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Global existence of weak solution to the heat and moisture transport system in fibrous porous media

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ABSTRACT

This paper is concerned with theoretical analysis of a heat and moisture transfer model arising from textile industries, which is described by a degenerate and strongly coupled parabolic system. We prove the global (in time) existence of weak solution by constructing an approximate solution with some standard smoothing. The proof is based on the physical nature of gas convection, in which the heat (energy) flux in convection is determined by the mass (vapor) flux in convection.

1. Introduction

Mathematical modeling for heat and moisture transport with phase change in porous textile materials was studied by many authors, e.g. see [5,6,11,14,18,22,24]. A typical application of these models is a clothing assembly, consisting of a thick porous fibrous batting sandwiched by two thin fabrics. The outside cover of the assembly is exposed to a cold environment with fixed temperature and relative humidity while the inside cover is exposed to a mixture of air and vapor at higher temperature and relative humidity. In general, the physical process can be viewed as a multiphase and single (or multi) component flow. In this process, the water vapor moves through the clothing assembly by convection which is induced by the pressure gradient. The heat is transferred by conduction in all phases (liquid, fiber and gas) and convection in gas. Phase changes occur in the form of evapora-

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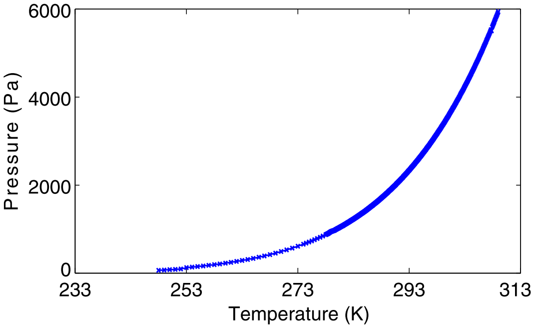

Fig. 1. Experimental measurements of saturation pressure [4].

tion/condensation and/or sublimation. Based on the conservation of mass and energy and the neglect of the water influence, the model can be described by

$$\frac{\partial}{\partial t}(\epsilon C_v) + \frac{\partial}{\partial x}(u\epsilon C_v) = -\Gamma_{ce}, \tag{1.1}$$

$$\frac{\partial}{\partial t}\big(\epsilon C_{vg} C_v T + (1-\epsilon) C_{vs} T\big) + \frac{\partial}{\partial x}(\epsilon C_{vg} u C_v T) = \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) + \lambda \Gamma_{ce}. \tag{1.2}$$

Here C_v is the vapor concentration (mol/m^3), T the temperature (K), ϵ the porosity of the fiber, and λ the latent heat of evaporation/condensation in the wet zone while in frozen zone, it represents the latent heat of sublimation. C_{vg} and C_{vs} are the heat capacities of the gas and mixture solid, respectively.

The evaporation/condensation (molar) rate of phase change per unit volume is defined by the Hertz–Knudsen equation [12]

$$\Gamma_{ce} = -\frac{E}{R_f}\sqrt{\frac{(1-\epsilon)(1-\epsilon')}{2\pi RM}}\left(\frac{P_{\text{sat}}(T) - P}{\sqrt{T}}\right) \tag{1.3}$$

where R is the universal gas constant, R_f is the radius of fibre, M the molecular weight of water and E is the nondimensional phase change coefficient. The vapor pressure is given by $P = RC_vT$ because of the ideal gases' assumption. The saturation pressure P_{sat} is determined from experimental measurements, see Fig. 1.

The vapor velocity (volumetric discharge) is given by the Darcy's law

$$u = -\frac{k}{\mu_g}\frac{\partial P}{\partial x} \tag{1.4}$$

where k and μ_g are the permeability and viscosity of the vapor, respectively. For compressible flows, μ_g is concentration-dependent with different forms in different applications, such as a linear form of $\mu_g := \mu C_v$ and a constant [1,6,9]. More general form can be found in [8,17]. Here we assume that μ_g is a constant. The extension to some other cases is straightforward.

Numerical methods and simulations for the heat and moisture transport in porous textile materials have been studied by many authors with various applications [2,19,22]. However, no theoretical analysis has been explored for the above system of nonlinear equations. A simple heat and moisture model was studied in [20], where the model was described by a pure diffusion process (without convection and condensation) with a non-symmetric parabolic part. There are several related porous media flow problems from other physical applications. A popular one is a compressible (or incompressible) flow in porous media with applications in oil and underground water industries, which is described by an elliptic pressure equation coupled with a parabolic concentration equation for incompressible case and a system of parabolic equations for compressible case. The existence of weak solution for the incompressible and compressible flows has been studied in [7,17] and [1,8,9], respectively. However, most of these works focus on isothermal case due to the nature of these applications, while both temperature and phase change (condensation/evaporation) play important roles in the textile model. Analysis for certain nonisothermal and incompressible flows in porous media was presented in [3,25], where the fluid flows were described by a heat (temperature) equation coupled with an elliptic pressure equation (when the Darcy's law is used).

For the textile model, the water content in the batting area usually is relative small and one often assumes that all these physical parameters involved in the system (1.1)–(1.2) are positive constants. With nondimensionalization, the system (1.1)–(1.2) reduces to

$$\begin{cases} \rho_t - \big((\rho\theta)_x\rho\big)_x = -\Gamma(\rho,\theta), \\ (\rho\theta)_t + \sigma\theta_t - \big((\rho\theta)_x\rho\theta\big)_x - \big(\kappa(\rho)\theta_x\big)_x = \lambda\Gamma(\rho,\theta), \end{cases} \tag{1.5}$$

for $x \in (0,1)$, $t > 0$, where $(\cdot)_\mu = \frac{\partial}{\partial\mu}$ for $\mu = x, t$, $\rho = \rho(x,t)$ and $\theta = \theta(x,t)$ represent the density of vapor and the temperature, respectively,

$$\Gamma(\rho,\theta) = \rho\theta^{1/2} - p_s(\theta)$$

and $p_s(\theta) \sim P_{\mathrm{sat}}(\theta)/\theta^{1/2}$, σ and λ are given positive constants and $\kappa(\rho) = \kappa_1 + \kappa_2\rho^2$ is the heat conductivity coefficient with κ_i $(i = 1,2)$ being positive constants. A more general form of $\kappa(\rho)$ can be found in [21]. We consider a class of commonly used Robin type boundary conditions [5,6,11,23] defined by

$$(\rho\theta)_x\rho|_{x=1} = \alpha^1\big(\bar{\rho}^1 - \rho(1,t)\big), \qquad (\rho\theta)_x\rho|_{x=0} = \alpha^0\big(\rho(0,t) - \bar{\rho}^0\big), \tag{1.6}$$

and

$$\kappa(\rho)\theta_x|_{x=1} = \beta^1\big(\bar{\theta}^1 - \theta(1,t)\big), \qquad \kappa(\rho)\theta_x|_{x=0} = \beta^0\big(\theta(0,t) - \bar{\theta}^0\big), \tag{1.7}$$

and the initial condition is

$$\rho(x,0) = \rho_0(x), \qquad \theta(x,0) = \theta_0(x), \quad x \in (0,1), \tag{1.8}$$

where α^0, α^1 represent the mass transfer coefficients, $\bar{\rho}^0, \bar{\rho}^1$ are the density of the gas in the inner background and outer background, respectively, β^0, β^1 the heat transfer coefficients, and $\bar{\theta}^0, \bar{\theta}^1$ the inner and outer background temperatures. We assume that all the parameters above are positive constants.

Based on the experimental data in Fig. 1, we assume that p_s is a smooth, increasing and nonnegative function defined on $\mathbb{R}^+$ which satisfies

$$\lim_{\theta\to 0}\frac{p_s(\theta)}{\theta} = 0, \qquad \lim_{\theta\to\infty}\frac{p_s(\theta)}{\theta^{1+\eta}} = \infty \tag{1.9}$$

for some $\eta > 0$. For physical reasons, we set $p_s(\theta) = 0$ for $\theta \leqslant 0$.

The objective of this paper is to establish the global existence of weak solution to the initial–boundary value problem (1.5)–(1.8) under the general physical hypotheses (1.9) for the saturation pressure function Γ. The difficulty lies on the strong nonlinearity and the coupling of equations. To the best of our knowledge, there are no theoretical results for the underlying model. More important is its significant applications in textile industries. Also analysis presented in this paper may provide a fundamental tool for theoretical analysis of existing numerical methods. Our proof is based on an energy method and the equivalence of mass and heat transfer in convection.

2. The main result

Before we present our main result, we introduce some notations. Let T be a given positive number in the following sections. We define

$$\Omega=(0,1),\qquad I=(0,T],\qquad Q_t=\Omega\times(0,t],\qquad Q_T=\Omega\times I,$$
$$W_2^{2,1}(Q_T)=\left\{f\in L^2(Q_T)\;\middle|\; f_t, f_x, f_{xx}\in L^2(Q_T)\right\}.$$

Let $\mathcal{D}(\overline{\Omega}\times[0,T))$ be the subspace of $C^\infty(\mathbb{R}^2)$ consisting of functions which have compact support in $\mathbb{R}\times[-\infty,T)$, restricted to $\overline{\Omega}\times[0,T)$.

Now we give the definition of weak solution to the system (1.5)–(1.8) and then, state our main result.

Definition 2.1 (*Weak solution*). We say that the measurable function pair (ρ,θ) defined on $\overline{\Omega}\times[0,T)$ is a global weak solution to (1.5)–(1.8) if $(\rho,\theta)\in(L^2(I;H^1(\Omega)))^2$ and the density ρ and the temperature θ are nonnegative functions satisfying

$$\int_0^T \alpha^0\big(\rho(0,t)-\overline{\rho}^0\big)\phi(0,t)\,dt+\int_0^T \alpha^1\big(\rho(1,t)-\overline{\rho}^1\big)\phi(1,t)\,dt$$
$$+\int_0^T\int_\Omega\big(-\rho\phi_t+(\rho\theta)_x\rho\phi_x+\Gamma\phi\big)\,dx\,dt=\int_\Omega\rho_0\phi_0\,dx \tag{2.1}$$

and

$$\int_0^T\big[\alpha^0\big(\rho(0,t)-\overline{\rho}^0\big)\theta(0,t)+\beta^0\big(\theta(0,t)-\overline{\theta}^0\big)\big]\psi(0,t)\,dt$$
$$+\int_0^T\big[\alpha^1\big(\rho(1,t)-\overline{\rho}^1\big)\theta(1,t)+\beta^1\big(\theta(1,t)-\overline{\theta}^1\big)\big]\psi(1,t)\,dt$$
$$+\int_0^\infty\int_\Omega\big[-(\rho\theta+\sigma\theta)\psi_t+(\rho\theta)_x\rho\theta\psi_x+\kappa\theta_x\psi_x-\lambda\Gamma\psi\big]\,dx\,dt$$
$$=\int_\Omega(\rho_0\theta_0+\sigma\theta_0)\psi_0\,dx \tag{2.2}$$

for any test functions $\phi,\psi\in\mathcal{D}(\overline{\Omega}\times[0,T))$.

Theorem 2.1. *If the initial data (ρ_0,θ_0) satisfies $\rho_0\in L^{1+\gamma}(\Omega)$ $(\forall\gamma>0)$, $\theta_0\in L^\infty(\Omega)$ and $\rho_0\geqslant 0$, $\theta_0\geqslant\underline{\theta}$ for some positive constant $\underline{\theta}$, then there exists a global weak solution (ρ,θ), in the sense of Definition* 2.1, *to the initial–boundary value problem* (1.5)–(1.8) *such that*

$$\rho\ln\rho\in L^\infty\big(0,T;L^1(\Omega)\big),\qquad \rho\in L^4(Q_T),\qquad \rho_x\in L^2(Q_T);$$
$$\theta,\theta^{-1}\in L^\infty(Q_T),\qquad (1+\rho)\theta_x\in L^2(Q_T). \tag{2.3}$$

Remark 1. Theorem 2.1 shows the global-in-time existence of weak solutions to (1.5)–(1.8) in the sense of Definition 2.1. However, the uniqueness of this kind of weak solutions is still open due to the degeneracy and strong coupling of the system and will be investigated in the future.

Remark 2. We shall present the proof of Theorem 2.1 in Sections 3–4 by an energy method. Here, we formally derive some energy estimates from which one can see clearly that the solution space presented in (2.3) is reasonable.

Adding Eq. $(1.5)_1$ multiplying by λ into $(1.5)_2$ and then, integrating the resulting equation over Q_t, we get

$$\int_0^1(\lambda\rho+\rho\theta+\sigma\theta)(x,t)\,dx-\int_0^t\big[(\lambda+\theta)\rho(\rho\theta)_x+\kappa\theta_x\big]\big|_{x=0}^{x=1}\,d\tau$$
$$\leqslant\int_0^1(\lambda\rho_0+\rho_0\theta_0+\sigma\theta_0)(x)\,dx \tag{2.4}$$

which with boundary conditions in (1.6)–(1.7) produces a basic energy estimate

$$\int_0^1(\lambda\rho+\rho\theta+\sigma\theta)(x,t)\,dx\leqslant C_T+C\int_0^t\big\|\theta(\cdot,\tau)\big\|_{C(\overline{\Omega})}\,d\tau\leqslant C_T,$$

where C_T is a positive constant depending on T, the initial values and the constants in boundary conditions.

Moreover, subtracting Eq. $(1.5)_2$ times $(l+1)\theta^l$ from Eq. $(1.5)_1$ times $l\theta^{l+1}$ and then, integrating the resulting equation over Q_t with the boundary conditions (1.6)–(1.7), we arrive at

$$\int_0^1(\rho+\sigma)\theta^{l+1}(x,t)\,dx-\int_0^t H_3(x,\tau)\big|_{x=0}^{x=1}\,d\tau+\int_0^t\int_0^1\kappa l(l+1)\theta^{l-1}|\theta_x|^2\,dx\,d\tau$$
$$+(l+1)\int_0^t\int_0^1(\lambda+\theta)p_s(\theta)\theta^l\,dx\,d\tau$$
$$=\int_0^1(\rho_0+\sigma)(\theta_0)^{l+1}(x)\,dx+\int_0^t\int_0^1\big[l\theta^{l+1}+\lambda(l+1)\theta^l\big]\rho\sqrt{\theta}\,dx\,d\tau$$
$$+\int_0^t\int_0^1 p_s(\theta)\theta^{l+1}\,dx\,d\tau. \tag{2.5}$$

In terms of the above inequality and some classical inequalities, we can obtain the following estimate for the temperature θ

$$\int_0^1 (\rho+\sigma)\theta^{l+1}(x,t)\,dx + l(l+1)\int_0^t\int_0^1 \kappa\theta^{l-1}|\theta_x|^2\,dx\,d\tau + l\int_0^t\int_0^1 (\lambda+\theta)p_s(\theta)\theta^l\,dx\,d\tau \leqslant C_T^{l+1}$$

for any $l \geqslant 0$, where C_T is independent of l, which implies

$$\|\theta\|_{L^\infty(Q_T)},\ \|\theta_x\|_{L^2(Q_T)},\ \|\rho\theta_x\|_{L^2(Q_T)} \leqslant C_T.$$

By applying the maximum principle to the temperature equation, we can also obtain a lower bound for θ so that

$$\left\|\theta^{-1}\right\|_{L^\infty(Q_T)} \leqslant C_T.$$

Furthermore, multiplying Eq. $(1.5)_1$ by $\ln\rho$, integrating the resulting equation over Q_t and using the boundary conditions (1.6)–(1.7), we can obtain

$$\int\limits_{[0,1]\cap\{\rho\geqslant 1\}} \rho\ln\rho(x,t)\,dx + \frac{1}{2}\int_0^t\int_0^1 \theta|\rho_x|^2\,dx\,d\tau + \iint\limits_{[0,1]\times[0,t]\cap\{\rho\geqslant 1\}} \rho\ln\rho\,dx\,d\tau \leqslant C_T. \tag{2.6}$$

From (2.4)–(2.6), one can obtain these desired estimates for the solution space presented in (2.3).

Remark 3. In Theorem 2.1, we assume that the initial density satisfies $\rho_0 \in L^{1+\gamma}(\Omega)$ for some $\gamma > 0$ due to some technical reason in our approximation procedure. A more reasonable condition on the initial density should be $\rho\ln\rho \in L^1(\Omega)$, but this requires a more complicated regularization procedure.

In the following sections, we denote by $C_{p_1,p_2,\ldots,p_k}$ a generic positive constant, which depends solely upon $p_1, p_2, \ldots, p_k$, the physical parameters κ_1, κ_2, σ and λ and the parameters involved in initial and boundary conditions. In addition, we denote by $C(p_1, p_2, \ldots, p_k)$ a generic positive function, dependent upon the physical parameters κ_1, κ_2, σ and λ and the parameters involved in boundary conditions, which is bounded when $p_1, p_2, \ldots, p_k$ are bounded.

3. Construction of approximate solutions

Throughout this section, we let ε be a fixed positive number which satisfies

$$0 < \varepsilon \leqslant \min\{\overline{\rho}^0, \overline{\rho}^1, \overline{\theta}^0, \overline{\theta}^1, 1\},$$

and $0 < \nu < \varepsilon$. To prove the existence of global weak solutions to the system (1.5)–(1.8), we introduce a regularized approximate system as follows:

$$\begin{aligned}
&\rho_t - \big(\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x\big)_x - \big(\rho(\rho_\varepsilon\theta_x)_\varepsilon\big)_x = -\rho\chi^\varepsilon(\sqrt{\theta}) + \chi^\varepsilon\big(p_s(\theta)\big),\\
&(\rho\theta+\sigma\theta)_t - \big(\kappa^\varepsilon\theta_x\big)_x - \big(\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x\theta\big)_x - \big(\rho(\rho_\varepsilon\theta_x)_\varepsilon\theta\big)_x\\
&\quad = \lambda\rho\chi^\varepsilon(\sqrt{\theta}) - \lambda\chi^\varepsilon\big(p_s(\theta)\big) + (\lambda+\theta)\big(\chi^\varepsilon\big(p_s(\theta)\big) - p_s(\theta)\big), \quad \text{in } Q_T,
\end{aligned} \tag{3.1}$$

where χ^ε is a cut-off function defined by

$$\chi^\varepsilon(h) = \begin{cases} h & \text{if } |h| \leqslant \varepsilon^{-1}, \\ \operatorname{sign}(h)\varepsilon^{-1} & \text{if } |h| \geqslant \varepsilon^{-1}, \end{cases}$$

and

$$\kappa^\varepsilon = \kappa_1 + \kappa_2(\rho_\varepsilon)^2,$$

and the subscriptions ε, ν define the smoothing operators in general by $f_\mu = \operatorname{Ext}(f) * \eta_\mu$ with $\mu = \nu, \varepsilon$. Here η_μ is the standard mollifier and $\operatorname{Ext}(\cdot)$ is the extension operator which extends any measurable functions defined on Ω_T to be zero on $\mathbb{R}^2 \setminus \Omega_T$.

Remark 4. We give a further remark on the parameters ε, ν and the regularizations introduced in (3.1). In order to treat the degenerate diffusion coefficient $\rho\theta$ of Eq. $(3.1)_1$ due to the possible appearance of the vacuum state, i.e. $\rho = 0$, we add an artificial viscosity term $-\varepsilon\rho_{xx}$. We expect that the term $\rho\theta$ is nonnegative so that $\varepsilon + (\rho\theta)_\nu$ is uniformly positive. Also we introduce a regularization of $\rho\theta$ by $(\rho\theta)_\nu$ such that the coefficient is smooth and the constructed approximate solution has a higher regularity. Although the equation for the energy equation $(1.5)_2$ is a strictly parabolic equation, the maximum principle cannot be used directly to the equation because of the strongly coupling with the density. Thus we truncate the temperature θ in the nonlinear terms on the right side of $(3.1)_2$ by the cut-off function χ^ε. Without the truncation, the term $p_s(\theta^0)$ will be unbounded for $\theta^0 \in L^2(I; H^1(\Omega))$ (see Section 3.1). Then we derive the uniform bound of θ by the classical energy method. Correspondingly, we modify some nonlinear terms in view of the low regularity of the solution. With those regularizations, we successfully construct a class of approximate solutions and prove the compactness of the approximate solutions to get the desired weak solution.

The system (3.1) can be rewritten as

$$\begin{cases} \rho_t - \big(\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x\big)_x - \big(\rho(\rho_\varepsilon\theta_x)_\varepsilon\big)_x + \rho\chi^\varepsilon(\sqrt{\theta}) = \chi^\varepsilon\big(p_s(\theta)\big), \\ (\rho+\sigma)\theta_t - \big(\kappa^\varepsilon\theta_x\big)_x - \big[\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x + \rho(\rho_\varepsilon\theta_x)_\varepsilon\big]\theta_x - \rho\chi^\varepsilon(\sqrt{\theta})\theta + (\lambda+\theta)p_s(\theta) \\ \quad = \lambda\rho\chi^\varepsilon(\sqrt{\theta}). \end{cases} \tag{3.2}$$

The corresponding initial and boundary conditions are given by

$$\begin{aligned} &\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x + \rho(\rho_\varepsilon\theta_x)_\varepsilon\big|_{x=1} = \alpha^1\big(\bar\rho^1 - \rho(1,t)\big), \\ &\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x + \rho(\rho_\varepsilon\theta_x)_\varepsilon\big|_{x=0} = \alpha^0\big(\rho(0,t) - \bar\rho^0\big), \\ &\rho(x,0) = \rho_{0\varepsilon}(x) := (\rho_0)_\varepsilon(x) + \varepsilon, \\ &\kappa^\varepsilon\theta_x\big|_{x=1} = \beta^1\big(\bar\theta^1 - \theta(1,t)\big), \\ &\kappa^\varepsilon\theta_x\big|_{x=0} = \beta^0\big(\theta(0,t) - \bar\theta^0\big), \\ &\theta(x,0) = \theta_{0\varepsilon}(x) := (\theta_0)_\varepsilon(x). \end{aligned} \tag{3.3}$$

We prove the existence of approximate solutions $(\rho^{\varepsilon,\nu}, \theta^{\varepsilon,\nu})(t,x)$ to the system (3.2)–(3.3) by using the Leray–Schauder fixed point theorem. The following two lemmas are useful in our proof. The first one can be found in [15] and [16]. The second one is given in [10] and its proof is only a slight variation of the proof and so we omit it.

Lemma 3.1 *(Aubin–Lions). Let $B_1 \hookrightarrow\hookrightarrow B_2 \hookrightarrow B_3$ be reflective and separable Banach spaces. Then*

$$\{u \in L^p(I; B_1) \mid u_t \in L^q(I; B_3)\} \hookrightarrow\hookrightarrow L^p(I; B_2), \quad 1 < p, q < \infty;$$
$$\{u \in L^q(I; B_2) \cap L^1(I; B_1) \mid u_t \in L^1(I; B_3)\} \hookrightarrow\hookrightarrow L^p(I; B_2), \quad 1 \leqslant p < q < \infty.$$

Lemma 3.2 *(Leray–Schauder). Let X be a Banach space, Y a closed convex cone of X centered at the origin ($y \in Y \Rightarrow ty \in Y$, $\forall t \geqslant 0$). Let $\mathbf{T} : Y \times [0, 1] \to Y$ be a completely continuous map such that $\mathbf{T}(x, 0) = 0$ for all $x \in Y$. Suppose the map $\mathbf{T}$ has the property that the elements $x \in Y$ which satisfy*

$$x = \mathbf{T}(x, \sigma) \tag{3.4}$$

for some $\sigma \in [0, 1]$ are uniformly bounded in X with respect to σ. Then the map $\mathbf{T}(\cdot, 1)$ has a fixed point in Y. In other words, there exists $x \in Y$ satisfying $x = \mathbf{T}(x, 1)$.

3.1. Existence of approximate solutions

We define

$$X = \{u \in L^2(I; H^1(\Omega)) \mid u \geqslant 0\}, \qquad Y = \{u \in W_2^{2,1}(Q_T) \mid u \geqslant 0\}.$$

By Aubin–Lions lemma, $Y \hookrightarrow\hookrightarrow X$. Let ε and ν be given positive constants and the parameter $s \in [0, 1]$. For any given $(\rho^0, \theta^0) \in X^2$, we define ρ to be the solution of the following linear parabolic equation

$$\rho_t - \big(\big(\varepsilon + (\rho^0\theta^0)_\nu\big)\rho_x\big)_x - \big(\rho\big(\rho^0_\varepsilon\theta^0_x\big)_\varepsilon\big)_x + s\rho\chi^\varepsilon\big(\sqrt{\theta^0}\big) = s\chi^\varepsilon\big(p_s(\theta^0)\big), \tag{3.5}$$

with the initial and boundary conditions

$$\begin{cases} \big(\varepsilon + (\rho^0\theta^0)_\nu\big)\rho_x + \rho\big(\rho^0_\varepsilon\theta^0_x\big)_\varepsilon = \alpha^1\big(s\bar\rho^1 - \rho\big), & \text{at } x = 1, \\ \big(\varepsilon + (\rho^0\theta^0)_\nu\big)\rho_x + \rho\big(\rho^0_\varepsilon\theta^0_x\big)_\varepsilon = \alpha^0\big(\rho - s\bar\rho^0\big), & \text{at } x = 0, \\ \rho(x, 0) = s\rho_{0\varepsilon}(x), & \text{for } x \in \Omega. \end{cases} \tag{3.6}$$

Note that (3.5) is a parabolic equation with $\varepsilon + (\rho^0\theta^0)_\nu \geqslant \varepsilon$ since $\rho^0, \theta^0 \in X$ implies $\rho^0\theta^0 \geqslant 0$. Now with ρ in hand, we define θ to be the solution of the semi-linear parabolic equation

$$\begin{aligned} &(\rho + \sigma)\theta_t - \big(\kappa^\varepsilon\theta_x\big)_x - \big[\big(\varepsilon + (\rho^0\theta^0)_\nu\big)\rho_x + \rho\big(\rho^0_\varepsilon\theta^0_x\big)_\varepsilon\big]\theta_x - s\rho\chi^\varepsilon\big(\sqrt{\theta^0}\big)\theta + s(\lambda + \theta)p_s(\theta) \\ &\quad = s\lambda\rho\chi^\varepsilon\big(\sqrt{\theta^0}\big), \end{aligned} \tag{3.7}$$

with the initial and boundary conditions

$$\begin{cases} \kappa^\varepsilon\theta_x = \beta^1\big(s\bar\theta^1 - \theta\big), & \text{at } x = 1, \\ \kappa^\varepsilon\theta_x = \beta^0\big(\theta - s\bar\theta^0\big), & \text{at } x = 0, \\ \theta(x, 0) = s\theta_{0\varepsilon}(x), & \text{for } x \in \Omega. \end{cases} \tag{3.8}$$

Note that the existence and uniqueness of the solution to the semi-linear equation (3.7)–(3.8) will be proved in Appendix A.

Now let M denote the mapping from (ρ^0, θ^0, s) to (ρ, θ). Then we have the following lemma.

Lemma 3.3. *The mapping $M : X^2 \times [0,1] \to X^2$ is well defined, continuous and compact.*

Proof. By the L^2-theory of linear parabolic equations [13], there exists a solution $\rho \in W_2^{2,1}(Q_T)$ for the system (3.5)–(3.6) such that

$$\|\rho\|_{W_2^{2,1}(Q_T)} \leqslant C\big(\varepsilon^{-1}, \big\|\big(\rho^0\theta^0\big)_\nu\big\|_{C^1(\overline{Q}_T)}, \big\|\big(\rho^0_\varepsilon\theta^0_x\big)_\varepsilon\big\|_{C^1(\overline{Q}_T)}, \|\rho_{0\varepsilon}\|_{H^1(\Omega)}, T\big).$$

By noting the fact

$$\big\|\rho^0_\varepsilon\big\|_{H^1(\Omega)} \leqslant C_\varepsilon \|\rho_0\|_{L^1(\Omega)}, \qquad \big\|\big(\rho^0\theta^0\big)_\nu\big\|_{C^1(\overline{Q}_T)} \leqslant C_{\nu,T}\big\|\rho^0\big\|_{L^2(Q_T)}\big\|\theta^0\big\|_{L^2(Q_T)},$$
$$\big\|\big(\rho^0_\varepsilon\theta^0_x\big)_\varepsilon\big\|_{C^1(\overline{Q}_T)} \leqslant C_{\varepsilon,T}\big\|\rho^0_\varepsilon\theta^0_x\big\|_{L^1(Q_T)} \leqslant C_{\varepsilon,T}\big\|\rho^0\big\|_{L^2(Q_T)}\big\|\theta^0_x\big\|_{L^2(Q_T)},$$

for the standard smoothing operator, we have

$$\|\rho\|_{W_2^{2,1}(Q_T)} \leqslant C\big(\varepsilon^{-1}, \nu^{-1}, \big\|\rho^0\big\|_X, \big\|\theta^0\big\|_X, T\big) \tag{3.9}$$

and therefore,

$$\|\rho\|_{L^\infty(Q_T)} \leqslant \|\rho\|_{W_2^{2,1}(Q_T)} \leqslant C\big(\varepsilon^{-1}, \nu^{-1}, \big\|\rho^0\big\|_X, \big\|\theta^0\big\|_X, T\big).$$

Let $\rho^+ = \max\{\rho, 0\}$, $\rho^- = \max\{-\rho, 0\}$. Then $\rho = \rho^+ - \rho^-$. By multiplying ρ^- on both sides of Eq. (3.5) and integrating the resulting equation over Q_t, we get

$$\begin{aligned}
&\int_0^1 \frac{|\rho^-|^2}{2}\,dx + \int_0^t\int_0^1 \big(\varepsilon + \big(\rho^0\theta^0\big)_\nu\big)\big|\rho^-_x\big|^2\,dx\,d\tau + \int_0^t\int_0^1 \big(s\chi^\varepsilon(\sqrt{\theta})|\rho^-|^2 + s\chi^\varepsilon\big(p_s(\theta)\big)\rho^-\big)\,dx\,d\tau \\
&\quad + \int_0^t \big(\alpha^0\big|\rho^-(0,\tau)\big|^2 + \alpha^0 s\bar{\rho}^0\rho^-(0,\tau)\big)\,d\tau + \int_0^t \big(\alpha^1\big|\rho^-(1,\tau)\big|^2 + \alpha^1 s\bar{\rho}^1\rho^-(1,\tau)\big)\,d\tau \\
&= -\int_0^t\int_0^1 \rho^-\rho^-_x\big(\rho^0_\varepsilon\theta^0_x\big)_\varepsilon\,dx\,d\tau \\
&\leqslant \int_0^t\int_0^1 \left(\frac{\|(\rho^0_\varepsilon\theta^0_x)_\varepsilon\|_{L^\infty(Q_T)}}{2\varepsilon}|\rho^-|^2 + \frac{\varepsilon}{2}\big|\rho^-_x\big|^2\right)dx\,d\tau.
\end{aligned}$$

Notice that $\rho^- \geqslant 0$. Thus we have that

$$\int_0^1 |\rho^-|^2\,dx \leqslant \frac{\|(\rho^0_\varepsilon\theta^0_x)_\varepsilon\|_{L^\infty(Q_T)}}{2\varepsilon}\int_0^t\int_0^1 |\rho^-|^2\,dx\,d\tau.$$

By Gronwall's inequality, we see that $\rho^- \equiv 0$. Thus $\rho = \rho^+ \geqslant 0$. This and (3.8) imply that $\rho \in Y \hookrightarrow\hookrightarrow X$.

Similarly, by the L^2-theory of quasi-linear parabolic equations [13], there exists a solution $\theta \in W_2^{2,1}(Q_T)$ for the system (3.7)–(3.8) and

$$\|\theta\|_{W_2^{2,1}(Q_T)} \leqslant C\big(\varepsilon^{-1}, \nu^{-1}, \|\rho^0\|_X, \|\theta^0\|_X, T\big). \tag{3.10}$$

Let $\theta^+ = \max\{\theta, 0\}$, $\theta^- = \max\{-\theta, 0\}$. Then $\theta = \theta^+ - \theta^-$. Multiplying $\theta^-/(\rho+\sigma)$ on both sides of Eq. (3.7) and integrating the resulting equation over Q_t, we can get

$$\begin{aligned}
&\int_0^1 \frac{|\theta^-|^2}{2} dx + \int_0^t\int_0^1 \frac{\kappa^\varepsilon}{\rho+\sigma}\big|\theta_x^-\big|^2 dx\,d\tau + \int_0^t\int_0^1 \frac{s(\lambda+\theta)p_s(\theta)}{(\rho+\sigma)}\theta^- dx\,d\tau \\
&\quad + \int_0^t\int_0^1 s\lambda\rho\chi^\varepsilon\big(\sqrt{\theta^0}\big)\frac{\theta^-}{\rho+\sigma} dx\,d\tau + \int_0^t \frac{\theta^-(1,\tau)}{\rho(1,\tau)+\sigma}\beta^1\big(s\bar{\theta}^1 + \theta^-(1,\tau)\big) d\tau \\
&\quad + \int_0^t \frac{\theta^-(0,\tau)}{\rho(0,\tau)+\sigma}\beta^1\big(s\bar{\theta}^1 + \theta^-(0,\tau)\big) d\tau \\
&= \int_0^t\int_0^1 s\rho\chi^\varepsilon\big(\sqrt{\theta^0}\big)\frac{|\theta^-|^2}{\rho+\sigma} dx\,d\tau + \int_0^t\int_0^1 \kappa^\varepsilon\theta_x^-\frac{\rho_x\theta^-}{(\rho+\sigma)^2} dx\,d\tau \\
&\quad + \int_0^t\int_0^1 \big[\big(\varepsilon + \big(\rho^0\theta^0\big)_\nu\big)\rho_x + \rho\big(\rho_\varepsilon^0\theta_x^0\big)_\varepsilon\big]\theta_x^-\frac{\theta^-}{\rho+\sigma} dx\,d\tau.
\end{aligned}$$

Since $p_s(\theta) = 0$ for $\theta \leqslant 0$, we observe that $(\lambda+\theta)p_s(\theta)\theta^- = 0$ a.e. in Ω_T. By Cauchy inequality and the estimations (3.8)–(3.9), we can estimate the terms in the right-hand side of the above equality. Thus we obtain

$$\int_0^1 |\theta^-|^2 dx \leqslant C\big(\varepsilon^{-1}, \nu^{-1}, \|\rho^0\|_X, \|\theta^0\|_X, T\big)\int_0^t\int_0^1 |\theta^-|^2 dx\,d\tau.$$

Gronwall's inequality gives that $\theta^- \equiv 0$. Thus $\theta = \theta^+ \geqslant 0$. This and (3.9) imply that $\theta \in Y \hookrightarrow\hookrightarrow X$.

We conclude that the mapping $M : X^2 \times [0,1] \to X^2$ is a compact mapping.

Now we prove the continuity of the mapping M. For any $(\hat{\rho}^0, \hat{\theta}^0, \hat{s}) \in X^2 \times [0,1]$, let $(\hat{\rho}, \hat{\theta}) = M(\hat{\rho}^0, \hat{\theta}^0, \hat{s})$. Then

$$\hat{\rho}_t - \big[\big(\varepsilon + \big(\hat{\rho}^0\hat{\theta}^0\big)_\nu\big)\hat{\rho}_x + \hat{\rho}\big(\hat{\rho}_\varepsilon^0\hat{\theta}_x^0\big)_\varepsilon\big]_x + \hat{s}\hat{\rho}\chi^\varepsilon\big(\sqrt{\hat{\theta}^0}\big) = \hat{s}\chi^\varepsilon\big(p_s\big(\hat{\theta}^0\big)\big), \tag{3.11}$$

$$\begin{aligned}
&(\hat{\rho}+\sigma)\hat{\theta}_t - \big(\hat{\kappa}^\varepsilon\hat{\theta}_x\big)_x - \big[\big(\varepsilon + \big(\hat{\rho}^0\hat{\theta}^0\big)_\nu\big)\hat{\rho}_x + \hat{\rho}\big(\hat{\rho}_\varepsilon^0\hat{\theta}_x^0\big)_\varepsilon\big]\hat{\theta}_x - \hat{s}\hat{\rho}\chi^\varepsilon\big(\sqrt{\hat{\theta}^0}\big)\hat{\theta} + \hat{s}(\lambda+\hat{\theta})p_s(\hat{\theta}) \\
&\quad = \hat{s}\lambda\hat{\rho}\chi^\varepsilon\big(\sqrt{\hat{\theta}^0}\big),
\end{aligned} \tag{3.12}$$

with the initial and boundary conditions

$$\begin{cases}
\big(\varepsilon + \big(\hat{\rho}^0\hat{\theta}^0\big)_\nu\big)\hat{\rho}_x + \hat{\rho}\big(\hat{\rho}_\varepsilon^0\hat{\theta}_x^0\big)_\varepsilon = \alpha^1\big(\hat{s}\bar{\rho}^1 - \hat{\rho}\big), & \text{at } x = 1, \\
\big(\varepsilon + \big(\hat{\rho}^0\hat{\theta}^0\big)_\nu\big)\hat{\rho}_x + \hat{\rho}\big(\hat{\rho}_\varepsilon^0\hat{\theta}_x^0\big)_\varepsilon = \alpha^0\big(\hat{\rho} - \hat{s}\bar{\rho}^0\big), & \text{at } x = 0, \\
\rho(x,0) = \hat{s}\rho_{0\varepsilon}(x), & \text{for } x \in \Omega,
\end{cases} \tag{3.13}$$

and

$$\begin{cases} \hat{\kappa}^{\varepsilon}\hat{\theta}_x = \beta^1\big(\hat{s}\bar{\theta}^1 - \hat{\theta}\big), & \text{at } x = 1, \\ \hat{\kappa}^{\varepsilon}\hat{\theta}_x = \beta^0\big(\hat{\theta} - \hat{s}\bar{\theta}^0\big), & \text{at } x = 0, \\ \theta(x,0) = \hat{s}\theta_{0\varepsilon}(x), & \text{for } x \in \Omega. \end{cases} \tag{3.14}$$

Denote $\tilde{\rho} = \rho - \hat{\rho}$ and $\tilde{\theta} = \theta - \hat{\theta}$. Then $\tilde{\rho}$ satisfies the following equation,

$$\begin{aligned} &\tilde{\rho}_t - (F - \hat{F})_x + s\tilde{\rho}\chi^{\varepsilon}\big(\sqrt{\theta^0}\big) + (s - \hat{s})\hat{\rho}\chi^{\varepsilon}\big(\sqrt{\theta^0}\big) + \hat{s}\hat{\rho}\big[\chi^{\varepsilon}\big(\sqrt{\theta^0}\big) - \chi^{\varepsilon}\big(\sqrt{\hat{\theta}^0}\big)\big] \\ &\quad = (s - \hat{s})\chi^{\varepsilon}\big(p_s\big(\theta^0\big)\big) + \hat{s}\big[\chi^{\varepsilon}\big(p_s\big(\theta^0\big)\big) - \chi^{\varepsilon}\big(p_s\big(\hat{\theta}^0\big)\big)\big], \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} F &= \big(\varepsilon + \big(\rho^0\theta^0\big)_{\nu}\big)\rho_x + \rho\big(\rho^0_{\varepsilon}\theta^0_x\big)_{\varepsilon}, \\ \hat{F} &= \big(\varepsilon + \big(\hat{\rho}^0\hat{\theta}^0\big)_{\nu}\big)\hat{\rho}_x + \hat{\rho}\big(\hat{\rho}^0_{\varepsilon}\hat{\theta}^0_x\big)_{\varepsilon}. \end{aligned}$$

Multiplying Eq. (3.15) by $\tilde{\rho}$ and integrating over Q_t gives

$$\int_0^1 \tilde{\rho}^2(x,t)\,dx + \int_0^t\int_0^1 \tilde{\rho}_x^2\,dx\,d\tau \leqslant C\left[\int_0^t\int_0^1 \tilde{\rho}^2\,dx\,d\tau + (s - \hat{s})^2 + \big\|\rho^0 - \hat{\rho}^0\big\|_X^2 + \big\|\theta^0 - \hat{\theta}^0\big\|_X^2\right]$$

with $C = C(\varepsilon^{-1}, \nu^{-1}, \|\rho_{0\varepsilon}\|_{L^2(\Omega)}, \|\rho^0\|_X, \|\theta^0\|_X, \|\hat{\rho}^0\|_X, \|\hat{\theta}^0\|_X, T)$.

Thus Gronwall inequality implies that

$$\|\tilde{\rho}\|_X^2 \leqslant C\big[(s - \hat{s})^2 + \big\|\rho^0 - \hat{\rho}^0\big\|_X^2 + \big\|\theta^0 - \hat{\theta}^0\big\|_X^2\big].$$

Similarly, we can derive the equation for $\tilde{\theta}$ and get

$$\|\tilde{\theta}\|_X^2 \leqslant C\big[(s - \hat{s})^2 + \big\|\rho^0 - \hat{\rho}^0\big\|_X^2 + \big\|\theta^0 - \hat{\theta}^0\big\|_X^2\big].$$

Thus, the mapping $M : X^2 \times [0,1] \to X^2$ is continuous. The proof of Lemma 3.2 is complete. □

In addition, for $s = 0$ we see that $M(\rho, \theta, 0) = 0$ for any $(\rho, \theta) \in X^2$. Thus, by the Leray–Schauder fixed point theorem, there exists a fixed point for the mapping $M(\cdot,\cdot,1) : X^2 \to X^2$ if all the functions $(\rho, \theta) \in X^2$ satisfying

$$(\rho, \theta) = M(\rho, \theta, s) \tag{3.16}$$

for some $s \in [0,1]$ are uniformly bounded in X^2. In fact, by the proof of Lemma 3.3, M maps $(\rho, \theta, s) \in X^2 \times [0,1]$ into Y^2. Therefore, if (ρ, θ) is a fixed point of $M(\cdot,\cdot,1)$, then $(\rho, \theta) \in W_2^{2,1}(Q_T)$.

So we have the following theorem for the existence of approximate solutions $(\rho^{\varepsilon,\nu}, \theta^{\varepsilon,\nu})(t,x)$.

Theorem 3.1. *Under the assumptions of Theorem* 2.1, *the system* (3.2)–(3.3) *has a* (*strong*) *solution* $(\rho, \theta) \in W_2^{2,1}(Q_T)$ *which satisfies*

$$\rho \geqslant \underline{\rho}_{\varepsilon,T} \quad \textit{and} \quad \underline{\theta}_T \leqslant \theta \leqslant \overline{\theta}_T \quad \textit{for } (x,t) \in Q_T. \tag{3.17}$$

$$\|\rho\|_{L^\infty(I;L^4(\Omega))}, \|\rho_x\|_{L^2(Q_T)}, \|\rho\rho_x\|_{L^2(Q_T)} \leqslant C_{\varepsilon,T},$$
$$\|\theta\|_{L^\infty(Q_T)}, \|\rho\|_{L^\infty(I;L^1(\Omega))}, \|\theta_x\|_{L^2(Q_T)}, \|\rho_\varepsilon\theta_x\|_{L^2(Q_T)} \leqslant C_T \tag{3.18}$$

where $\underline{\rho}_{\varepsilon,T}$ and $C_{\varepsilon,T}$ are positive constants which depend on ε and T, independent of ν; $\underline{\theta}_T$, and $\overline{\theta}_T$ and C_T are positive constants, dependent upon T and independent of ε and ν.

By the Leray–Schauder fixed point theorem, it suffices to prove the uniform boundedness of functions $(\rho,\theta) \in X^2$ satisfying Eqs. (3.16) and (3.17).

3.2. Uniform estimates

We assume that $(\rho,\theta) \in X^2$ and therefore, $(\rho,\theta) = M(\rho,\theta,s) \in Y^2$, for $s \in [0,1]$, i.e., (ρ,θ) is a (strong) solution of the following system,

$$\rho_t - \big(\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x\big)_x - \big(\rho(\rho_\varepsilon\theta_x)_\varepsilon\big)_x + s\rho\chi^\varepsilon(\sqrt{\theta}) = s\chi^\varepsilon\big(p_s(\theta)\big), \tag{3.19}$$

$$\begin{aligned}(\rho+\sigma)\theta_t - \big(\kappa^\varepsilon\theta_x\big)_x - \big[\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x + \rho(\rho_\varepsilon\theta_x)_\varepsilon\big]\theta_x - s\rho\chi^\varepsilon(\sqrt{\theta})\theta + s(\lambda+\theta)p_s(\theta)\\ = s\lambda\rho\chi^\varepsilon(\sqrt{\theta}),\end{aligned} \tag{3.20}$$

with the initial and boundary conditions

$$\begin{cases} \big(\varepsilon + (\rho\theta)_\nu\big)\rho_x + \rho(\rho_\varepsilon\theta_x)_\varepsilon = \alpha^1\big(s\overline{\rho}^1 - \rho\big), & \text{at } x=1,\\ \big(\varepsilon + (\rho\theta)_\nu\big)\rho_x + \rho(\rho_\varepsilon\theta_x)_\varepsilon = \alpha^0\big(\rho - s\overline{\rho}^0\big), & \text{at } x=0,\\ \rho(x,0) = s\rho_{0\varepsilon}(x), & \text{for } x\in\Omega, \end{cases} \tag{3.21}$$

and

$$\begin{cases} \kappa^\varepsilon\theta_x = \beta^1\big(s\overline{\theta}^1 - \theta\big), & \text{at } x=1,\\ \kappa^\varepsilon\theta_x = \beta^0\big(\theta - s\overline{\theta}^0\big), & \text{at } x=0,\\ \theta(x,0) = s\theta_{0\varepsilon}(x), & \text{for } x\in\Omega. \end{cases} \tag{3.22}$$

In this subsection, we derive some uniform estimates for solutions to the above initial–boundary value problems.

Firstly we add Eq. (3.19) multiplying by $(\lambda+\theta)$ into (3.20) and then integrate the resulting equation over Q_t. We arrive at

$$\int_0^1 (\lambda\rho + \rho\theta + \sigma\theta)(x,t)\,dx - \int_0^t H_2(x,\tau)\big|_{x=0}^{x=1}\,d\tau \leqslant \int_0^1 (\lambda\rho_{0\varepsilon} + \rho_{0\varepsilon}\theta_{0\varepsilon} + \sigma\theta_{0\varepsilon})(x)\,dx$$

where

$$H_2(x,\tau) = \big[\varepsilon\rho_x + (\rho\theta)_\nu\rho_x + \rho(\rho_\varepsilon\theta_x)_\varepsilon\big](\lambda+\theta) + \kappa^\varepsilon\theta_x.$$

With boundary conditions in (3.21)–(3.22), we have

$$\begin{aligned}-H_2(x,\tau)\big|_{x=0}^{x=1} &= \alpha^1\big(\rho(1,\tau)-s\overline{\rho}^1\big)\big(\lambda+\theta(1,\tau)\big)+\alpha^0\big(\rho(0,\tau)-s\overline{\rho}^0\big)\big(\lambda+\theta(0,\tau)\big)\\ &\quad+\beta^1\big(\theta(1,\tau)-s\overline{\theta}^1\big)+\beta^0\big(\theta(0,\tau)-s\overline{\theta}^0\big)\\ &\geqslant -\alpha^1 s\overline{\rho}^1\theta(1,\tau)-\alpha^0 s\overline{\rho}^0\theta(0,\tau)-\lambda s\big(\alpha^1\overline{\rho}^1+\alpha^0\overline{\rho}^0\big)-s\big(\beta^1\overline{\theta}^1+\beta^0\overline{\theta}^0\big)\end{aligned}$$

and therefore,

$$\int_0^1 (\lambda\rho+\rho\theta+\sigma\theta)(x,t)\,dx \leqslant C_T + C\int_0^t \big\|\theta(\cdot,\tau)\big\|_{C(\overline{\Omega})}\,d\tau, \tag{3.23}$$

where

$$C_T = \big(\lambda+\|\theta_{0\varepsilon}\|_{L^\infty}\big)\|\rho_{0\varepsilon}\|_{L^1}+\sigma\|\theta_{0\varepsilon}\|_{L^\infty}+\big[\lambda\big(\alpha^1\overline{\rho}^1+\alpha^0\overline{\rho}^0\big)+\big(\beta^1\overline{\theta}^1+\beta^0\overline{\theta}^0\big)\big]T.$$

Similarly, adding Eq. (3.20) multiplying by $(l+1)\theta^l$ with Eq. (3.19) multiplying by θ^{l+1} and integrating the resulting equation over Q_t, we arrive at

$$\begin{aligned}&\int_0^1 (\rho+\sigma)\theta^{l+1}(x,t)\,dx-\int_0^t H_3(x,\tau)\big|_{x=0}^{x=1}\,d\tau+\int_0^t\int_0^1 \kappa^\varepsilon l(l+1)\theta^{l-1}|\theta_x|^2\,dx\,d\tau\\ &\quad+s(l+1)\int_0^t\int_0^1 (\lambda+\theta)p_s(\theta)\theta^l\,dx\,d\tau\\ &=\int_0^1 (\rho_{0\varepsilon}+\sigma)(\theta_{0\varepsilon})^{l+1}(x)\,dx+s\int_0^t\int_0^1 \big[l\theta^{l+1}+\lambda(l+1)\theta^l\big]\rho\chi^\varepsilon(\sqrt{\theta})\,dx\,d\tau\\ &\quad+s\int_0^t\int_0^1 \chi^\varepsilon\big(p_s(\theta)\big)\theta^{l+1}\,dx\,d\tau,\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}-H_3(x,\tau)\big|_{x=0}^{x=1} &= \alpha^1\big(\rho(1,\tau)-s\overline{\rho}^1\big)\big[\theta(1,\tau)\big]^{l+1}+\alpha^0\big(\rho(0,\tau)-s\overline{\rho}^0\big)\big[\theta(0,\tau)\big]^{l+1}\\ &\quad+(l+1)\beta^1\big(\theta(1,\tau)-s\overline{\theta}^1\big)\big[\theta(1,\tau)\big]^l+(l+1)\beta^0\big(\theta(0,\tau)-s\overline{\theta}^0\big)\big[\theta(0,\tau)\big]^l\\ &=\big[\alpha^1\rho(1,\tau)+(l+1)\beta^1-\alpha^1 s\overline{\rho}^1\big]\big[\theta(1,\tau)\big]^{l+1}-(l+1)\beta^1 s\overline{\theta}^1\big[\theta(1,\tau)\big]^l\\ &\quad+\big[\alpha^0\rho(0,\tau)+(l+1)\beta^0-\alpha^0 s\overline{\rho}^0\big]\big[\theta(0,\tau)\big]^{l+1}-(l+1)\beta^0 s\overline{\theta}^0\big[\theta(0,\tau)\big]^l\\ &\geqslant -2^l(l+1)\big[\beta^1\big(s\overline{\theta}^1\big)^{l+1}+\beta^0\big(s\overline{\theta}^0\big)^{l+1}\big]\end{aligned}$$

when l is large enough. Since $\theta^l \leqslant \theta^{1/2}+\theta^{l+1}$ for any $\theta \geqslant 0$ and $l \geqslant 1$, by (3.23)–(3.24),

$$\int_0^1 (\rho+\sigma)\theta^{l+1}(x,t)\,dx + l(l+1)\int_0^t\int_0^1 \kappa^\varepsilon\theta^{l-1}|\theta_x|^2\,dx\,d\tau + sl\int_0^t\int_0^1 (\lambda+\theta)p_s(\theta)\theta^l\,dx\,d\tau$$
$$\leqslant C_{l,T} + C_0 s\int_0^t\int_0^1 l\big(1+\theta^{l+1/2}\big)\rho\theta\,dx\,d\tau$$
$$\leqslant C'_{l,T} + C_0 sl\int_0^t \big\|\theta(\cdot,\tau)\big\|_{L^\infty(\Omega)}^{l+3/2}\,d\tau \tag{3.25}$$

where

$$C_{l,T} = \int_0^1 (\rho_0+\sigma)\theta_0^{l+1}(x)\,dx + 2^l(l+1)\big[\beta^1\big(s\bar{\theta}^1\big)^{l+1} + \beta^0\big(s\bar{\theta}^0\big)^{l+1}\big]$$

and $C'_{l,T} = C_{l,T} + Cl$. Recall the Gagliardo–Nirenberg inequality

$$\|f\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^2(\Omega)} + C\|f\|_{L^2(\Omega)}^{1/2}\|f_x\|_{L^2(\Omega)}^{1/2}, \quad \forall f\in H^1(\Omega).$$

With $f=\theta^{\frac{l+1}{2}}$ in the above inequality, we obtain

$$\big\|\theta(\cdot,\tau)\big\|_{L^\infty(\Omega)}^{l+3/2} \leqslant \frac{C_2}{2}\int_0^1 \theta^{l+3/2}(x,\tau)\,dx + C_1\big\|\theta^{\frac{l+1}{2}}(\cdot,\tau)\big\|_{L^2(\Omega)}^{\frac{2l+3}{2l+2}}\big\|\big(\theta^{\frac{l+1}{2}}\big)_x(\cdot,\tau)\big\|_{L^2(\Omega)}^{\frac{2l+3}{2l+2}}$$

and by Hölder's inequality,

$$\int_0^t \big\|\theta^{\frac{l+1}{2}}(\cdot,\tau)\big\|_{L^2(\Omega)}^{\frac{2l+3}{2l+2}}\big\|\big(\theta^{\frac{l+1}{2}}\big)_x(\cdot,\tau)\big\|_{L^2(\Omega)}^{\frac{2l+3}{2l+2}}\,d\tau$$
$$\leqslant Cl\int_0^t\int_0^1 \big(\theta^{\frac{l+1}{2}}\big)^{\frac{4l+6}{2l+1}}\,dx\,d\tau + \frac{1}{(l+1)C_0C_1}\int_0^t\int_0^1 \kappa\big|\big(\theta^{\frac{l+1}{2}}\big)_x\big|^2\,dx\,d\tau$$
$$\leqslant Cl\int_0^t\int_0^1 \theta^{\frac{(l+1)(2l+3)}{2l+1}}\,dx\,d\tau + \frac{l+1}{4C_0C_1}\int_0^t\int_0^1 \kappa\theta^{l-1}|\theta_x|^2\,dx\,d\tau.$$

It follows that

$$\int_0^t \big\|\theta(\cdot,\tau)\big\|_{L^\infty(\Omega)}^{l+3/2}\,d\tau \leqslant \frac{C_2}{2}\int_0^t\int_0^1 \theta^{l+3/2}\,dx\,d\tau + Cl\int_0^t\int_0^1 \theta^{\frac{(l+1)(2l+3)}{2l+1}}\,dx\,d\tau$$
$$+ \frac{l+1}{4C_0}\int_0^t\int_0^1 \kappa_1\theta^{l-1}|\theta_x|^2\,dx\,d\tau.$$

By the assumption (1.9), we observe that $C_0C_2\theta^{l+\frac{3}{2}} \leqslant p_s(\theta)\theta^{l+1} + C$ for all $\theta \geqslant 0$. Substituting the last inequality into (3.25) gives

$$\int_0^1 (\rho+\sigma)\theta^{l+1}(x,t)\,dx + \frac{l(l+1)}{2}\int_0^t\int_0^1 \kappa^\varepsilon\theta^{l-1}|\theta_x|^2\,dx\,d\tau + \frac{sl}{2}\int_0^t\int_0^1 p_s(\theta)\theta^{l+1}\,dx\,d\tau$$

$$\leqslant C'_{l,T} + C_3 sl^2\int_0^t\int_0^1 \theta^{\frac{(l+1)(2l+3)}{2l+1}}\,dx\,d\tau, \tag{3.26}$$

for l being large enough. Let l_0 be a positive integer satisfying

$$\frac{(l_0+1)(2l_0+3)}{2l_0+1} = l_0+1+\frac{2l_0+2}{2l_0+1} < l_0+1+(1+\eta)$$

where η is defined in (1.9). By noting the fact

$$C_3\theta^{\frac{(l_0+1)(2l_0+3)}{2l_0+1}} \leqslant \frac{1}{4l_0}p_s(\theta)\theta^{l_0+1} + (Cl_0)^{l_1}$$

with $l_1 = 2(l_0+2+\eta)/\eta$, we have

$$\int_0^1 (\rho+\sigma)\theta^{l_0+1}(x,t)\,dx + \frac{l_0(l_0+1)}{2}\int_0^t\int_0^1 \kappa^\varepsilon\theta^{l_0-1}|\theta_x|^2\,dx\,d\tau + \frac{sl}{4}\int_0^t\int_0^1 p_s(\theta)\theta^{l_0+1}\,dx\,d\tau \leqslant C''_{l_0,T},$$

where $C''_{l_0,T} = C'_{l_0,T} + C_T(Cl_0)^{l_1}$ for some constant C_T independent of l_0. Furthermore,

$$\sup_{0\leqslant t\leqslant T}\int_0^1 \theta^{l_0+1}(x,t)\,dx + \int_0^T\int_0^1 \left|\left(\theta^{\frac{l_0+1}{2}}\right)_x\right|^2\,dx\,dt \leqslant C''_{l_0,T}$$

and by the Sobolev embedding inequality,

$$\int_0^T \|\theta\|^{l_0+1}_{L^\infty(\Omega)}\,dx\,dt \leqslant C_T^{l_0+1}C''_{l_0,T}.$$

Since l_0 is a fixed positive integer dependent solely upon η, we obtain the estimate

$$\int_0^T \|\theta\|_{L^\infty(\Omega)}\,dx\,dt \leqslant C_T. \tag{3.27}$$

From (3.23) and (3.25), we get

$$\sup_{0\leqslant t\leqslant T}\int_0^1 (\rho+\rho\theta)\,dx \leqslant C_T \tag{3.28}$$

and

$$\int_0^1 (\rho+\sigma)\theta^{l+1}(x,t)\,dx + l(l+1)\int_0^t\int_0^1 \kappa^\varepsilon\theta^{l-1}|\theta_x|^2\,dx\,d\tau + sl\int_0^t\int_0^1 (\lambda+\theta)p_s(\theta)\theta^l\,dx\,d\tau$$
$$\leqslant C_{l,T} + Csl\int_0^t\int_0^1 \rho\theta\,dx\,d\tau + Csl\int_0^t \|\theta\|_{L^\infty(\Omega)}^{1/2}\int_0^1 \rho\theta^{l+1}\,dx\,d\tau$$
$$\leqslant (C_{l,T}+C_T l) + Csl\int_0^t \|\theta\|_{L^\infty(\Omega)}^{1/2}\int_0^1 (\rho+\sigma)\theta^{l+1}\,dx\,d\tau.$$

Moreover, by using Gronwall's inequality,

$$\int_0^1 (\rho+\sigma)\theta^{l+1}(x,t)\,dx \leqslant (C_{l,T}+C_T l) + (C_{l,T}+C_T l)e^{C_T l}$$

and

$$\|\theta\|_{L^{l+1}(Q_T)} \leqslant \big[2(C_{l,T}+C_T l)\big]^{\frac{1}{l+1}} e^{C_T}.$$

By taking $l\to\infty$, we have

$$\|\theta\|_{L^\infty(Q_T)} \leqslant C_T \tag{3.29}$$

where we have noted the fact

$$C_{l,T}^{\frac{1}{l+1}} \leqslant C_T.$$

By taking $l=1$ in Eq. (3.25), we obtain

$$\int_0^1 (\rho+\sigma)\theta^2(x,t)\,dx + \frac{1}{2}\int_0^t\int_0^1 \big(\big(\kappa_1+\kappa_2|\rho_\varepsilon|^2\big)|\theta_x|^2 + s\theta^2 p_s(\theta)\big)\,dx\,d\tau \leqslant C_T,$$

which implies that

$$\|\theta_x\|_{L^2(Q_T)},\ \|\rho_\varepsilon\theta_x\|_{L^2(Q_T)} \leqslant C_T. \tag{3.30}$$

Secondly we present some estimates for ρ. Again by multiplying ρ on both sides of Eq. (3.19) and integrating the resulting equation over Q_T, with Gronwall's inequality we get

$$\sup_{0\leqslant t\leqslant T}\int_0^1 \rho^2\,dx + \int_0^T\int_0^1 |\rho_x|^2\,dx\,dt \leqslant C_{\varepsilon,T} + C\big(\varepsilon, \big\|(\rho_\varepsilon\theta_x)_\varepsilon\big\|_{L^\infty(Q_T)}\big) \leqslant C_{\varepsilon,T}, \tag{3.31}$$

which together with the Sobolev embedding inequality gives

$$\int_0^T \int_0^1 \rho^6 \, dx \, dt \leqslant C_{\varepsilon,T}.$$

Once again, multiplying ρ^3 on both sides of Eq. (3.19) and integrating the resulting equation over Q_T lead to

$$\sup_{0 \leqslant t \leqslant T} \int_0^1 \rho^4 \, dx + \int_0^T \int_0^1 \rho^2 |\rho_x|^2 \, dx \, dt \leqslant C_{\varepsilon,T}. \tag{3.32}$$

From (3.29), (3.30) and (3.31), we conclude that (ρ, θ) is uniformly bounded in X^2. Thus, by the Leray–Schauder fixed point theorem, there exists a fixed point $(\rho^{\varepsilon,\nu}, \theta^{\varepsilon,\nu})$ for the mapping $M(\cdot,\cdot,1): X^2 \to X^2$ and $(\rho^{\varepsilon,\nu}, \theta^{\varepsilon,\nu})$ is a solution of the system (3.2)–(3.3).

3.3. *Positivity of the approximate solutions*

Since θ is uniformly bounded by (3.29) with an upper bound independent of ε, the cut-off operator χ^ε can be removed from the system (3.2)–(3.3) because

$$\chi^\varepsilon\big(p_s(\theta)\big) = p_s(\theta) \quad \text{and} \quad \chi^\varepsilon(\sqrt{\theta}) = \sqrt{\theta}$$

for ε being small enough.

Finally we prove the positivity of the approximate solutions $(\rho^{\varepsilon,\nu}, \theta^{\varepsilon,\nu})$. Let $\tilde{\theta}^\delta = \theta e^t - \delta$. Then $\tilde{\theta}^\delta$ is the solution of the following problem,

$$\begin{aligned}
&(\rho+\sigma)\tilde{\theta}^\delta_t - \big(\kappa^\varepsilon \tilde{\theta}^\delta_x\big)_x - \big[\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x + \rho(\rho_\varepsilon\theta_x)_\varepsilon\big]\tilde{\theta}^\delta_x - (\rho+\sigma)\tilde{\theta}^\delta - \rho\sqrt{\theta}\tilde{\theta}^\delta + \tilde{q}\big(\theta e^t, \delta\big)\tilde{\theta}^\delta \\
&\quad = \rho\sqrt{\theta}\theta e^t + \lambda\rho\sqrt{\theta}e^t + (\rho+\sigma)\delta - \big(\lambda + e^{-t}\delta\big)p_s\big(e^{-t}\delta\big)e^t,
\end{aligned} \tag{3.33}$$

with the initial and boundary conditions

$$\begin{cases}
\kappa^\varepsilon \tilde{\theta}^\delta_x + \beta^1 \tilde{\theta}^\delta = \beta^1\big(\bar{\theta}^1 e^t - \delta\big), & \text{at } x = 1, \\
-\kappa^\varepsilon \tilde{\theta}^\delta_x + \beta^0 \tilde{\theta}^\delta = \beta^1\big(\bar{\theta}^0 e^t - \delta\big), & \text{at } x = 0, \\
\tilde{\theta}^\delta(x,0) = \theta_{0\varepsilon}(x) - \delta, & \text{for } x \in \Omega,
\end{cases} \tag{3.34}$$

where

$$\tilde{q}(\tilde{\theta}, \delta) = \frac{(\lambda + e^{-t}\tilde{\theta})p_s(e^{-t}\tilde{\theta}) - (\lambda + e^{-t}\delta)p_s(e^{-t}\delta)}{\tilde{\theta} - \delta} e^t \geqslant 0.$$

By the assumption (1.9), the right-hand sides of Eqs. (3.33)–(3.34) are nonnegative if δ is small enough (independent of ε and ν). Multiplying $(\tilde{\theta}^\delta)^-/(\rho+\sigma)$ on both sides of Eq. (3.33) and integrating the resulting equation over Q_t, we derive $\tilde{\theta}^\delta \geqslant 0$, i.e. $\theta \geqslant e^{-T}\delta$, which together with (3.29) implies that

$$\underline{\theta}_T \leqslant \theta(x,t) \leqslant \bar{\theta}_T \quad \text{for } (x,t) \in Q_T \tag{3.35}$$

where $\underline{\theta}_T$ and $\bar{\theta}_T$ are positive constants independent of ε and ν.

For ρ, we define $\rho^\delta = \rho - \delta$. Then ρ^δ is the solution of the following equation

$$\rho_t^\delta - \big(\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x^\delta\big)_x - \big(\rho^\delta(\rho_\varepsilon\theta_x)_\varepsilon\big)_x + \rho^\delta\sqrt{\theta} = p_s(\theta) + \delta\big[(\rho_\varepsilon\theta_x)_\varepsilon\big]_x - \delta\sqrt{\theta}, \tag{3.36}$$

with the initial and boundary conditions

$$\begin{cases} \big(\varepsilon + (\rho\theta)_\nu\big)\rho_x^\delta + \rho^\delta(\rho_\varepsilon\theta_x)_\varepsilon + \alpha^1\rho^\delta = \alpha^1\big(\overline{\rho}^1 - \delta\big) - \delta(\rho_\varepsilon\theta_x)_\varepsilon, & \text{at } x = 1, \\ -\big(\varepsilon + (\rho\theta)_\nu\big)\rho_x^\delta - \rho^\delta(\rho_\varepsilon\theta_x)_\varepsilon + \alpha^0\rho^\delta = \alpha^0\big(\overline{\rho}^0 - \delta\big) + \delta(\rho_\varepsilon\theta_x)_\varepsilon, & \text{at } x = 0, \\ \rho^\delta(x,0) = \rho_{0\varepsilon}(x) - \delta, & \text{for } x \in \Omega. \end{cases} \tag{3.37}$$

Note that by (3.35) we have $p_s(\theta) \geqslant p_s(\underline{\theta}_T)$, the right-hand sides of Eqs. (3.36)–(3.37) are nonnegative if

$$\delta = \min\left\{\frac{\varepsilon}{2}, \frac{p_s(\underline{\theta}_T)}{2\sqrt{\overline{\theta}_T}}, \frac{\min\{p_s(\underline{\theta}_T), \alpha^0\rho^0, \alpha^1\rho^1\}}{1 + 2\|(\rho_\varepsilon\theta_x)_\varepsilon\|_{C^1(\overline{Q}_T)}}\right\},$$

in which case $\rho^\delta \geqslant 0$, or equivalently $\rho \geqslant \delta$. On the other hand, from (3.30) we have

$$\big\|(\rho_\varepsilon\theta_x)_\varepsilon\big\|_{C^1(\overline{Q}_T)} \leqslant \frac{1}{\varepsilon^2}\|\rho_\varepsilon\theta_x\|_{L^1(Q_T)} \leqslant C_\varepsilon.$$

Thus, there exists a positive constant $\underline{\rho}_{\varepsilon,T}$ such that

$$\rho \geqslant \underline{\rho}_{\varepsilon,T} \quad \text{for } (x,t) \in Q_T. \tag{3.38}$$

The proof of Theorem 3.1 is complete.

4. Global existence

We have constructed an approximate solution $(\rho^{\varepsilon,\nu}, \theta^{\varepsilon,\nu})$ to the system (3.1) and (3.3) (or equivalently (3.2)–(3.3)) in the last section. In this section, we prove the global existence of weak solutions for the system (1.5)–(1.8).

Firstly we fix $\varepsilon > 0$ and study the convergence as $\nu \to 0$. Since the system (3.19)–(3.20) reduces to (3.2)–(3.3) when $s = 1$, the uniform estimates (3.29), (3.30), (3.31) and (3.35) given in the last section still hold for the approximate solution $(\rho^{\varepsilon,\nu}, \theta^{\varepsilon,\nu})$. We rewrite the first equation in (3.2) by

$$\rho_t = -f_x + g$$

with g uniformly bounded in $L^2(Q_T)$ and

$$f = \big(\varepsilon + (\rho\theta)_\nu\big)\rho_x + \rho(\rho_\varepsilon\theta_x)_\varepsilon.$$

Since ρ is uniformly bounded in $L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)) \hookrightarrow L^6(Q_T)$, we derive that

$$\big\|\rho(\rho_\varepsilon\theta_x)_\varepsilon\big\|_{L^2(Q_T)}, \big\|(\rho\theta)_\nu\big\|_{L^6(Q_T)}, \big\|(\rho\theta)_\nu\rho_x\big\|_{L^{\frac{5}{4}}(Q_T)} \leqslant C_{\varepsilon,T}$$

and

$$\|\rho_t\|_{L^{5/4}(I; W_0^{-1,5/4}(\Omega))} \leqslant C_{\varepsilon,T}.$$

From the first equation in (3.1) we derive that

$$\left\|(\rho\theta+\sigma\theta)_t\right\|_{L^{5/4}(I;W_0^{-1,5/4}(\Omega))}\leqslant C_{\varepsilon,T}$$

where we have noted (3.29), and moreover, from (3.31), we observe that $\rho^{\varepsilon,\nu}$ is uniformly bounded in $L^6(I;L^6(\Omega))\cap L^2(I;H^1(\Omega))$ and $\rho_t^{\varepsilon,\nu}$ is uniformly bounded in $L^{5/4}(I;W_0^{-1,5/4}(\Omega))$. Using Aubin–Lions lemma, we conclude that there exists a sequence $\nu_j\to 0$ such that

$$\begin{aligned}
&\rho^{\varepsilon,\nu_j}\to\rho^{\varepsilon}\quad\text{strongly in } L^p(Q_T)\ (\forall 1\leqslant p<6),\\
&\rho^{\varepsilon,\nu_j}\to\rho^{\varepsilon}\quad\text{strongly in } L^2\big(I;C(\overline{\Omega})\big),\\
&\rho^{\varepsilon,\nu_j}\rightharpoonup\rho^{\varepsilon}\quad\text{weakly in } L^2\big(I,H^1(\Omega)\big),\\
&\rho_t^{\varepsilon,\nu_j}\rightharpoonup\rho_t^{\varepsilon}\quad\text{weakly in } L^{5/4}\big(I;W_0^{-1,5/4}(\Omega)\big)
\end{aligned}\tag{4.1}$$

and

$$\rho^{\varepsilon,\nu_j}(0,\cdot)\to\rho^{\varepsilon}(0,\cdot)\quad\text{and}\quad\rho^{\varepsilon,\nu_j}(1,\cdot)\to\rho^{\varepsilon}(1,\cdot)\quad\text{strongly in } L^2(0,T).$$

Similarly, by noting the uniform estimates (3.29), (3.30) and (3.35), we conclude that there exists a subsequence of $\theta^{\varepsilon,\nu_j}$ (also denoted by $\theta^{\varepsilon,\nu_j}$) such that

$$\begin{aligned}
&\theta^{\varepsilon,\nu_j}\to\theta^{\varepsilon}\quad\text{strongly in } L^p(Q_T)\ (\forall 1\leqslant p<\infty),\\
&\theta^{\varepsilon,\nu_j}\to\theta^{\varepsilon}\quad\text{strongly in } L^2\big(I;C(\overline{\Omega})\big),\\
&\theta^{\varepsilon,\nu_j}\rightharpoonup\theta^{\varepsilon}\quad\text{weakly in } L^2\big(I,H^1(\Omega)\big),\\
&\big(\rho^{\varepsilon,\nu_j}\theta^{\varepsilon,\nu_j}+\sigma\theta^{\varepsilon,\nu_j}\big)_t\rightharpoonup\big(\rho^{\varepsilon}\theta^{\varepsilon}+\sigma\theta^{\varepsilon}\big)_t\quad\text{weakly in } L^{5/4}\big(I;W_0^{-1,5/4}(\Omega)\big)
\end{aligned}\tag{4.2}$$

and

$$\theta^{\varepsilon,\nu_j}(0,\cdot)\to\theta^{\varepsilon}(0,\cdot)\quad\text{and}\quad\theta^{\varepsilon,\nu_j}(1,\cdot)\to\theta^{\varepsilon}(1,\cdot)\quad\text{strongly in } L^p(0,T),\ 1\leqslant p<\infty.$$

Since $(\rho^{\varepsilon,\nu_j},\theta^{\varepsilon,\nu_j})$ is a strong solution of the system (3.1) and (3.3), it satisfies

$$\begin{aligned}
&\int_0^T\alpha^0\big(\rho^{\varepsilon,\nu_j}(0,t)-\overline{\rho}^0\big)\phi(0,t)\,dt+\int_0^T\alpha^1\big(\rho^{\varepsilon,\nu_j}(1,t)-\overline{\rho}^1\big)\phi(1,t)\,dt\\
&\quad+\int_0^T\int_\Omega\rho_t^{\varepsilon,\nu_j}\phi\,dx\,dt+\int_0^T\Big[\Big(\varepsilon+\big(\rho^{\varepsilon,\nu_j}\theta^{\varepsilon,\nu_j}\big)_\nu\Big)\rho_x^{\varepsilon,\nu_j}+\rho^{\varepsilon,\nu_j}\big(\rho_\varepsilon^{\varepsilon,\nu_j}\theta_x^{\varepsilon,\nu_j}\big)_\varepsilon\Big]\phi_x\,dx\,dt\\
&=\int_0^T\int_\Omega p_s\big(\theta^{\varepsilon,\nu_j}\big)\,dx\,dt-\int_0^T\int_\Omega\rho^{\varepsilon,\nu_j}\sqrt{\theta^{\varepsilon,\nu_j}}\phi\,dx\,dt
\end{aligned}$$

and

$$\int_0^T\int_0^1 \big[\big(\rho^{\varepsilon,\nu_j}+\sigma\big)\theta^{\varepsilon,\nu_j}\big]_t\psi\,dx\,dt+\int_0^T \beta^0\big(\theta^{\varepsilon,\nu_j}(0,t)-\overline{\theta}^0\big)\psi(0,t)\,dt$$
$$+\int_0^T \beta^1\big(\theta^{\varepsilon,\nu_j}(1,t)-\overline{\theta}^1\big)\psi(1,t)\,dt+\int_0^T \alpha^0\big(\rho^{\varepsilon,\nu_j}(0,t)-\overline{\rho}^0\big)\theta^{\varepsilon,\nu_j}(0,t)\psi(0,t)\,dt$$
$$+\int_0^T \alpha^1\big(\rho^{\varepsilon,\nu_j}(1,t)-\overline{\rho}^1\big)\theta^{\varepsilon,\nu_j}(1,t)\psi(1,t)\,dt+\int_0^T\int_0^1 \kappa^\varepsilon\theta_x^{\varepsilon,\nu_j}\psi_x\,dx\,dt$$
$$+\int_0^T\int_0^1 \big[\big(\varepsilon+\big(\rho^{\varepsilon,\nu_j}\theta^{\varepsilon,\nu_j}\big)_\nu\big)\rho_x^{\varepsilon,\nu_j}\theta^{\varepsilon,\nu_j}+\rho^{\varepsilon,\nu_j}\big(\rho_\varepsilon^{\varepsilon,\nu_j}\theta_x^{\varepsilon,\nu_j}\big)_\varepsilon\theta^{\varepsilon,\nu_j}\big]\psi_x\,dx\,dt$$
$$=\lambda\int_0^T\int_0^1 \big[\rho^{\varepsilon,\nu_j}\sqrt{\theta^{\varepsilon,\nu_j}}-p_s\big(\theta^{\varepsilon,\nu_j}\big)\big]\psi\,dx\,dt,$$

for any $\phi,\psi\in L^5(I;W^{1,5}(\Omega))$. By taking the limit $j\to\infty$, we obtain a global weak solution $(\rho^\varepsilon,\theta^\varepsilon)$ to the approximate system

$$\begin{aligned}\rho_t-\big((\varepsilon+\rho\theta)\rho_x\big)_x-\big(\rho(\rho_\varepsilon\theta_x)_\varepsilon\big)_x&=p_s(\theta)-\rho\sqrt{\theta},\\ (\rho\theta+\sigma\theta)_t-\big(\kappa^\varepsilon\theta_x\big)_x-\big((\varepsilon+\rho\theta)\rho_x\theta\big)_x-\big(\rho(\rho_\varepsilon\theta_x)_\varepsilon\theta\big)_x&=\lambda\big(\rho\sqrt{\theta}-p_s(\theta)\big),\end{aligned}\tag{4.3}$$

with the boundary and initial conditions

$$\begin{aligned}&(\varepsilon+\rho\theta)\rho_x+\rho(\rho\theta_x)_\varepsilon|_{x=1}=\alpha^1\big(\overline{\rho}^1-\rho(1,t)\big),\\ &(\varepsilon+\rho\theta)\rho_x+\rho(\rho\theta_x)_\varepsilon|_{x=0}=\alpha^0\big(\rho(0,t)-\overline{\rho}^0\big),\\ &\rho(x,0)=\rho_{0\varepsilon}(x):=\rho_0*\eta_\varepsilon(x)+\varepsilon,\\ &\kappa^\varepsilon\theta_x\big|_{x=1}=\beta^1\big(\overline{\theta}^1-\theta(1,t)\big),\\ &\kappa^\varepsilon\theta_x\big|_{x=0}=\beta^0\big(\theta(0,t)-\overline{\theta}^0\big),\\ &\theta(x,0)=\theta_{0\varepsilon}(x):=\theta_0*\eta_\varepsilon(x).\end{aligned}\tag{4.4}$$

Secondly, we study the convergence as $\varepsilon\to 0$. To take the limit $\varepsilon\to 0$, we need more uniform estimates for ρ with respect to ε.

Clearly the system (3.2)–(3.3) reduces to the system (4.3)–(4.4) when $\nu=0$. Then the uniform estimates (3.17) and (3.18) hold for the obtained solution $(\rho^\varepsilon,\theta^\varepsilon)$. From (3.32) we see that

$$\|\rho\theta\rho_x\|_{L^2(Q_T)}\leqslant C_{\varepsilon,T}$$

and from the first equation of (4.3) we deduce that $\rho_t\in L^2(I;H_0^{-1}(\Omega))$. Note that $\ln\rho\in L^2(I;H^1(\Omega))$. By multiplying the first equation of (4.3) by $\ln\rho$ and integrating the resulting equation over Q_t, we arrive at

$$\int_0^1 \rho\ln\rho(x,t)\,dx - \int_0^1 \rho(x,t)\,dx + \int_0^t \big[\varepsilon\rho_x + \rho\theta\rho_x + \rho(\rho_\varepsilon\theta_x)_\varepsilon\big]\ln\rho\Big|_{x=0}^{x=1}d\tau + \int_0^t\int_0^1 \theta\rho_x^2\,dx\,d\tau$$
$$\leqslant \int_0^1 \rho_{0\varepsilon}\ln\rho_{0\varepsilon}(x)\,dx - \int_0^1 \rho_{0\varepsilon}\,dx - \int_0^t\int_0^1 (\rho_\varepsilon\theta_x)_\varepsilon\rho_x\,dx\,d\tau - \int_0^t\int_0^1 \big(\rho\sqrt{\theta} - p_s(\theta)\big)\ln\rho\,dx\,d\tau.$$

Since

$$\int_0^t\int_0^1 \big|(\rho_\varepsilon\theta_x)_\varepsilon\rho_x\big|\,dx\,d\tau \leqslant \frac{1}{2}\int_0^t\int_0^1 \theta\rho_x^2\,dx\,d\tau + \frac{1}{2}\int_0^t\int_0^1 \frac{|(\rho_\varepsilon\theta_x)_\varepsilon|^2}{\theta}\,dx\,d\tau$$
$$\leqslant \frac{1}{2}\int_0^t\int_0^1 \theta\rho_x^2\,dx\,d\tau + C_T\|\rho_\varepsilon\theta_x\|^2_{L^2(Q_T)}$$
$$\leqslant \frac{1}{2}\int_0^t\int_0^1 \theta\rho_x^2\,dx\,d\tau + C_T,$$

we get

$$\int_{[0,1]\cap\{\rho\geqslant 1\}} \rho\ln\rho(x,t)\,dx + \frac{1}{2}\int_0^t\int_0^1 \theta\rho_x^2\,dx\,d\tau + \iint_{[0,1]\times[0,t]\cap\{\rho\geqslant 1\}} \rho\ln\rho\,dx\,d\tau$$
$$\leqslant \int_0^1 \rho_{0\varepsilon}|\ln\rho_{0\varepsilon}|(x)\,dx + \int_{[0,1]\cap\{\rho\leqslant 1\}} \rho|\ln\rho|(x,t)\,dx$$
$$+ \iint_{[0,1]\times[0,t]\cap\{\rho\leqslant 1\}} \rho|\ln\rho|\,dx\,d\tau + \iint_{[0,1]\times[0,t]\cap\{\rho\geqslant 1\}} p_s(\theta)\ln\rho\,dx\,d\tau + C_T$$
$$\leqslant C_T,$$

which, together with (3.35), leads to

$$\|\rho\ln\rho\|_{L^\infty(0,T;L^1(\Omega))},\ \|\rho_x\|_{L^2(Q_T)} \leqslant C_T. \tag{4.5}$$

From the inequalities (3.28) and (4.5) we derive that

$$\|\rho\|_{L^2(0,T;H^1(\Omega))} \leqslant C_T \tag{4.6}$$

and

$$\|\rho\|^3_{L^\infty(\Omega)} \leqslant \|\rho\|^3_{L^1(\Omega)} + \|\rho\|^{3/2}_{L^2(\Omega)}\|\rho_x\|^{3/2}_{L^2(\Omega)}$$
$$\leqslant C_T + C\|\rho\|^{3/4}_{L^1(\Omega)}\|\rho\|^{3/4}_{L^\infty(\Omega)}\|\rho_x\|^{3/2}_{L^2(\Omega)}$$
$$\leqslant C_T + \frac{1}{2}\|\rho\|^3_{L^\infty(\Omega)} + C_T\|\rho_x\|^2_{L^2(\Omega)},$$

which results in

$$\int_0^T \|\rho\|_{L^\infty(\Omega)}^3 \, dt \leqslant C_T + C_T \int_0^T \|\rho_x\|_{L^2(\Omega)}^2 \, dt \leqslant C_T.$$

It follows that

$$\int_0^T \int_0^1 \rho^4 \, dx \, dt \leqslant \left(\int_0^T \|\rho\|_{L^\infty(\Omega)}^3 \, dt \right) \left(\sup_{0 \leqslant t \leqslant T} \int_0^1 \rho \, dx \right) \leqslant C_T, \tag{4.7}$$

i.e. ρ is uniformly bounded in $L^4(Q_T)$.

Finally, we let

$$B_1 = H^1(\Omega), \qquad B_2 = L^4(\Omega), \qquad B_3 = W_0^{-1,6/5}(\Omega).$$

Then $B_1 \hookrightarrow\hookrightarrow B_2 \hookrightarrow B_3$ and $\{\rho^\varepsilon\}$ is uniformly bounded in $L^4(I; B_2) \cap L^2(I; B_1)$. From the first equation in (3.2), i.e.

$$\rho_t = \left[\varepsilon \rho_x + \rho\theta\rho_x + \rho(\rho\theta_x)_\varepsilon\right]_x - \rho\sqrt{\theta} + p_s(\theta),$$

we observe that $\{\rho_t^\varepsilon\}$ is uniformly bounded in $L^{6/5}(I; B_3)$. By Aubin–Lions lemma, $\{\rho^\varepsilon\}$ is relatively compact in $L^p(I; L^4(\Omega))$ for $1 \leqslant p < 4$. Thus, there exists a sequence ρ^{ε_j} such that $\lim_{j\to\infty} \varepsilon_j = 0$ and

$$\begin{aligned}
&\rho^{\varepsilon_j} \to \rho \quad \text{strongly in } L^p\big(I, L^4(\Omega)\big) \ (\forall 1 \leqslant p < 4),\\
&\rho^{\varepsilon_j} \to \rho \quad \text{strongly in } L^2\big(I, C(\overline{\Omega})\big),\\
&\rho^{\varepsilon_j} \rightharpoonup \rho \quad \text{weakly in } L^2\big(I, H^1(\Omega)\big),\\
&\rho_t^{\varepsilon_j} \rightharpoonup \rho_t \quad \text{weakly in } L^{6/5}\big(I; W_0^{-1,6/5}(\Omega)\big).
\end{aligned} \tag{4.8}$$

Similarly, by (3.17) and (3.18), there exists a subsequence of θ^{ε_j} (also denoted by θ^{ε_j}) such that

$$\begin{aligned}
&\theta^{\varepsilon_j} \to \theta \quad \text{strongly in } L^p(Q_T) \ (\forall 1 \leqslant p < \infty),\\
&\theta^{\varepsilon_j} \to \theta \quad \text{strongly in } L^2\big(I, C(\overline{\Omega})\big),\\
&\theta^{\varepsilon_j} \rightharpoonup \theta \quad \text{weakly in } L^2\big(I, H^1(\Omega)\big),\\
&\big(\rho^{\varepsilon_j}\theta^{\varepsilon_j} + \sigma\theta^{\varepsilon_j}\big)_t \rightharpoonup (\rho\theta + \sigma\theta)_t \quad \text{weakly in } L^{6/5}\big(I; W_0^{-1,6/5}(\Omega)\big).
\end{aligned} \tag{4.9}$$

Now we take the limit $j \to \infty$ and by (4.8) and (4.9), we obtain a weak solution (ρ, θ) which satisfies (2.1) and (2.2). The proof of Theorem 2.1 is complete.

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Appendix A. The solution to Eq. (3.7)–(3.8)

In Section 3.1, we claimed that the semi-linear parabolic equation (3.7)–(3.8) has a unique solution in $W_2^{2,1}(Q_T)$. Here, we present the detailed proof. The problem (3.7)–(3.8) can be viewed as in the following form:

$$\big(\rho(x,t)+\sigma\big)\theta_t - \big(\kappa^\varepsilon\theta_x\big)_x - b(x,t)\theta_x - c(x,t)\theta + p(\theta) = f(x,t) \tag{A.1}$$

with the initial and boundary conditions

$$\begin{cases} \kappa^\varepsilon\theta_x = \beta^1\big(\overline{\theta}^1 - \theta\big), & \text{at } x=1, \\ \kappa^\varepsilon\theta_x = \beta^0\big(\theta - \overline{\theta}^0\big), & \text{at } x=0, \\ \theta(x,0) = \theta_0(x), & \text{for } x\in\Omega, \end{cases} \tag{A.2}$$

where $\rho \in W_2^{2,1}(Q_T)$, $\kappa^\varepsilon \in C^\infty(Q_T)$, $c \in L^\infty(Q_T)$, $f \in L^\infty(Q_T)$, $|b| \leqslant C + C|\rho_x|$ and hence $b \in L^2(I; L^\infty(\Omega))$. The only nonlinearity is $p(\theta) = (\lambda+\theta)p_s(\theta)$, which is a continuous function of $\theta \in \mathbb{R}$ (see (1.9) for the assumption of the function $p_s(\theta)$).

Eq. (A.1) can be written as the classical form

$$\theta_t - \frac{\kappa^\varepsilon}{\rho+\sigma}\theta_{xx} - \frac{\kappa_x^\varepsilon + b}{\rho+\sigma}\theta_x - \frac{c}{\rho+\sigma}\theta + \frac{1}{\rho+\sigma}p(\theta) = \frac{f}{\rho+\sigma}. \tag{A.3}$$

We can obtain the existence of solution for the above equation by a fixed point argument. Let $\theta^0 \in L^\infty(Q_T)$ and θ be the solution of the following linear parabolic equation

$$\theta_t - \frac{\kappa^\varepsilon}{\rho+\sigma}\theta_{xx} - \frac{\kappa_x^\varepsilon + b}{\rho+\sigma}\theta_x - \frac{c}{\rho+\sigma}\theta = \frac{f}{\rho+\sigma} - \frac{1}{\rho+\sigma}p\big(\theta^0\big). \tag{A.4}$$

Then

$$\|\theta\|_{W_2^{2,1}(Q_T)} \leqslant C\|f\|_{L^\infty(Q_T)} + C\big\|p\big(\theta^0\big)\big\|_{L^\infty(Q_T)}.$$

Clearly the map M_1 from $\theta^0 \in L^\infty(Q_T)$ to $\theta \in W_2^{2,1}(Q_T)$ is continuous and compact. Because $W_2^{2,1}(Q_T) \hookrightarrow\hookrightarrow L^\infty(Q_T)$ for $\Omega \subset \mathbb{R}^1$, a fixed point of the map is a solution of Eq. (A.3) or (A.1).

By the Leray–Schauder fixed point theorem, it suffices to prove that all θ^0 which satisfy the following equation

$$\theta^0 = sM_1\theta^0 \tag{A.5}$$

for some $s \in [0,1]$ are uniformly bounded in $L^\infty(Q_T)$.

In fact, if we let $\theta = M_1\theta^0$, then $\theta^0 = s\theta$. It follows that θ is the solution of the following equation

$$\theta_t - \frac{\kappa^\varepsilon}{\rho+\sigma}\theta_{xx} - \frac{\kappa_x^\varepsilon + b}{\rho+\sigma}\theta_x - \frac{c}{\rho+\sigma}\theta + \frac{1}{\rho+\sigma}p(s\theta) = \frac{f}{\rho+\sigma}.$$

Converting the above equation into the divergence form, we have

$$\theta_t - \left(\frac{\kappa^\varepsilon}{\rho+\sigma}\theta_x\right)_x + \tilde{b}\theta_x - \frac{c}{\rho+\sigma}\theta + \frac{1}{\rho+\sigma}p(s\theta) = \frac{f}{\rho+\sigma}, \tag{A.6}$$

where

$$|\tilde{b}| = \left| \left(\frac{\kappa^{\varepsilon}}{\rho+\sigma} \right)_x - \frac{\kappa_x^{\varepsilon}+b}{\rho+\sigma} \right| \leqslant C + C|\rho_x|$$

and $\tilde{b} \in L^2(I; L^{\infty}(\Omega))$. By the definition of $p_s(\theta)$, $p_s(\theta)=0$ for $\theta \leqslant 0$ and $p_s(\theta)>0$ for $\theta>0$. Therefore, multiplying Eq. (A.6) by θ^k with k being an odd integer and integrating the result, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega} \frac{\theta^{k+1}}{k+1}\,dx + \int_{\Omega} \frac{\kappa^{\varepsilon}k}{\rho+\sigma}\theta^{k-1}|\theta_x|^2\,dx + \frac{\beta_0(\theta-\overline{\theta}^0)\theta^k}{\rho+\sigma}\bigg|_{x=0} + \frac{\beta_1(\theta-\overline{\theta}^1)\theta^k}{\rho+\sigma}\bigg|_{x=1} \\ &\quad \leqslant \sigma^{-1}\int_{\Omega} \big(\|f\|_{L^\infty(Q_T)}\theta^k + \|c\|_{L^\infty(Q_T)}\theta^{k+1}\big)\,dx + \frac{C}{\delta}\|\tilde{b}\|^2_{L^\infty(\Omega)}\int_{\Omega}\theta^{k+1}\,dx + \delta\int_{\Omega}\theta^{k-1}|\theta_x|^2\,dx \end{aligned}$$

which implies that

$$\frac{d}{dt}\int_{\Omega}\theta^{k+1}\,dx \leqslant (k+1)C^{k+1} + (k+1)\big(C + \|\tilde{b}\|^2_{L^\infty(\Omega)}\big)\int_{\Omega}\theta^{k+1}\,dx.$$

By using Gronwall's inequality, we further get

$$\max_{0\leqslant t\leqslant T}\int_{\Omega}\theta^{k+1}\,dx \leqslant e^{(k+1)\int_0^T (C+\|\tilde{b}\|^2_{L^\infty(\Omega)})\,dt}\left[(k+1)C^{k+1} + \int_{\Omega}\theta_0^{k+1}\,dx\right].$$

It follows that

$$\|\theta\|_{L^{k+1}(Q_T)} \leqslant T^{\frac{1}{k+1}} e^{\int_0^T (C+\|\tilde{b}\|^2_{L^\infty(\Omega)})\,dt}\left[(k+1)C^{k+1} + \int_{\Omega}\theta_0^{k+1}\,dx\right]^{\frac{1}{k+1}} \leqslant C,$$

where the constant C is independent of k. Let $k \to \infty$, we obtain

$$\|\theta\|_{L^\infty(Q_T)} \leqslant C. \tag{A.7}$$

Thus we have proved the existence of a solution $\theta \in W_2^{2,1}(Q_T)$ for Eq. (A.3), which is also a solution for Eq. (A.1), or equivalently a solution of Eq. (A.6) with $s=1$. The uniqueness follows immediately by noting the assumption (1.9) for the function $p_s(\theta)$.

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