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A global unique solvability of entropic weak solution to the one-dimensional pressureless Euler system with a flocking dissipation ☆

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Abstract

We present a unique global solvability and flocking estimate of an entropic weak solution to the one-dimensional pressureless Euler system with a flocking dissipation in all-to-all coupling setting. This model appears naturally as a quasi-equilibrium model for hydrodynamic description of Cucker–Smale flocking. For the unique global solvability, we adopt the variation approach from Ding and Huang's work [19] on the inhomogeneous pressureless gas dynamic model. When initial mass and velocity are locally integrable and bounded measurable functions, respectively, we give explicit representations for the global entropic weak solutions. Our results do not require any smallness of initial data except that initial mass density is almost everywhere positive.

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1. Introduction

Collective coherent motions of self-propelled agents (particles), e.g., flocking of birds, schooling of fish, swarming of bacteria and herding of cows, etc. are often found in our biological systems [30,35,41]. Recently, research on the collective coherent motions has received considerable attention from many scientific disciplines such as applied math, biology, computer science, statistical physics and control theory due to their diverse applications [22,31,36–38,40] in relation with the decentralized control of multi-agents such as UAVs and robots. The jargon "flocking" represents a phenomenon in which self-propelled agents are organized into the ordered motion using only limited environmental information and simple rules. In this paper, we are particularly interested in the velocity flocking, in which all agents move with the same velocity, but their spatial positions are distributed. So far, several phenomenological flocking models have been proposed in literature. Among them, our main interest lies on the simple ODE system proposed by Cucker and Smale [16].

Consider infinitely many Cucker–Smale (for short C–S) agents whose phase configuration is close to a flocking one. In this case, the dynamics of C–S agents can be effectively approximated by a quasi-equilibrium hydrodynamic model. Let ρ and u be the local mass and velocity densities of C–S agents exhibiting the flocking phenomenon, respectively. Then, the temporal-spatial evolution of (ρ, u) is governed by the Cauchy problem to the quasi-equilibrium C–S hydrodynamic model:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \quad x \in \mathbb{R},\ t > 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = -K\rho \displaystyle\int_{\mathbb{R}} \psi\big(|x-y|\big)\big(u(x,t) - u(y,t)\big)\rho(y,t)dy, \\ (\rho, u)(x, 0) = (\rho_0, u_0), \end{cases} \tag{1.1}$$

where K is the positive coupling strength and $\psi(|x-y|)$ is a Lipschitz continuous communication weight satisfying

$$\psi \geq 0, \quad \|\psi\|_{L^\infty} + \|\psi\|_{\mathrm{Lip}} < \infty, \qquad \big(\psi(r_1) - \psi(r_2)\big)(r_1 - r_2) \leq 0, \quad r_1, r_2 > 0.$$

The system (1.1) appears as an approximate model for the hydrodynamic Cucker–Smale model [27], when the continuum flocking group is close to a flocking configuration (see Section 2.1). Note that in the zero coupling limit $K \to 0$, the system (1.1) reduces to the pressureless Euler system, which has been extensively studied in the hyperbolic conservation law community in last decades. For a detailed discussion on literature, we refer to Section 2.2.

The purpose of this paper is to address a unique solvability and flocking estimate of entropic weak solution to the system (1.1) in one-dimensional and all-to-all coupling setting ($\psi = 1$) using the variational arguments of [19,20,42–44]. In this all-to-all coupling case, the system (1.1) can be reduced to the pressureless Euler system with a damping under some suitable normalization. So we might have some possibility to apply the variational method to construct entropic weak solutions as in [19] for the pressureless Euler system with some specific source term.

The main results of this paper are two-fold. First, we present a global existence and flocking estimate of an entropic weak solution to the Cauchy problem (1.1) with initial data in some Lebesgue space (see (3.2) in Section 3). More precisely, when initial mass density ρ_0 is locally integrable and positive a.e., and initial velocity is uniformly bounded, we explicitly construct an entropic weak solutions using the minimizer of some potential function, and then show that explicit representations of (ρ, u) satisfy the defining relations of entropic weak solutions (see Definition 3.1). Second, we show that any entropic weak solution in the sense of Definition 3.1 is unique so that the explicit solution constructed by the variational method is in fact the unique solution to the Cauchy problem (1.1).

The rest of this paper is organized as follows. In Section 2, we briefly discuss a bottom-up modeling for C–S flocking from microscopic scale to the macroscopic scale and a relation between our proposed model (1.1) and hydrodynamic description in [27]. We also present a priori flocking estimate for our proposed model under some regularity and decay conditions. In Section 3, we briefly recall the concept of entropic weak solution and present our main results on the global existence, flocking estimates and uniqueness of an entropic weak solution using the variational method without proofs. In Section 4, we study several preparatory lemmas to be crucially used in the following two sections. In Section 5, we explicitly construct a global entropic weak solution and present the proof of Theorem 3.1. In Section 6, we show that the entropic weak solution constructed in Section 5 is in fact unique. Finally, Section 7 is devoted to the summary of our main results and some future directions. In Appendix A, we present the proof of Lemma 4.5.

2. Preliminaries

In this section, we discuss hierarchical Cucker–Smale models from the view point of microscopic to macroscopic scales and discuss the relation between macroscopic Cucker–Smale model and our quasi-equilibrium model. We also provide several a priori estimates for our quasi-equilibrium model. Throughout the paper, as long as there is no confusion, we suppress t-dependence in functions for the simplicity of presentation.

2.1. A quasi-equilibrium C–S model

In [16], Cucker and Smale introduced a Newton type system for an interacting particle system exhibiting a flocking phenomenon, and derived sufficient conditions for a global flocking in terms of initial configuration and the decay rate of ψ. Then, their results were immediately generalized to several directions, e.g., stochastic noise effects [2,15,24], collision avoidance [1,10], steering toward preferred directions [14], extra forces [34], the mean-field limit [3,11,18,21,25,27] and relation with mechanical oscillator model [26] etc. In particular, an unexpected application was devised by Perea, Gómez and Elosegui [33] who suggested to use C–S flocking mechanism [16] to make the several spacecrafts fly in formation for the Darwin space mission of European space agency. Below, we will briefly discuss several C–S flocking models in microscopic, mesoscopic and macroscopic levels, and motivate our quasi-equilibrium hydrodynamic model.

Let x_i and v_i be the position and velocity of the i-th C–S particle, respectively. Then, a generalized Cucker–Smale model [25] can be written as follows.

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \; i = 1, \cdots, N,$$

$$\frac{dv_i}{dt} = \frac{K}{N}\sum_{j=1}^{N} \psi\big(|x_j - x_i|\big)(v_j - v_i).$$

In [1,16,25], several sufficient conditions for a global flocking were proposed in terms of initial configuration, K and ψ. When the number of particles goes to infinity ($N \to \infty$), it is reasonable to introduce a kinetic density $f = f(x, v, t)$, which corresponds to the one-particle distribution function for the C–S ensemble. Then the dynamics of f is governed by the Vlasov–Mckean type equation [25,27]:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \big(F[f]f\big) = 0, \quad x \in \mathbb{R}^d,\ t > 0,$$
$$F[f](x, v, t) = -K \int\limits_{\mathbb{R}^{2d}} \psi\big(|x - y|\big)(v - v_*) f(y, v_*)\, dv_* dy. \tag{2.1}$$

The system (2.1) admits a global C^1-solution as long as initial datum is compactly supported in x and v and C^1-regular (see [27]), and when the initial datum is a Radon measure, the global unique solvability of measure valued solutions to (2.1) was also studied in [25]. It is well known that the velocity moments of f gives the macroscopic observables. For a given $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$, we set

$$\rho := \int\limits_{\mathbb{R}^d} f\, dv : \quad \text{local mass density},$$
$$\rho u := \int\limits_{\mathbb{R}^d} v f\, dv : \quad \text{local momentum density},$$
$$\rho E := \rho e + \frac{1}{2}\rho|u|^2 : \quad \text{local energy density}, \tag{2.2}$$

where $\rho e := \frac{1}{2}\int_{\mathbb{R}^d} |v - u(x)|^2 f\, dv$ is the internal energy. Then, macroscopic observables (2.2) satisfy the following hydrodynamic equations [27]:

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d,\ t > 0,$$
$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + P) = S^{(1)},$$
$$\partial_t(\rho E) + \nabla_x \cdot (\rho E u + P u + q) = S^{(2)}, \tag{2.3}$$

where $P = (p_{ij})$ and $q = (q_1, \cdots, q_d)$ are stress tensor and heat flow, respectively.

$$p_{ij} := \int\limits_{\mathbb{R}^d} (v_i - u_i)(v_j - u_j) f\, dv, \qquad q_i := \int\limits_{\mathbb{R}^d} (v_i - u_i)|v - u|^2 f\, dv, \tag{2.4}$$

and the source terms are given by the following relations:

$$S^{(1)} := -K \int_{\mathbb{R}^d} \psi\big(|x-y|\big)\big(u(x)-u(y)\big)\rho(x)\rho(y)dy,$$

$$S^{(2)} := -K \int_{\mathbb{R}^d} \psi\big(|x-y|\big)\big(E(x)+E(y)-u(x)\cdot u(y)\big)\rho(x)\rho(y)dy. \tag{2.5}$$

Of course, the moment system (2.3) is not closed as it is, because we need to know the third velocity moment of f to calculate the heat flux q in (2.4). So far, suitable closure conditions for (2.3) (e.g., the local Maxwellian for the Botlzmann equation) are not known. Thus, we consider a situation where the configuration for the C–S ensemble is close to a flocking state so that we can assume that the spatial configuration is *collisionless*. In this regime, we may assume the mono-kinetic ansatz for f:

$$f(x,v,t) = \rho(x,t)\delta\big(v-u(x,t)\big), \quad x, v \in \mathbb{R}^d,\ t > 0. \tag{2.6}$$

Then, under the assumption (2.6), the stress tensor $P = (P_{ij})$ and heat flux q in (2.4) become

$$p_{ij} = 0, \qquad q_i = 0, \quad 1 \le i, j \le d.$$

Thus, the system (2.3)–(2.5) becomes the pressureless Euler system with a flocking dissipation:

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d,\ t > 0, \\ &\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) = -K\rho \int_{\mathbb{R}^d} \psi\big(|x-y|\big)\rho(y)\big(u(x)-u(y)\big)dy. \end{aligned} \tag{2.7}$$

Note that the energy equation in (2.3) can be derivable from the equations for ρ and ρu, and the condition (2.6) will be valid only for the collisionless regime. However, when particles with different microscopic velocities collide, the mono-kinetic ansatz (2.6) will break down. Therefore, our system (2.7) should be regarded as a quasi equilibrium model for the hydrodynamic description of the C–S ensemble.

2.2. *A pressureless Euler system with a flocking dissipation*

In the absence of coupling ($K = 0$), the system (2.7) is simply the pressureless Euler system:

$$\begin{aligned} &\partial_t \rho + \nabla_x (\rho u) = 0, \quad x \in \mathbb{R},\ t > 0, \\ &\partial_t (\rho u) + \nabla_x (\rho u \otimes u) = 0. \end{aligned} \tag{2.8}$$

This model has been used as a modeling of the formation of large-scale structures of the universe [39,45]. The main feature of the system (2.8) is the formation of delta shock waves, no matter how smooth the initial data are. This provides a new challenge to the well-posedness issue of non-strictly hyperbolic systems like (2.8). The global existence of entropic weak solution to (2.8) was first obtained independently by two groups Brenier–Grenier [8] and Weinan–Rykov–Sinai [20], whereas an explicit formula of the weak solution was obtained by [20] using a generalized variational principle. Then, Wang, Huang and Ding [44] extended their results to the general case, when initial velocity u_0 is discontinuous. Similar results were also obtained by Chen, Li and Zhang [13]. Boudin [7] showed that the weak solution can also be obtained as the vanishing viscosity limit from the viscous pressureless Euler equations (2.8). As far as the uniqueness of the weak solution is concerned, the system (2.8) is more subtle than that of strictly hyperbolic systems. It is well known that the weak solutions to hyperbolic system of the conservation laws is not unique without the entropy condition, even for one-dimensional case. In contrast, for the pressureless Euler system (2.8), it is shown in [4,8,20] that the Lax entropy condition is not sufficient to guarantee the uniqueness of the system (2.8) even for one-dimension. Weinan, Rykov and Sinai pointed this out in [20] that the Oleinik entropy condition might be necessary for the uniqueness of the weak solution. Along this line, Wang and Ding [42] established the uniqueness of the weak solution in the case when the initial density ρ_0 is a locally integrable function. Similar results were also obtained by Bouchut and James [5,6] using a different argument based on functional analysis. When the initial density ρ_0 is a Radon measure, Huang and Wang [29] showed that besides the Oleinik entropy condition, it is also necessary to require that the energy should be weakly continuous initially, which is called the energy condition, to ensure the uniqueness of the weak solution. For recent works on the pressureless Euler system, we refer to [9,12,32] and references therein.

We now return to the pressureless Euler system with a flocking dissipation which is discussed in previous subsection:

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d,\ t > 0, \\ &\partial_t(\rho u) + \nabla_x(\rho u \otimes u) = -K\rho \int_{\mathbb{R}^d} \psi\big(|x-y|\big)\big(u(x)-u(y)\big)\rho(y)dy. \end{aligned} \tag{2.9}$$

We set velocity moments:

$$m_0(t) := \int_{\mathbb{R}^d} \rho dx, \qquad m_1(t) := \int_{\mathbb{R}^d} \rho u dx, \qquad m_2(t) := \int_{\mathbb{R}^d} \rho |u|^2 dx,$$

and study the time-evolution of them in the following lemma.

Lemma 2.1. *Suppose that $T \in (0,\infty)$ and initial data satisfy*

$$m_0(0) := \int_{\mathbb{R}^d} \rho_0(x)dx < \infty, \qquad m_1(0) := \int_{\mathbb{R}^d} (\rho_0 u_0)(x)dx < \infty,$$

and let (ρ, u) be a smooth solution in the time-interval $[0,T)$ to (2.9) decaying sufficiently fast enough at $|x| = \infty$. Then, we have

(i)

$$m_0(t) = m_0(0), \qquad m_1(t) = m_1(0), \quad t \in (0,T).$$

(ii)

$$\frac{dm_2}{dt} = -K \int_{\mathbb{R}^{2d}} \psi\big(|x-y|\big)\rho(x)\rho(y)\big|u(x)-u(y)\big|^2 dydx.$$

Proof. (i) The conservation of mass and momentum follow from the direct integration of system (2.9).

(ii) We first note that for any smooth solutions, the system (2.9) is equivalent to the following quasi-linear system:

$$\begin{aligned} &\partial_t\rho + \nabla_x\cdot(\rho u) = 0, \quad x\in\mathbb{R}^d,\ t>0, \\ &\rho\partial_t u + \rho(u\cdot\nabla_x)u = -K\rho\int_{\mathbb{R}^d}\psi\big(|x-y|\big)\big(u(x)-u(y)\big)\rho(y)dy. \end{aligned} \tag{2.10}$$

By direct calculation using (2.10), we have

$$\begin{aligned} \partial_t\big(\rho|u|^2\big) &= (\partial_t\rho)|u|^2 + 2(\rho\partial_t u)\cdot u = -\nabla_x(\rho u)|u|^2 + 2(\rho\partial_t u)\cdot u \\ &= -\nabla_x\cdot\big(\rho u|u|^2\big) + 2u\cdot\big[\rho(u\cdot\nabla_x)u + \rho\partial_t u\big] \\ &= -\nabla_x\cdot\big(\rho u|u|^2\big) - 2K\rho u\cdot\int_{\mathbb{R}^d}\psi\big(|x-y|\big)\rho(y)\big(u(x)-u(y)\big)dy. \end{aligned}$$

Finally, we integrate the above relation to obtain

$$\begin{aligned} \frac{dm_2}{dt} &= -2K\int_{\mathbb{R}^{2d}}\psi\big(|x-y|\big)\rho(x)\rho(y)\big[\big|u(x)\big|^2 - u(x)\cdot u(y)\big]dydx \\ &= -2K\int_{\mathbb{R}^{2d}}\psi\big(|x-y|\big)\rho(x)\rho(y)\big[\big|u(y)\big|^2 - u(x)\cdot u(y)\big]dydx \\ &= -K\int_{\mathbb{R}^{2d}}\psi\big(|x-y|\big)\rho(x)\rho(y)\big|u(x)-u(y)\big|^2 dydx. \qquad \square \end{aligned}$$

Remark 2.1. Note that the non-local source term $S^{(1)}$ can be rewritten as

$$\begin{aligned} S^{(1)} &:= -K\rho\int_{\mathbb{R}^d}\psi\big(|x-y|\big)\big(u(x)-u(y)\big)\rho(y)dy \\ &= -K\bigg(\int_{\mathbb{R}^d}\psi\big(|x-y|\big)\rho(y)dy\bigg)\rho u + K\bigg(\int_{\mathbb{R}^d}\psi\big(|x-y|\big)(\rho u)(y)dy\bigg)\rho \\ &=: -Km_0^\psi(\rho u) + Km_1^\psi\rho, \end{aligned}$$

where m_0^ψ and m_1^ψ are weighted mass and momentum respectively:

$$m_0^\psi(x) := \int_{\mathbb{R}^d} \psi\big(|x-y|\big)\rho(y)dy, \qquad m_1^\psi := \int_{\mathbb{R}^d} \psi\big(|x-y|\big)(\rho u)(y)dy.$$

For $\psi \equiv 1$, we have the conservation of mass and momentum to see

$$S^{(1)} = -Km_0(0)\rho u + Km_1(0)\rho. \tag{2.11}$$

We are now ready to study the asymptotic flocking estimate for (2.9) via a Lyapunov functional approach. For this, we introduce a functional Λ which corresponds to an energy dissipation rate:

$$\begin{aligned}\Lambda[t] &:= \int_{\mathbb{R}^2} \big|u(x)-u(y)\big|^2\rho(x)\rho(y)dydx \\ &= 2m_0(0)m_2(t) - 2\big|m_1(0)\big|^2.\end{aligned} \tag{2.12}$$

Proposition 2.1. *Suppose that $T \in (0,\infty)$ and initial data satisfy*

$$m_0(0) := \int_{\mathbb{R}^d} \rho_0(x)dx < \infty, \qquad m_1(0) := \int_{\mathbb{R}^d} (\rho_0 u_0)(x)dx < \infty,$$

and let (ρ, u) be a smooth solution in the time-interval $[0,T)$ to (2.9) decaying sufficiently fast enough at $|x| = \infty$, and the diameter of support for spatial configuration is smaller than D in the time interval $[0,T)$. Then, we have a time-asymptotic flocking:

$$\Lambda[t] \le \Lambda[0]e^{-2m_0(0)\psi(D)Kt}, \quad t \ge 0.$$

Proof. It follows from (2.12) and Lemma 2.1 that

$$\begin{aligned}\frac{d\Lambda[t]}{dt} &= 2m_0(0)\frac{dm_2(t)}{dt} \\ &= -2m_0(0)K\int_{\mathbb{R}^{2d}} \psi\big(|x-y|\big)\rho(x)\rho(y)\big|u(x)-u(y)\big|^2dydx \\ &= -2m_0(0)K\int_{\mathrm{spt}(\rho)\times\mathrm{spt}(\rho)} \psi\big(|x-y|\big)\rho(x)\rho(y)\big|u(x)-u(y)\big|^2dydx \\ &\le -2m_0(0)\psi(D)K\int_{\mathbb{R}^{2d}} \rho(x)\rho(y)\big|u(x)-u(y)\big|^2dydx \\ &= -2m_0(0)\psi(D)K\Lambda[t]. \qquad \square\end{aligned}$$

Remark 2.2. Note that in order to use the a priori flocking result given in Proposition 2.1, we need sufficiently regular and decaying solution.

From now on, we assume that

$$m_0(0) = 1, \qquad m_1(0) = 0, \qquad K \equiv 1, \qquad \psi \equiv 1. \tag{2.13}$$

Under the condition (2.13), system (2.9) becomes the pressureless Euler system with a damping term:

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d, \ t > 0, \\ &\partial_t (\rho u) + \nabla_x (\rho u \otimes u) = -\rho u. \end{aligned} \tag{2.14}$$

3. Discussion of main results

In this section, we briefly summarize our main results on the global existence and uniqueness of an entropic weak solution to (2.14) with $d = 1$, and suitable normalization (2.11) and (2.13). The proofs of our main results will be treated in the following two sections.

Consider the one-dimensional pressureless Euler system with a damping:

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \quad x \in \mathbb{R}, \ t > 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2) = -\rho u, \\ (\rho, u)(x, 0) = (\rho_0, u_0)(x), \end{cases} \tag{3.1}$$

where the initial data (ρ_0, u_0) satisfy

$$\rho_0 \in L^1_{loc}(\mathbb{R}), \qquad u_0 \in L^\infty(\mathbb{R}), \qquad \rho_0(x) > 0, \quad \text{a.e. } x \in \mathbb{R}. \tag{3.2}$$

As discussed in Section 2.1, the main feature of the pressureless Euler system (3.1) is the formation of the delta-shock wave for generic smooth initial data. Motivated by the previous literature on the pressureless gas dynamics, a measure valued solution to (3.1) is expected. Note that the continuity equation can be rewritten as

$$\partial_t \rho - \partial_x (-\rho u) = 0, \quad \text{i.e., the vector field } (\rho, -\rho u) \text{ is conservative.}$$

Thus, if ρ and u are bounded measurable functions, then the potential function m can be defined by

$$m(x,t) := \int_{(0,0)}^{(x,t)} \rho dx - \rho u dt. \tag{3.3}$$

Then, it follows from (3.3) that

$$\partial_x m = \rho \quad \text{and} \quad \partial_t m = -\rho u.$$

Thus, the system (3.1) can be rewritten for the variables (m, u):

$$\begin{aligned}&\partial_t m + u\partial_x m = 0, \quad x \in \mathbb{R},\ t > 0,\\ &\partial_t\big((\partial_x m)u\big) + \partial_x\big((\partial_x m)u^2\big) = -(\partial_x m)u.\end{aligned} \tag{3.4}$$

Before we present our main results, we first recall the definition of the entropic weak solution to (3.4) in the sense of Lebesgue–Stieltjes integral.

Definition 3.1. (See [43].) Let $m = m(x,t)$ and $u = u(x,t)$ be functions satisfying the following regularity conditions:

(i) m is locally bounded variation in x.
(ii) u is locally bounded and measurable with respect to $\partial_x m$.
(iii) $\partial_x m$ and $u\partial_x m$ are weakly continuous in t.

Then, $(\rho, u) = (\partial_x m, u)$ is called an entropic weak solution of (3.1) or (m, u) is called an entropic weak solution to (3.4) if and only if the following relations hold.

(1) (m, u) satisfies the system (3.4) in distribution sense:

$$\begin{aligned}&\iint\limits_{\mathbb{R}\times\mathbb{R}_+} \partial_t\varphi m\,dx\,dt - \iint\limits_{\mathbb{R}\times\mathbb{R}_+} \varphi u\,dm\,dt = 0,\\ &\iint\limits_{\mathbb{R}\times\mathbb{R}_+} \big(\partial_t\varphi u + \partial_x\varphi u^2 - u\big)\,dm\,dt = 0, \quad \varphi \in \mathcal{C}_0^\infty\big(\mathbb{R}\times\mathbb{R}^+\big).\end{aligned} \tag{3.5}$$

(2) The initial values (3.2) are understood in the following sense: as $t \to 0+$,

$$m(x,t) \to \int_0^x \rho_0\,d\eta, \quad \int_{0\pm 0}^{x\pm 0} u\,dm \to \int_0^x \rho_0 u_0\,d\eta, \quad \text{in } L^\infty_{loc}(\mathbb{R}). \tag{3.6}$$

Here $\iint_{\mathbb{R}\times\mathbb{R}_+}\cdots dm\,dt$ denotes the Lebesgue–Stieltjes integral.

(3) (m, u) satisfies an Oleinik type entropy condition: for any $x_1 < x_2$ and almost everywhere $t > 0$

$$\frac{u(x_2,t) - u(x_1,t)}{x_2 - x_1} \le \frac{e^{-t}}{1 - e^{-t}}. \tag{3.7}$$

Remark 3.1. The R.H.S. of (3.7) is due to the effect of the nonhomogeneous term $-\rho u$, while it is exactly $1/t$ for the homogeneous case. Note that

$$\text{R.H.S.} \sim \frac{1}{t}, \quad \text{as } t \to 0+.$$

We are now ready to state our main results on the existence of variational solution which turns out to be a unique entropic weak solution to system (3.1). For a smooth solution (ρ, u)

with $\rho(x,t) > 0$, the momentum equation in (3.1) is equivalent to Burgers' equation with a linear damping:

$$\partial_t u + u\partial_x u = -u, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{3.8}$$

In a smooth and non-vacuum regime, the system (3.1) is also equivalent to the system:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \quad x \in \mathbb{R},\ t > 0, \\ &\partial_t u + u\partial_x u = -u. \end{aligned} \tag{3.9}$$

Note that the second equation for u is completely decoupled from the first equation. Therefore, we first solve the second equation using the method of characteristics and then with an explicit u, we solve the continuity equation with a known velocity field u. Let $(x(t), t)$ be a characteristics issued from $(x_0, 0)$ at time t. Then along the characteristics $x = x(t)$ starting from x_0, Eq. (3.8) can be written as a system of ODEs:

$$\begin{aligned} &\frac{dx}{dt} = u(x,t), \qquad \frac{du}{dt} = -u, \\ &\big(x(0), u(0)\big) = \big(x_0, u(x_0)\big). \end{aligned}$$

Then, we integrate the above ODEs to derive an explicit formula:

$$x(t) = x_0 + u_0(x_0)\big(1 - e^{-t}\big), \qquad u\big(x(t), t\big) = u_0(x_0)e^{-t}, \quad t \geq 0. \tag{3.10}$$

Based on the above formula (3.10) for u, we introduce a potential function F: for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$,

$$F(y; x, t) := \int_0^y \rho_0(\eta)\big(\eta + u_0(\eta)\big(1 - e^{-t}\big) - x\big)\,d\eta. \tag{3.11}$$

Our first result says that system (3.1) admits a variational solution which turns out to be entropic weak solution satisfying a flocking estimate.

Theorem 3.1 *(Existence). Suppose that initial data $\rho_0(x)$ and $u_0(x)$ satisfy the condition (3.2). Then, $(\rho, \rho u) = (m_x, m_x u)$ defined by*

$$\rho(x,t) := -\partial_x^2\Big(\min_{y\in\mathbb{R}} F(y; x, t)\Big), \qquad \rho(x,t)u(x,t) := \partial_x\partial_t\Big(\min_{y\in\mathbb{R}} F(y; x, t)\Big)$$

is the entropic weak solution of system (3.1) in the sense of Definition 3.1. Moreover, the solution exhibits the asymptotic flocking estimate:

$$\big\|u(\cdot,t)\big\|_{L^\infty} \leq \|u_0\|_{L^\infty} e^{-t}.$$

Remark 3.2. 1. Note that the global existence in Theorem 3.1 does not require any smallness of initial data, periodicity and differentiability due to the variational method. Recently, the first author and his collaborators [23] studied the global existence of small and smooth solutions for the multi-dimensional system (2.14) using the standard energy estimates for very restricted class of the initial data satisfying periodicity, high Sobolev regularity and non-vacuum condition, smallness in suitable Sobolev space. Hence our result in Theorem 3.1 is orthogonal to the corresponding result in [23].

2. For inhomogeneous pressureless Euler system (3.1) with source terms, as long as the momentum equation can be reduced a solvable inhomogeneous Burgers equation:

$$\partial_t u + u\partial_x u = \frac{1}{\rho} S(x, t, \rho, u)$$

by the method of characteristics, the variational approach can be applied. In fact, Ding and Huang [19] considered the source term ρx, so that the momentum equation can be reduced to

$$\partial_t u + u\partial_x u = x,$$

which can be solvable by the method of characteristics.

Our second result is concerned with the uniqueness of an entropic weak solution in the sense of Definition 3.1.

Theorem 3.2 *(Uniqueness). Suppose that* (m_1, u_1) *and* (m_2, u_2) *are any two entropy weak solutions of* (3.1) *with the same initial data* (ρ_0, u_0) *in the sense of Definition 3.1. Then, we have*

$$m_1 = m_2, \qquad u_1 = u_2 \quad a.e.\ (x, t) \in \mathbb{R} \times \mathbb{R}_+$$

with respect to the measure $\partial_x m_1 = \partial_x m_2$.

Remark 3.3. Since the solution of (3.1) is usually a Radon measure, it is natural to study the general case, when the initial data is a Radon measure. We conjecture that the solution for the measure initial data can also be constructed by a similar procedure as in Theorem 3.1. Since the initial data is no longer measurable function, the initial conditions (3.6) is too strong and the usual initial condition should be introduced instead. Similar to the homogeneous pressureless flow, we guess that the Oleinik type entropy condition (3.7) and an energy condition introduced in [29] (i.e. the measure ρu^2 weakly converges to $\rho_0 u_0^2$ as $t \to 0$) will be enough to guarantee the uniqueness for the nonhomogeneous system (3.1). This will be investigated in future work.

4. Preparatory lemmas

In this section, we provide several basic properties of the potential function F in (3.11). Although similar lemmas in this section can also be found in literature [19,42,44], for reader's convenience, we provide the proofs of lemmas.

Note that for a constant initial density $\rho_0 = 1$, the potential $F(y)$ behaves like $\frac{|y|^2}{2}$ for $|y| \gg 1$. This can be seen as follows. For a fixed $(x, t) \in \mathbb{R} \times \mathbb{R}_+$,

$$F(y;x,t) \sim \begin{cases} \frac{(y-x)^2}{2} - \frac{x^2}{2}, & 0 < t \ll 1, \\ \frac{(y-x)^2}{2} - \frac{x^2}{2} + \int_0^y u_0(\eta)d\eta, & t \gg 1. \end{cases}$$

Since $u_0 \in L^\infty(\mathbb{R})$, the functional F will behave like $\frac{y^2}{2}$ for $|y| \gg 1$.

Lemma 4.1. *For a fixed $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, the function $F(y) = F(y;x,t)$ is bounded below, thus it has a minimum.*

Proof. Let (x,t) be fixed and we set

$$M := \|u_0\|_{L^\infty}, \qquad \underline{y} := x - M\big(1-e^{-t}\big), \qquad \bar{y} := x + M\big(1-e^{-t}\big). \tag{4.1}$$

Then, the integrand of $-F$ satisfies

$$\rho_0(\eta)(\underline{y} - \eta) \le \rho_0(\eta)\big(x - u_0(\eta)\big(1-e^{-t}\big) - \eta\big) \le \rho_0(\eta)(\bar{y} - \eta). \tag{4.2}$$

To obtain the desired result, it suffices to show that F is non-increasing and non-decreasing in the intervals $(-\infty, \underline{y}]$ and $[\bar{y}, \infty)$, respectively. Thus, F will have a minimum in the closed interval $I := [\underline{y}, \bar{y}]$, i.e.,

$$\min_{y\in\mathbb{R}} F(y;x,t) = \min_{y\in I} F(y;x,t).$$

• Case A (F is non-increasing in the interval $(-\infty, \underline{y}]$): We set

$$y_1 < y_2 \le \underline{y}.$$

Then, we use (4.2) to obtain

$$\begin{aligned} F(y_1;x,t) - F(y_2;x,t) &= \int_{y_1}^{y_2} \rho_0(\eta)\big(x - u_0(\eta)\big(1-e^{-t}\big) - \eta\big)d\eta \\ &\ge \int_{y_1}^{y_2} \rho_0(\eta)(\underline{y} - \eta)d\eta \ge 0. \end{aligned}$$

• Case B (F is non-decreasing in the interval $[\bar{y}, \infty)$): We set

$$\bar{y} \le y_1 < y_2 < \infty,$$

then, by the same argument, we have

$$F(y_1;x,t)-F(y_2;x,t)=\int_{y_1}^{y_2}\rho_0(\eta)\big(x-u_0(\eta)\big(1-e^{-t}\big)-\eta\big)d\eta$$
$$\le\int_{y_1}^{y_2}\rho_0(\eta)(\bar{y}-\eta)d\eta\le 0. \qquad \Box$$

In the sequel, we study the basic properties of the set consisting of minimum points of F. For a given $(x,t)\in\mathbb{R}\times\mathbb{R}_+$, we set

$$v(x,t):=\min_{y\in\mathbb{R}}F(y;x,t), \qquad S(x,t):=\big\{y\;\big|\;F(y;x,t)=v(x,t)\big\},$$
$$y_*(x,t):=\min_{y\in S(x,t)}\{y\}, \qquad y^*(x,t):=\max_{y\in S(x,t)}\{y\}. \tag{4.3}$$

It is easy to see from (4.1) and Lemma 4.1 that

$$\underline{y}\le y_*(x,t)\le y^*(x,t)\le\bar{y}. \tag{4.4}$$

Lemma 4.2. *The extremal functions $y_*(x,t)$ and $y^*(x,t)$ are non-decreasing in x, and moreover for $x_1<x_2$,*

$$y^*(x_1,t)\le y_*(x_2,t).$$

Proof. We set

$$y_1\in S(x_1,t), \qquad y_2\in S(x_2,t), \quad x_1<x_2. \tag{4.5}$$

Then, we have

$$F(y_2;x_1,t)-F(y_1;x_1,t)=\int_{y_1}^{y_2}\rho_0(\eta)\big(\eta+u_0(\eta)\big(1-e^{-t}\big)-x_1\big)d\eta\ge 0,$$
$$F(y_1;x_2,t)-F(y_2;x_2,t)=\int_{y_2}^{y_1}\rho_0(\eta)\big(\eta+u_0(\eta)\big(1-e^{-t}\big)-x_2\big)d\eta\ge 0, \tag{4.6}$$

where we used (4.5):

$$F(y_1;x_1,t)=\min_{y\in\mathbb{R}}F(y;x_1,t), \qquad F(y_2;x_2,t)=\min_{y\in\mathbb{R}}F(y;x_2,t).$$

We combine two inequalities in (4.6) to obtain

$$(x_2-x_1)\int_{y_1}^{y_2}\rho_0(\eta)d\eta\ge 0.$$

Since $\rho > 0$ a.e. and $x_1 < x_2$, the above relation implies

$$y_2 \geq y_1. \tag{4.7}$$

The relations (4.5) and (4.7) yield desired estimates:

$$y_*(x_2, t) \geq y_*(x_1, t), \qquad y^*(x_2, t) \geq y^*(x_1, t), \qquad y_*(x_2, t) \geq y^*(x_1, t). \qquad \square$$

Lemma 4.3. *Let $y_*(x, t)$ and $y^*(x, t)$ be extremal points of the minimum point set $S(x, t)$, respectively. Then, we have the following assertions:*

(1) *$y_*(x, t)$ and $y^*(x, t)$ as functions of (x, t) are lower and upper semi-continuous, respectively.*
(2) *At the point where $y_*(x, t) = y^*(x, t)$, both functions are continuous with respect to (x, t).*

Proof. (i) We only show that $y_*(x, t)$ is lower semi-continuous. The case for $y^*(x, t)$ can be proved similarly. Suppose that

$$(x_n, t_n) \to (x_0, t_0), \qquad \underline{\lim}_{n\to\infty} y_*(x_n, t_n) = y, \quad \text{as } n \to \infty.$$

Then, we claim:

$$y_*(x_0, t_0) \leq y,$$

that is, $y_*(x, t)$ is lower semi-continuous with respect to (x, t). *Proof of claim:* Note that $\nu(x, t)$ in (4.3) is continuous, then it holds that

$$\nu(x_0, t_0) = \lim_{n\to+\infty} \nu(x_n, t_n) = \lim_{n\to+\infty} F\big(y_*(x_n, t_n); x_n, t_n\big) = F(y; x_0, t_0).$$

Thus, $y \in S(x_0, t_0)$, and this implies

$$y_*(x_0, t_0) \leq y \leq y^*(x_0, t_0).$$

This proves the claim.

(ii) Since the lower and upper continuity is equivalent to the continuity, the statement is true. $\square$

Note that for a smooth solution, the system (3.1) can be reduced to the partially decoupled system (3.9) and Dafermos' generalized characteristic method [17] can be applied here. For each point $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}_+$, we define left and right backward generalized characteristics ℓ_L, ℓ_R as follows: for $t < t_0$,

$$\begin{aligned} \ell_L(x_0, t_0): \quad x_L(t) &= \frac{1 - e^{-t}}{1 - e^{-t_0}} x_0 + y_*(x_0, t_0) \frac{e^{-t} - e^{-t_0}}{1 - e^{-t_0}}, \\ \ell_R(x_0, t_0): \quad x_R(t) &= \frac{1 - e^{-t}}{1 - e^{-t_0}} x_0 + y^*(x_0, t_0) \frac{e^{-t} - e^{-t_0}}{1 - e^{-t_0}}. \end{aligned} \tag{4.8}$$

We have the following properties of the backward generalized characteristics.

Lemma 4.4. *Along the backward generalized characteristics, the set $S(x,t)$ is singleton:*

$$S\big(x_L(t),t\big)=\big\{y_*(x_0,t_0)\big\} \quad \textit{and} \quad S\big(x_R(t),t\big)=\big\{y^*(x_0,t_0)\big\}, \quad t<t_0.$$

Proof. We only provide the proof of the first assertion. The second assertion can be treated similarly. We first set $y_0 := y_*(x_0,t_0)$. For $y \neq y_0$, note that

$$\begin{aligned}
&F\big(y;x_L(t),t\big)-F\big(y_0;x_L(t),t\big)\\
&=\int_{y_0}^{y}\rho_0(\eta)\big(\eta+u_0(\eta)\big(1-e^{-t}\big)-x_L(t)\big)d\eta\\
&=\frac{1-e^{-t}}{1-e^{-t_0}}\int_{y_0}^{y}\rho_0(\eta)\big(\eta+u_0(\eta)\big(1-e^{-t_0}\big)-x_0\big)d\eta+\frac{e^{-t}-e^{-t_0}}{1-e^{-t_0}}\int_{y_0}^{y}\rho_0(\eta)(\eta-y_0)d\eta\\
&=\frac{1-e^{-t}}{1-e^{-t_0}}\big[F(y;x_0,t_0)-F(y_0;x_0,t_0)\big]+\frac{e^{-t}-e^{-t_0}}{1-e^{-t_0}}\int_{y_0}^{y}\rho_0(\eta)(\eta-y_0)d\eta\\
&\geq\frac{e^{-t}-e^{-t_0}}{1-e^{-t_0}}\int_{y_0}^{y}\rho_0(\eta)(\eta-y_0)d\eta>0.
\end{aligned}$$

Thus, we have

$$F\big(y;x_L(t),t\big)>F\big(y_0;x_L(t),t\big)>0, \quad \text{for } y\neq y_0,\ t<t_0.$$

This implies

$$S\big(x_L(t),t\big)=\big\{y_*(x_0,t_0)\big\}. \qquad \square$$

Remark 4.1. The result of Lemma 4.4 is equivalent to say

$$\begin{aligned}
y_*(x,t)=y^*(x,t)=y_*(x_0,t_0), &\quad \text{on } \ell_L,\\
y_*(x,t)=y^*(x,t)=y^*(x_0,t_0), &\quad \text{on } \ell_R.
\end{aligned}$$

For a given point (x_0,t_0), $t_0>0$, we denote the characteristic region $\Delta(x_0,t_0)$ to be the region enclosed by the curves $\ell_L(x_0,t_0)$, $\ell_R(x_0,t_0)$ and x-axis (see Fig. 1). Then it follows from Lemmas 4.2–4.4 that for any two points (x_0,t_0), (x_0',t_0'), there are only three possibilities:

$$\Delta(x_0,t_0)\cap\Delta\big(x_0',t_0'\big)=\emptyset, \qquad \Delta(x_0,t_0)\subset\Delta\big(x_0',t_0'\big) \quad \text{and} \quad \Delta\big(x_0',t_0'\big)\subset\Delta(x_0,t_0).$$

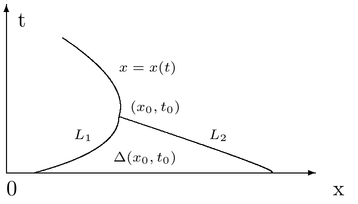

Fig. 1. Characteristic region of (x_0, t_0).

Lemma 4.5. *For each point (x_0, t_0) with $t_0 > 0$, there exists a unique Lipschitz continuous curve $x = x(t)$, $t \geq t_0$ such that*

$$x_0 = x(t_0), \qquad \lim_{t_2,t_1 \to t+0} \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \begin{cases} \dfrac{e^{-t}}{1 - e^{-t}}(x(t) - y_*), & \text{if } y_* = y^*, \\ \dfrac{\int_{y_*}^{y^*} \rho_0(\eta) u_0(\eta) d\eta}{\int_{y_*}^{y^*} \rho_0(\eta) d\eta} e^{-t}, & \text{if } y_* \neq y^*, \end{cases} \tag{4.9}$$

where $y_* = y_*(x(t), t)$ and $y^* = y^*(x(t), t)$.

Proof. Since the proof is rather lengthy, we leave it to Appendix A. □

5. Global existence of an entropic weak solution

In this section, we provide a global existence of an entropic weak solution to the system (3.1) using the variational formulation in [19] where the pressureless Euler system with some specific source term is studied. We first construct an explicit representation for m and u, and then we show that the explicit representation formula satisfy the defining relations in Definition 3.1.

5.1. Representation of a variational solution

Our strategy to construct the variational solution to (3.1) is to look for the minimum of the potential function $F(y; x, t)$ in y:

$$F(y; x, t) := \int_0^y \rho_0(\eta)\big(\eta + u_0(\eta)\big(1 - e^{-t}\big) - x\big) d\eta.$$

Note that the potential function F depends only on the initial data and is motivated by Hopf's work [28] for the Burgers' equation. As in [19], we can expect that the entropy weak solution of (3.9) can be constructed by the procedure of taking the minimum of $F(y; x, t)$ in y. For a notational simplicity, we suppress (x, t)-dependence in y_* and y^*:

$$y_* = y_*(x, t), \qquad y^* = y^*(x, t).$$

For a given point $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, we define

$$m(x,t) := \int_0^{y_*} \rho_0(\eta)d\eta \quad \text{and}$$

$$u(x,t) := \begin{cases} \dfrac{e^{-t}}{1-e^{-t}}(x-y_*), & \text{if } y_* = y^*, \\ \dfrac{e^{-t}\int_{y_*}^{y^*} \rho_0(\eta)u_0(\eta)d\eta}{\int_{y_*}^{y^*} \rho_0(\eta)d\eta} & \text{if } y_* \neq y^*. \end{cases} \tag{5.1}$$

By Lemma 4.2 and (5.1), both $m(x,t)$ and $u(x,t)$ are of bounded variation locally in x. We expect that $(m(x,t), u(x,t))$ or $(\rho(x,t) = \partial_x m(x,t), u(x,t))$ is an entropy solution of the system (3.9) in the sense of Definitions 3.1. This will be justified in Section 5.3.

5.2. *Basic estimates*

In this subsection, we will study some basic estimates to be used in the verification procedure in Section 5.3. Let $\nu(x,t)$ be the minimum of the functional $F(y;x,t)$ with respect to y-variable whose existence is guaranteed by Lemma 4.1. With $y_*(x,t)$, $y^*(x,t)$ defined in (4.3), we set

$$q(x,t) := e^{-t} \int_0^{y_*(x,t)} \rho_0(\eta)u_0(\eta)d\eta \quad \text{or} \quad \int_0^{y_*(x,t)} \rho_0(\eta)u_0(\eta)d\eta = q(x,t)e^t. \tag{5.2}$$

In the following lemma, we study the spatial and temporal variations of ν.

Lemma 5.1. *For $x_i \in \mathbb{R}$, $t_i \in \mathbb{R}_+$, $i = 1, 2$, we have*

(i)

$$\int_{x_1}^{x_2} m(x,t)dx = \nu(x_1,t) - \nu(x_2,t),$$

(ii)

$$\int_{t_1}^{t_2} q(x,t)dt = \nu(x,t_2) - \nu(x,t_1).$$

Proof. (i) Let $[x, x']$ be a subdivision of the interval (x_1, x_2) and $t \in \mathbb{R}_+$. We set

$$y_* = y_*(x,t) \quad \text{and} \quad y'_* = y_*\big(x',t\big).$$

Then, we use the monotonicity of $F(\cdot;x,t)$ in Lemma 4.2 to obtain

$$\begin{aligned}
\nu(x,t)-\nu(x',t) &= F(y_*; ,x,t)-F(y'_*;x',t)\\
&= \big[F(y_*; ,x,t)-F(y'_*;x,t)\big]+\big[F(y'_*;x,t)-F(y'_*;x',t)\big]\\
&\le F(y'_*;x,t)-F(y'_*;x',t)\\
&= \int_0^{y'_*}\rho_0(\eta)d\eta(x'-x)=(x'-x)m(x',t),
\end{aligned}$$

i.e., we have

$$\nu(x,t)-\nu(x',t)\le(x'-x)m(x',t),\quad \forall[x,x']\subset[x_1,x_2].$$

Then, we use the interpolation argument to obtain

$$\int_{x_1}^{x_2} m(x,t)dx\ge\nu(x_1,t)-\nu(x_2,t). \tag{5.3}$$

On the other hand, we also have the same argument as above and (5.1) to obtain

$$\begin{aligned}
\nu(x,t)-\nu(x',t) &= F(y_*; ,x,t)-F(y'_*;x',t)\\
&= \big[F(y_*; ,x,t)-F(y_*;x',t)\big]+\big[F(y_*;x',t)-F(y'_*;x',t)\big]\\
&\ge F(y_*;x,t)-F(y_*;x',t)\\
&= (x'-x)\int_0^{y_*}\rho_0(\eta)d\eta\\
&= (x'-x)m(x,t).
\end{aligned}$$

Thus, we have

$$\int_{x_1}^{x_2} m(x,t)dx\le\nu(x_1,t)-\nu(x_2,t). \tag{5.4}$$

Finally we combine (5.3) and (5.4) to obtain the first relation.

(ii) Let $[t,t']$ be any subdivision of the interval $[t_1,t_2]$ and $x\in\mathbb{R}$. Then, by the same argument as in (i), we have

$$\begin{aligned}
&\nu(x,t)-\nu(x,t')\\
&\quad= F(y_*;x,t)-F(y'_*;x,t')\\
&\quad= \underbrace{F(y_*;x,t)-F(y'_*;x,t)}_{\le 0}+F(y'_*;x,t)-F(y'_*;x,t')
\end{aligned}$$

$$
\begin{aligned}
&\le F\big(y'_*; x, t\big) - F\big(y'_*; x, t'\big) \\
&= \int_0^{y'_*} \rho_0(\eta)\big(\eta + u_0(\eta)\big(1 - e^{-t}\big) - x\big)d\eta - \int_0^{y'_*} \rho_0(\eta)\big(\eta + u_0(\eta)\big(1 - e^{-t'}\big) - x\big)d\eta \\
&= \int_0^{y'_*} \big(e^{-t'} - e^{-t}\big)\rho(\eta)u_0(\eta)d\eta \\
&= \big(e^{-t'} - e^{-t}\big)\int_0^{y'_*} \rho(\eta)u_0(\eta)d\eta \\
&= q\big(x, t'\big)e^{t'}\big(e^{-t'} - e^{-t}\big) \\
&= q\big(x, t'\big)\big(1 - e^{t'-t}\big),
\end{aligned}
$$

where we used the monotonicity of $F(\cdot; x, t)$ in $y_* = y_*(x, t)$ (see Lemma 4.2) and $y'_* = y_*(x, t')$. Therefore, we have

$$
\nu(x, t) - \nu\big(x, t'\big) \le q\big(x, t'\big)\big(1 - e^{t'-t}\big).
$$

By the similar argument as in (i), we can also have

$$
\nu(x, t) - \nu\big(x, t'\big) \ge q\big(x, t'\big)\big(1 - e^{t'-t}\big).
$$

Thus, we have

$$
\nu\big(x, t'\big) - \nu(x, t) = q\big(x, t'\big)\big(e^{t'-t} - 1\big).
$$

This yields

$$
\nu(x, t_2) - \nu(x, t_1) = \int_{t_1}^{t_2} q(x, t)dt. \qquad \square
$$

Remark 5.1. The result of Lemma 5.1 implies

$$
m = -\partial_x \nu \quad \text{and} \quad q = \partial_t \nu \quad \text{in the sense of distributions.}
$$

In order to establish (3.5), we need to investigate the relation between m and q.

Lemma 5.2. *Let $m = m(x, t)$ and $q = q(x, t)$ be given by the defining relations in (5.1) and (5.2), respectively. Then we have*

$$
\frac{dq}{dm} = u \quad \text{in the sense of Radon–Nikodym derivatives in } x. \tag{5.5}
$$

Proof. Without loss of generality, we may assume that y_* is not always a constant in any neighborhood of (x,t). Let

$$x_1 < x < x_2, \quad y_1 = y_*(x_1,t), \qquad y_2 = y_*(x_2,t).$$

Then, we have

$$y_1 < y_2, \quad y_1 \to y_*(x,t), \qquad y_2 \to y^*(x,t) \quad \text{as } x_1, x_2 \to x.$$

Note that the estimate (5.5) is equivalent to

$$\lim_{x_2,x_1\to x\pm 0} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} = u(x,t). \tag{5.6}$$

• Case A ($y_* < y^*$): By direct calculation, we have

$$\lim_{x_2,x_1\to x\pm 0} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} = \lim_{x_2,x_1\to x\pm 0} \frac{e^{-t}\int_{y_1}^{y_2}\rho_0(\eta)u_0(\eta)d\eta}{\int_{y_1}^{y_2}\rho_0(\eta)d\eta} = u(x,t).$$

• Case B ($y_* = y^*$): It follows from the definition of y_* that we have

$$F(y_2;x_2,t) \le F(y_1;x_2,t), \quad \text{i.e., } e^{-t}\int_{y_1}^{y_2}\rho_0(\eta)u_0(\eta)d\eta \le \frac{e^{-t}}{1-e^{-t}}\int_{y_1}^{y_2}\rho_0(\eta)(x_2-\eta)d\eta.$$

Thus, we have

$$\frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} \le \frac{e^{-t}}{1-e^{-t}} \frac{\int_{y_1}^{y_2}\rho_0(\eta)(x_2-\eta)d\eta}{\int_{y_1}^{y_2}\rho_0(\eta)d\eta}.$$

Let $x_1 \to x-0$, $x_2 \to x+0$ to obtain

$$\lim_{x_2,x_1\to x\pm 0} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} \le \frac{x-y_*}{1-e^{-t}}e^{-t} = u(x,t). \tag{5.7}$$

Similarly we have

$$\lim_{x_2,x_1\to x\pm 0} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} \ge u(x,t). \tag{5.8}$$

Thus, it follows from (5.7) and (5.8) that

$$\lim_{x_2,x_1\to x\pm 0} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} = u(x,t).$$

Finally, we combine Case A and Case B to derive the estimate (5.6). □

We next introduce another functional which is crucial to prove (3.9)$_2$ for u given by the relation (5.1). For each $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, it follows from the same method in Lemma 4.5 that there is a Lipschitz curve $x = X(\eta, t)$ for each point $(\eta, 0)$ on x-axis except at most a countable of points:

$$\begin{cases} \dfrac{\partial X(\eta,s)}{\partial s} = u\big(X(\eta,s),s\big), & 0 < s < t, \\ X(x,t) = x, \qquad X(\eta,0) = \eta, & \text{a.e. } \eta. \end{cases}$$

By the construction of X, we have

$$X(\eta,t) = x, \quad \forall \eta \in \big[y_*, y^*\big].$$

Note that $\partial_\eta X(\eta,t)$ satisfies

$$\frac{d}{dt}\partial_\eta X = \partial_x u\big(X(\eta,t),t\big)\partial_\eta X, \quad \partial_\eta X(\eta,0) = 1.$$

Therefore, we have

$$\partial_\eta X(\eta,t) = \exp\left(\int_0^t \partial_x u\big(X(\eta,s),s\big)ds\right) > 0.$$

Thus, $X(\cdot,t)$ is strictly increasing in η. Since $u_0 \in L^\infty(\mathbb{R})$, we can choose k sufficiently large to satisfy

$$k > \|u_0\|_{L^\infty}.$$

Then, for such k, we define

$$G(y;x,t) := e^{-t}\int_0^y \rho_0(\eta)\underbrace{\big(u_0(\eta)+k\big)}_{>0}\big(X(\eta,t)-x\big)d\eta, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

Lemma 5.3. *For each $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, the function $G(\cdot;x,t)$ has a minimum and satisfies*

$$\mu(x,t) := \min_{y \in \mathbb{R}} G(y;x,t) = G(y_*;x,t) = G\big(y^*;x,t\big).$$

Proof. We first use

$$X(\eta,t) = x, \quad \eta \in \big[y_*, y^*\big]$$

to obtain the second equality:

$$\begin{aligned}G\big(y^*;x,t\big) &= e^{-t}\int_0^{y_*}\rho_0(\eta)\big(u_0(\eta)+k\big)\big(X(\eta,t)-x\big)d\eta \\ &\quad + e^{-t}\int_{y_*}^{y^*}\rho_0(\eta)\big(u_0(\eta)+k\big)\underbrace{\big(X(\eta,t)-x\big)}_{=0}d\eta \\ &= G(y_*;x,t). \end{aligned} \tag{5.9}$$

For the first equality, we consider three cases:

$$y \le y_*, \qquad y_* < y < y^*, \qquad y \ge y^*.$$

• Case A ($y \le y_*$): In this case, we use

$$X(\eta,t) \le X(y_*,t) = x, \quad \text{for } \eta \in [y, y_*]$$

to see

$$\begin{aligned}G(y_*;x,t) &= e^{-t}\int_0^{y}\rho_0(\eta)\big(u_0(\eta)+k\big)\big(X(\eta,t)-x\big)d\eta \\ &\quad + e^{-t}\int_{y}^{y_*}\rho_0(\eta)\big(u_0(\eta)+k\big)\underbrace{\big(X(\eta,t)-x\big)}_{\le 0}d\eta \\ &\le G(y;x,t). \end{aligned}$$

• Case B ($y_* < y < y^*$): We use

$$X(y,t) \ge X(y_*,t) = x$$

to obtain

$$\begin{aligned}G(y;x,t) &= e^{-t}\int_0^{y_*}\rho_0(\eta)\big(u_0(\eta)+k\big)\big(X(\eta,t)-x\big)d\eta \\ &\quad + e^{-t}\int_{y_*}^{y}\rho_0(\eta)\big(u_0(\eta)+k\big)\underbrace{\big(X(\eta,t)-x\big)}_{=0}d\eta \\ &\ge G(y_*;x,t). \end{aligned}$$

• Case C ($y \ge y^*$): We use (5.9) and

$$X(\eta,t) \ge X\big(y^*,t\big) = x, \quad \text{for } \eta \in [y^*, \infty)$$

to obtain

$$\begin{aligned} G(y;x,t) &= e^{-t}\int_0^{y^*}\rho_0(\eta)\big(u_0(\eta)+k\big)\big(X(\eta,t)-x\big)d\eta \\ &\quad + e^{-t}\int_{y^*}^{y}\rho_0(\eta)\big(u_0(\eta)+k\big)\underbrace{\big(X(\eta,t)-x\big)}_{\geq 0}d\eta \\ &\geq G\big(y^*;x,t\big) = G(y_*;x,t). \end{aligned}$$

Finally, we combine all three cases to obtain the first equality. □

We set

$$\begin{aligned} E(x,t) &:= \frac{1}{2}e^{-t}\int_0^{y_*}\rho_0(\eta)u_0(\eta)u\big(X(\eta,t),t\big)d\eta, \\ \theta(x,t) &:= e^{-t}\int_0^{y_*}\rho_0(\eta)u_0(\eta)\big(X(\eta,t)-x\big)d\eta. \end{aligned} \tag{5.10}$$

Then, by the same argument as in the proof of Lemma 5.1 and (5.10), we have

$$\partial_x\theta = -q, \qquad \partial_t\big(e^t\theta\big) = 2Ee^t. \tag{5.11}$$

5.3. Verification procedure

In this subsection, we provide a sketchy proof of Theorem 3.1. For that, it suffices to check that (m,u) given in (5.1) satisfies the defining relations in Definition 3.1.

- (Verification of relation (3.5)): It follows from Lemmas 5.1 and 5.2 that we have

$$\partial_x\nu = -m, \qquad \partial_t\nu = q \quad \text{and} \quad \partial_x q = u\partial_x m.$$

Let $\varphi \in C_0^\infty(\mathbb{R}_+^2)$ be any test function. Then, we have

$$\iint \varphi_t m\,dx\,dt - \iint \varphi u\,dm\,dt = \iint (m\varphi_t + q\varphi_x)dx\,dt = \iint (\nu_t\varphi_x - \nu_x\varphi_t)dx\,dt = 0.$$

On the other hand, it follows from (5.11) that

$$\iint \big(qe^t\varphi_t + 2Ee^t\varphi_x\big)dx\,dt = \iint \big(-\big(e^t\theta\big)_x\phi_t + \big(e^t\theta\big)_t\phi_x\big)dx\,dt = 0. \tag{5.12}$$

As in Lemma 5.2, we get $\partial_x E = \frac{1}{2}u^2\partial_x m$ in the sense of Radon–Nikodym derivatives in x. Thus (5.12) gives for any $\bar{\psi} \in C_0^\infty(\mathbb{R}_+^2)$,

$$\begin{aligned}
&\iint \left(\bar{\psi}_t u + \bar{\psi}_x u^2 - \bar{\psi} u\right) dm\, dt \\
&= \iint \left[\left(\bar{\psi} e^{-t}\right)_t u e^t + \left(\bar{\psi} e^{-t}\right)_x u^2 e^t\right] dm dt \\
&= \iint \left(\bar{\psi} e^{-t}\right)_t e^t dq dt + \iint 2\left(\bar{\psi} e^{-t}\right)_x e^t dE dt \\
&= -\iint \left[q e^t \left(\bar{\psi} e^{-t}\right)_{xt} + 2E e^t \left(\bar{\psi} e^{-t}\right)_{xx}\right] dx dt \\
&= 0. \qquad (5.13)
\end{aligned}$$

Thus, (5.12) and (5.13) yield the relations (3.5).

• (Verification of relation (3.6)): Since $y_*(x,t)$ and $y^*(x,t)$ converge to x as $t \to 0$, the formulas of m and q naturally indicates the initial condition (3.6). Therefore, the functions $(m(x,t), u(x,t))$ constructed in (5.1) is indeed a weak solution of (3.9). By (5.1), Lemmas 5.1 and 5.2, it is easy to check that

$$\begin{aligned}
\rho(x,t) &= \partial_x m(x,t) = -\partial_x^2 \min_y F(y;x,t), \\
\rho(x,t) u(x,t) &= \left(\partial_x m(x,t)\right) u(x,t) = \partial_x \partial_t \min_y F(y;x,t). \qquad (5.14)
\end{aligned}$$

Moreover, it should be noted that $\rho(x,t) = \partial_x m(x,t)$ is usually a Radon measure.

For the entropy condition, we only need to justify (3.7). From the construction of $u(x,t)$, we have

$$u(x-0,t) = \frac{x - y_*}{1 - e^{-t}} e^{-t}, \qquad u(x+0,t) = \frac{x - y^*}{1 - e^{-t}} e^{-t}. \qquad (5.15)$$

On the other hand, (A.2) and (A.3) imply

$$u(x+0,t) \le u(x,t) \le u(x-0,t). \qquad (5.16)$$

Hence we calculate from (5.15) and (5.16) that

$$\frac{u(x_2,t) - u(x_1,t)}{x_2 - x_1} \le \frac{u(x_2 - 0,t) - u(x_1 + 0,t)}{x_2 - x_1} \le \frac{e^{-t}}{1 - e^{-t}}, \quad x_1 \ne x_2.$$

This shows that $(\rho(x,t), u(x,t))$ constructed in (5.14) is indeed an entropy solution of the system (3.9). Finally, we point out from (4.4) that

$$\left|x - y_*(x,t)\right| \le M\left(1 - e^{-t}\right)$$

and

$$\left|x - y^*(x,t)\right| \le M\left(1 - e^{-t}\right).$$

Therefore, (5.15) and (5.16) yield that

$$\big\|u(\cdot,t)\big\|_{L^\infty} \le Me^{-t} = \|u_0\|_{L^\infty} e^{-t}.$$

This completes the proof of Theorem 3.1.

6. Uniqueness of an entropy solution

In this section, we study the uniqueness of an entropy solution to the nonhomogeneous system (3.9) following the idea of [19,29]. Our strategy for uniqueness is to verify that any entropy solution (ρ, u) coincides with the standard entropy solution constructed in Theorem 3.1. This naturally leads to the uniqueness Theorem 3.2.

Suppose that (ρ, u) or (m, u) is any entropy solution in the sense of Definitions 3.1. Then we have

$$m_t + u(x,t)m_x = 0, \qquad m_0(x) = \int_0^x \rho_0(\xi)\,d\xi, \tag{6.1}$$

where $u(x,t)$ satisfies

$$\frac{u(x_2,t) - u(x_1,t)}{x_2 - x_1} \le \frac{e^{-t}}{1 - e^{-t}},$$

for any $x_1 < x_2$ and almost all $t > 0$. We set

$$M := \|u_0\|_{L^\infty} \quad \text{and} \quad u_\varepsilon = u * j_\varepsilon, \quad \text{for the standard mollifier } j_\varepsilon.$$

Then, we have

$$|u_\varepsilon| \le M \quad \text{and} \quad u_{\varepsilon x} \le \frac{e^{-t}}{e^t - e^{-t}}.$$

Let $x = X^\varepsilon(\xi, t)$ be the characteristic curve satisfying

$$\frac{dx}{dt} = u_\varepsilon, \quad x(0) = \xi. \tag{6.2}$$

The characteristics $x = X^\varepsilon(\xi, t)$ has already been studied in [42]. We state some properties of them in the following two lemmas.

Lemma 6.1. *(See [42].) Let $X^\varepsilon(\xi, t)$ be a characteristic curve given by (6.2). Then $X^\varepsilon(\xi, t)$ has the following properties.*

(1) *There exist a subsequence $X^{\varepsilon_i}(\xi, t)$ such that*

$$\lim_{\varepsilon_i \to 0} X^{\varepsilon_i}(\xi, t) = X(\xi, t), \quad \text{for all } \xi \text{ and } t.$$

Furthermore, $X(\xi, t)$ is Lipschitz continuous with respect to t and is increasing with respect to ξ.

(2) *If* $X(\xi_1, t_0) = X(\xi_2, t_0)$ *for some* $\xi_1 < \xi_2$ *and* $t_0 > 0$, *then* $X(\xi_1, t) = X(\xi_2, t)$ *for all* $t \geq t_0$.

(3) *Let* $U := \{\xi\colon \exists t > 0,\ s.t.\ X(\xi - 0, t) \neq X(\xi + 0, t)\}$, *then for any* $\xi \in \mathbb{R}/U$ *and almost all* $t > 0$,

$$X'(\xi, t) = u\big(X(\xi, t), t\big), \qquad u\big(X(\xi, t) - 0, t\big) \leq u\big(X(\xi, t), t\big) \leq u\big(X(\xi, t) + 0, t\big),$$

where $'$ *denotes the upper derivative with respect to* t.

(4) *Let* $\xi(x, t) := \sup\{\xi : X(\xi, t) < x\}$, *and* $\eta(x, t) := \inf\{\xi : X(\xi, t) > x\}$, *then the set* $\Gamma := \{(x, t) : \xi(x, t) \neq \eta(x, t), t \geq 0\}$ *consists of at most countable Lipschitz continuous curves. Furthermore,*

$$\xi(x - 0, t) = \xi(x, t), \qquad \eta(x, t) = \xi(x + 0, t) = \eta(x + 0, t),$$
$$X\big(\xi(x, t) - 0, t\big) \leq x \leq X\big(\xi(x, t) + 0, t\big).$$

(5) *For any point* (x_0, t_0), *there exists at least one curve* L' *through* (x_0, t_0) *such that* $\xi(x, t)$ *keeps constant along the curve.*

Below, we first study the uniqueness of the transport equation (6.1), when the transport velocity $u = u(x, t)$ is a given bounded measurable function.

Lemma 6.2. *(See [42].) Suppose that the function* $u(x, t)$ *is a given bounded measurable function. Then the solution to* (6.1) *is unique and it is given by the following formula.*

$$m(x, t) = \int_0^{\xi(x,t)} \rho_0(\xi)\, d\xi.$$

Proof of Theorem 3.2. For a fixed time t, we define

$$C_t := \big\{x;\ (x, t) \in \mathbb{R}/\Gamma,\ X\big(\xi(x, t) - 0, \tau\big) = X\big(\xi(x, t) + 0, \tau\big), 0 \leq \tau \leq t\big\}.$$

It is easy to verify that for any $0 < \tau \leq t$, $\xi(x(\tau), \tau)$ keeps constant along the curve $x(\tau) = X(\xi(x, t), \tau)$. For any $x_1, x_2 \in C_t$ satisfying $x_1 < x_2$, we set

$$\xi(x_1, t) = \xi_1, \qquad \xi(x_2, t) = \xi_2, \qquad x_1(\tau) = X(\xi_1, \tau), \qquad x_2(\tau) = X(\xi_2, \tau), \quad 0 \leq \tau \leq t.$$

Let $\phi_{1\varepsilon}(x), \phi_{2\varepsilon}(x) \in C^\infty(\mathbb{R})$ satisfy

$$\phi_{1\varepsilon}(x) = \begin{cases} 1, & x \leq -\varepsilon, \\ 0, & x \geq 0, \end{cases} \qquad \phi_{2\varepsilon}(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x \geq \varepsilon. \end{cases}$$

For any $\varphi(\tau) \in C_0^\infty[0, t)$, we also set

$$\psi_\varepsilon = \bar{\phi}_\varepsilon(x, \tau)\varphi(\tau), \qquad \bar{\phi}_\varepsilon(x, \tau) = \phi_{2\varepsilon}\big(x - x_2(\tau)\big) - \phi_{1\varepsilon}\big(x - x_1(\tau)\big).$$

Then by (3.5), we have

$$\iint \big(\psi_{\varepsilon\tau} u + \psi_{\varepsilon x} u^2 - u\psi_\varepsilon\big) dm\, dt + \int \psi_\varepsilon(x,0) u_0 \rho_0 d\eta = 0.$$

This yields

$$\iint (\varphi_\tau \bar{\phi}_\varepsilon u - u\varphi\bar{\phi}_\varepsilon) dm d\tau + \int \psi_\varepsilon(x,0) u_0 \rho_0 d\eta$$

$$= -\int_0^t \int_{x_1(\tau)-\varepsilon}^{x_1(\tau)} \varphi\phi'_{1\varepsilon}\big(x'_1(\tau) - u\big) u dm d\tau + \int_0^t \int_{x_2(\tau)}^{x_2(\tau)} +\varepsilon\varphi\phi'_{2\varepsilon}\big(x'_2(\tau) - u\big) u\, dm d\tau. \tag{6.3}$$

By Lemma 6.1(3) and the fact that $x_1, x_2 \in C_t$, we have for almost every $0 < \tau < t$,

$$u\big(x_i(\tau) - 0, \tau\big) = x'_i(\tau) = u\big(x_i(\tau) + 0, \tau\big), \quad i = 1, 2.$$

Thus the R.H.S. of (6.3) tends to zero, when ε tends to zero. We let $\varepsilon \to 0$ in (6.3) to obtain

$$\int_0^t \int_{x_1(\tau)-0}^{x_2(\tau)+0} \varphi_\tau u - u\varphi\, dm d\tau + \int_{\xi_1}^{\xi_2} \varphi(0) u_0 \rho_0 d\eta = 0. \tag{6.4}$$

For any $0 \le s < t$, we choose $0 < \delta < t - s$. Let $\varphi(\tau) = \varphi_\delta(\tau)$ satisfying

$$\varphi_\delta(\tau) = \begin{cases} 1, & 0 \le \tau \le s, \\ 0, & \tau \ge s + \delta. \end{cases} \tag{6.5}$$

We substitute (6.5) into (6.4) and let $\delta \to 0$ obtain

$$\int_{\xi_1}^{\xi_2} \rho_0(\eta) u_0(\eta) d\eta - \int_{\xi_1}^{\xi_2} X'(\eta, s) \rho_0(\eta)\, d\eta - \int_0^s \int_{\xi_1}^{\xi_2} X'(\eta, \tau) \rho_0(\eta) d\eta d\tau = 0,$$

or equivalently,

$$\int_{\xi_1}^{\xi_2} \rho_0(\eta) u_0(\eta) d\eta - \int_{\xi_1}^{\xi_2} X'(\eta, s) \rho_0(\eta)\, d\eta - \int_{\xi_1}^{\xi_2} \big(X(\eta, s) - \eta\big) \rho_0(\eta) d\eta = 0, \tag{6.6}$$

where we have used the fact that for any measurable function f with respect to $\partial_x m$,

$$\int_{x_1(s)-0}^{x_2(s)+0} f(x) dm(x, s) = \int_{\xi_1}^{\xi_2} f\big(X(\eta, s)\big) \rho_0(\eta) d\eta.$$

This is due to Lemma 6.2, $\xi(x_1(\tau), \tau) = \xi_1$ and $\xi(x_2(\tau), \tau) = \xi_2$ for any $0 \le \tau \le t$. We set

$$h(s) := \int_0^s \int_{\xi_1}^{\xi_2} X(\eta, \tau)\rho_0(\eta)\,d\eta\,d\tau.$$

Then, it follows from (6.6) that

$$h''(s) + h'(s) - \int_{\xi_1}^{\xi_2} \rho_0(\eta)\big(u_0(\eta) + \eta\big)\,d\eta = 0,$$

$$h(0) = 0, \qquad h'(0) = \int_{\xi_1}^{\xi_2} \eta\rho_0(\eta)\,d\eta. \tag{6.7}$$

Straightforward computation on (6.7) yields that for any $0 \le s \le t$ and for any $x_1, x_2 \in C_t$ with $x_1 < x_2$,

$$h'(s) = \int_{\xi_1}^{\xi_2} \rho_0(\eta)X(\eta, s)\,d\eta = \int_{\xi_1}^{\xi_2} \rho_0(\eta)\big[\eta + u_0(\eta)\big(1 - e^{-s}\big)\big]\,d\eta,$$

$$h''(s) = \int_{\xi_1}^{\xi_2} \rho_0(\eta)X'(\eta, s)\,d\eta = \int_{\xi_1}^{\xi_2} \rho_0(\eta)u_0(\eta)e^{-s}\,d\eta. \qquad \square \tag{6.8}$$

Furthermore, we have the following lemma.

Lemma 6.3. *For any $x_1 < x_2$, we set*

$$\xi_1 = \xi(x_1, t) \quad \textit{and} \quad \xi_2 = \xi(x_2, t).$$

Then, for any $0 \le s \le t$, we have

$$\int_{\xi_1}^{\xi_2} \rho_0(\eta)X(\eta, s)\,d\eta = \int_{\xi_1}^{\xi_2} \rho_0(\eta)\big[\eta + u_0(\eta)\big(1 - e^{-s}\big)\big]\,d\eta,$$

$$\int_{\xi_1}^{\xi_2} \rho_0(\eta)X'(\eta, s)\,d\eta = \int_{\xi_1}^{\xi_2} \rho_0(\eta)u_0(\eta)e^{-s}\,d\eta. \tag{6.9}$$

Proof. Without loss of generality, we assume that

$$x_1 \in \mathbb{R}/C_t \quad \text{and} \quad x_2 \in C_t,$$

since if $x_1, x_2 \in C_t$ then (6.9) is already proved in (6.8), and if $x_1, x_2 \in \mathbb{R}/C_t$, then (6.9) can be achieved by a similar procedure as in the following. Then, there are two cases.

• Case A ($x_1 = X(\xi_1 - 0, t)$): In this case, we define

$$V_t := \{x; \exists\xi \text{ s.t. } X(\xi - 0, t) < x < X(\xi + 0, t)\}.$$

It is easy to see that V_t consists of at most countable and disjoint intervals. Since in any such interval, $\xi(x, t)$ is a constant, it follows from Lemma 6.2 that for any $x \in V(t)$,

$$\rho(x, t) = m_x(x, t) = 0.$$

By definition of C_t and V_t, the set $\mathbb{R}/(C_t \cup V_t)$ has a measure zero. Thus, there exist a sequence of $x_{1n} \in C_t$ satisfying

$$x_{1n} < x_1 \quad \text{and} \quad x_{1n} \to x_1 \quad \text{as } n \to \infty.$$

We set $\xi_{1n} = \xi(x_{1n}, t)$, then the equality (6.8) yields

$$\int_{\xi_{1n}}^{\xi_2} \rho_0(\eta) X(\eta, s) d\eta = \int_{\xi_{1n}}^{\xi_2} \rho_0(\eta)\big[\eta + u_0(\eta)\big(1 - e^{-s}\big)\big] d\eta.$$

We let $n \to \infty$ and use Lemma 6.1(4) to obtain (6.9). The second equality can be treated similarly.

• Case B ($X(\xi_1 - 0, t) < x_1 \le X(\xi_1 + 0, t)$): We set $\bar{x}_1 = X(\xi_1 + 0, t)$. Then, we have

$$\xi(\bar{x}_1, t) = \xi(x_1, t) = \xi_1.$$

We follows the same argument of Case A to obtain (6.9). □

Now we define

$$q(x, t) := \left(\int_0^{\xi(x,t)} \rho_0(\eta) u_0(\eta) d\eta \right) e^{-t}.$$

Lemma 6.4. *$\partial_x q = u(x, t)\partial_x m$ holds in the sense of Radon–Nykodym in x.*

Proof. We set

$$\Gamma_t := \{x; (x, t) \in \Gamma\} \quad \text{and} \quad A = C_t \cup V_t \cup \Gamma_t.$$

Then, the set $\mathbb{R}/A$ consists of at most countably many points where $m(x, t)$ is continuous, and note that ρ is vacuum in V_t. Therefore we only need to show Lemma 6.4 holds in the set Γ_t everywhere and the set C_t almost everywhere. Let $x \in C_t$ satisfying

$$\lim_{\varepsilon \to 0} \frac{\int_{\xi-\varepsilon}^{\xi+\varepsilon} \rho_0 u_0 d\eta}{\int_{\xi-\varepsilon}^{\xi+\varepsilon} \rho_0 d\eta} = u_0(\xi), \quad \xi = \xi(x, t).$$

It is easy to check that such points are dense in C_t. Then there exist two sequences of $x_{1n}, x_{2n} \in C_t$ satisfying $x \in (x_{1n}, x_{2n})$, $x_{1n} \to x_1$ and $x_{2n} \to x_2$. From Lemma 6.3, we have

$$\int_{\xi(x_{1n},t)}^{\xi(x_{2n},t)} \rho_0(\eta) X'(\eta, s) d\eta = \int_{\xi(x_{1n},t)}^{\xi(x_{2n},t)} \rho_0(\eta) u_0(\eta) e^{-s} d\eta.$$

We let $n \to \infty$ obtain

$$X'(\xi, s) = u_0(\xi) e^{-s}, \quad 0 \le s \le t.$$

For $s = t$, we have

$$q_x = u(x,t) m_x, \quad u(x,t) = X'(\xi, t) = u_0(\xi) e^{-t}.$$

We use the same argument to obtain that for any $x \in \Gamma_t$ and $0 \le s \le t$,

$$\int_{\xi(x,t)}^{\eta(x,t)} \rho_0(\eta) X'(\eta, s) d\eta = \int_{\xi(x,t)}^{\eta(x,t)} \rho_0(\eta) u_0(\eta) e^{-s} d\eta.$$

For $s = t$, we use Lemma 6.1 to see

$$X'(\eta, t) = u(x,t) = \int_{\xi(x,t)}^{\eta(x,t)} \rho_0 u_0 e^{-t} d\eta \quad \text{holds for any } \eta \in \big(\xi(x,t), \eta(x,t)\big).$$

This yields

$$\partial_x q = u(x,t) \partial_x m. \qquad \square$$

From Lemma 6.4, Eq. (6.1) yields a conservation law:

$$\partial_t m + \partial_x q = 0.$$

Note that m and q are left continuous, we introduce the following generalized potential $\Phi(x,t)$:

$$\Phi(x,t) = \oint_{(0,0)}^{(x,t)} m(x,t) dx - q(x,t) dt.$$

Obviously, the potential $\Phi(x,t)$ is independent of the integral path and the relations:

$$\partial_x \Phi = m, \qquad \partial_t \Phi = -q \quad \text{in the sense of distributions.}$$

Suppose that

$$\Phi(x,t) = -\min_y F(y;x,t), \tag{6.10}$$

where $F(y;x,t)$ is a functional introduced in (3.11). Since $F(y;x,t)$ depends only on the initial data, the relation (6.10) implies our second result Theorem 3.2. In the sequel, we will establish the relation (6.10) by exploiting two integral paths. We first present a lemma which will be used in the verification of the relation (6.10).

Lemma 6.5. *Suppose that*

$$x(\tau) = y_0 + v_0\big(1-e^{-\tau}\big) \quad \textit{and} \quad y(\tau) = \xi\big(x(\tau),\tau\big), \quad 0 \le \tau \le t.$$

Then for any $0 < \tau < t$, *we have*

$$\int_{y_0}^{y(\tau)} \rho_0(\eta)\,d\eta\, x'(\tau) \ge \int_{y_0}^{y(\tau)} \rho_0(\eta) u_0(\eta) e^{-\tau}\,d\eta. \tag{6.11}$$

Proof. Since $x'(\tau) = v_0 e^{-\tau}$, to prove (6.11), it is sufficient to prove that

$$\int_{y_0}^{y(\tau)} \rho_0(\eta)\big[v_0 - u_0(\eta)\big]\,d\eta \ge 0.$$

By definition, it holds that $v_0 = \frac{x(\tau)-y_0}{1-e^{-\tau}}$. Then, it is sufficient to prove

$$\int_{y_0}^{y(\tau)} \rho_0(\eta)\big[x(\tau) - y_0 - u_0(\eta)\big(1-e^{-\tau}\big)\big]\,d\eta \ge 0. \tag{6.12}$$

By the first equality in (6.9), we have

$$\int_{y_0}^{y(\tau)} \rho_0(\eta) X(\eta,\tau)\,d\eta = \int_{y_0}^{y(\tau)} \rho_0(\eta)\big[\eta + u_0(\eta)\big(1-e^{-\tau}\big)\big]\,d\eta.$$

Therefore, to prove (6.12), it suffices to obtain

$$\int_{y_0}^{y(\tau)} \rho_0(\eta)\big[x(\tau) - X(\eta,\tau)\big]\,d\eta + \int_{y_0}^{y(\tau)} \rho_0(\eta)(\eta - y_0)\,d\eta \ge 0. \tag{6.13}$$

It is obvious that the second term in (6.13) is non-negative. By the facts that

$$X\big(y(\tau)-0,\tau\big) \le x(\tau) \le X\big(y(\tau)+0,\tau\big)$$

in Lemma 6.1(4) and $X(\eta,\tau)$ is increasing with respect to η in Lemma 6.1(1), we can see that the first term in (6.13) is also non-negative. This completes the proof of (6.11). □

Proof of (6.10). For any point $(x_0,t_0)\in\mathbb{R}^2_+$, there exists at least one curve L' through (x_0,t_0) and $(\xi(x_0,t_0),0)$ such that $\xi(x,t)$ keeps constant along L' due to Lemma 6.1(5). The first integral path is chosen from $(0,0)$ to $(\xi(x_0,t_0),0)$ along x-axis and then from $(\xi(x_0,t_0),0)$ to (x_0,t_0) along the curve L'. We set $\xi_0=\xi(x_0,t_0)$, and compute

$$\begin{aligned}\Phi(x_0,t_0)&=\int_0^{\xi_0} m_0(\eta)d\eta+\int_{L'} m(x,t)dx-q(x,t)dt\\&=\int_0^{\xi_0}(\xi_0-\eta)\rho_0 d\eta-\int_0^{t_0}\int_0^{\xi_0}\rho_0(\eta)u_0(\eta)e^{-s}d\eta ds+(x_0-\xi_0)\int_0^{\xi_0}\rho_0 d\eta\\&=\int_0^{\xi_0}\rho_0(\eta)\big[x_0-\eta-u_0(\eta)\big(1-e^{-t_0}\big)\big]d\eta.\end{aligned}\tag{6.14}$$

On the other hand, we choose another integral path which is from $(0,0)$ to $(y_0,0)$ along the x-axis and from $(y_0,0)$ to (x_0,t_0) along the curve L:

$$L: x=x(t)=y_0+v_0\big(1-e^{-t}\big),\quad 0\le t\le t_0,$$

where $v_0=\frac{x_0-y_0}{1-e^{-t_0}}$ and y_0 is any constant. Then we have

$$\begin{aligned}\Phi(x_0,t_0)&=\int_0^{y_0} m_0(\eta)d\eta+\int_L m dx-q dt\\&=\int_0^{y_0} m_0(\eta)d\eta+\int_0^{t_0} x'(t)m\big(x(t),t\big)-q\big(x(t),t\big)dt.\end{aligned}\tag{6.15}$$

For the estimate of the last term in (6.15), we use Lemma 6.5 to obtain

$$\begin{aligned}&x'(t)m\big(x(t),t\big)-q\big(x(t),t\big)\\&\quad=\int_0^{y(t)}\rho_0(\eta)\big(x'(t)-u_0(\eta)e^{-t}\big)d\eta\ge\int_0^{y_0}\rho_0\big(x'(t)-u_0(\eta)e^{-t}\big)d\eta.\end{aligned}\tag{6.16}$$

We substitute (6.16) into (6.15) to obtain

$$\Phi(x_0,t_0) \geq \int_0^{y_0} \rho_0(y_0-\eta)d\eta + \int_0^{y_0}\int_0^{t_0} \rho_0\big[x'(t) - u_0(\eta)e^{-t}\big]d\eta dt$$
$$= \int_0^{y_0} \rho_0(\eta)\big(x_0 - \eta - u_0(\eta)\big(1-e^{-t_0}\big)\big)d\eta.$$

Since y_0 is arbitrary, we also have

$$\Phi(x_0,t_0) \geq \max_y \int_0^{y} \rho_0(\eta)\big[x_0 - \eta - u_0(\eta)\big(1-e^{-t_0}\big)\big]d\eta. \tag{6.17}$$

We combine (6.14) and (6.17) to obtain

$$\Phi(x_0,t_0) = \max_y \int_0^{y} \rho_0\big(x_0 - \eta - u_0\big(1-e^{-t_0}\big)\big)d\eta$$
$$= -\min_y F(y;x_0,t_0). \tag{6.18}$$

On the other hand, it follows from Theorem 3.1 that we can construct an entropy solution (m_s, u_s) satisfying $m_s = \partial_x \Phi$, $q_s = -\partial_t \Phi$, where $\partial_x(q_s) = u_s \partial_x m_s$. Hence, by (6.18) we have

$$m = m_s, \qquad q = q_s \quad \text{a.e. in } t > 0.$$

This yields the proof of Theorem 3.2. □

7. Conclusion

In this paper, we provide a unique solvability and flocking estimate on entropic weak solutions to the one-dimensional pressureless Euler system with a flocking dissipation. The pressureless Euler system with a flocking dissipation naturally appears as a quasi-equilibrium hydrodynamic model for the Cucker–Smale particles. So far, most studies on the pressureless gas dynamics are restricted on the homogeneous case. The pressureless Euler system has been extensively studied in previous literature due to the novel phenomenon such as the finite-time formation of δ-shock wave and its application in the formation of large-scale structure of universe in astrophysics. When the communication weight between self-propelled C–S particles is constant (i.e., all-to-all interaction), a quasi-equilibrium hydrodynamic model becomes the pressureless Euler system with a damping. For the unique solvability of an entropic weak solution to the system (3.1) with a constant communication weight, we employed the variational approach from Ding and Huang [19]. Unfortunately, it seems that the variational approach in [19] works only for the constant communication weight as it is. Hence, the challenging remaining case will be the non-constant communication weight, for example, algebraically decaying communication weights used by Cucker–Smale in their seminal work [16]. Thus, we leave this interesting case as a future work.

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Appendix A. Proof of Lemma 4.5

In this section, we provide the proof of Lemma 4.5.

Since the proof is rather long and complicated, we splits its proof into several steps. Let (x_0, t_0), $t_0 > 0$ be given.

• Step A: For each $t_1 > t_0$, we claim that the time-strip

$$\Lambda(t_1) := \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ \colon -\infty < x < \infty,\ 0 < t < t_1\}$$

is completely covered by disjoint characteristic regions:

$$\Lambda(t_1) = \dot{\bigcup_{-\infty < x < \infty}} \Delta(x, t_1). \tag{A.1}$$

The proof of (A.1). Since $\dot{\bigcup}_{-\infty<x<\infty} \Delta(x,t_1) \subset \Lambda(t_1)$ is obvious, it suffices to show that $\Lambda(t_1) \subset \dot{\bigcup}_{-\infty<x<\infty} \Delta(x,t_1)$, i.e., for any (x_0,t_0), $t_0 \le t_1$, there exists a characteristic region $\Lambda(x,t_1)$ such that

$$(x_0, t_0) \in \Lambda(x, t_1), \quad \text{for some } x \in \mathbb{R}.$$

For this, we first note that for any line $t = t_1 > t_0$, it follows from Lemma 4.2 that

$$\Delta(x_1, t_1) \cap \Delta(x_2, t_2) = \phi, \quad x_1 < x_2.$$

We set

$$\mathcal{L}(x_0,t_0) := \{x \in \mathbb{R} \colon\ \Delta(x,t_1) \text{ is located at the left side of } (x_0,t_0)\},$$
$$\mathcal{R}(x_0,t_0) := \{x \in \mathbb{R} \colon\ \Delta(x,t_1) \text{ is located at the right side of } (x_0,t_0)\},$$
$$x_m(x_0,t_0) := \sup \mathcal{L}(x_0,t_0), \qquad x_i(x_0,t_0) := \inf \mathcal{R}(x_0,t_0).$$

Then, we have

$$x_m(x_0,t_0) \le x_i(x_0,t_0).$$

We next show that

$$x_m(x_0,t_0) = x_i(x_0,t_0).$$

Suppose not, then, there exists $x_* \in (x_m, x_i)$ such that

$$(x_0,t_0) \in \Delta(x_*, t_1).$$

Since characteristic regions are disjoint, there exists $x_{**} \in (x_*, x_i)$ such that

$$\Delta(x_{**}, t_1) \text{ is located at the right side of } (x_0, t_0), \quad \text{i.e., } x_{**} \in \mathcal{R}(x_0, t_0).$$

However this contradicts to the fact that $x_{**} < x_i(x_0, t_0) := \inf \mathcal{R}(x_0, t_0)$. Hence we set

$$\bar{x} := x_m(x_0, t_0) = x_i(x_0, t_0).$$

⋄ Case A.1: If $(x_0, t_0) \in \Delta(\bar{x}, t_1)$, then we are done.
⋄ Case A.2: If not, without loss of generality, we consider the case that $\Delta(\bar{x}, t_1)$ is located on the left side of (x_0, t_0). By (4.8), there is a positive constant δ such that $y^*(\bar{x}, t_1) + \delta < y_*(y, t_1)$ holds for any $y \in \mathcal{R}(x_0, t_0)$. On the other hand, Lemma 4.3 infers that $\lim_{y \to \bar{x}} y_*(y, t_1) = y^*(\bar{x}, t_1)$. This is a contradiction. □

• Step B (defining the curve $x = x(t)$, $t \geq t_0$): In terms of the above argument, there is a unique point $x = x(t_1)$ on the line $t = t_1$ such that the point $(x_0, t_0) \in \Delta(x(t_1), t_1)$. Since $t_1 > t_0$ is arbitrary, we obtain a curve $x(t)$ with $x_0 = x(t_0)$ in $t > t_0$. See Fig. 1. Furthermore, $x(t)$ is continuous due to (4.8) and Lemmas 4.2 and 4.3.
• Step C (verification of (4.9)): Let $x' = x(t')$, $x'' = x(t'')$, $t'' > t' > t$ and

$$y_{\prime} = y_*(x', t'), \qquad y' = y^*(x', t'), \qquad y_{\prime\prime} = y_*(x'', t''), \qquad y'' = y^*(x'', t'').$$

Then Lemmas 4.2 and 4.3 imply

$$y_{\prime\prime} \leq y_{\prime} \leq y_* \leq y^* \leq y' \leq y'' \quad \text{and} \quad y_{\prime\prime} \to y_*, \quad y'' \to y^* \quad \text{as} \quad t'' \to t.$$

⋄ Case C.1 ($y_* = y^*$): We set

$$x_1 := \frac{1 - e^{-t'}}{1 - e^{-t''}} x'' + y_{\prime\prime} \frac{e^{-t'} - e^{-t''}}{1 - e^{-t''}}$$

Then (x_1, t') is located on the left backward characteristic belonging to (x'', t'') and satisfies $x_1 < x'$. Note that

$$\begin{aligned} \frac{x'' - x'}{t'' - t'} \leq \frac{x'' - x_1}{t'' - t'} &\leq \frac{x''}{1 - e^{-t''}} \frac{e^{-t'} - e^{-t''}}{t'' - t'} - \frac{y_{\prime\prime}}{1 - e^{-t''}} \frac{e^{-t'} - e^{-t''}}{t'' - t'} \\ &= \frac{x'' - y_{\prime\prime}}{1 - e^{-t''}} \frac{e^{-t'} - e^{-t''}}{t'' - t'}. \end{aligned} \tag{A.2}$$

Let $t'', t' \to t + 0$, we get the RHS of (A.2) converges to $\frac{e^{-t}}{1 - e^{-t}}(x(t) - y_*)$. Similarly, we get

$$\lim_{t'', t' \to t+0} \frac{x'' - x'}{t'' - t'} \geq \frac{e^{-t}}{1 - e^{-t}} \big(x(t) - y_*\big). \tag{A.3}$$

⋄ Case C.2 ($y_* < y^*$): It follows from definition of y_* and y^* that we have

$$F\big(y''; x'', t''\big) + F\big(y_{\prime}; x', t'\big) \leq F\big(y''; x', t'\big) + F\big(y_{\prime}; x'', t''\big),$$

then, we have

$$F\big(y''; x'', t''\big) - F\big(y_{\prime}; x'', t''\big) \le F\big(y''; x', t'\big) - F\big(y_{\prime}; x', t'\big).$$

That is,

$$\int_{y_{\prime}}^{y''} \rho_0(\eta)\big(\eta + u_0(\eta)\big(1 - e^{-t''}\big) - x''\big)\, d\eta \le \int_{y_{\prime}}^{y''} \rho_0(\eta)\big(\eta + u_0(\eta)(\eta)\big(1 - e^{-t'}\big) - x'\big)\, d\eta.$$

This yields

$$\frac{x'' - x'}{t'' - t'} \int_{y_{\prime}}^{y''} \rho_0(\eta) d\eta \ge \left(\int_{y_{\prime}}^{y''} \rho_0(\eta) u_0(\eta) d\eta\right) \frac{e^{-t'} - e^{-t''}}{t'' - t'}. \tag{A.4}$$

Similarly, it follows from the inequality

$$F\big(y_{\prime\prime}; x'', t''\big) - F\big(y'; x'', t''\big) \le F\big(y_{\prime\prime}; x', t'\big) - F\big(y'; x', t'\big),$$

that we have

$$\frac{x'' - x'}{t'' - t'} \int_{y_{\prime\prime}}^{y'} \rho_0(\eta) d\eta \le \left(\int_{y_{\prime\prime}}^{y'} \rho_0(\eta) u_0(\eta) d\eta\right) \frac{e^{-t'} - e^{-t''}}{t'' - t'}. \tag{A.5}$$

We let $t'', t' \to t + 0$ in (A.4) and (A.5) to obtain (4.9) for the case $y_* < y^*$. Therefore Lemma 4.5 is proved.

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