furthermore, Lemma 3.7 implies that it is sufficient to prove the equality when there is exactly one insertion of the form $\tau_a([D_\infty]\cdot\delta)$ and all other insertions are of the form $\tau_a(\delta)$, where the cohomology class δ is pulled back from $H^*(D, \mathbb{Q})$. As discussed in Section 3, it is enough to prove the following lemma for genus zero invariants.

Lemma 4.2. *Let p be a non-orbifold and non-relative marked point. For r sufficiently large, we have*

(34)

$$\epsilon_*^{\mathrm{orb}}\left(\operatorname{ev}_p^*([D_\infty]) \cap [\overline{M}_{0,\vec{k},n,\vec{\mu},d}(Y_{D_0,r}, D_\infty)]^{\mathrm{vir}}\right) \cong \epsilon_*^{\mathrm{rel}}\left([\overline{M}_{0,\vec{k},n,\vec{\mu},d}(Y, D_0 \cup D_\infty)^\sim]^{\mathrm{vir}}\right).$$

Proof. Following the proof of Lemma 3.4, the localization formula is

$$\mathrm{ev}_p^*([D_\infty])\cap[\overline{M}_{0,\vec{k},n,\vec{\mu},d}(Y_{D_0,r},D_\infty)]^{\mathrm{vir}} = \sum_\Gamma \frac{1}{|\mathrm{Aut}(\Gamma)|\prod_{e\in E(\Gamma)}d_e}\cdot\iota_*\left(\left(-\mathrm{ev}_p^*(c_1(L))-t\right)\cdot\frac{[\overline{M}_\Gamma]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}\right). \tag{35}$$

The inverse of the virtual normal bundle $\frac{1}{e(\mathrm{Norm}^{\mathrm{vir}})}$ can be written as the product of the following factors:

- for each stable vertex v over the zero section, there is a factor

$$\begin{aligned}&\prod_{e\in E(v)}\frac{rd_e}{t+\mathrm{ev}_e^*c_1(L)-d_e\bar{\psi}_{(e,v)}}\left(\sum_{i=0}^\infty (t/r)^{-1+|E(v)|-i}c_i(-R^*\pi_*\mathcal{L})\right)\\ &=\frac{r}{t}\prod_{e\in E(v)}\frac{d_e}{1+\frac{\mathrm{ev}_e^*c_1(L)-d_e\bar{\psi}_{(e,v)}}{t}}\left(\sum_{i=0}^\infty (t/r)^{-i}c_i(-R^*\pi_*\mathcal{L})\right);\end{aligned} \tag{36}$$

- if the target expands over the ∞-section, there is a factor

$$\frac{\prod_{e\in E(\Gamma)}d_e}{-t-\psi_\infty}=-\frac{1}{t}\frac{\prod_{e\in E(\Gamma)}d_e}{1+\frac{\psi_\infty}{t}}. \tag{37}$$

Each factor contains only negative powers of t and contributes at least one t^{-1}. In order to extract t^0-coefficient from (35), there can only be one stable vertex in the decorated graph Γ. Since the non-orbifold and non-relative marked point p has to land on the infinity divisor D_∞, the only stable vertex is over ∞. Therefore, the decorated graph Γ is of a stable vertex of full genus g over ∞ and m unstable vertices over 0. Since every ψ_∞ class comes with an extra factor of t^{-1}, no term with ψ_∞ class appears in the coefficient of t^0. What is left is exactly the right hand side of (34). $\square$

Remark 4.3. The proof does not work for higher genus invariants due to the fact that the contributions from stable vertices over zero section contain nonnegative power of t. Therefore, the coefficient of t^0 does not get simplified as in genus zero case. Hence, for higher genus invariants, one needs to pushforward to the moduli space of stable maps to X and also take the coefficient of r^0, as discussed in Lemma 3.4.

5. Stationary Gromov-Witten Theory of curves.

In this section, we prove Theorem 2.4 for the equality between stationary Gromov-Witten invariants of $(\mathbb{P}^1[r],\infty)$ and stationary Gromov-Witten invariants of $(\mathbb{P}^1,0,\infty)$. The proof is based on the degeneration formula in the proof of Theorem 2.3 and the equality for genus zero invariants.

5.1. **The Proof of Theorem 2.4.** We consider the degeneration (29) in the proof of Lemma 3.8 such that all stationary marked points are distributed to the component containing ∞. Therefore, the proof of Theorem 2.4 is reduced to the case of orbifold-relative stationary invariants of $(\mathbb{P}^1[r], \infty)$ with no stationary marked points, that is,

$$\langle|\mu\rangle^{(\mathbb{P}^1[r],\infty)}_{g,\vec{k},0,\vec{\mu},d}.$$

There are no insertions, therefore the virtual dimension $\overline{M}_{g,\vec{k},0,\vec{\mu},d}(\mathbb{P}^1[r],\infty)$ has to be zero. That is,

$$2g-2+m+l(\mu)=0.$$

This means $g=0$, $m=1$ and $l(\mu)=1$. This is genus 0 invariants of $(\mathbb{P}^1[r],\infty)$ when there is only one relative marked point, one orbifold marked point and, no non-relative and non-orbifold marked points.

Similarly for relative invariants of $(\mathbb{P}^1,0,\infty)$. We only need to consider genus zero invariants of $(\mathbb{P}^1,0,\infty)$ with single relative marked point at 0 and ∞ respectively; and no non-relative marked points.

Hence, it is sufficient to prove the following equality

$$\langle\,|(d)\rangle^{(\mathbb{P}^1[r],\infty)}_{0,(d),0,(d),d}=\langle(d)||(d)\rangle^{(\mathbb{P}^1,0,\infty)}_{0,(d),0,(d),d},$$

where (d) represents the trivial partition of d with only one part. It is simply a special case of the equality for genus zero invariants. This completes the proof of Theorem 2.4.

5.2. **Application: Stationary Orbifold Invariants as Hurwitz Numbers.** In the celebrated paper [27] by Okounkov-Pandharipande, stationary relative Gromov-Witten invariants of target curves are proven to be equal to Hurwitz numbers with completed cycles, that is, the sum of the Hurwitz numbers obtained by replacing $\tau_a(\omega)$ by the associated ramification conditions. The ramification conditions associated to $\tau_a(\omega)$ are universal, independent of all factors including the target curve. This is known as GW/H correspondence for relative theory of target curves. The equality (11) between stationary relative invariants and orbifold invariants implies orbifold Gromov-Witten invariants of r-th root stack of target curves with orbifold conditions given by the partitions of the degree d are equal to Hurwitz numbers with completed cycles when r is sufficiently large.

We briefly review the theory in [27]. Hurwitz numbers $H^C_d(\vec{\eta}^1,\dots,\vec{\eta}^l)$ can be extended to all degree d and all partitions $\vec{\eta}^i$ as follows:

- $H^C_0(\varnothing,\dots,\varnothing)=1$, where $\varnothing$ stands for empty partition.
- If $|\vec{\eta}^i|>d$ for some i, then the Hurwitz number vanishes.

- If $|\vec{\eta}^i| \leq d$ for all $1 \leq i \leq l$, then the Hurwitz number is defined as

$$H_d^C(\vec{\eta}^1, \dots, \vec{\eta}^l) = \prod_{i=1}^{l} \binom{m_1(\vec{\eta}^i_+)}{m_1(\vec{\eta}^i)} \cdot H_d^C(\vec{\eta}^1_+, \dots, \vec{\eta}^l_+), \tag{38}$$

where $\vec{\eta}^i_+$ is the partition of d determined by adjoining $d - |\vec{\eta}^i|$ parts of size 1

$$\vec{\eta}^i_+ = (\eta^i_1, \dots, \eta^i_n, 1, \dots, 1);$$

$m_1(\vec{\eta})$ is the multiplicity of the 1 in $\vec{\eta}$.

Let $S(d)$ be the symmetric group. The class algebra $\mathcal{Z}(d) \subset \mathbb{Q}S(d)$ is the center of the group algebra $\mathbb{Q}S(d)$. Let $C_\eta \in \mathcal{Z}(d)$ be the conjugacy class corresponding to the partition $\vec{\eta}$. Let λ be an irreducible representation of $S(d)$. The conjugacy class $C_{\vec{\eta}}$ acts as a scalar operator with eigenvalue

$$f_{\vec{\eta}}(\lambda) = \binom{|\lambda|}{|\vec{\eta}|} |C_\eta| \frac{\chi^\lambda_\eta}{\dim \lambda},$$

where $\chi^\lambda_{\vec{\eta}}$ is the character of any element of $C_{\vec{\eta}}$ in the representation λ and $\dim \lambda$ is the dimension of the representation λ.

Let $\mathcal{P}$ be the set of all partitions. There is a linear, injective Fourier transform

$$\phi : \bigoplus_{d=0}^{\infty} \mathcal{Z}(d) \to \mathcal{Q}^{\mathcal{P}}$$
$$C_{\vec{\eta}} \mapsto f_{\vec{\eta}}.$$

The image of ϕ is the so-called shifted symmetric functions Λ^*. An element f of the algebra of shifted symmetric functions Λ^* can be concretely given as a sequence of polynomials

$$f = \{f^{(n)}\}, \quad f^{(n)} \in \mathbb{Q}[\lambda_1, \dots, \lambda_n]^{*S(n)},$$

where $\mathbb{Q}[\lambda_1, \dots, \lambda_n]^{*S(n)}$ is the invariants of the shifted action of the symmetric group $S(n)$ on the algebra $\mathbb{Q}[\lambda_1, \dots, \lambda_n]$. The shifted action is defined by permutation of the variables λ_i. The sequence $\{f^{(n)}\}$ satisfies

- $f^{(n)}$ are of uniformly bounded degree,
- $f^{(n)}$ are stable under restriction, that is, $f^{(n+1)}|_{\lambda_{n+1}=0} = f^{(n)}$.

The shifted symmetric power sum $p_k \in \Lambda^*$ is defined by

$$p_k(\lambda) = \sum_{i=1}^{\infty} \left[(\lambda_i - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k \right] + (1 - 2^{-k})\zeta(-k).$$

For each partition $\vec{\eta}$, define $p_{\vec{\eta}} \in \Lambda^*$ as

$$p_{\vec{\eta}} = \prod p_{\eta_i}.$$

The completed conjugacy classes are defined by

$$\overline{C}_{\vec{\eta}} = \frac{1}{\prod_i \eta_i}\phi^{-1}(p_{\vec{\eta}}) \in \bigoplus_{d=0}^{|\eta|}\mathcal{Z}(d).$$

The completed cycles are defined by

$$\overline{(a)} = \overline{C}_{(a)}, \quad a = 1, 2, \dots.$$

More concretely, completed cycle $\overline{(a)}$ is obtained from the cycle (a) by adding multiples of constant terms and nonnegative multiples of nontrivial conjugacy classes of strictly smaller size. More details can be found in [27, Section 0.4].

The following GW/H correspondence is proved in [27]:

Theorem 5.1. *([27], Theorem 1) Let C be a smooth target curve in any genus. The GW/H correspondence for the relative Gromov-Witten theory of C is*

(39)

$$\langle\prod_{i=1}^{n}\tau_{a_i}(\omega)|\eta^1|\dots|\eta^l\rangle^{\bullet,(C,q_1,\dots,q_l)}_{g,n,\vec{\eta}^1,\dots,\vec{\eta}^l,d} = \frac{1}{\prod(a_i!)}H^C_d(\overline{(a_1+1)},\dots,\overline{(a_n+1)},\vec{\eta}^1,\dots,\vec{\eta}^l).$$

Theorem 1.9 and Theorem 5.1 together imply the following GW/H correspondence for orbifolds.

Corollary 5.2. *Let C be a smooth target curve in any genus. Let $C[r_1,\dots,r_l]$ be the root stack over C by taking r_i-th root at the point $q_i \in C$, for the l distinct points $q_1,\dots,q_l$ of C. When r_i are sufficiently large for all $1 \le i \le l$, we have the following GW/H correspondence:*

$$\langle\prod_{i=1}^{n}\tau_{a_i}(\omega)\rangle^{\bullet,C[r_1,\dots,r_l]}_{g,n,\vec{\eta}^1,\dots,\vec{\eta}^l,d} = \frac{1}{\prod a_i!}H^C_d(\overline{(a_1+1)},\dots,\overline{(a_n+1)},\vec{\eta}^1,\dots,\vec{\eta}^l). \tag{40}$$

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