

Hydrodynamic Limit of the Boltzmann Equation with Contact Discontinuities

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Abstract: The hydrodynamic limit for the Boltzmann equation is studied in the case when the limit system, that is, the system of Euler equations contains contact discontinuities. When suitable initial data is chosen to avoid the initial layer, we prove that there exist a family of solutions to the Boltzmann equation globally in time for small Knudsen number. And this family of solutions converge to the local Maxwellian defined by the contact discontinuity of the Euler equations uniformly away from the discontinuity as the Knudsen number ε tends to zero. The proof is obtained by an appropriately chosen scaling and the energy method through the micro-macro decomposition.

1. Introduction

Consider the Boltzmann equation with slab symmetry

$$f_t + \xi_1 f_x = \frac{1}{\varepsilon} Q(f, f), \quad (f, x, t, \xi) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^3, \quad (1.1)$$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$, $f(x, t, \xi)$ is the density distribution function of the particles at time t and space x with velocity ξ , and $\varepsilon > 0$ is the Knudsen number which is proportional to the mean free path.

Equation (1.1) was established by Boltzmann [4] in 1872 to describe the motion of rarefied gases and it is a fundamental equation in statistics physics. For monatomic gas, the rotational invariance of the particles leads to the following bilinear form for the collision operator:

$$Q(f, g)(\xi) \equiv \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{S}_+^2} (f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) - f(\xi_*)g(\xi)) \\ \times B(|\xi - \xi_*|, \hat{\theta}) d\xi_* d\Omega,$$

where ξ', ξ'_* are the velocities after an elastic collision of two particles with velocities ξ, ξ_* before the collision. Here, $\hat{\theta}$ is the angle between the relative velocity $\xi - \xi_*$ and the unit vector Ω in $\mathbf{S}^2_+ = \{\Omega \in \mathbf{S}^2 : (\xi - \xi_*) \cdot \Omega \geq 0\}$. The conservation of momentum and energy gives the following relation between the velocities before and after collision:

$$\begin{cases} \xi' = \xi - [(\xi - \xi_*) \cdot \Omega] \Omega, \\ \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega] \Omega. \end{cases}$$

In this paper, we consider the Boltzmann equation for the two basic models, that is, the hard sphere model and the hard potential including Maxwellian molecules under the assumption of angular cut-off. That is, we assume that the collision kernel $B(|\xi - \xi_*|, \hat{\theta})$ takes one of the following two forms:

$$B(|\xi - \xi_*|, \hat{\theta}) = |\xi - \xi_*|, \Omega|,$$

and

$$B(|\xi - \xi_*|, \hat{\theta}) = |\xi - \xi_*|^{\frac{n-5}{n-1}} b(\hat{\theta}), \quad b(\hat{\theta}) \in L^1([0, \pi]), \quad n \geq 5.$$

Here, n is the index in the inverse power potential which is proportional to r^{1-n} with r being the distance between two particles.

Formally, when the Knudsen number ε tends to zero, the limit of the Boltzmann equation (1.1) is the classical system of Euler equations

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = 0, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = 0, \quad i = 2, 3, \\ [\rho(E + \frac{|u|^2}{2})]_t + [\rho u_1(E + \frac{|u|^2}{2}) + p u_1]_x = 0, \end{cases} \tag{1.2}$$

where

$$\begin{cases} \rho(x, t) = \int_{\mathbf{R}^3} \varphi_0(\xi) f(x, t, \xi) d\xi, \\ \rho u_i(x, t) = \int_{\mathbf{R}^3} \varphi_i(\xi) f(x, t, \xi) d\xi, \quad i = 1, 2, 3, \\ \rho(E + \frac{|u|^2}{2})(x, t) = \int_{\mathbf{R}^3} \varphi_4(\xi) f(x, t, \xi) d\xi. \end{cases} \tag{1.3}$$

Here, ρ is the density, $u = (u_1, u_2, u_3)$ is the macroscopic velocity, E is the internal energy and the pressure $p = R\rho\theta$ with R being the gas constant. The temperature θ is related to the internal energy by $E = \frac{3}{2}R\theta$, and $\varphi_i(\xi) (i = 0, 1, 2, 3, 4)$ are the collision invariants given by

$$\begin{cases} \varphi_0(\xi) = 1, \\ \varphi_i(\xi) = \xi_i \quad \text{for } i = 1, 2, 3, \\ \varphi_4(\xi) = \frac{1}{2}|\xi|^2, \end{cases}$$

that satisfy

$$\int_{\mathbf{R}^3} \varphi_i(\xi) Q(h, g) d\xi = 0, \quad \text{for } i = 0, 1, 2, 3, 4.$$

How to justify the above limit, that is, the Euler equation (1.2) from the Boltzmann equation (1.1) when the Knudsen number tends to zero is an open problem going way back to the time of Maxwell. For this, Hilbert introduced the famous Hilbert expansion to show formally that the first order approximation of the Boltzmann equation gives the Euler equations. On the other hand, it is important to verify this limit process rigorously in mathematics. For the case when the Euler equation has smooth solutions, the zero Knudsen number limit of the Boltzmann equation has been studied even in the case with an initial layer, cf. Asona-Ukai [1], Caflish [5], Lachowicz [21] and Nishida [29], etc. However, as is well-known, solutions of the Euler equation (1.2) in general develop singularities, such as shock waves and contact discontinuities. Therefore, how to verify the hydrodynamic limit from the Boltzmann equation to the Euler equations with basic wave patterns is a natural problem. In this direction, Yu [35] showed that when the solution of the Euler equation (1.2) contains non-interacting shocks, there exists a sequence of solutions to the Boltzmann equation that converge to the local Maxwellian defined by the solution of the Euler equation (1.2) uniformly away from the shock. In this work, the inner and outer expansions developed by Goodman-Xin [12] for conservation laws and the Hilbert expansion were skillfully and cleverly used.

The main purpose of this paper is to study the hydrodynamic limit of the Boltzmann equation when the corresponding Euler equation contains contact discontinuities. More precisely, given a solution of the Euler equation (1.2) with contact discontinuities, we will show that there exists a family of solutions to the Boltzmann equation that converge to a local Maxwellian defined by the Euler solution uniformly away from the contact discontinuity as $\varepsilon \rightarrow 0$. Moreover, a uniform convergence rate in ε is also given. The proof is obtained by a scaling transformation of the independent variables and the perturbation together with the energy method introduced by Liu-Yang-Yu [24].

For later use, we now briefly introduce the micro-macro decomposition around the local Maxwellian defined by the solution to the Boltzmann equation, cf. [24]. For a solution $f(x, t, \xi)$ of the Boltzmann equation (1.1), we decompose it into

$$f(x, t, \xi) = \mathbf{M}(x, t, \xi) + \mathbf{G}(x, t, \xi),$$

where the local Maxwellian $\mathbf{M}(x, t, \xi) = \mathbf{M}_{[\rho, u, \theta]}(\xi)$ represents the macroscopic (fluid) component of the solution, which is naturally defined by the five conserved quantities, i.e., the mass density $\rho(x, t)$, the momentum $\rho u(x, t)$, and the total energy $\rho(E + \frac{1}{2}|u|^2)(x, t)$ in (1.3), through

$$\mathbf{M} = \mathbf{M}_{[\rho, u, \theta]}(x, t, \xi) = \frac{\rho(x, t)}{\sqrt{(2\pi R\theta(x, t))^3}} e^{-\frac{|\xi - u(x, t)|^2}{2R\theta(x, t)}}. \tag{1.4}$$

And $\mathbf{G}(x, t, \xi)$ being the difference between the solution and the above local Maxwellian represents the microscopic (non-fluid) component.

For convenience, we denote the inner product of h and g in $L^2_{\xi}(\mathbf{R}^3)$ with respect to a given Maxwellian $\tilde{\mathbf{M}}$ by:

$$\langle h, g \rangle_{\tilde{\mathbf{M}}} \equiv \int_{\mathbf{R}^3} \frac{1}{\tilde{\mathbf{M}}} h(\xi) g(\xi) d\xi.$$

If $\tilde{\mathbf{M}}$ is the local Maxwellian \mathbf{M} defined in (1.4), with respect to the corresponding inner product, the macroscopic space is spanned by the following five pairwise orthogonal base:

$$\begin{cases} \chi_0(\xi) \equiv \frac{1}{\sqrt{\rho}}\mathbf{M}, \\ \chi_i(\xi) \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}}\mathbf{M} \text{ for } i = 1, 2, 3, \\ \chi_4(\xi) \equiv \frac{1}{\sqrt{6\rho}}\left(\frac{|\xi - u|^2}{R\theta} - 3\right)\mathbf{M}, \\ \langle \chi_i, \chi_j \rangle = \delta_{ij}, \quad i, j = 0, 1, 2, 3, 4. \end{cases}$$

In the following, if $\tilde{\mathbf{M}}$ is the local Maxwellian \mathbf{M} , we just use the simplified notation $\langle \cdot, \cdot \rangle$ to denote the inner product $\langle \cdot, \cdot \rangle_{\mathbf{M}}$. We can now define the macroscopic projection \mathbf{P}_0 and microscopic projection \mathbf{P}_1 as follows:

$$\begin{cases} \mathbf{P}_0 h = \sum_{j=0}^4 \langle h, \chi_j \rangle \chi_j, \\ \mathbf{P}_1 h = h - \mathbf{P}_0 h. \end{cases} \tag{1.5}$$

The projections \mathbf{P}_0 and \mathbf{P}_1 are orthogonal and satisfy

$$\mathbf{P}_0 \mathbf{P}_0 = \mathbf{P}_0, \mathbf{P}_1 \mathbf{P}_1 = \mathbf{P}_1, \mathbf{P}_0 \mathbf{P}_1 = \mathbf{P}_1 \mathbf{P}_0 = 0.$$

We remark that a function $h(\xi)$ is called microscopic or non-fluid if

$$\int h(\xi) \varphi_i(\xi) d\xi = 0, \quad i = 0, 1, 2, 3, 4,$$

where $\varphi_i(\xi)$ is the collision invariants.

Under the above micro-macro decomposition, the solution $f(x, t, \xi)$ of the Boltzmann equation (1.1) satisfies

$$\mathbf{P}_0 f = \mathbf{M}, \quad \mathbf{P}_1 f = \mathbf{G},$$

and the Boltzmann equation (1.1) becomes

$$(\mathbf{M} + \mathbf{G})_t + \xi_1 (\mathbf{M} + \mathbf{G})_x = \frac{1}{\varepsilon} [2Q(\mathbf{M}, \mathbf{G}) + Q(\mathbf{G}, \mathbf{G})]. \tag{1.6}$$

If we multiply Eq. (1.6) by the collision invariants $\varphi_i(\xi)$ ($i = 0, 1, 2, 3, 4$) and integrate the resulting equations with respect to ξ over \mathbf{R}^3 , then we can get the following fluid-type system for the fluid components:

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = - \int \xi_1^2 \mathbf{G}_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = - \int \xi_1 \xi_i \mathbf{G}_x d\xi, \quad i = 2, 3, \\ \left[\rho \left(E + \frac{|u|^2}{2} \right) \right]_t + \left[\rho u_1 \left(E + \frac{|u|^2}{2} \right) + p u_1 \right]_x = - \int \frac{1}{2} \xi_1 |\xi|^2 \mathbf{G}_x d\xi. \end{cases} \tag{1.7}$$

Note that the above fluid-type system is not closed and we need one more equation for the non-fluid component \mathbf{G} which can be obtained by applying the projection operator \mathbf{P}_1 to Eq. (1.6):

$$\mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{M}_x) + \mathbf{P}_1(\xi_1 \mathbf{G}_x) = \frac{1}{\varepsilon} [\mathbf{L}_M \mathbf{G} + Q(\mathbf{G}, \mathbf{G})]. \tag{1.8}$$

Here \mathbf{L}_M is the linearized collision operator of $Q(f, f)$ with respect to the local Maxwellian \mathbf{M} :

$$\mathbf{L}_M h = 2Q(\mathbf{M}, h) = Q(\mathbf{M}, h) + Q(h, \mathbf{M}).$$

And the null space \mathcal{N} of \mathbf{L}_M is spanned by the macroscopic variables:

$$\chi_j(\xi), \quad j = 0, 1, 2, 3, 4.$$

Furthermore, there exists a positive constant $\sigma_0(\rho, u, \theta) > 0$ such that for any function $h(\xi) \in \mathcal{N}^\perp$, cf. [13],

$$\langle h, \mathbf{L}_M h \rangle \leq -\sigma_0 \langle \nu(|\xi|) h, h \rangle,$$

where $\nu(|\xi|)$ is the collision frequency. For the hard sphere and the hard potential with angular cut-off, the collision frequency $\nu(|\xi|)$ has the following property:

$$0 < \nu_0 < \nu(|\xi|) \leq c(1 + |\xi|)^\kappa,$$

for some positive constants ν_0, c and $0 \leq \kappa \leq 1$.

Consequently, the linearized collision operator \mathbf{L}_M is a dissipative operator on $L^2(\mathbf{R}^3)$, and its inverse \mathbf{L}_M^{-1} exists and is a bounded operator in $L^2(\mathbf{R}^3)$.

It follows from (1.8) that

$$\mathbf{G} = \varepsilon \mathbf{L}_M^{-1} [\mathbf{P}_1(\xi_1 \mathbf{M}_x)] + \Theta, \tag{1.9}$$

with

$$\Theta = \mathbf{L}_M^{-1} [\varepsilon(\mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{G}_x)) - Q(\mathbf{G}, \mathbf{G})]. \tag{1.10}$$

Plugging Eq. (1.9) into (1.7) gives

$$\left\{ \begin{array}{l} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = \frac{4\varepsilon}{3} (\mu(\theta) u_{1x})_x - \int \xi_1^2 \Theta_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = \varepsilon (\mu(\theta) u_{ix})_x - \int \xi_1 \xi_i \Theta_x d\xi, \quad i = 2, 3, \\ [\rho(E + \frac{|u|^2}{2})]_t + [\rho u_1(E + \frac{|u|^2}{2}) + p u_1]_x = \varepsilon (\lambda(\theta) \theta_x)_x + \frac{4\varepsilon}{3} (\mu(\theta) u_1 u_{1x})_x \\ + \varepsilon \sum_{i=2}^3 (\mu(\theta) u_i u_{ix})_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Theta_x d\xi, \end{array} \right. \tag{1.11}$$

where the viscosity coefficient $\mu(\theta) > 0$ and the heat conductivity coefficient $\lambda(\theta) > 0$ are smooth functions of the temperature θ , and we normalize the gas constant R to be $\frac{2}{3}$ so that $E = \theta$ and $p = \frac{2}{3} \rho \theta$. The explicit formula of $\mu(\theta)$ and $\lambda(\theta)$ can be found for example in [36], we omit it here for brevity.

Since the problem considered in this paper is one dimensional in the space variable $x \in \mathbf{R}$, in the macroscopic level, it is more convenient to rewrite Eq. (1.1) and the system (1.2) in the *Lagrangian* coordinates as in the study of conservation laws. That is, set the coordinate transformation:

$$x \Rightarrow \int_0^x \rho(y, t) dy, \quad t \Rightarrow t.$$

We will still denote the *Lagrangian* coordinates by (x, t) for simplicity of notation. Then (1.1) and (1.2) in the Lagrangian coordinates become, respectively,

$$f_t - \frac{u_1}{v} f_x + \frac{\xi_1}{v} f_x = \frac{1}{\varepsilon} Q(f, f), \tag{1.12}$$

and

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = 0, \\ u_{it} = 0, \quad i = 2, 3, \\ \left(\theta + \frac{|u|^2}{2}\right)_t + (pu_1)_x = 0. \end{cases} \tag{1.13}$$

Also, (1.7)–(1.11) take the form

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = - \int \xi_1^2 \mathbf{G}_x d\xi, \\ u_{it} = - \int \xi_1 \xi_i \mathbf{G}_x d\xi, \quad i = 2, 3, \\ \left(\theta + \frac{|u|^2}{2}\right)_t + (pu_1)_x = - \int \frac{1}{2} \xi_1 |\xi|^2 \mathbf{G}_x d\xi, \end{cases} \tag{1.14}$$

$$\mathbf{G}_t - \frac{u_1}{v} \mathbf{G}_x + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_x) + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{G}_x) = \frac{1}{\varepsilon} (\mathbf{L}_M \mathbf{G} + Q(\mathbf{G}, \mathbf{G})), \tag{1.15}$$

with

$$\mathbf{G} = \varepsilon \mathbf{L}_M^{-1} \left(\frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_x) \right) + \Theta_1, \tag{1.16}$$

$$\Theta_1 = \mathbf{L}_M^{-1} \left[\varepsilon \left(\mathbf{G}_t - \frac{u_1}{v} \mathbf{G}_x + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{G}_x) \right) - Q(\mathbf{G}, \mathbf{G}) \right], \tag{1.17}$$

and

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = \frac{4\varepsilon}{3} \left(\frac{\mu(\theta)}{v} u_{1x} \right)_x - \int \xi_1^2 \Theta_{1x} d\xi, \\ u_{it} = \varepsilon \left(\frac{\mu(\theta)}{v} u_{ix} \right)_x - \int \xi_1 \xi_i \Theta_{1x} d\xi, \quad i = 2, 3, \\ \left(\theta + \frac{|u|^2}{2}\right)_t + (pu_1)_x = \varepsilon \left(\frac{\lambda(\theta)}{v} \theta_x \right)_x + \frac{4\varepsilon}{3} \left(\frac{\mu(\theta)}{v} u_1 u_{1x} \right)_x \\ + \varepsilon \sum_{i=2}^3 \left(\frac{\mu(\theta)}{v} u_i u_{ix} \right)_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Theta_{1x} d\xi. \end{cases} \tag{1.18}$$

In the following sections, we will apply some scaling and energy method to these equations.

2. Main Result

We will state the main result in this section. For this, we first recall the construction of the contact wave $(\bar{v}, \bar{u}, \bar{\theta})(x, t)$ for the Boltzmann equation in [18]. Consider the Euler system (1.13) with a Riemann initial data

$$(v, u, \theta)(x, 0) = \begin{cases} (v_-, 0, \theta_-), & x < 0, \\ (v_+, 0, \theta_+), & x > 0, \end{cases} \tag{2.1}$$

where v_{\pm}, θ_{\pm} are positive constants. It is well-known (cf. [30]) that the Riemann problem (1.13), (2.1) admits a contact discontinuity solution

$$(\bar{V}, \bar{U}, \bar{\Theta})(x, t) = \begin{cases} (v_-, 0, \theta_-), & x < 0, \\ (v_+, 0, \theta_+), & x > 0, \end{cases} \tag{2.2}$$

provided that

$$p_- := \frac{R\theta_-}{v_-} = p_+ := \frac{R\theta_+}{v_+}. \tag{2.3}$$

Motivated by (2.2) and (2.3), we expect that for the contact wave $(\bar{v}, \bar{u}, \bar{\theta})(x, t)$,

$$\bar{p} = \frac{R\bar{\theta}}{\bar{v}} \approx p_+, \quad |\bar{u}|^2 \ll 1.$$

Then the leading order of the energy equation (1.18)₄ is

$$\theta_t + p_+ u_{1x} = \varepsilon \left(\frac{\lambda(\theta)\theta_x}{v} \right)_x. \tag{2.4}$$

By using the mass equation (1.18)₁ and $v \approx \frac{R\theta}{p_+}$, we obtain the following nonlinear diffusion equation:

$$\theta_t = \varepsilon (a(\theta)\theta_x)_x, \quad a(\theta) = \frac{9p_+\lambda(\theta)}{10\theta}. \tag{2.5}$$

From [2,9], we know that the nonlinear diffusion equation (2.5) admits a unique self-similar solution $\hat{\Theta}(\eta)$, $\eta = \frac{x}{\sqrt{\varepsilon(1+t)}}$ with the following boundary conditions:

$$\hat{\Theta}(-\infty, t) = \theta_-, \quad \hat{\Theta}(+\infty, t) = \theta_+.$$

Let $\delta = |\theta_+ - \theta_-|$. $\hat{\Theta}(x, t)$ has the property

$$\hat{\Theta}_x(x, t) = \frac{O(1)\delta}{\sqrt{\varepsilon(1+t)}} e^{-\frac{cx^2}{\varepsilon(1+t)}}, \quad \text{as } x \rightarrow \pm\infty, \tag{2.6}$$

with some positive constant c depending only on θ_{\pm} .

Now the contact wave $(\bar{v}, \bar{u}, \bar{\theta})(x, t)$ can be defined by

$$\bar{v} = \frac{2}{3p_+} \hat{\Theta}, \quad \bar{u}_1 = \frac{2\varepsilon a(\hat{\Theta})}{3p_+} \hat{\Theta}_x, \quad \bar{u}_i = 0, (i = 2, 3), \quad \bar{\theta} = \hat{\Theta} - \frac{|\bar{u}|^2}{2}. \quad (2.7)$$

Note that $(\bar{v}, \bar{u}, \bar{\theta})(x, t)$ satisfies the following system:

$$\begin{cases} \bar{v}_t - \bar{u}_{1x} = 0, \\ \bar{u}_{1t} + \bar{p}_x = \frac{4\varepsilon}{3} \left(\frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{1x} \right)_x + R_{1x}, \\ \bar{u}_{it} = \varepsilon \left(\frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{ix} \right)_x, i = 2, 3, \\ \left(\bar{\theta} + \frac{|\bar{u}|^2}{2} \right)_t + (\bar{p}\bar{u}_1)_x = \varepsilon \left(\frac{\lambda(\bar{\theta})}{\bar{v}} \bar{\theta}_x \right)_x + \frac{4\varepsilon}{3} \left(\frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_1 \bar{u}_{1x} \right)_x \\ + \varepsilon \left(\sum_{i=2}^3 \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_i \bar{u}_{ix} \right)_x + R_{2x}, \end{cases} \quad (2.8)$$

where

$$R_1 = \frac{2\varepsilon}{3p_+} a(\hat{\Theta}) \hat{\Theta}_t + (\bar{p} - p_+) - \frac{4\varepsilon\mu(\bar{\theta})}{3\bar{v}} \bar{u}_{1x} = O(1)\delta\varepsilon(1+t)^{-1} e^{-\frac{cx^2}{\varepsilon(1+t)}}, \quad (2.9)$$

$$\begin{aligned} R_2 &= \frac{\varepsilon}{v} (\lambda(\hat{\Theta}) \hat{\Theta}_x - \lambda(\bar{\theta}) \bar{\theta}_x) + (\bar{p} - p_+) \bar{u}_1 - \frac{4\varepsilon\mu(\bar{\theta})}{3\bar{v}} \bar{u}_1 \bar{u}_{1x} \\ &= O(1)\delta\varepsilon^{3/2} (1+t)^{-3/2} e^{-\frac{cx^2}{\varepsilon(1+t)}}, \end{aligned} \quad (2.10)$$

with some positive constant $c > 0$ depending only on θ_{\pm} .

From (2.6), we have

$$\begin{cases} |\hat{\Theta} - \theta_-| = O(1)\delta e^{-\frac{cx^2}{2\varepsilon(1+t)}}, & \text{if } x < 0, \\ |\hat{\Theta} - \theta_+| = O(1)\delta e^{-\frac{cx^2}{2\varepsilon(1+t)}}, & \text{if } x > 0. \end{cases} \quad (2.11)$$

Therefore,

$$\begin{cases} |(\bar{v}, \bar{u}, \bar{\theta})(x, t) - (v_-, 0, \theta_-)| = O(1)\delta e^{-\frac{cx^2}{2\varepsilon(1+t)}}, & \text{if } x < 0, \\ |(\bar{v}, \bar{u}, \bar{\theta})(x, t) - (v_+, 0, \theta_+)| = O(1)\delta e^{-\frac{cx^2}{2\varepsilon(1+t)}}, & \text{if } x > 0. \end{cases} \quad (2.12)$$

We are now ready to state the main result as follows.

Theorem 2.1. *Given a contact discontinuity solution $(\bar{V}, \bar{U}, \bar{\Theta})(x, t)$ of the Euler system (1.13), there exist small positive constants δ_0, ε_0 and a global Maxwellian $\mathbf{M}_* = \mathbf{M}_{[\rho_*, u_*, \theta_*]}$, such that if $\delta \leq \delta_0, \varepsilon \leq \varepsilon_0$, then the Boltzmann equation (1.1) admits a unique global solution $f^\varepsilon(x, t, \xi)$ satisfying*

$$\int_{\mathbf{R}^3} \frac{|f^\varepsilon(x, t, \xi) - \mathbf{M}_{[\bar{v}, \bar{u}, \bar{\theta}]}(x, t, \xi)|^2}{\mathbf{M}_*} d\xi \leq \tilde{C}_1 \delta_0 \varepsilon^{\frac{1}{2}} + \tilde{C}_2 \delta_0 e^{-\frac{\tilde{C}_3 x^2}{\varepsilon(1+t)}}, \quad (2.13)$$

with some positive constants $\tilde{C}_i (i = 1, 2, 3)$ independent of ε .

Consequently, we have for any $T > 0$ and $t \in [0, T]$,

$$\sup_{|x| \geq h} \|f^\varepsilon(x, t, \xi) - \mathbf{M}_{[\bar{v}, \bar{u}, \bar{\theta}]}(x, t, \xi)\|_{L^2_\xi(\frac{1}{\sqrt{M_*}})} \leq C_h \delta_0 \varepsilon^{\frac{1}{4}}, \quad \forall h > 0, \quad (2.14)$$

where the norm $\|\cdot\|_{L^2_\xi(\frac{1}{\sqrt{M_*}})}$ is $\|\cdot\|_{\sqrt{M_*}} \| \cdot \|_{L^2_\xi(\mathbf{R}^3)}$.

Remark. Theorem 2.1 shows that, away from the contact discontinuity located at $x = 0$, for the Knudsen number $\varepsilon < \varepsilon_0$, there exists a unique global solution $f^\varepsilon(x, t, \xi)$ of the Boltzmann equation (1.1) which tends to $\mathbf{M}_{[\bar{v}, \bar{u}, \bar{\theta}]}(x, t, \xi)$ as two global Maxwellians with a jump at $x = 0$ when $\varepsilon \rightarrow 0$. Moreover, a uniform convergence rate $\varepsilon^{\frac{1}{4}}$ in the norm $L_x^\infty L^2_\xi(\frac{1}{\sqrt{M_*}})$ holds. Here, $\varepsilon_0 > 0$ is a small constant.

3. Reformulated System

In this section, we will reformulate the system and introduce a scaling for the independent variable and the perturbation. Firstly, we define the scaled independent variables

$$y = \varepsilon^{-\frac{1}{2}}x, \quad \tau = \varepsilon^{-\frac{1}{2}}t. \quad (3.1)$$

Correspondingly, set the scaled perturbation as

$$\begin{aligned} v(x, t) &= \bar{v}(x, t) + \varepsilon^{\frac{1}{2}}\phi(y, \tau), \\ u(x, t) &= \bar{u}(x, t) + \varepsilon^{\frac{1}{2}}\psi(y, \tau), \\ \theta(x, t) &= \bar{\theta}(x, t) + \varepsilon^{\frac{1}{2}}\zeta(y, \tau), \\ \left(\theta + \frac{|u|^2}{2}\right)(x, t) &= \left(\bar{\theta} + \frac{|\bar{u}|^2}{2}\right)(x, t) + \varepsilon^{\frac{1}{2}}\omega(y, \tau), \\ \mathbf{G}(x, t, \xi) &= \varepsilon^{\frac{1}{2}}\bar{\mathbf{G}}(y, \tau, \xi), \\ \Theta_1(x, t, \xi) &= \varepsilon^{\frac{1}{2}}\bar{\Theta}_1(y, \tau, \xi). \end{aligned} \quad (3.2)$$

We remark that the above scaling transformation plays an important role in the following proof.

Under this scaling, the hydrodynamic limit problem is now transferred into a scaled time-asymptotic stability of the viscous contact wave to the Boltzmann equation. In fact, this scaling is suitable for the contact wave because of its parabolic structure. Notice that the hydrodynamic limit proved by this method is globally in time unlike the case with shock profile proved in [35] which is locally in time. However, we do not know whether there exists some appropriate scaling for the shock profile so that this method can be applied.

With the above scaling, the proof of Theorem 2.1 will be given by the energy method as [18] for the scaled perturbation $(\phi, \psi, \zeta)(y, \tau)$ and $\bar{\mathbf{G}}(y, \tau, \xi)$.

From the construction of the contact wave $(\bar{v}, \bar{u}, \bar{\theta})$, the relation between the viscous contact wave $(\bar{v}, \bar{u}, \bar{\theta})$ to the Boltzmann equation and the inviscid contact discontinuity $(\bar{V}, \bar{U}, \bar{\Theta})$ is given by (2.12). Thus, in order to prove Theorem 2.1, it is sufficient to consider the convergence of the solution $f(y, \tau, \xi)$ of the Boltzmann equation to the Maxwellian $\mathbf{M}_{[\bar{v}, \bar{u}, \bar{\theta}]}(y, \tau, \xi)$ defined by the contact wave $(\bar{v}, \bar{u}, \bar{\theta})$ as the Knudsen number ε tends to zero.

For this, as in [18], we introduce the following anti-derivative of the perturbation:

$$(\Phi, \Psi, \bar{W})(y, \tau) = \int_{-\infty}^y (\phi, \psi, \omega)(y', \tau) dy'. \tag{3.3}$$

Obviously,

$$(\Phi, \Psi, \bar{W})_y(y, \tau) = (\phi, \psi, \omega)(y, \tau).$$

From (1.18) and (2.8), we have the following system for (Φ, Ψ, \bar{W}) :

$$\begin{cases} \Phi_\tau - \Psi_{1y} = 0, \\ \Psi_{1\tau} + \varepsilon^{-\frac{1}{2}}(p - \bar{p}) = \frac{4}{3} \left(\frac{\mu(\theta)}{v} u_{1y} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{1y} \right) - \varepsilon^{-\frac{1}{2}} R_1 - \int \xi_1^2 \bar{\Theta}_1 d\xi, \\ \Psi_{i\tau} = \left(\frac{\mu(\theta)}{v} u_{iy} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{iy} \right) - \int \xi_1 \xi_i \bar{\Theta}_1 d\xi, \quad i = 2, 3, \\ \bar{W}_\tau + \varepsilon^{-\frac{1}{2}}(p u_1 - \bar{p} \bar{u}_1) = \left(\frac{\lambda(\theta)}{v} \theta_y - \frac{\lambda(\bar{\theta})}{\bar{v}} \bar{\theta}_y \right) + \frac{4}{3} \left(\frac{\mu(\theta)}{v} u_1 u_{1y} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_1 \bar{u}_{1y} \right) \\ + \sum_{i=2}^3 \frac{\mu(\theta)}{v} u_i u_{iy} - \varepsilon^{-\frac{1}{2}} R_2 - \int \frac{1}{2} \xi_1 |\xi|^2 \bar{\Theta}_1 d\xi, \end{cases} \tag{3.4}$$

where the error terms R_i ($i = 1, 2$) are given in (2.9) and (2.10).

Introduce a new variable

$$W = \bar{W} - \bar{u}_1 \Psi_1. \tag{3.5}$$

It follows that

$$\zeta = W_y - Y, \quad \text{with } Y = \frac{1}{2} \varepsilon^{\frac{1}{2}} |\Psi_y|^2 - \bar{u}_{1y} \Psi_1. \tag{3.6}$$

By using the new variable W and linearizing the system (3.4), we have

$$\begin{cases} \Phi_\tau - \Psi_{1y} = 0, \\ \Psi_{1\tau} - \frac{p_+}{\bar{v}} \Phi_y + \frac{2}{3\bar{v}} W_y = \frac{4}{3} \varepsilon^{\frac{1}{2}} \frac{\mu(\bar{\theta})}{\bar{v}} \Psi_{1yy} - \int \xi_1^2 \bar{\Theta}_1 d\xi + Q_1, \\ \Psi_{i\tau} = \varepsilon^{\frac{1}{2}} \frac{\mu(\bar{\theta})}{\bar{v}} \Psi_{iy} - \int \xi_1 \xi_i \bar{\Theta}_1 d\xi + Q_i, \quad i = 2, 3, \\ W_\tau + p_+ \Psi_{1y} = \varepsilon^{\frac{1}{2}} \frac{\lambda(\bar{\theta})}{\bar{v}} W_{yy} - \int \frac{1}{2} \xi_1 |\xi|^2 \bar{\Theta}_1 d\xi + \bar{u}_1 \int \xi_1^2 \bar{\Theta}_1 d\xi + Q_4, \end{cases} \tag{3.7}$$

where

$$\begin{aligned} Q_1 &= \frac{4}{3} \left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) u_{1y} + J_1 + \frac{2}{3\bar{v}} Y - \varepsilon^{-\frac{1}{2}} R_1, \\ Q_i &= \left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) u_{iy}, \quad i = 2, 3, \\ Q_4 &= \left(\frac{\lambda(\theta)}{v} - \frac{\lambda(\bar{\theta})}{\bar{v}} \right) \theta_y + \frac{4\varepsilon^{\frac{1}{2}}}{3} \frac{\mu(\theta)}{v} u_{1y} \Psi_{1y} - \varepsilon^{-\frac{1}{2}} R_2 - \bar{u}_{1\tau} \Psi_1 + \varepsilon^{-\frac{1}{2}} \bar{u}_1 R_1 \\ &\quad + \sum_{i=2}^3 \frac{\mu(\theta)}{v} u_i u_{iy} + J_2 - \varepsilon^{\frac{1}{2}} \frac{\lambda(\bar{\theta})}{\bar{v}} Y_y, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} J_1 &= \frac{p - p_+}{\bar{v}} \Phi_y = O(1)(\varepsilon^{\frac{1}{2}} \Phi_y^2 + \varepsilon^{\frac{1}{2}} W_y^2 + \varepsilon^{\frac{1}{2}} Y^2 + |\bar{u}|^4), \\ J_2 &= (p_+ - p) \Psi_{1y} = O(1)(\varepsilon^{\frac{1}{2}} \Phi_y^2 + \varepsilon^{\frac{1}{2}} W_y^2 + \varepsilon^{\frac{1}{2}} \Psi_{1y}^2 + \varepsilon^{\frac{1}{2}} Y^2 + |\bar{u}|^4). \end{aligned} \quad (3.9)$$

We now derive the equation for the scaled non-fluid component $\bar{\mathbf{G}}(y, \tau, \xi)$. From (1.15), we have

$$\bar{\mathbf{G}}_\tau - \frac{u_1}{v} \bar{\mathbf{G}}_y + \varepsilon^{-\frac{1}{2}} \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_y) + \frac{1}{v} \mathbf{P}_1(\xi_1 \bar{\mathbf{G}}_y) = \varepsilon^{-\frac{1}{2}} \mathbf{L}_M \bar{\mathbf{G}} + Q(\bar{\mathbf{G}}, \bar{\mathbf{G}}). \quad (3.10)$$

Thus, we obtain

$$\bar{\mathbf{G}} = \frac{1}{v} \mathbf{L}_M^{-1} [\mathbf{P}_1(\xi_1 \mathbf{M}_y)] + \bar{\Theta}_1, \quad (3.11)$$

and

$$\bar{\Theta}_1(y, \tau, \xi) = \varepsilon^{\frac{1}{2}} \mathbf{L}_M^{-1} \left[\bar{\mathbf{G}}_\tau - \frac{u_1}{v} \bar{\mathbf{G}}_y + \frac{1}{v} \mathbf{P}_1(\xi_1 \bar{\mathbf{G}}_y) - Q(\bar{\mathbf{G}}, \bar{\mathbf{G}}) \right]. \quad (3.12)$$

Let

$$\bar{\mathbf{G}}_0(y, \tau, \xi) = \frac{3}{2v\theta} \mathbf{L}_M^{-1} \left\{ \mathbf{P}_1 \left[\xi_1 \left(\frac{|\xi - u|^2}{2\theta} \bar{\theta}_y + \xi \cdot \bar{u}_y \right) \mathbf{M} \right] \right\}, \quad (3.13)$$

and

$$\bar{\mathbf{G}}_1(y, \tau, \xi) = \bar{\mathbf{G}}(y, \tau, \xi) - \bar{\mathbf{G}}_0(y, \tau, \xi). \quad (3.14)$$

Then $\bar{\mathbf{G}}_1(y, \tau, \xi)$ satisfies

$$\begin{aligned} \bar{\mathbf{G}}_{1\tau} - \varepsilon^{-\frac{1}{2}} \mathbf{L}_M \bar{\mathbf{G}}_1 &= -\frac{3}{2v\theta} \mathbf{P}_1 \left[\xi_1 \left(\frac{|\xi - u|^2}{2\theta} \zeta_y + \xi \cdot \psi_y \right) \mathbf{M} \right] \\ &\quad + \frac{u_1}{v} \bar{\mathbf{G}}_y - \frac{1}{v} \mathbf{P}_1(\xi_1 \bar{\mathbf{G}}_y) + Q(\bar{\mathbf{G}}, \bar{\mathbf{G}}) - \bar{\mathbf{G}}_{0\tau}. \end{aligned} \quad (3.15)$$

Notice that in (3.14) and (3.15), $\bar{\mathbf{G}}_0$ is subtracted from $\bar{\mathbf{G}}$ because $\|\bar{\theta}_y\|^2 \sim (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1/2}$ is not integrable with respect to τ .

Finally, from (1.12) and the scaling (3.1), we have

$$f_\tau - \frac{u_1}{v} f_y + \frac{\xi_1}{v} f_y = \varepsilon^{-\frac{1}{2}} Q(f, f). \quad (3.16)$$

In the following, we will derive the energy estimate on the scaled Boltzmann equation (3.16). Indeed, to prove Theorem 2.1, it is sufficient to prove the following theorem:

Theorem 3.1. *There exist small positive constants δ_1, ε_1 and a global Maxwellian $\mathbf{M}_* = \mathbf{M}_{[v_*, u_*, \theta_*]}$ such that if the initial data and the wave strength δ satisfy*

$$E_6(\tau)|_{\tau=0} + \delta \leq \delta_1, \quad (3.17)$$

and the Knudsen number ε satisfies $\varepsilon \leq \varepsilon_1$, then the problem (3.16) admits a unique global solution $f^\varepsilon(y, \tau, \xi)$ satisfying

$$\sup_y \|f^\varepsilon(y, \tau, \xi) - \mathbf{M}_{[\bar{v}, \bar{u}, \bar{\theta}]}(y, \tau, \xi)\|_{L^2_\xi(\frac{1}{\sqrt{\mathbf{M}_*})} \leq C\delta_1\varepsilon^{\frac{1}{4}}. \tag{3.18}$$

Here $E_6(\tau)$ will be defined in (5.1) satisfying

$$\begin{aligned} E_6(\tau) \sim & \|(\Phi, \Psi, W)\|^2 + \|(\phi, \psi, \zeta)\|^2 + \varepsilon\|(\phi_y, \psi_y, \zeta_y)\|^2 + \int \int \frac{\bar{\mathbf{G}}_1^2}{\mathbf{M}_*} d\xi dy \\ & + \varepsilon \sum_{|\alpha|=1} \int \int \frac{|\partial^{\alpha'} \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy + \varepsilon \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha f|^2}{\mathbf{M}_*} d\xi dy. \end{aligned} \tag{3.19}$$

From now on, $\partial^\alpha, \partial^{\alpha'}$ denote the derivatives with respect to y or τ , and $\|\cdot\|^2$ represents $\|\cdot\|_{L^2}^2$ for simplicity of notations.

Remark. In particular, if we choose the initial value of the Boltzmann equation (3.16) as

$$f^\varepsilon(y, 0, \xi) = \mathbf{M}_{[\bar{v}, \bar{u}, \bar{\theta}]}(y, 0, \xi) = \mathbf{M}_{[\bar{v}(y,0), \bar{u}(y,0), \bar{\theta}(y,0)]}(\xi), \tag{3.20}$$

then

$$E_6(\tau)|_{\tau=0} = O(1) \left[\|(\bar{\theta}_y, \bar{u}_y)\|^2 + \varepsilon\|(\bar{v}_{yy}, \bar{\theta}_{yy}, \bar{u}_{yy})\|^2 \right] \Big|_{\tau=0} = O(1)\delta. \tag{3.21}$$

In fact, the initial data $f(y, 0, \xi)$ can be chosen such that the initial perturbation $E_6(\tau)|_{\tau=0}$ is suitably small and of order $O(1)$ with respect to ε . This is the reason why we use the scaled variables y, τ in (3.1), otherwise, the initial perturbation $E_6(\tau)|_{\tau=0}$ is not uniform with respect to ε .

4. A Priori Estimate

We will focus on the reformulated system (3.7) and (3.15). Since the local existence of the solution to (3.7) and (3.15) is now standard, cf. [31] or [36], to prove the global existence, we only need to close the following a priori estimate by the continuity argument:

$$\begin{aligned} N(T) = \sup_{0 \leq \tau \leq T} & \left\{ \|(\Phi, \Psi, W)\|_{L^\infty}^2 + \|(\phi, \psi, \zeta)\|^2 + \varepsilon\|(\phi_y, \psi_y, \zeta_y)\|^2 + \int \int \frac{\bar{\mathbf{G}}_1^2}{\mathbf{M}_*} d\xi dy \right. \\ & \left. + \varepsilon \sum_{|\alpha|=1} \int \int \frac{|\partial^{\alpha'} \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy + \varepsilon \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha f|^2}{\mathbf{M}_*} d\xi dy \right\} \leq \gamma^2, \end{aligned} \tag{4.1}$$

where γ is a small positive constant depending on the initial data and the strength of the contact wave, and \mathbf{M}_* is a global Maxwellian chosen later.

We now briefly explain the a priori assumption $\|(\Phi, \Psi, W)\|_{L^\infty}^2 \leq \gamma^2$ in (4.1). Roughly speaking, based on the observation in [18] that the energy estimate involving $\|(\Phi, \Psi, W)\|_{L^2}^2$ may grow at a rate $(1 + \varepsilon^{\frac{1}{2}}\tau)^{\frac{1}{2}}$, the decay of $\|(\Phi_x, \Psi_x, W_x)\|_{L^2}^2$ in the order of $(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}}$ is needed to compensate this growth. This yields a uniform boundedness of $\|(\Phi, \Psi, W)\|_{L^\infty}$, which is essential to close the a priori estimate.

Note that the a priori assumption (4.1) also gives

$$\varepsilon^{\frac{1}{2}} \|(\phi, \psi, \zeta)\|_{L^\infty}^2 \leq C\gamma^2, \tag{4.2}$$

$$\varepsilon^{\frac{1}{2}} \left\| \int \frac{\bar{\mathbf{G}}_1^2}{\mathbf{M}_*} d\xi \right\|_{L^\infty} \leq C\varepsilon^{\frac{1}{2}} \left(\int \int \frac{\bar{\mathbf{G}}_1^2}{\mathbf{M}_*} d\xi dy \right)^{\frac{1}{2}} \cdot \left(\int \int \frac{|\bar{\mathbf{G}}_{1y}|^2}{\mathbf{M}_*} d\xi dy \right)^{\frac{1}{2}} \leq C(\delta + \gamma)^2, \tag{4.3}$$

and for $|\alpha'| = 1$,

$$\begin{aligned} \varepsilon^{\frac{3}{2}} \left\| \int \frac{|\partial^{\alpha'} \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi \right\|_{L^\infty} &\leq C\varepsilon^{\frac{3}{2}} \left(\int \int \frac{|\partial^{\alpha'} \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy \right)^{\frac{1}{2}} \\ &\cdot \left(\int \int \frac{|\partial^{\alpha'} \bar{\mathbf{G}}_y|^2}{\mathbf{M}_*} d\xi dy \right)^{\frac{1}{2}} \leq C(\delta + \gamma)^2. \end{aligned} \tag{4.4}$$

From (1.14) and (2.8), we have

$$\left\{ \begin{aligned} \phi_\tau - \psi_{1y} &= 0, \\ \psi_{1\tau} + \varepsilon^{-\frac{1}{2}}(p - \bar{p})_y &= -\frac{4\varepsilon^{\frac{1}{2}}}{3} \left(\frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{1y} \right)_y - \varepsilon^{-\frac{1}{2}} R_{1y} - \int \xi_1^2 \bar{\mathbf{G}}_y d\xi, \\ \psi_{i\tau} &= -\varepsilon^{-\frac{1}{2}} \left(\frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{iy} \right)_y - \int \xi_1 \xi_i \bar{\mathbf{G}}_y d\xi, \quad i = 2, 3, \\ \zeta_\tau + \varepsilon^{-\frac{1}{2}}(pu_{1y} - \bar{p}\bar{u}_{1y}) &= -\varepsilon^{-\frac{1}{2}} \left(\frac{\lambda(\bar{\theta})}{\bar{v}} \bar{\theta}_y \right)_y - \frac{4\varepsilon^{\frac{1}{2}}}{3} \left(\frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_1 \bar{u}_{1y} \right)_y - \varepsilon^{-\frac{1}{2}} R_{2y} \\ &\quad + \varepsilon^{-\frac{1}{2}} \left(\frac{|\bar{u}|^2}{2} \right)_\tau - \varepsilon^{-\frac{1}{2}} \bar{p}_y \bar{u}_1 - \frac{1}{2} \int \xi_1 |\xi|^2 \bar{\mathbf{G}}_y d\xi + \sum_{i=1}^3 u_i \int \xi_1 \xi_i \bar{\mathbf{G}}_y d\xi. \end{aligned} \right. \tag{4.5}$$

Thus

$$\varepsilon \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 \leq C(\delta + \gamma)^2. \tag{4.6}$$

Hence, we have

$$\|(v_\tau, u_\tau, \theta_\tau)\|^2 \leq C\varepsilon \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 + C\|(\bar{v}_\tau, \bar{u}_\tau, \bar{\theta}_\tau)\|^2 \leq C(\delta + \gamma)^2. \tag{4.7}$$

In addition, (4.1) also implies that

$$\|(v_y, u_y, \theta_y)\|^2 \leq C\varepsilon \|(\phi_y, \psi_y, \zeta_y)\|^2 + C\|(\bar{v}_y, \bar{u}_y, \bar{\theta}_y)\|^2 \leq C(\delta + \gamma)^2. \tag{4.8}$$

Since

$$\varepsilon \|\partial^\alpha \left(\rho, \rho u, \rho \left(E + \frac{|u|^2}{2} \right) \right)\|^2 \leq C\varepsilon \int \int \frac{|\partial^\alpha f|^2}{\mathbf{M}_*} d\xi dy \leq C\gamma^2, \tag{4.9}$$

Eqs. (4.7)–(4.9) give

$$\begin{aligned} \varepsilon \|\partial^\alpha(v, u, \theta)\|^2 &\leq C\varepsilon \|\partial^\alpha\left(\rho, \rho u, \rho\left(E + \frac{|u|^2}{2}\right)\right)\|^2 \\ &\quad + C\varepsilon \sum_{|\alpha'|=1} \int |\partial^{\alpha'}\left(\rho, \rho u, \rho\left(E + \frac{|u|^2}{2}\right)\right)|^4 dy \\ &\leq C(\delta + \gamma)^2. \end{aligned} \tag{4.10}$$

Thus, for $|\alpha| = 2$, we have

$$\varepsilon^2 \|\partial^\alpha(\phi, \psi, \zeta)\|^2 \leq C\varepsilon (\|\partial^\alpha(v, u, \theta)\|^2 + \|\partial^\alpha(\bar{v}, \bar{u}, \bar{\theta})\|^2) \leq C(\delta + \gamma)^2. \tag{4.11}$$

Finally, from the fact that $f = \mathbf{M} + \varepsilon^{\frac{1}{2}} \bar{\mathbf{G}}$, we can obtain for $|\alpha| = 2$,

$$\begin{aligned} \varepsilon^2 \int \int \frac{|\partial^\alpha \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy &\leq C\varepsilon \int \int \frac{|\partial^\alpha f|^2}{\mathbf{M}_*} d\xi dy + C\varepsilon \int \int \frac{|\partial^\alpha \mathbf{M}|^2}{\mathbf{M}_*} d\xi dy \\ &\leq C\varepsilon \int \int \frac{|\partial^\alpha f|^2}{\mathbf{M}_*} d\xi dy + C\varepsilon \|\partial^\alpha(v, u, \theta)\|^2 + C\varepsilon \sum_{|\alpha'|=1} \int |\partial^{\alpha'}(v, u, \theta)|^4 dy \\ &\leq C(\delta + \gamma)^2. \end{aligned} \tag{4.12}$$

Before proving the a priori estimate (4.1), we list some basic lemmas based on the celebrated H-theorem for later use. The first lemma is from [13].

Lemma 4.1. *There exists a positive constant C such that*

$$\int \frac{\nu(|\xi|)^{-1} \mathcal{Q}(f, g)^2}{\tilde{\mathbf{M}}} d\xi \leq C \left\{ \int \frac{\nu(|\xi|) f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int \frac{g^2}{\tilde{\mathbf{M}}} d\xi + \int \frac{f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int \frac{\nu(|\xi|) g^2}{\tilde{\mathbf{M}}} d\xi \right\},$$

where $\tilde{\mathbf{M}}$ can be any Maxwellian so that the above integrals are well defined.

Based on Lemma 4.1, the following three lemmas are proved in [25]. The proofs are straightforward by using the Cauchy inequality.

Lemma 4.2. *If $\theta/2 < \theta_* < \theta$, then there exist two positive constants $\sigma = \sigma(v, u, \theta; v_*, u_*, \theta_*)$ and $\eta_0 = \eta_0(v, u, \theta; v_*, u_*, \theta_*)$ such that if $|v - v_*| + |u - u_*| + |\theta - \theta_*| < \eta_0$, we have for $h(\xi) \in \mathcal{N}^\perp$,*

$$-\int \frac{h \mathbf{L}_M h}{\mathbf{M}_*} d\xi \geq \sigma \int \frac{\nu(|\xi|) h^2}{\mathbf{M}_*} d\xi.$$

Lemma 4.3. *Under the assumptions in Lemma 4.2, we have for each $h(\xi) \in \mathcal{N}^\perp$,*

$$\begin{cases} \int \frac{\nu(|\xi|)}{\mathbf{M}} |\mathbf{L}_M^{-1} h|^2 d\xi \leq \sigma^{-2} \int \frac{\nu(|\xi|)^{-1} h^2}{\mathbf{M}} d\xi, \\ \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\mathbf{L}_M^{-1} h|^2 d\xi \leq \sigma^{-2} \int \frac{\nu(|\xi|)^{-1} h^2}{\mathbf{M}_*} d\xi. \end{cases}$$

Lemma 4.4. *Under the conditions in Lemma 4.2, for any positive constants k and λ , it holds that*

$$\left| \int \frac{g_1 \mathbf{P}_1(|\xi|^k g_2)}{\mathbf{M}_*} d\xi - \int \frac{g_1 |\xi|^k g_2}{\mathbf{M}_*} d\xi \right| \leq C_{k,\lambda} \int \frac{\lambda |g_1|^2 + \lambda^{-1} |g_2|^2}{\mathbf{M}_*} d\xi,$$

where the constant $C_{k,\lambda}$ depends on k and λ .

4.1. *Lower order estimate.* Now we will derive the lower order estimates of (Φ, Ψ, W) . By multiplying (3.7)₁ by $p_+ \Phi$, (3.7)₂ by $\bar{v} \Psi_1$, (3.7)₃ by Ψ_i , (3.7)₄ by $\frac{2}{3p_+} W$ respectively and adding all the resulting equations, we have

$$\begin{aligned} & \left(\frac{p_+}{2} \Phi^2 + \frac{\bar{v}}{2} \Psi_1^2 + \frac{1}{2} \sum_{i=2}^3 \Psi_i^2 + \frac{W^2}{3p_+} \right)_\tau + \frac{4\varepsilon^{\frac{1}{2}}}{3} \mu(\bar{\theta}) \Psi_{1y}^2 + \sum_{i=2}^3 \varepsilon^{\frac{1}{2}} \frac{\mu(\bar{\theta})}{\bar{v}} \Psi_{iy}^2 + \frac{2\varepsilon^{\frac{1}{2}}}{3p_+} \frac{\lambda(\bar{\theta})}{\bar{v}} W_y^2 \\ &= \frac{1}{2} \bar{v}_\tau \Psi_1^2 - \frac{4\varepsilon^{\frac{1}{2}}}{3} [\mu(\bar{\theta})]_y \Psi_1 \Psi_{1y} - \sum_{i=2}^3 \varepsilon^{\frac{1}{2}} \left(\frac{\mu(\bar{\theta})}{\bar{v}} \right)_y \Psi_i \Psi_{iy} - \frac{2\varepsilon^{\frac{1}{2}}}{3p_+} \left(\frac{\lambda(\bar{\theta})}{\bar{v}} \right)_y W W_y \\ &+ \bar{v} Q_1 \Psi_1 + \sum_{i=2}^3 Q_i \Psi_i + \frac{2W}{3p_+} Q_4 + N_1 + (\dots)_y, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} N_1 &= -\bar{v} \Psi_1 \int \xi_1^2 \bar{\Theta}_1 d\xi - \sum_{i=2}^3 \Psi_i \int \xi_1 \xi_i \bar{\Theta}_1 d\xi \\ &+ \frac{2W}{3p_+} \left(\bar{u}_1 \int \xi_1^2 \bar{\Theta}_1 d\xi - \int \frac{1}{2} \xi_1 |\xi|^2 \bar{\Theta}_1 d\xi \right). \end{aligned} \tag{4.14}$$

From now on, $(\dots)_y$ denotes the term in the conservative form so that it vanishes after integration with respect to y over \mathbf{R} . Let

$$\begin{aligned} E_1 &= \int \left(\frac{p_+}{2} \Phi^2 + \frac{\bar{v}}{2} \Psi_1^2 + \frac{1}{2} \sum_{i=2}^3 \Psi_i^2 + \frac{W^2}{3p_+} \right) dy, \\ K_1 &= \int \left(\frac{4\varepsilon^{\frac{1}{2}}}{3} \mu(\bar{\theta}) \Psi_{1y}^2 + \sum_{i=2}^3 \varepsilon^{\frac{1}{2}} \frac{\mu(\bar{\theta})}{\bar{v}} \Psi_{iy}^2 + \frac{2\varepsilon^{\frac{1}{2}}}{3p_+} \frac{\lambda(\bar{\theta})}{\bar{v}} W_y^2 \right) dy. \end{aligned} \tag{4.15}$$

We estimate the right hand side of (4.13) term by term as follows. First,

$$\int \frac{1}{2} \bar{v}_\tau \Psi_1^2 dy \leq C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} E_1, \tag{4.16}$$

$$\int \frac{4\varepsilon^{\frac{1}{2}}}{3} [\mu(\bar{\theta})]_y \Psi_1 \Psi_{1y} dy \leq \beta K_1 + C_\beta \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} E_1, \tag{4.17}$$

where β is a small positive constant to be chosen later.

Now we estimate $\int \bar{v} Q_1 \Psi_1 dy$ by

$$\begin{aligned} \int \bar{v} Q_1 \Psi_1 dy &\leq \int \left| \frac{4}{3} \bar{v} \left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) u_{1y} \Psi_1 \right| dy + \int |\bar{v} J_1 \Psi_1| dy \\ &+ \int |\varepsilon^{-\frac{1}{2}} \bar{v} R_1 \Psi_1| dy + \int \left| \frac{2}{3} Y \Psi_1 \right| dy := \sum_{i=1}^4 I_i. \end{aligned} \tag{4.18}$$

Note that

$$\begin{aligned}
 I_1 &\leq C\varepsilon^{\frac{1}{2}} \int |(\Phi_y, \zeta)\bar{u}_{1y}\Psi_1|dy + C\varepsilon \int |(\Phi_y, \zeta)\psi_{1y}\Psi_1|dy \\
 &\leq C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}E_1 + C(\delta + \gamma)(\varepsilon^{\frac{1}{2}}\|\Phi_y\|^2 + K_1) + C\gamma\varepsilon^{\frac{3}{2}}\|\psi_{1y}\|^2,
 \end{aligned}
 \tag{4.19}$$

$$\begin{aligned}
 I_2 &\leq C \int (\varepsilon^{\frac{1}{2}}|\Phi_y|^2 + \varepsilon^{\frac{1}{2}}|W_y|^2 + \varepsilon^{\frac{1}{2}}Y^2 + |\bar{u}|^4)|\Psi_1|dy \\
 &\leq C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}E_1 + C\gamma(\varepsilon^{\frac{1}{2}}\|\Phi_y\|^2 + K_1) + C\delta\varepsilon^{\frac{5}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{5}{2}},
 \end{aligned}
 \tag{4.20}$$

$$I_3 \leq C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}E_1 + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}},
 \tag{4.21}$$

and

$$I_4 \leq C \int |(\varepsilon^{\frac{1}{2}}|\Psi_y|^2 + \bar{u}_{1y}\Psi_1)\Psi_1|dy \leq C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}E_1 + C\gamma K_1.
 \tag{4.22}$$

Substituting (4.18)–(4.22) into (4.17) yields

$$\begin{aligned}
 \int \bar{v}Q_1\Psi_1dy &\leq C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}E_1 + C(\delta + \gamma)(\varepsilon^{\frac{1}{2}}\|\Phi_y\|^2 + K_1) \\
 &\quad + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}} + C\gamma\varepsilon^{\frac{3}{2}}\|\psi_{1y}\|^2.
 \end{aligned}
 \tag{4.23}$$

Similarly, we can estimate

$$\int Q_i\Psi_i dy \ (i = 2, 3) \quad \text{and} \quad \int \frac{2W}{3p_+}Q_4dy.$$

Now we estimate $\int N_1dy$. We only need to estimate $T_1 =: -\int \bar{v}\Psi_1 \int \xi_1^2 \bar{\Theta}_1 d\xi dy$ because other terms in $\int N_1dy$ can be estimated similarly. Let \mathbf{M}_* be a global Maxwellian with the state (v_*, u_*, θ_*) satisfying $\frac{1}{2}\theta < \theta_* < \theta$ and $|v - v_*| + |u - u_*| + |\theta - \theta_*| \leq \eta_0$ so that Lemma 4.2 holds. By the definition of $\bar{\Theta}_1$, cf. (3.12), we have

$$\begin{aligned}
 T_1 &= -\varepsilon^{\frac{1}{2}} \int \bar{v}\Psi_1 \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\bar{\mathbf{G}}_\tau) d\xi dy + \varepsilon^{\frac{1}{2}} \int \frac{u_1 \bar{v}\Psi_1}{v} \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\bar{\mathbf{G}}_y) d\xi dy \\
 &\quad - \varepsilon^{\frac{1}{2}} \int \frac{\bar{v}\Psi_1}{v} \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}[\mathbf{P}_1(\xi_1 \bar{\mathbf{G}}_y)] d\xi dy + \varepsilon^{\frac{1}{2}} \int \bar{v}\Psi_1 \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}[Q(\bar{\mathbf{G}}, \bar{\mathbf{G}})] d\xi dy \\
 &=: \sum_{i=1}^4 T_1^i.
 \end{aligned}
 \tag{4.24}$$

For the integral T_1^1 , we have

$$\begin{aligned}
 T_1^1 &= -\varepsilon^{\frac{1}{2}} \int \bar{v}\Psi_1 \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\bar{\mathbf{G}}_{1\tau}) d\xi dy - \varepsilon^{\frac{1}{2}} \int \bar{v}\Psi_1 \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\bar{\mathbf{G}}_{0\tau}) d\xi dy \\
 &=: T_1^{11} + T_1^{12}.
 \end{aligned}
 \tag{4.25}$$

Note that the linearized operator $\mathbf{L}_{\mathbf{M}}^{-1}$ satisfies, for any $h \in \mathcal{N}^\perp$,

$$\begin{aligned}
 (\mathbf{L}_{\mathbf{M}}^{-1}h)_\tau &= \mathbf{L}_{\mathbf{M}}^{-1}(h_\tau) - 2\mathbf{L}_{\mathbf{M}}^{-1}\{Q(\mathbf{L}_{\mathbf{M}}^{-1}h, M_\tau)\}, \\
 (\mathbf{L}_{\mathbf{M}}^{-1}h)_y &= \mathbf{L}_{\mathbf{M}}^{-1}(h_y) - 2\mathbf{L}_{\mathbf{M}}^{-1}\{Q(\mathbf{L}_{\mathbf{M}}^{-1}h, M_y)\}.
 \end{aligned}
 \tag{4.26}$$

Then we have

$$\begin{aligned}
 T_1^{11} &= -\varepsilon^{\frac{1}{2}} \int \bar{v} \Psi_1 \int \xi_1^2 (\mathbf{L}_M^{-1} \bar{\mathbf{G}}_1)_\tau d\xi dy - 2\varepsilon^{\frac{1}{2}} \int \bar{v} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1} \{Q(\mathbf{L}_M^{-1} \bar{\mathbf{G}}_1, M_\tau)\} d\xi dy \\
 &= -(\varepsilon^{\frac{1}{2}} \int \bar{v} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1} (\bar{\mathbf{G}}_1) d\xi dy)_\tau + \varepsilon^{\frac{1}{2}} \int (\bar{v} \Psi_1)_\tau \int \xi_1^2 \mathbf{L}_M^{-1} (\bar{\mathbf{G}}_1) d\xi dy \\
 &\quad - 2\varepsilon^{\frac{1}{2}} \int \bar{v} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1} \{Q(\mathbf{L}_M^{-1} \bar{\mathbf{G}}_1, M_\tau)\} d\xi dy.
 \end{aligned} \tag{4.27}$$

The Hölder inequality and Lemma 4.3 yield

$$\left| \int \xi_1^2 \mathbf{L}_M^{-1} (\bar{\mathbf{G}}_1) d\xi \right|^2 \leq C \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi. \tag{4.28}$$

Moreover, from Lemmas 4.1-4.3, we have

$$\begin{aligned}
 \int \xi_1^2 \mathbf{L}_M^{-1} \{Q(\mathbf{L}_M^{-1} \bar{\mathbf{G}}_1, M_\tau)\} d\xi &\leq C \left(\int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\mathbf{L}_M^{-1} \{Q(\mathbf{L}_M^{-1} \bar{\mathbf{G}}_1, M_\tau)\}|^2 d\xi \right)^{\frac{1}{2}} \\
 &\leq C \left(\int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\mathbf{L}_M^{-1} \bar{\mathbf{G}}_1|^2 d\xi \right)^{\frac{1}{2}} \cdot \left(\int \frac{\nu(|\xi|)}{\mathbf{M}_*} |M_\tau|^2 d\xi \right)^{\frac{1}{2}} \\
 &\leq C |(v_\tau, u_\tau, \theta_\tau)| \left(\int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi \right)^{\frac{1}{2}}.
 \end{aligned} \tag{4.29}$$

Combining (4.27)-(4.29) gives

$$\begin{aligned}
 T_1^{11} &\leq -(\varepsilon^{\frac{1}{2}} \int \bar{v} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1} \bar{\mathbf{G}}_1 d\xi dy)_\tau + C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} E_1 \\
 &\quad + C \beta \varepsilon^{\frac{1}{2}} \|\Psi_{1\tau}\|^2 + C \varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy + C \gamma \varepsilon^{\frac{3}{2}} \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2.
 \end{aligned} \tag{4.30}$$

On the other hand, by (3.13), we have

$$\begin{aligned}
 T_1^{12} &= -\varepsilon^{\frac{1}{2}} \int \bar{v} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1} (\bar{\mathbf{G}}_{0\tau}) d\xi dy \\
 &\leq C \varepsilon^{\frac{1}{2}} \int |\Psi_1| (|(\bar{\theta}_{y\tau}, \bar{u}_{y\tau})| + |(\bar{\theta}_y, \bar{u}_y)|) |(v_\tau, u_\tau, \theta_\tau)| dy \\
 &\leq C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} E_1 + C \delta \varepsilon (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} + C \delta \varepsilon^{\frac{3}{2}} \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2,
 \end{aligned} \tag{4.31}$$

which, together with (4.30), imply

$$\begin{aligned}
 T_1^1 &\leq -(\varepsilon^{\frac{1}{2}} \int \bar{v} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1} \bar{\mathbf{G}}_1 d\xi dy)_\tau + C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} E_1 + C \delta \varepsilon (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} \\
 &\quad + C \beta \varepsilon^{\frac{1}{2}} \|\Psi_{1\tau}\|^2 + C \varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy + C (\delta + \gamma) \varepsilon^{\frac{3}{2}} \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2.
 \end{aligned} \tag{4.32}$$

The estimation on T_1^i , $i = 2, 4$ is relatively easy by using the Cauchy inequality and Lemmas 4.1-4.3. In fact, direct computation gives

$$\begin{aligned}
 T_1^2 &\leq C \varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_y|^2 d\xi dy + C \varepsilon^{\frac{1}{2}} \int \Psi_1^2 u_1^2 dy \\
 &\leq C \delta \varepsilon^2 (1 + \varepsilon^{\frac{1}{2}} \tau)^{-2} E_1 + C \gamma \varepsilon K_1 + C \varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_y|^2 d\xi dy.
 \end{aligned} \tag{4.33}$$

On the other hand,

$$\begin{aligned}
 T_1^4 &\leq C\gamma\varepsilon^{\frac{1}{2}} \int \left(\int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\mathbf{L}_M^{-1}\{Q(\bar{\mathbf{G}}, \bar{\mathbf{G}})\}|^2 d\xi \right)^{\frac{1}{2}} dy \\
 &\leq C\gamma\varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}|^2 d\xi dy \\
 &\leq C\gamma\varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}}.
 \end{aligned} \tag{4.34}$$

The estimation on T_1^3 is similar to the one for T_1^1 . First, notice that

$$\mathbf{P}_1(\xi_1 \bar{\mathbf{G}}_y) = [\mathbf{P}_1(\xi_1 \bar{\mathbf{G}})]_y + \sum_{j=0}^4 \langle \xi_1 \bar{\mathbf{G}}, \chi_j \rangle \mathbf{P}_1(\chi_{jy}). \tag{4.35}$$

Then, it follows from (3.46), (3.55) and Lemmas 3.1–3.4 that

$$\begin{aligned}
 T_1^3 &= \varepsilon^{\frac{1}{2}} \int \left(\frac{\bar{v}\Psi_1}{v} \right)_y \int \xi_1^2 \mathbf{L}_M^{-1} [\mathbf{P}_1(\xi_1 \bar{\mathbf{G}})] d\xi dy \\
 &\quad - \varepsilon^{\frac{1}{2}} \int \frac{\bar{v}\Psi_1}{v} \int \xi_1^2 \mathbf{L}_M^{-1} \left[\sum_{j=0}^4 \langle \xi_1 \bar{\mathbf{G}}, \chi_j \rangle \mathbf{P}_1(\chi_{jy}) \right] d\xi dy \\
 &\quad - 2\varepsilon^{\frac{1}{2}} \int \frac{\bar{v}\Psi_1}{v} \int \xi_1^2 \mathbf{L}_M^{-1} \{Q(\mathbf{L}_M^{-1}[\mathbf{P}_1(\xi_1 \bar{\mathbf{G}})], \mathbf{M}_y)\} d\xi dy \\
 &\leq C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1} E_1 + C(\gamma + \beta)K_1 + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}} \\
 &\quad + C\gamma\varepsilon^{\frac{3}{2}} \|(\phi_y, \psi_y, \zeta_y)\|^2 + C\varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy.
 \end{aligned} \tag{4.36}$$

By (4.25), (4.27)-(4.34) and (4.36), we have

$$\begin{aligned}
 T_1 &\leq -(\varepsilon^{\frac{1}{2}} \int \bar{v}\Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1}(\bar{\mathbf{G}}_1) d\xi dy)_\tau + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1} E_1 + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}} \\
 &\quad + C\beta\varepsilon^{\frac{1}{2}} \|\Psi_{1\tau}\|^2 + C(\gamma + \beta)(K_1 + \varepsilon^{\frac{1}{2}}\|\Phi_y\|^2) + C\varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy \\
 &\quad + C\varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_y|^2 d\xi dy + C(\delta + \gamma)\varepsilon^{\frac{3}{2}} \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2.
 \end{aligned} \tag{4.37}$$

The estimates on the other terms of $\int N_1 dy$ are similar and we omit the details for brevity. Therefore, collecting the above inequalities gives

$$\begin{aligned}
 E_{1\tau} &+ (\varepsilon^{\frac{1}{2}} \int \int \hat{A}(\xi, \Phi, \Psi, W) \mathbf{L}_M^{-1}(\bar{\mathbf{G}}_1) d\xi dy)_\tau + \frac{1}{2} K_1 \leq C_1\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1} E_1 \\
 &\quad + C_1\beta\varepsilon^{\frac{1}{2}} \|(\Psi_\tau, W_\tau)\|^2 + C_1(\delta + \gamma)\varepsilon^{\frac{1}{2}} \|\Phi_y\|^2 + C_1\varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy \\
 &\quad + C_1\varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_y|^2 d\xi dy + C_1(\delta + \gamma)\varepsilon^{\frac{3}{2}} \\
 &\quad \times \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 + C_1\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}},
 \end{aligned} \tag{4.38}$$

where we have used the smallness of δ , β and γ . Here $\hat{A}(\xi, \Phi, \Psi, W)$ is a linear function of (Φ, Ψ, W) which is a polynomial of ξ .

Note that the dissipation term K_1 does not contain the term $\varepsilon^{\frac{1}{2}} \|\Phi_y\|^2$. To complete the lower order inequality, we have to estimate Φ_y . From (3.8)₂, we have

$$\frac{4\varepsilon^{\frac{1}{2}}}{3} \frac{\mu(\bar{\theta})}{\bar{v}} \Phi_{y\tau} - \Psi_{1\tau} + \frac{P_+}{\bar{v}} \Phi_y = \frac{2}{3\bar{v}} W_y - Q_1 + \int \xi_1^2 \bar{\Theta}_1 d\xi. \quad (4.39)$$

Multiplying (4.39) by $\varepsilon^{\frac{1}{2}} \Phi_y$ yields

$$\begin{aligned} & \left(\varepsilon \frac{2\mu(\bar{\theta})}{3\bar{v}} \Phi_y^2 - \varepsilon^{\frac{1}{2}} \Phi_y \Psi_1 \right)_\tau + \varepsilon^{\frac{1}{2}} \frac{P_+}{\bar{v}} \Phi_y^2 \\ &= \varepsilon \left(\frac{2\mu(\bar{\theta})}{3\bar{v}} \right)_\tau \Phi_y^2 + \varepsilon^{\frac{1}{2}} \Psi_{1y}^2 + \varepsilon^{\frac{1}{2}} \left(\frac{2}{3\bar{v}} W_y - Q_1 + \int \xi_1^2 \bar{\Theta}_1 d\xi \right) \Phi_y + (\cdots)_y, \end{aligned} \quad (4.40)$$

where we have used

$$\Phi_y \Psi_{1\tau} = (\Phi_y \Psi_1)_\tau - (\Phi_\tau \Psi_1)_y + \Psi_{1y}^2.$$

Integrating (4.40) with respect to y gives

$$\begin{aligned} & \left(\int \varepsilon \frac{2\mu(\bar{\theta})}{3\bar{v}} \Phi_y^2 - \varepsilon^{\frac{1}{2}} \Phi_y \Psi_1 dy \right)_\tau + \int \varepsilon^{\frac{1}{2}} \frac{P_+}{2\bar{v}} \Phi_y^2 dy \\ & \leq CK_1 + C\varepsilon^{\frac{1}{2}} \int Q_1^2 dy + C\varepsilon^{\frac{1}{2}} \int \left| \int \xi_1^2 \bar{\Theta}_1 d\xi \right|^2 dy. \end{aligned} \quad (4.41)$$

By (3.8) and the Cauchy inequality, one has

$$\begin{aligned} \varepsilon^{\frac{1}{2}} \int Q_1^2 dy & \leq C\delta\varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} E_1 + C\gamma\varepsilon (K_1 + \varepsilon^{\frac{1}{2}} \|\Phi_y\|^2) \\ & \quad + C\delta\varepsilon^{\frac{3}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} + C(\delta + \gamma)\varepsilon^2 \|\psi_{1y}\|^2. \end{aligned} \quad (4.42)$$

On the other hand, Lemmas 4.1–4.3 imply

$$\begin{aligned} \varepsilon^{\frac{1}{2}} \int \left| \int \xi_1^2 \bar{\Theta}_1 d\xi \right|^2 dy & \leq C\varepsilon^{\frac{3}{2}} \sum_{|\alpha'|=1} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy \\ & \quad + C\gamma\varepsilon \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy + C\delta\varepsilon^{\frac{3}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}}. \end{aligned} \quad (4.43)$$

Thus combining (4.42)–(4.43) yields

$$\begin{aligned} & \left(\int \varepsilon \frac{2\mu(\bar{\theta})}{3\bar{v}} \Phi_y^2 - \varepsilon^{\frac{1}{2}} \Phi_y \Psi_1 dy \right)_\tau + \int \varepsilon^{\frac{1}{2}} \frac{P_+}{4\bar{v}} \Phi_y^2 dy \\ & \leq C_2\delta\varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} E_1 + C_2K_1 + C_2\delta\varepsilon^{\frac{3}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} + C(\delta + \gamma)\varepsilon^2 \|\psi_{1y}\|^2 \\ & \quad + C_2\varepsilon^{\frac{3}{2}} \sum_{|\alpha'|=1} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy + C_2\gamma\varepsilon \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy. \end{aligned} \quad (4.44)$$

The microscopic component $\bar{\mathbf{G}}_1$ can be estimated through Eq. (3.15). Multiplying (3.15) by $\frac{\bar{\mathbf{G}}_1}{\mathbf{M}_*}$ gives

$$\begin{aligned} \left(\frac{\bar{\mathbf{G}}_1^2}{2\mathbf{M}_*}\right)_\tau - \varepsilon^{-\frac{1}{2}} \frac{\bar{\mathbf{G}}_1}{\mathbf{M}_*} \mathbf{L}_M \bar{\mathbf{G}}_1 = & \left\{ -\frac{1}{Rv\theta} \mathbf{P}_1 \left[\xi_1 \left(\frac{|\xi - u|^2}{2\theta} \zeta_y + \xi \cdot \psi_y \right) \mathbf{M} \right] \right. \\ & \left. + \frac{u_1}{v} \bar{\mathbf{G}}_y - \frac{1}{v} \mathbf{P}_1 (\xi_1 \bar{\mathbf{G}}_y) + \mathcal{Q}(\bar{\mathbf{G}}, \bar{\mathbf{G}}) - \bar{\mathbf{G}}_{0\tau} \right\} \frac{\bar{\mathbf{G}}_1}{\mathbf{M}_*}. \end{aligned} \quad (4.45)$$

Integrating (4.45) with respect to ξ and y and using the Cauchy inequality and Lemma 4.1–4.4 yield that

$$\begin{aligned} & \left(\int \int \frac{\bar{\mathbf{G}}_1^2}{2\mathbf{M}_*} d\xi dy \right)_\tau + \frac{\sigma}{2} \varepsilon^{-\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy \\ & \leq C_3 \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{1}{2}} + C_3 \varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 + C_3 \varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_y|^2 d\xi dy. \end{aligned} \quad (4.46)$$

On the other hand, from the fluid-type system (3.7), we can get an estimate for $\varepsilon^{\frac{1}{2}} \|(\Psi_\tau, W_\tau)\|^2$ as follows:

$$\begin{aligned} \varepsilon^{\frac{1}{2}} \|(\Psi_\tau, W_\tau)\|^2 \leq & C_4 \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} E_1 + C_4 K_1 + C_4 \varepsilon^{\frac{1}{2}} \|\Phi_y\|^2 + C_4 \varepsilon^{\frac{3}{2}} \|(\psi_y, \zeta_y)\|^2 \\ & + C_4 \delta \varepsilon^{\frac{3}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} + C_4 (\delta + \gamma) \varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy \\ & + C_4 \varepsilon^{\frac{3}{2}} \sum_{|\alpha'|=1} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy. \end{aligned} \quad (4.47)$$

We can now complete the lower order estimate. Since $\hat{A}(\xi, \Phi, \Psi, W)$ is a linear function of the vector (Φ, Ψ, W) which is a polynomial of ξ , we get

$$|\varepsilon^{\frac{1}{2}} \int \int \hat{\mathbf{A}} \mathbf{L}_M^{-1} \bar{\mathbf{G}}_1 d\xi dy| \leq \frac{1}{4} E_1 + C\varepsilon \int \int \frac{\bar{\mathbf{G}}_1^2}{\mathbf{M}_*} d\xi dy.$$

We choose large constants $\bar{C}_1 > 1, \bar{C}_2 > 1, \bar{C}_3 > 1$ and small constant β such that

$$\begin{aligned} E_2 = & \bar{C}_1 E_1 + \bar{C}_1 \varepsilon^{\frac{1}{2}} \int \int \hat{\mathbf{A}} \mathbf{L}_M^{-1} \bar{\mathbf{G}}_1 d\xi dy + \bar{C}_2 \int \int \varepsilon \frac{2\mu(\bar{\theta})}{3\bar{v}} \Phi_y^2 - \varepsilon^{\frac{1}{2}} \Phi_y \Psi_1 dy \\ & + \bar{C}_3 \int \int \frac{\bar{\mathbf{G}}_1^2}{2\mathbf{M}_*} d\xi dy \\ \geq & \frac{1}{2} \bar{C}_1 E_1 + \bar{C}_2 \int \int \varepsilon \frac{\mu(\bar{\theta})}{3\bar{v}} \Phi_y^2 dy + \frac{\bar{C}_3}{4} \int \int \frac{\bar{\mathbf{G}}_1^2}{\mathbf{M}_*} d\xi dy, \end{aligned} \quad (4.48)$$

and

$$\begin{aligned} & \left(\frac{\bar{C}_1}{2} - C_2 \bar{C}_2 - \bar{C}_1 C_1 \beta C_4 \right) K_1 + \int \varepsilon^{\frac{1}{2}} (\bar{C}_2 \frac{P_+}{4\bar{v}} - \bar{C}_1 C_1 \beta (1 + C_4)) \Phi_y^2 dy \\ & \geq \frac{\bar{C}_1}{4} K_1 + \bar{C}_2 \int \varepsilon^{\frac{1}{2}} \frac{P_+}{8\bar{v}} \Phi_y^2 dy. \end{aligned} \quad (4.49)$$

Hence, multiplying (4.38) by \bar{C}_1 , (4.44) by \bar{C}_2 , (4.46) by \bar{C}_3 , (4.47) by $C_1(\delta + \gamma)\bar{C}_1$ and adding all these inequalities imply

$$E_{2\tau} + K_2 \leq C_5 \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} E_2 + C_5 \varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy + C_5 \varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 + C_5 \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{1}{2}}, \tag{4.50}$$

where

$$K_2 = \frac{\bar{C}_1}{4} K_1 + \bar{C}_2 \int \varepsilon^{\frac{1}{2}} \frac{p_+}{8\bar{v}} \Phi_y^2 dy + \varepsilon^{\frac{1}{2}} \|(\Psi_\tau, W_\tau)\|^2 + \frac{\sigma}{4} \bar{C}_3 \varepsilon^{-\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy. \tag{4.51}$$

4.2. *Higher order estimate.* In this subsection, we shall estimate the derivatives of (Φ, Ψ, W) . Applying ∂_y to the system (3.4) gives

$$\left\{ \begin{aligned} \phi_\tau - \psi_{1y} &= 0, \\ \psi_{1\tau} + \varepsilon^{-\frac{1}{2}}(p - \bar{p})_y &= \frac{4}{3} \left(\frac{\mu(\theta)}{v} u_{1y} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{1y} \right)_y - \varepsilon^{-\frac{1}{2}} R_{1y} - \int \xi_1^2 \bar{\Theta}_{1y} d\xi, \\ \psi_{i\tau} &= \left(\frac{\mu(\theta)}{v} u_{iy} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{iy} \right)_y - \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi, \quad i = 2, 3, \\ \zeta_\tau + \varepsilon^{-\frac{1}{2}}(p u_{1y} - \bar{p} \bar{u}_{1y}) &= \left(\frac{\lambda(\theta)}{v} \theta_y - \frac{\lambda(\bar{\theta})}{\bar{v}} \bar{\theta}_y \right)_y + Q_5 \\ &\quad + \sum_{i=1}^3 u_i \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi - \frac{1}{2} \int \xi_1 |\xi|^2 \bar{\Theta}_{1y} d\xi, \end{aligned} \right. \tag{4.52}$$

where

$$Q_5 = \frac{4}{3} \left(\frac{\mu(\theta)}{v} u_{1y}^2 - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{1y}^2 \right) + \sum_{i=2}^3 \frac{\mu(\theta)}{v} u_{iy}^2 - \varepsilon^{-\frac{1}{2}} R_{2y} - \varepsilon^{-\frac{1}{2}} R_{1y} \bar{u}_1 = O(1) \left[\varepsilon |\psi_y|^2 + \varepsilon^{\frac{1}{2}} |\bar{u}_{1y}|^2 |(\phi, \zeta)| + \varepsilon^{\frac{1}{2}} |\psi_{1y}| |\bar{u}_{1y}| + \varepsilon^{-\frac{1}{2}} |R_{2y}| + \varepsilon^{-\frac{1}{2}} |R_{1y} \bar{u}_1| \right]. \tag{4.53}$$

Multiplying (4.52)₂ by ψ_1 , (4.52)₃ by ψ_i ($i = 2, 3$) respectively and adding them together yield

$$\left(\sum_{i=1}^3 \frac{1}{2} \psi_i^2 \right)_\tau - \varepsilon^{-\frac{1}{2}}(p - \bar{p}) \psi_{1y} + \frac{4}{3} \left(\frac{\mu(\theta)}{v} u_{1y} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{1y} \right) \psi_{1y} + \sum_{i=2}^3 \left(\frac{\mu(\theta)}{v} u_{iy} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{iy} \right) \psi_{iy} = -\varepsilon^{-\frac{1}{2}} R_{1y} \psi_1 - \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi + (\dots)_y.$$

Since $p - \bar{p} = \varepsilon^{\frac{1}{2}} \frac{R\zeta}{v} + R\bar{\theta} \left(\frac{1}{v} - \frac{1}{\bar{v}} \right)$, we obtain

$$\begin{aligned} & \left(\sum_{i=1}^3 \frac{1}{2} \psi_i^2 \right)_\tau - \varepsilon^{-\frac{1}{2}} R\bar{\theta} \left(\frac{1}{v} - \frac{1}{\bar{v}} \right) \phi_\tau - \frac{R\zeta}{v} \psi_{1y} + \frac{4}{3} \varepsilon^{\frac{1}{2}} \frac{\mu(\theta)}{v} \psi_{1y}^2 + \sum_{i=2}^3 \varepsilon^{\frac{1}{2}} \frac{\mu(\theta)}{v} \psi_{iy}^2 \\ &= -\frac{4}{3} \left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) \bar{u}_{1y} \psi_{1y} - \varepsilon^{-\frac{1}{2}} R_{1y} \psi_1 - \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi + (\dots)_y. \end{aligned} \tag{4.54}$$

Set

$$\hat{\Phi}(s) = s - 1 - \ln s.$$

Then

$$\left\{ R\bar{\theta} \hat{\Phi} \left(\frac{v}{\bar{v}} \right) \right\}_\tau = -\varepsilon^{\frac{1}{2}} R\bar{\theta} \left(\frac{1}{v} - \frac{1}{\bar{v}} \right) \phi_\tau - \bar{p} \hat{\Psi} \left(\frac{v}{\bar{v}} \right) \bar{v}_\tau + \bar{v} \bar{p}_\tau \hat{\Phi} \left(\frac{v}{\bar{v}} \right), \tag{4.55}$$

where

$$\hat{\Psi}(s) = s^{-1} - 1 + \ln s.$$

It is easy to check that $\hat{\Phi}(1) = \hat{\Phi}'(1) = \hat{\Psi}(1) = \hat{\Psi}'(1) = 0$ and $\hat{\Phi}(s)$ is strictly convex around $s = 1$. Substituting (4.55) into (4.54) yields

$$\begin{aligned} & \left(\sum_{i=1}^3 \frac{1}{2} \psi_i^2 + \varepsilon^{-1} R\bar{\theta} \hat{\Phi} \left(\frac{v}{\bar{v}} \right) \right)_\tau - \frac{R}{v} \zeta \psi_{1y} + \frac{4}{3} \varepsilon^{\frac{1}{2}} \frac{\mu(\theta)}{v} \psi_{1y}^2 + \sum_{i=2}^3 \varepsilon^{\frac{1}{2}} \frac{\mu(\theta)}{v} \psi_{iy}^2 \\ &= -\varepsilon^{-1} \bar{p} \hat{\Psi} \left(\frac{v}{\bar{v}} \right) \bar{v}_\tau + \varepsilon^{-1} \bar{v} \bar{p}_\tau \hat{\Phi} \left(\frac{v}{\bar{v}} \right) - \frac{4}{3} \left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) \bar{u}_{1y} \psi_{1y} \\ & \quad - \varepsilon^{-\frac{1}{2}} R_{1y} \psi_1 - \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi + (\dots)_y. \end{aligned} \tag{4.56}$$

Note that

$$\left\{ \bar{\theta} \hat{\Phi} \left(\frac{\theta}{\bar{\theta}} \right) \right\}_\tau = \varepsilon^{\frac{1}{2}} \left(1 - \frac{\bar{\theta}}{\theta} \right) \zeta_\tau - \hat{\Psi} \left(\frac{\theta}{\bar{\theta}} \right) \bar{\theta}_\tau, \tag{4.57}$$

and

$$\begin{aligned} \varepsilon^{\frac{1}{2}} \left(1 - \frac{\bar{\theta}}{\theta} \right) \zeta_\tau &= \varepsilon \frac{\zeta}{\theta} \left\{ -\varepsilon^{-\frac{1}{2}} (p u_{1y} - \bar{p} \bar{u}_{1y}) + \left(\frac{\lambda(\theta)}{v} \theta_y - \frac{\lambda(\bar{\theta})}{\bar{v}} \bar{\theta}_y \right)_y + Q_5 \right. \\ & \quad \left. + \sum_{i=1}^3 u_i \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi - \frac{1}{2} \int \xi_1 |\xi|^2 \bar{\Theta}_{1y} d\xi \right\} \\ &= -\varepsilon \frac{R\zeta}{v} \psi_{1y} + \varepsilon \left\{ \varepsilon^{-\frac{1}{2}} \frac{\zeta}{\theta} (\bar{p} - p) \bar{u}_{1y} - \varepsilon^{\frac{1}{2}} \frac{\lambda(\theta)}{v\theta} \zeta_y^2 - \frac{\zeta_y}{\theta} \left(\frac{\lambda(\theta)}{v} - \frac{\lambda(\bar{\theta})}{\bar{v}} \right) \bar{\theta}_y + \frac{\zeta}{\theta} Q_5 \right. \\ & \quad \left. + \frac{\zeta \theta_y}{\theta^2} \left(\frac{\lambda(\theta) \theta_y}{v} - \frac{\lambda(\bar{\theta}) \bar{\theta}_y}{\bar{v}} \right) + \frac{\zeta}{\theta} \left(\sum_{i=1}^3 u_i \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi \right. \right. \\ & \quad \left. \left. - \frac{1}{2} \int \xi_1 |\xi|^2 \bar{\Theta}_{1y} d\xi \right) + (\dots)_y \right\}. \end{aligned} \tag{4.58}$$

Substituting (4.57) and (4.58) into (4.56) gives

$$\begin{aligned} & \left(\sum_{i=1}^3 \frac{1}{2} \psi_i^2 + \varepsilon^{-1} R \bar{\theta} \hat{\Phi} \left(\frac{v}{\bar{v}} \right) + \varepsilon^{-1} \bar{\theta} \hat{\Phi} \left(\frac{\theta}{\bar{\theta}} \right) \right)_{\tau} + \frac{4\varepsilon^{\frac{1}{2}}}{3} \frac{\mu(\theta)}{v} \psi_{1y}^2 + \sum_{i=2}^3 \varepsilon^{\frac{1}{2}} \frac{\mu(\theta)}{v} \psi_{iy}^2 \\ & + \varepsilon^{\frac{1}{2}} \frac{\lambda(\theta)}{v\theta} \zeta_y^2 = -\varepsilon^{-1} \bar{p} \hat{\Psi} \left(\frac{v}{\bar{v}} \right) \bar{v}_{\tau} + \varepsilon^{-1} \bar{v} \bar{p}_{\tau} \hat{\Phi} \left(\frac{v}{\bar{v}} \right) - \varepsilon^{-1} \hat{\Psi} \left(\frac{\theta}{\bar{\theta}} \right) \bar{\theta}_{\tau} \\ & - \frac{4}{3} \left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) \bar{u}_{1y} \psi_{1y} + \varepsilon^{-\frac{1}{2}} R_{1y} \psi_1 + \varepsilon^{-\frac{1}{2}} \frac{\zeta}{\theta} (\bar{p} - p) \bar{u}_{1y} \\ & - \frac{\zeta_y}{\theta} \left(\frac{\lambda(\theta)}{v} - \frac{\lambda(\bar{\theta})}{\bar{v}} \right) \bar{\theta}_y + \frac{\zeta \theta_y}{\theta^2} \left(\frac{\lambda(\theta) \theta_y}{v} - \frac{\lambda(\bar{\theta}) \bar{\theta}_y}{\bar{v}} \right) + \frac{\zeta}{\theta} Q_5 + N_2 + (\dots)_y, \end{aligned} \quad (4.59)$$

where

$$N_2 = - \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi + \frac{\zeta}{\theta} \left(\sum_{i=1}^3 u_i \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi - \frac{1}{2} \int \xi_1 |\xi|^2 \bar{\Theta}_{1y} d\xi \right). \quad (4.60)$$

Let

$$\begin{aligned} E_3 &= \int \left(\frac{1}{2} \sum_{i=1}^3 \psi_i^2 + \varepsilon^{-1} R \bar{\theta} \hat{\Phi} \left(\frac{v}{\bar{v}} \right) + \varepsilon^{-1} \bar{\theta} \hat{\Phi} \left(\frac{\theta}{\bar{\theta}} \right) \right) dy, \\ K_3 &= \int \left(\frac{4\varepsilon^{\frac{1}{2}}}{3} \frac{\mu(\theta)}{v} \psi_{1y}^2 + \sum_{i=2}^3 \varepsilon^{\frac{1}{2}} \frac{\mu(\theta)}{v} \psi_{iy}^2 + \varepsilon^{\frac{1}{2}} \frac{\lambda(\theta)}{v\theta} \zeta_y^2 \right) dy. \end{aligned} \quad (4.61)$$

Integrating (4.59) with respect to y yields

$$E_{3\tau} + \frac{1}{2} K_3 \leq C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} \|(\Phi_y, \Psi_y, W_y)\|^2 + C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} + \int N_2 dy. \quad (4.62)$$

Here, we only consider the term $-\int \psi_1 \int \xi_1^2 \bar{\Theta}_{1y} d\xi dy$ because other terms in $\int N_2 dy$ can be estimated similarly. By (4.43), one has

$$\begin{aligned} & - \int \psi_1 \int \xi_1^2 \bar{\Theta}_{1y} d\xi dy = \int \psi_{1y} \int \xi_1^2 \bar{\Theta}_1 d\xi dy \\ & \leq \frac{1}{8} K_3 + C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} + C \varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy \\ & \quad + C(\delta + \gamma) \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy. \end{aligned} \quad (4.63)$$

Combining (4.62) and (4.63) yields

$$\begin{aligned} E_{3\tau} + \frac{1}{4} K_3 & \leq C_6 \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} \|(\Phi_y, \Psi_y, W_y)\|^2 + C_6 \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} \\ & \quad + C_6 \varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy + C_6(\delta + \gamma) \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy. \end{aligned} \quad (4.64)$$

We need to estimate $\varepsilon^{\frac{1}{2}} \|\phi_y\|^2$ which is not contained in K_3 . Following the same way as in estimating $\varepsilon^{\frac{1}{2}} \|\Phi_y\|^2$ in the previous subsection, we first rewrite Eq. (4.52)₂ as

$$\begin{aligned} & \frac{4}{3} \varepsilon^{\frac{1}{2}} \frac{\mu(\bar{\theta})}{\bar{v}} \phi_{y\tau} - \psi_{1\tau} - \varepsilon^{-\frac{1}{2}} (p - \bar{p})_y \\ &= -\frac{4\varepsilon^{\frac{1}{2}}}{3} \left(\frac{\mu(\bar{\theta})}{\bar{v}} \right)_y \psi_{1y} - \frac{4}{3} \left[\left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) u_{1y} \right]_y + \varepsilon^{-\frac{1}{2}} R_{1y} + \int \xi_1^2 \bar{\Theta}_{1y} d\xi, \end{aligned} \tag{4.65}$$

by using the equation of conservation of the mass (4.52)₁.

Since

$$-(p - \bar{p})_y = \varepsilon^{\frac{1}{2}} \frac{\bar{p}}{\bar{v}} \phi_y - \varepsilon^{\frac{1}{2}} \frac{R}{\bar{v}} \zeta_y + \left(\frac{p}{v} - \frac{\bar{p}}{\bar{v}} \right) v_y - \left(\frac{R}{v} - \frac{R}{\bar{v}} \right) \theta_y,$$

and

$$\phi_y \psi_{1\tau} = (\phi_y \psi_1)_\tau - (\phi_\tau \psi_1)_y + \psi_{1y}^2,$$

then by multiplying (4.65) by $\varepsilon^{\frac{1}{2}} \phi_y$, we get

$$\begin{aligned} & \left(\frac{2\mu(\bar{\theta})}{3\bar{v}} \varepsilon \phi_y^2 - \varepsilon^{\frac{1}{2}} \phi_y \psi_1 \right)_\tau + \varepsilon^{\frac{1}{2}} \frac{\bar{p}}{\bar{v}} \phi_y^2 = \varepsilon \left(\frac{2\mu(\bar{\theta})}{3\bar{v}} \right)_\tau \phi_y^2 + \varepsilon^{\frac{1}{2}} \psi_{1y}^2 + \varepsilon^{\frac{1}{2}} \frac{R}{\bar{v}} \zeta_y \phi_y \\ & - \left(\frac{p}{v} - \frac{\bar{p}}{\bar{v}} \right) v_y \phi_y + \left(\frac{R}{v} - \frac{R}{\bar{v}} \right) \theta_y \phi_y - \frac{4}{3} \varepsilon \left(\frac{\mu(\bar{\theta})}{\bar{v}} \right)_y \psi_{1y} \phi_y \\ & - \frac{4}{3} \varepsilon^{\frac{1}{2}} \left[\left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) u_{1y} \right]_y \phi_y + R_{1y} \phi_y + \varepsilon^{\frac{1}{2}} \int \xi_1^2 \bar{\Theta}_{1y} d\xi \phi_y. \end{aligned} \tag{4.66}$$

Integrating (4.66) with respect to y and using the Cauchy inequality yield

$$\begin{aligned} & \left(\int \frac{2\mu(\bar{\theta})}{3\bar{v}} \varepsilon \phi_y^2 - \varepsilon^{\frac{1}{2}} \phi_y \psi_1 dy \right)_\tau + \int \varepsilon^{\frac{1}{2}} \frac{\bar{p}}{2\bar{v}} \phi_y^2 dy \leq C_7 K_3 + C_7 \delta \varepsilon^{\frac{1}{2}} \\ & \times (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} \|(\Phi_y, \Psi_y, W_y)\|^2 + C_7 \delta \varepsilon^{\frac{3}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} + C_7 \gamma \varepsilon^{\frac{3}{2}} \|\psi_{1yy}\|^2 \\ & + C_7 (\delta + \gamma) \varepsilon^{\frac{1}{2}} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy + C_7 (\delta + \gamma) \varepsilon \sum_{|\alpha'|=1} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy \\ & + C_7 \varepsilon^{\frac{3}{2}} \sum_{|\alpha|=2} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy. \end{aligned} \tag{4.67}$$

Here we have used

$$\begin{aligned} & \varepsilon^{\frac{1}{2}} \int \left| \int \xi_1^2 \bar{\Theta}_{1y} d\xi \right|^2 dy \leq C \varepsilon^{\frac{3}{2}} \sum_{|\alpha|=2} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy \\ & + C (\delta + \gamma) \varepsilon \sum_{|\alpha'|=1} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy + C (\delta + \gamma) \varepsilon^{\frac{1}{2}} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy \\ & + C \delta \varepsilon^2 (\varepsilon^{\frac{1}{2}} \|\phi_y\|^2 + K_3) + C \delta \varepsilon^{\frac{3}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{5}{2}}. \end{aligned} \tag{4.68}$$

To estimate $\varepsilon^{\frac{1}{2}} \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2$, we need to use Eq. (4.5). By multiplying (4.5)₁ by $\varepsilon^{\frac{1}{2}}\phi_\tau$, (4.5)₂ by $\varepsilon^{\frac{1}{2}}\psi_{1\tau}$, (4.5)₃ by $\varepsilon^{\frac{1}{2}}\psi_{i\tau}$ ($i = 2, 3$) and (4.5)₄ by $\varepsilon^{\frac{1}{2}}\zeta_\tau$ respectively, and adding them together, after integrating with respect to y , we have

$$\begin{aligned} \varepsilon^{\frac{1}{2}} \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 &\leq C_8(\varepsilon^{\frac{1}{2}}\|\phi_y\|^2 + K_3) + C_8\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1} \|(\Phi_y, \Psi_y, W_y)\|^2 \\ &\quad + C_8\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{3}{2}} + C_8\varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_y|^2 d\xi dy. \end{aligned} \quad (4.69)$$

Choose large constants $\bar{C}_4, \bar{C}_5 > 1$ such that

$$\bar{C}_4 E_3 + \bar{C}_5 \int \left(\frac{2\mu(\bar{\theta})}{3\bar{\nu}} \varepsilon \phi_y^2 - \varepsilon^{\frac{1}{2}} \phi_y \psi_1 \right) dy \geq \frac{\bar{C}_4}{2} E_3 + \bar{C}_5 \int \frac{\mu(\bar{\theta})}{3\bar{\nu}} \varepsilon \phi_y^2 dy,$$

and

$$\begin{aligned} \frac{\bar{C}_4}{4} K_3 - \bar{C}_4 C_7 K_3 - C_3(C_8 + 1) K_3 &\geq \frac{\bar{C}_4}{8} K_3, \\ \bar{C}_5 \int \varepsilon^{\frac{1}{2}} \frac{\bar{p}}{2\bar{\nu}} \phi_y^2 dy - C_3(C_8 + 1) \varepsilon^{\frac{1}{2}} \|\phi_y\|^2 &\geq \frac{C_5}{2} \int \varepsilon^{\frac{1}{2}} \frac{\bar{p}}{2\bar{\nu}} \phi_y^2 dy. \end{aligned}$$

Let

$$E_4 = \bar{C}_4 E_3 + \bar{C}_5 \int \left(\frac{2\mu(\bar{\theta})}{3\bar{\nu}} \varepsilon \phi_y^2 - \varepsilon^{\frac{1}{2}} \phi_y \psi_1 \right) dy + \int \int \frac{\bar{\mathbf{G}}_1^2}{2\mathbf{M}_*} d\xi dy, \quad (4.70)$$

$$K_4 = \frac{\bar{C}_4}{8} K_3 + \frac{\bar{C}_5}{2} \int \varepsilon^{\frac{1}{2}} \frac{\bar{p}}{2\bar{\nu}} \phi_y^2 dy + \varepsilon^{\frac{1}{2}} \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 + \frac{\sigma}{4} \varepsilon^{-\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{2\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy, \quad (4.71)$$

then from (4.46), (4.64), (4.67) and (4.69), we have

$$\begin{aligned} E_{4\tau} + K_4 &\leq C_9\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1} \|(\Phi_y, \Psi_y, W_y)\|^2 + C_9\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{3}{2}} \\ &\quad + C_9\gamma\varepsilon^{\frac{3}{2}} \|\psi_{1yy}\|^2 + C_9\varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy \\ &\quad + C_9\varepsilon^{\frac{3}{2}} \sum_{|\alpha|=2} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy. \end{aligned} \quad (4.72)$$

Next we derive the estimate on the higher order derivatives. By multiplying (4.52)₂ by $-\varepsilon\psi_{1yy}$, (4.52)₃ by $-\varepsilon\psi_{iyy}$ ($i = 2, 3$), (4.52)₄ by $-\varepsilon\zeta_{yy}$, and adding them together, we obtain

$$\begin{aligned}
 & \left(\sum_{i=1}^3 \varepsilon \frac{\psi_{1y}^2}{2} + \varepsilon \frac{\zeta_y^2}{2} \right)_\tau + \frac{4}{3} \varepsilon^{\frac{3}{2}} \frac{\mu(\theta)}{v} \psi_{1yy}^2 + \sum_{i=2}^3 \varepsilon^{\frac{3}{2}} \frac{\mu(\theta)}{v} \psi_{iyy}^2 + \varepsilon^{\frac{3}{2}} \frac{\lambda(\theta)}{v} \zeta_{yy}^2 \\
 &= -\frac{4}{3} \varepsilon^{\frac{3}{2}} \left(\frac{\mu(\theta)}{v} \right)_y \psi_{1y} \psi_{1yy} - \sum_{i=2}^3 \varepsilon^{\frac{3}{2}} \left(\frac{\mu(\theta)}{v} \right)_y \psi_{iy} \psi_{iyy} - \varepsilon^{\frac{3}{2}} \left(\frac{\lambda(\theta)}{v} \right)_y \zeta_y \zeta_{yy} \\
 &\quad - \frac{4}{3} \varepsilon \left[\left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) \bar{u}_{1y} \right]_y \psi_{1yy} - \varepsilon \left[\left(\frac{\lambda(\theta)}{v} - \frac{\lambda(\bar{\theta})}{\bar{v}} \right) \bar{\theta}_y \right]_y \zeta_{yy} + \varepsilon^{\frac{1}{2}} (p - \bar{p})_y \psi_{1yy} \\
 &\quad + \varepsilon^{\frac{1}{2}} R_{1y} \psi_{1yy} + \varepsilon^{\frac{1}{2}} (p u_{1y} - \bar{p} \bar{u}_{1y}) \zeta_{yy} - \varepsilon Q_5 \zeta_{yy} + \varepsilon \sum_{i=1}^3 \psi_{iyy} \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi \\
 &\quad - \varepsilon \zeta_{yy} \left(\sum_{i=1}^3 u_i \int \xi_1 \xi_i \bar{\Theta}_{1y} d\xi - \frac{1}{2} \int \xi_1 |\xi|^2 \bar{\Theta}_{1y} d\xi \right), \tag{4.73}
 \end{aligned}$$

where Q_5 is defined in (4.53).

Integrating (4.73) with respect to y yields

$$\begin{aligned}
 & \left(\int \varepsilon \sum_{i=1}^3 \frac{\psi_{iy}^2}{2} + \varepsilon \frac{\zeta_y^2}{2} dy \right)_\tau + \int \left(\frac{4}{3} \varepsilon^{\frac{3}{2}} \frac{\mu(\theta)}{v} \psi_{1yy}^2 + \sum_{i=2}^3 \varepsilon^{\frac{3}{2}} \frac{\mu(\theta)}{v} \psi_{iyy}^2 + \varepsilon^{\frac{3}{2}} \frac{\lambda(\theta)}{v} \zeta_{yy}^2 \right) dy \\
 & \leq C(\varepsilon^{\frac{1}{2}} \|\phi_y\|^2 + K_3) + C\delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} \|(\Phi_y, \Psi_y, W_y)\|^2 + C\delta \varepsilon^{\frac{3}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} \\
 & \quad + C(\delta + \gamma) \varepsilon^{\frac{1}{2}} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy + C(\delta + \gamma) \varepsilon \sum_{|\alpha'|=1} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy \\
 & \quad + C\varepsilon^{\frac{3}{2}} \sum_{|\alpha|=2} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy. \tag{4.74}
 \end{aligned}$$

Now we get the estimation of $\varepsilon^{\frac{3}{2}} \|\phi_{yy}\|^2$. By applying ∂_y to (4.5)₂, we get

$$\psi_{1y\tau} + \varepsilon^{-\frac{1}{2}} (p - \bar{p})_{yy} = -\frac{4}{3} \left(\frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{1y} \right)_{yy} - \varepsilon^{-\frac{1}{2}} R_{1yy} - \int \xi_1^2 \bar{\mathbf{G}}_{yy} d\xi. \tag{4.75}$$

Note that

$$\begin{aligned}
 (p - \bar{p})_{yy} &= -\varepsilon^{\frac{1}{2}} \frac{p}{v} \phi_{yy} + \varepsilon^{\frac{1}{2}} \frac{R}{v} \zeta_{yy} - \frac{1}{v} (p - \bar{p}) \bar{v}_{yy} - \varepsilon^{\frac{1}{2}} \frac{\phi}{v} \bar{p}_{yy} \\
 &\quad - \frac{2v_y}{v} (p - \bar{p})_y - \varepsilon^{\frac{1}{2}} \frac{2\bar{p}_y}{v} \phi_y. \tag{4.76}
 \end{aligned}$$

Multiplying (4.75) by $-\varepsilon^{\frac{3}{2}} \phi_{yy}$ and using (4.76) imply

$$\begin{aligned}
 & -\left(\int \varepsilon^{\frac{3}{2}} \psi_{1y} \phi_{yy} dy \right)_\tau + \int \varepsilon^{\frac{3}{2}} \frac{p}{2v} \phi_y^2 dy \leq C\varepsilon^{\frac{3}{2}} \|(\psi_{1yy}, \zeta_{yy})\|^2 + C\delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} \\
 & \quad + C\delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} \|(\Phi_y, \Psi_y, W_y)\|^2 + C(\delta + \gamma) (\varepsilon^{\frac{1}{2}} \|\phi_y\|^2 + K_3) \\
 & \quad + C\varepsilon^{\frac{3}{2}} \sum_{|\alpha|=2} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy. \tag{4.77}
 \end{aligned}$$

To estimate $\varepsilon^{\frac{3}{2}}\|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^2$ and $\varepsilon^{\frac{3}{2}}\|(\phi_{\tau\tau}, \psi_{\tau\tau}, \zeta_{\tau\tau})\|^2$, we use the system (4.5) again. By applying ∂_y to (4.5), and multiplying the four equations of (4.5) by $\varepsilon^{\frac{3}{2}}\phi_{y\tau}$, $\varepsilon^{\frac{3}{2}}\psi_{1y\tau}$, $\varepsilon^{\frac{3}{2}}\psi_{iy\tau}$ ($i = 2, 3$), $\varepsilon^{\frac{3}{2}}\zeta_{y\tau}$ respectively, then adding them together and integrating with respect to y give

$$\begin{aligned} \varepsilon^{\frac{3}{2}}\|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^2 &\leq C\varepsilon^{\frac{3}{2}}\|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^2 + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{3}{2}} \\ &\quad + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}\|(\Phi_y, \Psi_y, W_y)\|^2 + C(\delta + \gamma)(\varepsilon^{\frac{1}{2}}\|\phi_y\|^2 + K_3) \\ &\quad + C\varepsilon^{\frac{1}{2}}\int\int\frac{\nu(|\xi|)}{\mathbf{M}_*}|\bar{\mathbf{G}}_y|^2d\xi dy + C\varepsilon^{\frac{3}{2}}\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{\mathbf{M}_*}|\partial^\alpha\bar{\mathbf{G}}|^2d\xi dy. \end{aligned} \quad (4.78)$$

Similarly, we have

$$\begin{aligned} \varepsilon^{\frac{3}{2}}\|(\phi_{\tau\tau}, \psi_{\tau\tau}, \zeta_{\tau\tau})\|^2 &\leq C\varepsilon^{\frac{3}{2}}\|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^2 + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{3}{2}} \\ &\quad + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}\|(\Phi_y, \Psi_y, W_y)\|^2 + C(\delta + \gamma)\varepsilon^{\frac{1}{2}}\sum_{|\alpha|=1}\|\partial^\alpha(\phi, \psi, \zeta)\|^2 \\ &\quad + C\varepsilon^{\frac{1}{2}}\int\int\frac{\nu(|\xi|)}{\mathbf{M}_*}|\bar{\mathbf{G}}_y|^2d\xi dy + C\varepsilon^{\frac{3}{2}}\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{\mathbf{M}_*}|\partial^\alpha\bar{\mathbf{G}}|^2d\xi dy. \end{aligned} \quad (4.79)$$

By choosing \bar{C}_6 and \bar{C}_7 to be large enough, we have

$$\begin{aligned} &\left(\bar{C}_6\int\varepsilon\sum_{i=1}^3\frac{\psi_{iy}^2}{2} + \varepsilon\frac{\zeta_y^2}{2}dy - \bar{C}_7\int\varepsilon^{\frac{3}{2}}\psi_{1y}\phi_{yy}dy\right)_\tau + \varepsilon^{\frac{3}{2}}\sum_{|\alpha|=2}\|\partial^\alpha(\phi, \psi, \zeta)\|^2 \\ &\leq C\varepsilon^{\frac{3}{2}}\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{\mathbf{M}_*}|\partial^\alpha\bar{\mathbf{G}}|^2d\xi dy + C\varepsilon^{\frac{1}{2}}\sum_{|\alpha|=1}\int\int\frac{\nu(|\xi|)}{\mathbf{M}_*}|\partial^\alpha\bar{\mathbf{G}}|^2d\xi dy \\ &\quad + C(\delta + \gamma)\varepsilon^{\frac{1}{2}}\sum_{|\alpha|=1}\|\partial^\alpha(\phi, \psi, \zeta)\|^2 + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}\|(\Phi_y, \Psi_y, W_y)\|^2 \\ &\quad + C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{3}{2}}. \end{aligned} \quad (4.80)$$

To close the a priori estimate, we also need to estimate the derivatives on the non-fluid component $\bar{\mathbf{G}}$, i.e., $\partial^\alpha\bar{\mathbf{G}}$, ($|\alpha| = 1, 2$). Applying ∂_y on (3.15), we have

$$\begin{aligned} \bar{\mathbf{G}}_{y\tau} - \left(\frac{u_1}{\nu}\bar{\mathbf{G}}_y\right)_y + \varepsilon^{-\frac{1}{2}}\left\{\frac{1}{\nu}\mathbf{P}_1(\xi_1\mathbf{M}_y)\right\}_y + \left\{\frac{1}{\nu}\mathbf{P}_1(\xi_1\bar{\mathbf{G}}_y)\right\}_y \\ = \varepsilon^{-\frac{1}{2}}\mathbf{L}_M\bar{\mathbf{G}}_y + 2\varepsilon^{-\frac{1}{2}}Q(\mathbf{M}_y, \bar{\mathbf{G}}) + 2Q(\bar{\mathbf{G}}_y, \bar{\mathbf{G}}). \end{aligned} \quad (4.81)$$

Since

$$\mathbf{P}_1(\xi_1\mathbf{M}_y) = \frac{1}{R\nu\theta}\mathbf{P}_1\left[\xi_1\left(\frac{|\xi - u|^2}{2\theta}\theta_y + \xi \cdot u_y\right)\mathbf{M}\right],$$

we have

$$\left|\left\{\frac{1}{\nu}\mathbf{P}_1(\xi_1\mathbf{M}_y)\right\}_y\right| \leq C(\nu_y^2 + u_y^2 + \theta_y^2 + |\theta_{yy}| + |u_{yy}|)|\hat{B}(\xi)|\mathbf{M},$$

where $\hat{B}(\xi)$ is a polynomial of ξ . This yields that

$$\varepsilon^{\frac{1}{2}} \iint \left| \left\{ \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_y) \right\}_y \frac{\bar{\mathbf{G}}_y}{\mathbf{M}_*} \right| d\xi dy \leq \frac{\sigma}{8} \varepsilon^{\frac{1}{2}} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_y|^2 d\xi dy + C \varepsilon^{\frac{3}{2}} \|(\psi_{yy}, \zeta_{yy})\|^2 + C(\delta + \gamma)(\varepsilon^{\frac{1}{2}} \|\phi_y\|^2 + K_3) + C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}}.$$

Thus, multiplying (4.81) by $\varepsilon \frac{\bar{\mathbf{G}}_y}{\mathbf{M}_*}$ and using the Cauchy inequality and Lemmas 4.1–4.4, we get

$$\begin{aligned} & \left(\iint \varepsilon \frac{\bar{\mathbf{G}}_y^2}{2\mathbf{M}_*} d\xi dy \right)_\tau + \frac{\sigma}{2} \varepsilon^{\frac{1}{2}} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_y|^2 d\xi dy \leq C \varepsilon^{\frac{3}{2}} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_{yy}|^2 d\xi dy \\ & + C(\delta + \gamma) \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy + C(\delta + \gamma)(\varepsilon^{\frac{1}{2}} \|\phi_y\|^2 + K_3) \\ & + C \varepsilon^{\frac{3}{2}} \|(\phi_{yy}, \zeta_{yy})\|^2 + C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}}. \end{aligned} \quad (4.82)$$

Similarly,

$$\begin{aligned} & \left(\iint \varepsilon \frac{\bar{\mathbf{G}}_\tau^2}{2\mathbf{M}_*} d\xi dy \right)_\tau + \frac{\sigma}{2} \varepsilon^{\frac{1}{2}} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_\tau|^2 d\xi dy \leq C \varepsilon^{\frac{3}{2}} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_{y\tau}|^2 d\xi dy \\ & + C(\delta + \gamma) \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy + C(\delta + \gamma) \varepsilon^{\frac{1}{2}} \iint \frac{\nu(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_y|^2 d\xi dy \\ & + C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} + C(\delta + \gamma) \varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 + C \varepsilon^{\frac{3}{2}} \|(\psi_{y\tau}, \zeta_{y\tau})\|^2. \end{aligned} \quad (4.83)$$

Finally, we estimate the estimate on the highest order derivatives, that is, $\int \varepsilon^{\frac{3}{2}} \psi_{1y} \phi_{yy} dy$ and $\varepsilon^{\frac{3}{2}} \iint \frac{\nu(|\xi|) |\partial^\alpha \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy$ with $|\alpha| = 2$ in (4.80). To do so, it is sufficient to study the estimate for $\varepsilon \int \int \frac{|\partial^\alpha f|^2}{\mathbf{M}_*} d\xi dy$ ($|\alpha| = 2$) because of (4.10)–(4.13). For this, from (3.16) we have

$$v f_\tau - u_1 f_y + \xi_1 f_y = \varepsilon^{-\frac{1}{2}} v Q(f, f) = v[\mathbf{L}_M \bar{\mathbf{G}} + \varepsilon^{\frac{1}{2}} Q(\bar{\mathbf{G}}, \bar{\mathbf{G}})].$$

Applying ∂^α ($|\alpha| = 2$) to the above equation gives

$$\begin{aligned} & v(\partial^\alpha f)_\tau - v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}} - u_1 (\partial^\alpha f)_y + \xi_1 (\partial^\alpha f)_y \\ & = -\partial^\alpha v f_\tau + \partial^\alpha u_1 f_y - \sum_{|\alpha'|=1} [\partial^{\alpha-\alpha'} v \partial^{\alpha'} f_\tau - \partial^{\alpha-\alpha'} u_1 \partial^{\alpha'} f_y] \\ & + [\partial^\alpha (v \mathbf{L}_M \bar{\mathbf{G}}) - v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}}] + \varepsilon^{\frac{1}{2}} \partial^\alpha [v Q(\bar{\mathbf{G}}, \bar{\mathbf{G}})]. \end{aligned} \quad (4.84)$$

Multiplying (4.84) by $\varepsilon \frac{\partial^\alpha f}{\mathbf{M}_*} = \varepsilon \frac{\partial^\alpha \mathbf{M}}{\mathbf{M}_*} + \varepsilon^{\frac{3}{2}} \frac{\partial^\alpha \bar{\mathbf{G}}}{\mathbf{M}_*}$ yields

$$\begin{aligned} & \left(\varepsilon \frac{v|\partial^\alpha f|^2}{2\mathbf{M}_*} \right)_\tau - \varepsilon^{\frac{3}{2}} v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}} \cdot \frac{\partial^\alpha \bar{\mathbf{G}}}{\mathbf{M}_*} \\ &= \varepsilon \frac{\partial^\alpha f}{\mathbf{M}_*} \left\{ -\partial^\alpha v f_\tau + \partial^\alpha u_1 f_y - \sum_{|\alpha'|=1} [\partial^{\alpha-\alpha'} v \partial^{\alpha'} f_\tau - \partial^{\alpha-\alpha'} u_1 \partial^{\alpha'} f_y] \right. \\ & \quad \left. + [\partial^\alpha (v \mathbf{L}_M \bar{\mathbf{G}}) - v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}}] + \varepsilon^{\frac{1}{2}} \partial^\alpha [v Q(\bar{\mathbf{G}}, \bar{\mathbf{G}})] \right\} + \varepsilon v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}} \cdot \frac{\partial^\alpha \mathbf{M}}{\mathbf{M}_*} + (\dots)_y. \end{aligned} \quad (4.85)$$

We can compute that

$$\begin{aligned} & \varepsilon \int \int |\partial^\alpha v f_\tau| \frac{\partial^\alpha f}{\mathbf{M}_*} |d\xi dy| \\ & \leq \varepsilon \int |\partial^\alpha v| \int (|\mathbf{M}_\tau| + \varepsilon^{\frac{1}{2}} |\bar{\mathbf{G}}_\tau|) \frac{|\partial^\alpha \mathbf{M}| + \varepsilon^{\frac{1}{2}} |\partial^\alpha \bar{\mathbf{G}}|}{\mathbf{M}_*} d\xi dy \\ & \leq C(\delta + \gamma) \varepsilon^{\frac{3}{2}} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 + \frac{\sigma}{16} \varepsilon^{\frac{3}{2}} \int \int \frac{v|\partial^\alpha \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy \\ & \quad + C(\delta + \gamma)^2 \varepsilon^{\frac{1}{2}} \int \int \frac{|\bar{\mathbf{G}}_\tau|^2}{\mathbf{M}_*} d\xi dy \\ & \quad + C(\delta + \gamma) \varepsilon \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 + C\delta \varepsilon^{\frac{1}{2}} \left(1 + \varepsilon^{\frac{1}{2}} \tau\right)^{-\frac{3}{2}}, \end{aligned}$$

and

$$\begin{aligned} & \varepsilon \sum_{|\alpha'|=1} \int \int |\partial^{\alpha-\alpha'} v \partial^{\alpha'} f_\tau| \frac{\partial^\alpha f}{\mathbf{M}_*} |d\xi dy| \\ & \leq \varepsilon \sum_{|\alpha'|=1} \int |\partial^{\alpha-\alpha'} v| \int (|\partial^{\alpha'} \mathbf{M}_\tau| + |\partial^{\alpha'} \mathbf{G}_\tau|) \frac{|\partial^\alpha \mathbf{M}| + \varepsilon^{\frac{1}{2}} |\partial^\alpha \mathbf{G}|}{\mathbf{M}_*} d\xi dy \\ & \leq \frac{\sigma}{16} \varepsilon^{\frac{3}{2}} \int \int \frac{v|\partial^\alpha \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy + C(\delta + \gamma) \varepsilon^{\frac{3}{2}} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 + C\delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}}. \end{aligned}$$

Similar estimates can be obtained for the terms $\varepsilon \partial^\alpha u_1 f_y \frac{\partial^\alpha f}{\mathbf{M}_*}$ and $\varepsilon \sum_{|\alpha'|=1} \partial^{\alpha-\alpha'} u_1 \partial^{\alpha'} f_y \frac{\partial^\alpha f}{\mathbf{M}_*}$.

Also, we have

$$\begin{aligned} & \partial^\alpha (v \mathbf{L}_M \bar{\mathbf{G}}) - v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}} = (\partial^\alpha v) \mathbf{L}_M \bar{\mathbf{G}} + 2v Q(\partial^\alpha \mathbf{M}, \bar{\mathbf{G}}) \\ & \quad + \sum_{|\alpha'|=1} \left\{ 2v Q(\partial^{\alpha-\alpha'} \mathbf{M}, \partial^{\alpha'} \bar{\mathbf{G}}) + \partial^{\alpha-\alpha'} v [\mathbf{L}_M \partial^{\alpha'} \bar{\mathbf{G}} + 2Q(\partial^{\alpha'} \mathbf{M}, \bar{\mathbf{G}})] \right\}, \end{aligned}$$

and

$$\begin{aligned} & \varepsilon^{\frac{1}{2}} \partial^\alpha [v Q(\bar{\mathbf{G}}, \bar{\mathbf{G}})] = \varepsilon^{\frac{1}{2}} (\partial^\alpha v) Q(\bar{\mathbf{G}}, \bar{\mathbf{G}}) + \varepsilon^{\frac{1}{2}} 2v Q(\partial^\alpha \bar{\mathbf{G}}, \bar{\mathbf{G}}) \\ & \quad + \sum_{|\alpha'|=1} \{ v Q(\partial^{\alpha-\alpha'} \bar{\mathbf{G}}, \partial^{\alpha'} \bar{\mathbf{G}}) + 2(\partial^{\alpha-\alpha'} v) Q(\partial^{\alpha'} \bar{\mathbf{G}}, \bar{\mathbf{G}}) \}. \end{aligned}$$

We only compute one of the above terms as follows, the other terms can be calculated similarly:

$$\begin{aligned}
& \varepsilon^2 \int \int \frac{v \partial^\alpha \bar{\mathbf{G}} \cdot \mathcal{Q}(\partial^\alpha \bar{\mathbf{G}}, \bar{\mathbf{G}})}{\mathbf{M}_*} d\xi dy \\
& \leq \frac{\sigma}{16} \varepsilon^{\frac{3}{2}} \int \int \frac{v |\partial^\alpha \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy \\
& \quad + C \varepsilon^{\frac{5}{2}} \int \left(\int \frac{v(|\xi|) |\partial^\alpha \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi \cdot \int \frac{|\bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi + \int \frac{|\partial^\alpha \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi \cdot \int \frac{v(|\xi|) |\bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi \right) dy \\
& \leq \frac{\sigma}{8} \varepsilon^{\frac{3}{2}} \int \int \frac{v(|\xi|)}{\mathbf{M}_*} v |\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy + C \varepsilon^{\frac{5}{2}} \sup_y \int \frac{v(|\xi|) |\bar{\mathbf{G}}_1|^2}{\mathbf{M}_*} d\xi \cdot \int \int \frac{|\partial^\alpha \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy \\
& \leq \frac{\sigma}{8} \varepsilon^{\frac{3}{2}} \int \int \frac{v(|\xi|)}{\mathbf{M}_*} v |\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy + C(\delta + \gamma)^2 \varepsilon^{\frac{1}{2}} \int \int \frac{v(|\xi|) [|\bar{\mathbf{G}}_{1y}|^2 + |\bar{\mathbf{G}}_1|^2]}{\mathbf{M}_*} d\xi dy \\
& \leq \frac{\sigma}{8} \varepsilon^{\frac{3}{2}} \int \int \frac{v(|\xi|)}{\mathbf{M}_*} v |\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy + C(\delta + \gamma)^2 \varepsilon^{\frac{1}{2}} \int \int \frac{v(|\xi|) [|\bar{\mathbf{G}}_y|^2 + |\bar{\mathbf{G}}_1|^2]}{\mathbf{M}_*} d\xi dy \\
& \quad + C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} + C(\delta + \gamma)^2 \varepsilon^{\frac{3}{2}} \|(\phi_y, \psi_y, \zeta_y)\|^2.
\end{aligned}$$

Now we estimate the term $\varepsilon \int \int v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}} \cdot \frac{\partial^\alpha \mathbf{M}}{\mathbf{M}_*} d\xi dy$ in (4.85). Firstly, note that $\mathbf{P}_1(\partial^\alpha \mathbf{M})$ does not contain the term $\partial^\alpha(v, u, \theta)$ for $|\alpha| = 2$. Thus, we have

$$\begin{aligned}
& \varepsilon \int \int \frac{v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}} \cdot \partial^\alpha \mathbf{M}}{\mathbf{M}} d\xi dy = \varepsilon \int \int \frac{v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}} \cdot \mathbf{P}_1(\partial^\alpha \mathbf{M})}{\mathbf{M}} d\xi dy \\
& \leq \frac{\sigma}{16} \varepsilon^{\frac{3}{2}} \int \int \frac{v |\partial^\alpha \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy + C(\delta + \gamma) \varepsilon \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 + C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}}.
\end{aligned} \tag{4.86}$$

Also we can get

$$\begin{aligned}
& \varepsilon \int \int v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}} \cdot \partial^\alpha \mathbf{M} \left(\frac{1}{\mathbf{M}_*} - \frac{1}{\mathbf{M}} \right) d\xi dy \leq \frac{\sigma}{16} \varepsilon^{\frac{3}{2}} \int \int \frac{v(|\xi|)}{\mathbf{M}_*} v |\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy \\
& \quad + C \eta_0^2 \varepsilon^{\frac{3}{2}} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 + C(\delta + \gamma) \varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 + C \delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}},
\end{aligned} \tag{4.87}$$

where the small constant η_0 is defined in Lemma 4.2. The combination of (4.86) and (4.87) gives the estimation of $\varepsilon \int \int v \mathbf{L}_M \partial^\alpha \bar{\mathbf{G}} \cdot \frac{\partial^\alpha \mathbf{M}}{\mathbf{M}_*} d\xi dy$.

Thus integrating (4.85) and recalling all the above estimates imply

$$\begin{aligned}
 & \left(\int \int \varepsilon \frac{v|\partial^\alpha f|^2}{2\mathbf{M}_*} d\xi dy \right)_\tau + \frac{\sigma}{2} \varepsilon^{\frac{3}{2}} \int \int \frac{v(|\xi|)}{\mathbf{M}_*} v|\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy \\
 & \leq C(\delta + \gamma) \varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 + C(\eta_0 + \delta + \gamma) \varepsilon^{\frac{3}{2}} \sum_{|\alpha|=2} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 \\
 & \quad + C(\delta + \gamma) \varepsilon \sum_{|\alpha'|=1} \int \int \frac{v(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy + C\delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} \\
 & \quad + C(\delta + \gamma) \varepsilon^{\frac{1}{2}} \int \int \frac{v(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy. \tag{4.88}
 \end{aligned}$$

By (4.10)-(4.12), we can choose suitable constants $\hat{C}_i > 1, i = 1, 2, 3, 4$ so that

$$\begin{aligned}
 E_5 & = \hat{C}_1 E_4 + \hat{C}_2 (\bar{C}_6 \int \varepsilon \sum_{i=1}^3 \frac{\psi_{iy}^2}{2} + \varepsilon \frac{\zeta_y^2}{2} dy - \bar{C}_7 \int \varepsilon^{\frac{3}{2}} \psi_{1y} \phi_{yy} dy) \\
 & \quad + \hat{C}_3 \varepsilon \sum_{|\alpha'|=1} \int \int \frac{|\partial^{\alpha'} \bar{\mathbf{G}}|^2}{2\mathbf{M}_*} d\xi dy + \hat{C}_4 \varepsilon \sum_{|\alpha|=2} \int \int \frac{v|\partial^\alpha f|^2}{2\mathbf{M}_*} d\xi dy \\
 & \geq C \left[\|\phi, \psi, \zeta\|^2 + \varepsilon \|\phi_y, \psi_y, \zeta_y\|^2 + \int \int \frac{|\bar{\mathbf{G}}_1|^2}{\mathbf{M}_*} d\xi dy + \varepsilon \sum_{|\alpha'|=1} \int \int \frac{|\partial^{\alpha'} \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy \right. \\
 & \quad \left. + \varepsilon \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha f|^2}{\mathbf{M}_*} d\xi dy \right] - C\delta \varepsilon (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}}. \tag{4.89}
 \end{aligned}$$

Let

$$\begin{aligned}
 K_5 & = C^{-1} \left[\varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 + \varepsilon^{\frac{3}{2}} \sum_{|\alpha|=2} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 \right. \\
 & \quad \left. + \varepsilon^{-\frac{1}{2}} \int \int \frac{v(|\xi|)}{\mathbf{M}_*} |\bar{\mathbf{G}}_1|^2 d\xi dy + \varepsilon^{\frac{1}{2}} \sum_{|\alpha'|=1} \int \int \frac{v(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \bar{\mathbf{G}}|^2 d\xi dy \right. \\
 & \quad \left. + \varepsilon^{\frac{3}{2}} \sum_{|\alpha|=2} \int \int \frac{v(|\xi|)}{\mathbf{M}_*} |\partial^\alpha \bar{\mathbf{G}}|^2 d\xi dy \right]. \tag{4.90}
 \end{aligned}$$

Then by the estimates (4.72), (4.80), (4.82), (4.83), (4.88), we obtain

$$E_{5\tau} + K_5 \leq C\delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-\frac{3}{2}} + C\delta \varepsilon^{\frac{1}{2}} (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1} \|(\Phi_y, \Psi_y, W_y)\|^2. \tag{4.91}$$

5. The Proof of Theorem 3.1

Choose a large constant \hat{C}_5 and set

$$E_6 = E_2 + \hat{C}_5 E_5, \quad K_6 = K_2 + \hat{C}_5 K_5. \tag{5.1}$$

By combining (4.50) and (4.91), we have

$$\begin{aligned}
 E_{6\tau} + K_6 &\leq C_0\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}E_2 + C_0\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}\|(\Phi_y, \Psi_y, W_y)\|^2 + C_0\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}} \\
 &\leq C_0\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-1}E_6 + C_0\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}}.
 \end{aligned} \tag{5.2}$$

Then the Gronwall inequality implies that

$$E_6(\tau) \leq C(E_6(0) + \delta)(1 + \varepsilon^{\frac{1}{2}}\tau)^{\frac{1}{2}}, \quad \int_0^\tau K_6(y, s)ds \leq C(E_6(0) + \delta)(1 + \varepsilon^{\frac{1}{2}}\tau)^{\frac{1}{2}}. \tag{5.3}$$

Now multiplying (4.91) by $(1 + \varepsilon^{\frac{1}{2}}\tau)$ gives

$$\begin{aligned}
 [(1 + \varepsilon^{\frac{1}{2}}\tau)E_5]_\tau &\leq (1 + \varepsilon^{\frac{1}{2}}\tau)E_{5\tau} + \varepsilon^{\frac{1}{2}}E_5 \\
 &\leq C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}} + C\delta\varepsilon^{\frac{1}{2}}\|(\Phi_y, \Psi_y, W_y)\|^2 + \varepsilon^{\frac{1}{2}}E_5 \\
 &\leq C\delta\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}} + CK_6.
 \end{aligned} \tag{5.4}$$

Integrating (5.4) with respect to τ and using (5.3) yield that

$$E_5(\tau) \leq C(E_6(0) + \delta)(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}}.$$

Thus, we have

$$\|(\Phi, \Psi, W)\|_{L_y^\infty}^2 \leq C\|(\Phi, \Psi, W)\| \|(\Phi_y, \Psi_y, W_y)\| \leq CE_6^{\frac{1}{2}}E_5^{\frac{1}{2}} \leq C(E_6(0) + \delta),$$

and

$$\begin{aligned}
 &\|(\phi, \psi, \zeta)\|^2 + \varepsilon\|(\phi_y, \psi_y, \zeta_y)\|^2 + \int \int \frac{\bar{\mathbf{G}}_1^2}{\mathbf{M}_*} d\xi dy \\
 &\quad + \varepsilon \sum_{|\alpha'|=1} \int \int \frac{|\partial^{\alpha'} \bar{\mathbf{G}}|^2}{\mathbf{M}_*} d\xi dy + \varepsilon \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha f|^2}{\mathbf{M}_*} d\xi dy \\
 &\leq C(E_6(0) + \delta)(1 + \varepsilon^{\frac{1}{2}}\tau)^{-\frac{1}{2}} \\
 &\leq C(E_6(0) + \delta).
 \end{aligned} \tag{5.5}$$

And this closes the a priori estimate (4.1).

Now it remains to prove the decay rate of (3.18). By (5.5), we have

$$\varepsilon^{\frac{1}{2}}\|(\phi, \psi, \zeta)\|_{L_y^\infty}^2 \leq C\varepsilon^{\frac{1}{2}}\|(\phi, \psi, \zeta)\| \|(\phi_y, \psi_y, \zeta_y)\| \leq C(E_6(0) + \delta),$$

and

$$\begin{aligned}
 \varepsilon^{\frac{1}{2}}\| \int \frac{\bar{\mathbf{G}}^2}{\mathbf{M}_*} d\xi \|_{L_y^\infty} &\leq C\varepsilon^{\frac{1}{2}} \left(\int \int \frac{\bar{\mathbf{G}}^2}{\mathbf{M}_*} d\xi dy \right)^{\frac{1}{2}} \cdot \left(\int \int \frac{|\bar{\mathbf{G}}_y|^2}{\mathbf{M}_*} d\xi dy \right)^{\frac{1}{2}} \\
 &\leq C \left[\left(\int \int \frac{|\bar{\mathbf{G}}_1|^2}{\mathbf{M}_*} d\xi dy \right)^{\frac{1}{2}} + \|(\bar{\theta}_y, \bar{u}_y)\| \right] (E_6(0) + \delta)^{\frac{1}{2}} \\
 &\leq C(E_6(0) + \delta).
 \end{aligned}$$

Finally,

$$\begin{aligned}
 & \sup_y \int \frac{|f(y, \tau, \xi) - \mathbf{M}_{[\bar{v}, \bar{u}, \bar{\theta}]}(y, \tau, \xi)|^2}{\mathbf{M}_*} d\xi \\
 & \leq C \sup_y \int \frac{|\mathbf{M}(y, \tau, \xi) - \mathbf{M}_{[\bar{v}, \bar{u}, \bar{\theta}]}(y, \tau, \xi)|^2}{\mathbf{M}_*} d\xi + \sup_y \int \frac{\mathbf{G}^2}{\mathbf{M}_*} d\xi \\
 & \leq C\varepsilon \|(\phi, \psi, \zeta)\|_{L^\infty}^2 + C\varepsilon \sup_y \int \frac{\bar{\mathbf{G}}^2}{\mathbf{M}_*} d\xi \\
 & \leq C(E_6(0) + \delta)\varepsilon^{\frac{1}{2}}, \tag{5.6}
 \end{aligned}$$

which gives (3.18), and this completes the proof of Theorem 3.1.

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