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Vanishing viscosity of isentropic Navier-Stokes equations for interacting shocks

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Abstract We study the vanishing viscosity of the Navier-Stokes equations for interacting shocks. Given an entropy solution to p-system which consists of two different families of shocks interacting at some positive time, we show that such entropy solution is the vanishing viscosity limit of a family of global smooth solutions to the isentropic Navier-Stokes equations. The key point of the proofs is to derive the estimates separately before and after the interaction time and connect the incoming and outgoing viscous shock profiles.

Keywords isentropic Navier-Stokes equations, isentropic Euler equations, interacting shock, vanishing viscosity, entropy solution

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1 Introduction

The vanishing viscosity limit of solutions to compressible Navier-Stokes system to those of Euler system has been an open and challenging problem with a long history. In the case that the solution to the Euler system is smooth, it is easy to solve this problem by scaling method. However, it is well known that the Euler system is a typical hyperbolic system of conservation laws and its solution generally develops singularity in finite time, for example, shock waves, which has so far prevented solving the problem in the general setting by means of known analytic techniques and tools. Essential new ideas are needed to deal with this open problem. Therefore, any attempt on this problem that involves the singularity in the entropy solution to Euler equations can be viewed as progress for the general case.

Let us recall some related works on the vanishing viscosity limit of the viscous fluid system. For a system of hyperbolic conservation laws with artificial viscosity

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \tag{1.1}$$

Goodman and Xin [7] proved the limit in the L^∞-norm for piecewise smooth solutions separated by non-interacting shock waves by introducing a matched asymptotic expansion method. Later, Yu [25] extended the result to the case admitting both shock and initial layers. Serre [20] showed the vanishing viscosity limit associated with interacting shock waves in the L^2-norm, and the method can be adapted to the

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physical viscosity system. In 2005, a breakthrough was made by Bianchini and Bressan [1], in which they justified the vanishing viscosity limit in BV space for small BV initial data, but without convergence rate. This result was improved by Bressan and Yang [3] for the case where all characteristic fields are genuinely nonlinear that u^ε converges to u in the L^1-norm with almost optimal rate $O(1)\cdot\sqrt{\varepsilon}|\ln\varepsilon|$. Recently, Bressan et al. [2] further extended the result of [3] to the general systems where each characteristic field is either genuinely non-linear or linearly degenerate. For large L^∞ initial data, the vanishing viscosity limit to the γ-law gas dynamics was obtained by the compensated compactness argument [5, 14, 17].

However, the vanishing viscosity limit of physical systems like compressible Navier-Stokes equations is very challenging. For the isentropic Navier-Stokes equations, Hoff and Liu [8] first proved the vanishing viscosity limit in the case of single shock with initial layer. Later, Xin [23] did similar result for rarefaction wave, and also obtained a convergence rate in the case without initial layer. For the superposition of two shock waves, we refer to [27]. For the limit of nonisentropic Navier-Stokes equations, we refer to Jiang et al. [15] for the rarefaction wave, Wang [22] for the shock wave, Ma [18] for the contact discontinuity. It is known that the Riemann solution to the nonisentropic Euler equations is the linear superposition of three basic wave patterns: Shock, rarefaction wave and contact discontinuity, see [21]. The Riemann solution plays fundamental role in the theory of Euler equations as it captures both the local and global behaviors of general solutions. Recently, there are some progress on the vanishing viscosity limit to the Riemann solution for Navier-Stokes system, see [13] for the superposition of rarefaction wave and shock wave, and [9] for the superposition of rarefaction wave and contact discontinuity, see [28] for the superposition of two shock waves. For the hydrodynamic limit of the Boltzmann equation to the compressible Euler equations as the Knudsen number tends to zero, we refer to [10–12, 24, 26] and reference therein.

However, all of the above results only concern with the case that all distinct waves are separated in any positive time. To develop the theory of the vanishing viscosity limit of the physical system, it is important to understand the interaction of basic wave patterns. As a starting point along this direction, we begin to study the vanishing viscosity limit in the case of two different families of interacting shock waves in the present paper. More precisely, given an entropy solution to p-system which consists of two different families of shocks interacting at some positive time, our purpose is to construct a family of global smooth solutions of the isentropic Navier-Stokes equations and expect that these solutions converge to the given entropy solution.

Let us formulate our main result. The 1-D isentropic Navier-Stokes system in the Lagrangian coordinates reads

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \varepsilon\left(\dfrac{u_x}{v}\right)_x, \end{cases} \quad x \in \mathbb{R}, \quad t > 0, \tag{1.2}$$

where the functions $v(t,x) > 0, u(t,x)$ represent the specific volume, velocity of the gas respectively, and $\varepsilon > 0$ is the viscosity coefficient. The pressure $p = p(v)$ is assumed to satisfy

$$p'(v) < 0 < p''(v), \quad \text{for } v > 0. \tag{1.3}$$

For example, (1.3) holds for polytropic gases, for which $p = v^{-\gamma}$ and $\gamma > 1$ is the so called adiabatic exponent. Formally as the coefficient ε tends to zero, the limiting system of (1.2) is the p-system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \end{cases} \quad x \in \mathbb{R}, \quad t > 0, \tag{1.4}$$

where the first eigenvalue $\lambda_1 = -\sqrt{-p'(v)} < 0$ and the second eigenvalue $\lambda_2 = \sqrt{-p'(v)} > 0$.

Given two incoming shocks denoted by S_1, S_2, respectively, where S_1 is of the 1-shock which connects (v_m, u_m) as the left state and (v_+, u_+) as the right state with the speed $s_1 < 0$, while S_2 is of the 2-shock which connects (v_-, u_-) as the left state and (v_m, u_m) as the right state with the speed $s_2 > 0$, see Figure 1 below. The intermediate state between the two incoming shocks is (v_m, u_m). Since the shock

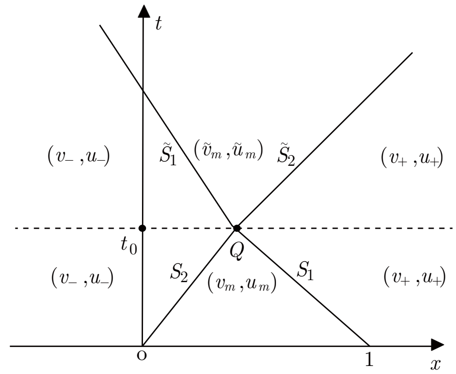

Figure 1 Entropy solution

wave S_2 propagates faster than S_1, the two shocks must interact at some point $Q \doteq (x_0, t_0)$ with $t_0 > 0$. After the interacting time, two outgoing shocks are generated, and denoted by $\tilde{S}_1$ and $\tilde{S}_2$. In this case, the intermediate state between the two outgoing shocks is no longer (v_m, u_m), but a new state $(\tilde{v}_m, \tilde{u}_m)$ determined by the R-H conditions (1.11) below. In this setting, without loss of generality, we assume the initial data of p-system (1.4) is

$$(v, u)(x, t = 0) = \begin{cases} (v_-, u_-), & x < 0, \\ (v_m, u_m), & 0 < x < 1, \\ (v_+, u_+), & x > 1. \end{cases} \tag{1.5}$$

We denote (v^R, u^R) as the unique entropy solution to (1.4) and (1.5). It is obvious that the formula of (v^R, u^R) before the interaction time t_0 is given by

$$(v^R, u^R)(x, t) = \begin{cases} (v_-, u_-), & x < s_2 t, \quad t \leqslant t_0, \\ (v_m, u_m), & s_2 t < x < s_1 t + 1, \quad t \leqslant t_0, \\ (v_+, u_+), & x > s_1 t + 1, \quad t \leqslant t_0. \end{cases} \tag{1.6}$$

By a simple calculation, the two shocks hit at the point $Q =: (x_0, t_0) = (\frac{s_2}{s_2 - s_1}, \frac{1}{s_2 - s_1})$, where $s_i,\ i = 1, 2$ are determined by the following R-H conditions:

$$\begin{cases} s_1(v_+ - v_m) = -(u_+ - u_m), \\ s_1(u_+ - u_m) = p(v_+) - p(v_m), \end{cases} \quad \text{and} \quad \begin{cases} s_2(v_m - v_-) = -(u_m - u_-), \\ s_2(u_m - u_-) = p(v_m) - p(v_-), \end{cases} \tag{1.7}$$

and the entropy conditions

$$\lambda_1(v_+) < s_1 < \lambda_1(v_m) < 0 < \lambda_2(v_m) < s_2 < \lambda_2(v_-). \tag{1.8}$$

After the interaction time t_0, it is equivalent to resolve the Riemann problem of the p-system (1.4) at time $t = t_0$ with the data

$$(v, u)(x, t = t_0) = \begin{cases} (v_-, u_-), & x < x_0, \\ (v_+, u_+), & x > x_0. \end{cases} \tag{1.9}$$

The explicit formula of (v^R, u^R) at $t > t_0$ is

$$(v^R, u^R)(x, t) = \begin{cases} (v_-, u_-), & x - x_0 < \tilde{s}_1(t - t_0), \quad t > t_0, \\ (\tilde{v}_m, \tilde{u}_m), & \tilde{s}_1(t - t_0) < x - x_0 < \tilde{s}_2(t - t_0), \quad t > t_0, \\ (v_+, u_+), & x - x_0 > \tilde{s}_2(t - t_0), \quad t > t_0. \end{cases} \tag{1.10}$$

The two shock speeds $\tilde{s}_i,\ i=1,2$ are given by R-H conditions

$$\begin{cases}\tilde{s}_1(\tilde{v}_m-v_-)=-(\tilde{u}_m-u_-),\\ \tilde{s}_1(\tilde{u}_m-u_-)=p(\tilde{v}_m)-p(v_-),\end{cases} \quad \text{and} \quad \begin{cases}\tilde{s}_2(v_+-\tilde{v}_m)=-(u_+-\tilde{u}_m),\\ \tilde{s}_2(u_+-\tilde{u}_m)=p(v_+)-p(\tilde{v}_m),\end{cases} \tag{1.11}$$

and the entropy conditions

$$\lambda_1(\tilde{v}_m)<\tilde{s}_1<\lambda_1(v_-)<0<\lambda_2(v_+)<\tilde{s}_2<\lambda_2(\tilde{v}_m). \tag{1.12}$$

We set

$$\delta_1=|v_+-v_m| \quad \text{and} \quad \delta_2=|v_--v_m| \tag{1.13}$$

be the wave strength of the two incoming shocks, respectively, and $\delta=\min\{\delta_1,\delta_2\}$. Then, if it holds

$$\delta_1+\delta_2\leqslant C\delta, \quad \delta_1+\delta_2\to 0, \tag{1.14}$$

for a positive constant C, we call the strengths of the shock waves "small with the same order". In what follows, we always assume (1.14). For later use, we denote

$$\tilde{\delta}_1=|v_--\tilde{v}_m| \quad \text{and} \quad \tilde{\delta}_2=|v_+-\tilde{v}_m|, \tag{1.15}$$

and from [21] it holds that

$$\tilde{\delta}_1=\delta_1+O(1)\delta_1\delta_2 \quad \text{and} \quad \tilde{\delta}_2=\delta_2+O(1)\delta_1\delta_2, \quad \text{if } \delta \text{ is small}, \tag{1.16}$$

which gives a relation of the wave strength between the incoming and outgoing shock waves.

The main result of this paper is the following.

Theorem 1.1. *Let $(v^R,u^R)(t,x)$ be the entropy solution to p-system (1.4), (1.5) defined in (1.6), (1.10). There exists a small positive constant δ_0, such that if the wave strength satisfies $\delta\leqslant\delta_0$, the Navier-Stokes system (1.2) admits a family of global smooth solutions $(v^\varepsilon,u^\varepsilon)(t,x)$ with well-prepared initial data (3.2) below for any $\varepsilon>0$. Moreover, before the interaction time t_0, it holds that*

$$\|(v^\varepsilon-v^R,u^\varepsilon-u^R)\|_{L^\infty(\Sigma_h)}\leqslant C\delta^{\frac{1}{3}}\exp\left(-\frac{c_0\delta h}{\varepsilon}\right), \tag{1.17}$$

where $\Sigma_h=\{(t,x)\mid 0\leqslant t\leqslant t_0-h, |x-s_2t|\geqslant h, |x-s_1t-1|\geqslant h\}$ for any give positive constant $h>0$. After the interaction time t_0, it holds that

$$\|(v^\varepsilon-v^R,u^\varepsilon-u^R)\|_{L^\infty(\tilde{\Sigma}_{\tilde{h}})}\to 0, \quad \textit{uniformly on the set } \tilde{\Sigma}_{\tilde{h}}, \quad \textit{as } \varepsilon\to 0, \tag{1.18}$$

where $\tilde{\Sigma}_{\tilde{h}}=\{(t,x)\mid t_0+\tilde{h}\leqslant t\leqslant+\infty, |(x-x_0)-\tilde{s}_1(t-t_0)|\geqslant\tilde{h}, |(x-x_0)-\tilde{s}_2(t-t_0)|\geqslant\tilde{h}\}$ for any given positive constant $\tilde{h}>0$.

Remark 1.2. The key point of the proofs is to derive the estimates separately before and after the interaction time and connect the incoming and outgoing viscous shocks, and this is mainly due to our key Lemma 4.1 below which enables us to construct a suitable profile after the interaction time.

Remark 1.3. Since Theorem 1.1 is concerned with the L^∞*-norm* which gives a detailed description on the vanishing viscosity limit, we require the initial data of Navier-Stokes system (1.2) depends on the viscosity coefficient which converges to the initial data (1.5) of the p-system as $\varepsilon\to 0$. If instead we consider the Navier-Stokes system (1.2) with the same initial data (1.5) as the inviscid solution, we can also get the vanishing viscosity limit by the same method in the L^2-norm (but not in L^∞-norm), i.e., $\|(v^\varepsilon-v^R,u^\varepsilon-u^R)\|_{L^2_x(\mathbb{R})}\to 0$ as $\varepsilon\to 0$. However, this is not the main concern in the present paper.

Remark 1.4. Theorem 1.1 only concerns the case of the interaction of two different family of shocks. If the interacting shocks are of the same family, the problem become very difficult because the interaction of the same family of shocks for gas dynamics is complicated itself.

Remark 1.5. It is interesting to study the problem of the vanishing viscosity of non-isentropic Navier-Stokes system for the interacting shocks. Even in the simplest case that the incoming shocks are of different families, the interaction in general generates not only two outgoing shocks, but also a contact discontinuity. It is also interesting to consider the hydrodynamic limit to the interacting shocks for the Boltzmann equation.

On the other hand, Chen and Perepelitsa [4] recently proved the vanishing viscosity limit of isentropic Navier-Stokes system to the isentropic gas dynamics by using the compensated compactness for large initial data if the far field does not contain vacuum. Note that this result is very general because it allows initial data containing vacuum in the interior domain. However, the limit in [4] was obtained in the sense of $L^1_{\rm loc}$ convergence without rate, and the structure of the limit solution is not clear yet.

The rest of this paper is arranged as follows. In Section 2, we give some known results on the viscous shock wave. In Section 3, we reformulate the original system (1.2) and study it before the interacting time. In Section 4, we first prove the key Lemma 4.1, and then use it to construct a suitable profile which is close to the original solution at the interacting time, and obtain the desired estimates after the interacting time.

Notation. Through out this paper, C_0, C_i, et al. always denote some specific positive constants, and C, $O(1)$ denotes the generic positive constant. $L^2(\mathbb{R})$ is the space of square integrable real valued function defined on $\mathbb{R}$ with the norm $\|\cdot\|$, and $H^k(\mathbb{R})$ (H^k without any ambiguity) denotes the usual Sobolev space with the norm $\|\cdot\|_k$, especially $\|\cdot\|_0 = \|\cdot\|$.

2 Preliminaries

The viscous shock wave of the Navier-Stokes system with the viscosity $\varepsilon = 1$

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \left(\dfrac{u_x}{v}\right)_x, \end{cases} \quad x \in \mathbb{R}, \quad t > 0, \tag{2.1}$$

takes the formula $(V,U)(x-st)$ where s is the shock speed. The 1-viscous shock wave $(V_1,U_1)(x-s_1t)$ connecting (v_m,u_m) and (v_+,u_+) exists uniquely up to a shift and satisfies

$$\begin{cases} -s_1V_1' - U_1' = 0, \\ -s_1U_1' + p(V_1)' = \left(\dfrac{U_1'}{V_1}\right)', \\ (V_1,U_1)(-\infty) = (v_m,u_m), \\ (V_1,U_1)(+\infty) = (v_+,u_+), \end{cases} \tag{2.2}$$

where $' = \frac{d}{d\xi}$, $\xi = x - s_1t$. To fix the viscous shock profile, we require

$$V_1(0) = \frac{v_m + v_+}{2}. \tag{2.3}$$

By $(2.2)_1$, it holds that

$$\begin{aligned} U_1(0) &= -s_1(V_1(0) - v_m) + u_m = -\frac{1}{2}s_1(v_+ - v_m) + u_m \\ &= \frac{1}{2}(u_+ - u_m) + u_m = \frac{1}{2}(u_+ + u_m). \end{aligned} \tag{2.4}$$

We will use these properties (2.3) and (2.4) in the proof of the key Lemma 4.1 below. Similarly, the 2-viscous shock wave $(V_2,U_2)(x-s_2t)$ connecting (v_-,u_-) as the left state and (v_m,u_m) as the right state satisfies

$$\begin{cases} -s_2 V_2' - U_2' = 0, \\ -s_2 U_2' + p(V_2)' = \left(\dfrac{U_2'}{V_2}\right)', \\ (V_2, U_2)(-\infty) = (v_-, u_-), \\ (V_2, U_2)(+\infty) = (v_m, u_m), \end{cases} \tag{2.5}$$

with

$$(V_2(0), U_2(0)) = \left(\frac{v_- + v_m}{2}, \frac{u_- + u_m}{2}\right). \tag{2.6}$$

We also denote $(\tilde{V}_1, \tilde{U}_1)(x - \tilde{s}_1 t)$ as the 1-viscous shock wave connecting (v_-, u_-) and $(\tilde{v}_m, \tilde{u}_m)$, and $(\tilde{V}_2, \tilde{U}_2)(x - \tilde{s}_2 t)$ as the 2-viscous shock wave connecting $(\tilde{v}_m, \tilde{u}_m)$ and (v_+, u_+). Similarly, we can impose

$$(\tilde{V}_1(0), \tilde{U}_1(0)) = \left(\frac{v_- + \tilde{v}_m}{2}, \frac{u_- + \tilde{u}_m}{2}\right), \tag{2.7}$$

and

$$(\tilde{V}_2(0), \tilde{U}_2(0)) = \left(\frac{v_+ + \tilde{v}_m}{2}, \frac{u_+ + \tilde{u}_m}{2}\right). \tag{2.8}$$

For any $\varepsilon > 0$, we define the corresponding viscous shock waves of (1.2) in the following:

$$(V_1^\varepsilon, U_1^\varepsilon)(x,t) =: (V_1, U_1)\left(\frac{x - s_1 t}{\varepsilon}\right) \quad \text{and} \quad (V_2^\varepsilon, U_2^\varepsilon)(x,t) =: (V_2, U_2)\left(\frac{x - s_2 t}{\varepsilon}\right), \tag{2.9}$$

$$(\tilde{V}_1^\varepsilon, \tilde{U}_1^\varepsilon)(x,t) =: (\tilde{V}_1, \tilde{U}_1)\left(\frac{x - \tilde{s}_1 t}{\varepsilon}\right) \quad \text{and} \quad (\tilde{V}_2^\varepsilon, \tilde{U}_2^\varepsilon)(x,t) =: (\tilde{V}_2, \tilde{U}_2)\left(\frac{x - \tilde{s}_2 t}{\varepsilon}\right). \tag{2.10}$$

From [16, Lemma 2.2], these viscous shock waves satisfy the following.

Proposition 2.1. *There are positive constants C_0 and c_0 depending only on (v_-, u_-) such that*

$$\begin{aligned}
&|(V_1 - v_m, U_1 - u_m)| \leqslant C_0 \delta_1 \mathrm{e}^{-c_0 \delta_1 |x - s_1 t|}, \quad x < s_1 t, \quad t \geqslant 0, \\
&|(V_1 - v_+, U_1 - u_+)| \leqslant C_0 \delta_1 \mathrm{e}^{-c_0 \delta_1 |x - s_1 t|}, \quad x > s_1 t, \quad t \geqslant 0, \\
&|(V_2 - v_-, U_2 - u_-)| \leqslant C_0 \delta_2 \mathrm{e}^{-c_0 \delta_2 |x - s_2 t|}, \quad x < s_2 t, \quad t \geqslant 0, \\
&|(V_2 - v_m, U_2 - u_m)| \leqslant C_0 \delta_2 \mathrm{e}^{-c_0 \delta_2 |x - s_2 t|}, \quad x > s_2 t, \quad t \geqslant 0, \\
&|(\tilde{V}_1 - \tilde{v}_m, \tilde{U}_1 - \tilde{u}_m)| \leqslant C_0 \tilde{\delta}_1 \mathrm{e}^{-c_0 \tilde{\delta}_1 |x - \tilde{s}_1 t|}, \quad x > \tilde{s}_1 t, \quad t \geqslant 0, \\
&|(\tilde{V}_1 - v_-, \tilde{U}_1 - u_-)| \leqslant C_0 \tilde{\delta}_1 \mathrm{e}^{-c_0 \tilde{\delta}_1 |x - \tilde{s}_1 t|}, \quad x < \tilde{s}_1 t, \quad t \geqslant 0, \\
&|(\tilde{V}_2 - \tilde{v}_m, \tilde{U}_2 - \tilde{u}_m)| \leqslant C_0 \tilde{\delta}_2 \mathrm{e}^{-c_0 \tilde{\delta}_2 |x - \tilde{s}_2 t|}, \quad x < \tilde{s}_2 t, \quad t \geqslant 0, \\
&|(\tilde{V}_2 - v_+, \tilde{U}_2 - u_+)| \leqslant C_0 \tilde{\delta}_2 \mathrm{e}^{-c_0 \tilde{\delta}_2 |x - \tilde{s}_2 t|}, \quad x > \tilde{s}_2 t, \quad t \geqslant 0,
\end{aligned} \tag{2.11}$$

$$|\partial_x (V_i, U_i)| \leqslant C_0 \delta_i^2 \mathrm{e}^{-c_0 \delta_i |x - s_i t|}, \quad |\partial_x (\tilde{V}_i, \tilde{U}_i)| \leqslant C_0 \tilde{\delta}_i^2 \mathrm{e}^{-c_0 \tilde{\delta}_i |x - \tilde{s}_i t|}, \quad x \in \mathbb{R}, \quad t \geqslant 0, \tag{2.12}$$

and $U_{ix} < 0$, $\tilde{U}_{ix} < 0, i = 1, 2$.

3 Estimates before the interacting time

This section is devoted to the estimates before the interacting time. In order to approximate the incoming shock waves, we define for $0 \leqslant t \leqslant t_0$,

$$\begin{pmatrix} \bar{V}^\varepsilon \\ \bar{U}^\varepsilon \end{pmatrix}(x,t) = \begin{pmatrix} V_1^\varepsilon(x - x_0 - s_1(t - t_0)) + V_2^\varepsilon(x - x_0 - s_2(t - t_0)) - v_m \\ U_1^\varepsilon(x - x_0 - s_1(t - t_0)) + U_2^\varepsilon(x - x_0 - s_2(t - t_0)) - u_m \end{pmatrix}. \tag{3.1}$$

Let the above approximate solutions at $t = 0$ to be the initial data of the Navier-Stokes system (1.2), i.e.,

$$(v, u)|_{t=0} = (\bar{V}^\varepsilon, \bar{U}^\varepsilon)(x, 0). \tag{3.2}$$

Then we reformulate the system by a scaling argument. Set

$$y = \frac{x - x_0}{\varepsilon}, \quad \tau = \frac{t - t_0}{\varepsilon}, \tag{3.3}$$

the system (1.2) becomes

$$\begin{cases} v_\tau - u_y = 0, \\ u_\tau + p_y = \left(\dfrac{u_y}{v}\right)_y, \end{cases} \quad y \in \mathbb{R}, \quad \tau > -\frac{t_0}{\varepsilon}. \tag{3.4}$$

Define

$$\begin{pmatrix} \bar V \\ \bar U \end{pmatrix}(y,\tau) =: \begin{pmatrix} \bar V^\varepsilon \\ \bar U^\varepsilon \end{pmatrix}(x,t) = \begin{pmatrix} V_1(y - s_1\tau) + V_2(y - s_2\tau) - v_m \\ U_1(y - s_1\tau) + U_2(y - s_2\tau) - u_m \end{pmatrix}, \tag{3.5}$$

it is easy to check that the profile $(\bar V, \bar U)$ satisfies

$$\begin{cases} \bar V_\tau - \bar U_y = 0, \\ \bar U_\tau + p(\bar V)_y = \left(\dfrac{\bar U_y}{\bar V}\right)_y + \bar R_y, \end{cases} \tag{3.6}$$

where

$$\bar R = (p(\bar V) - p(V_1) - p(V_2) + p_m) - \left(\frac{\bar U_y}{\bar V} - \frac{\bar U_{1y}}{\bar V_1} - \frac{\bar U_{2y}}{\bar V_2}\right). \tag{3.7}$$

Motivated by [16, 19], we study the system (3.4) by the antiderivative technique. Let

$$\phi(y,\tau) = (v - \bar V)(y,\tau), \quad \psi(y,\tau) = (u - \bar U)(y,\tau), \tag{3.8}$$

and define the antiderivative

$$\Phi(y,\tau) = \int_{-\infty}^{y} \phi(z,\tau)dz, \quad \Psi(y,\tau) = \int_{-\infty}^{y} \psi(z,\tau)dz. \tag{3.9}$$

By our initial data (3.2), we have

$$(\Phi, \Psi)\left(y, \tau = -\frac{t_0}{\varepsilon}\right) = 0. \tag{3.10}$$

Then (3.4) and (3.6) imply

$$\begin{cases} \Phi_\tau - \Psi_y = 0, \\ \Psi_\tau + p(v) - p(\bar V) = \left(\dfrac{u_y}{v} - \dfrac{\bar U_y}{\bar V}\right) + \bar R. \end{cases} \tag{3.11}$$

We linearize (3.11) around the approximate profile $(\bar V, \bar U)$ to obtain

$$\begin{cases} \Phi_\tau - \Psi_y = 0, \\ \Psi_\tau + p'(\bar V)\Phi_y = \dfrac{1}{\bar V}\Psi_{yy} + Q + \bar R, \end{cases} \tag{3.12}$$

where

$$\begin{aligned} Q &= -(p(v) - p(\bar V) - p'(\bar V)(v - \bar V)) + \left(\frac{1}{v} - \frac{1}{\bar V}\right)(\Psi_{yy} + \bar U_y) \\ &= O(1)(\Phi_y^2 + |\Phi_y \Psi_{yy}| + |\bar U_y \Phi_y|). \end{aligned} \tag{3.13}$$

We look for the solution to (3.12) in the following functional space:

$$X(I) = \{(\Phi, \Psi) \in C(I; H^2); \Psi_y \in L^2(I; H^2)\}, \tag{3.14}$$

where $I \subset \mathbb{R}$ is any interval. First, we show the existence results of the system (3.12) with the initial data (3.10) until the interaction time, i.e.,

Theorem 3.1 (Existence result before the interacting time)**.** *There are positive constants $\bar{\delta}_0$ and C independent of ε such that, if $\delta \leqslant \bar{\delta}_0$, then there exists a unique solution $(\Phi, \Psi)(\tau) \in X([-\frac{t_0}{\varepsilon}, 0])$ to* (3.12) *and* (3.10)*. Furthermore, it holds that*

$$\|(\Phi,\Psi)(\tau)\|_{H^2}^2 + \int_{-\frac{t_0}{\varepsilon}}^{\tau} \int (|\bar{U}_{1y}| + |\bar{U}_{2y}|)\Psi^2 dy ds + \int_{-\frac{t_0}{\varepsilon}}^{\tau} \|\Phi_y(s)\|_{H^1}^2 + \|\Psi_y(s)\|_{H^2}^2 ds \leqslant C\delta^{\frac{1}{3}} \exp\left(- \frac{c\delta|t-t_0|}{\varepsilon}\right). \tag{3.15}$$

Since the local existence is well known (e.g., see [7]), we omit it here for brevity. Following the arguments in [7], in order to prove Theorem 3.1, we only need to close the following *a priori* assumption:

$$N(T) =: \sup_{\tau \in [-\frac{t_0}{\varepsilon}, T]} \|(\Phi,\Psi)(\tau)\|_{H^2}^2 \leqslant \eta_0, \quad \text{for} \ \ T \leqslant 0, \tag{3.16}$$

where $[-\frac{t_0}{\varepsilon}, T]$ is a time interval on which the solution is supposed to exist, and η_0 is a positive small constant which is to be determined.

Proposition 3.2 (A priori estimates before the interacting time)**.** *Assume that there exists a solution $(\Phi, \Psi) \in X([-\frac{t_0}{\varepsilon}, T])$ to* (3.12) *and* (3.10) *with $T \leqslant 0$. Then there exist positive constants $\bar{\delta}_0$, η_0 and C independent of ε such that, if $\delta \leqslant \bar{\delta}_0$ and $N(T) \leqslant \eta_0$, then (Φ, Ψ) satisfies for $\tau \in [-\frac{t_0}{\varepsilon}, T]$,*

$$\|(\Phi,\Psi)(\tau)\|_{H^2}^2 + \int_{-\frac{t_0}{\varepsilon}}^{\tau} \int (|\bar{U}_{1y}| + |\bar{U}_{2y}|)\Psi^2 dy ds + \int_{-\frac{t_0}{\varepsilon}}^{\tau} \|\Phi_y(s)\|_{H^1}^2 + \|\Psi_y(s)\|_{H^2}^2 ds \leqslant C\delta^{\frac{1}{3}} \exp\left(- \frac{c\delta|t-t_0|}{\varepsilon}\right). \tag{3.17}$$

Proposition 3.2 is proved by the following lemmas. From the a priori assumption (3.16) with $\eta_0 > 0$ small enough, it holds that

$$\frac{v_-}{4} \leqslant v \leqslant 2v_-.$$

First, we have the following lemma.

Lemma 3.3. *If $\bar{\delta}_0$ and η_0 are suitably small, for $-\frac{t_0}{\varepsilon} \leqslant \tau \leqslant T \leqslant 0$, it holds for $\delta \leqslant \bar{\delta}_0$ that*

$$\begin{aligned} &\|(\Phi,\Psi)(\tau)\|_{L^2}^2 + \int_{-\frac{t_0}{\varepsilon}}^{\tau} \int (|\bar{U}_{1y}| + |\bar{U}_{2y}|)\Psi^2 dy ds + \int_{-\frac{t_0}{\varepsilon}}^{\tau} \|\Psi_y(s)\|_{L^2}^2 ds \\ &\quad \leqslant C\delta^{\frac{1}{3}} \exp\left(- \frac{c\delta|t-t_0|}{2\varepsilon}\right) + C(\delta+\eta_0) \int_{-\frac{t_0}{\varepsilon}}^{\tau} \|(\Psi_{yy}, \Phi_y)(s)\|_{L^2}^2 ds. \end{aligned} \tag{3.18}$$

Proof. Multiplying $(3.12)_1$ by Φ, $(3.12)_2$ by $-\frac{\Psi}{p'(\bar{V})}$, we obtain

$$\left(\frac{1}{2}\Phi^2\right)_\tau - \Phi\Psi_y = 0,$$

and

$$\begin{aligned} &\left(- \frac{1}{2p'(\bar{V})}\Psi^2\right)_\tau - \frac{p''(\bar{V})}{2|p'(\bar{V})|^2}\bar{V}_\tau \Psi^2 - \Phi_y\Psi + \frac{\Psi_y^2}{\bar{V}|p'(\bar{V})|} \\ &\quad = \left(\frac{1}{\bar{V}p'(\bar{V})}\right)_y \Psi_y\Psi - \frac{\Psi}{p'(\bar{V})}(Q+\bar{R}) - \left(\frac{\Psi_y\Psi}{\bar{V}p'(\bar{V})}\right)_y. \end{aligned}$$

Adding the above two equations, we have

$$\left(\frac{1}{2}\Phi^2 + \frac{1}{2|p'(\bar{V})|}\Psi^2\right)_\tau - \frac{p''(\bar{V})}{2|p'(\bar{V})|^2}\bar{V}_\tau \Psi^2 + \frac{\Psi_y^2}{\bar{V}|p'(\bar{V})|}$$

$$= \left(\frac{1}{\bar{V}p'(\bar{V})}\right)_y \Psi_y\Psi - \frac{\Psi}{p'(\bar{V})}(Q+\bar{R}) + \left(-\frac{\Psi_y\Psi}{\bar{V}p'(\bar{V})} + \Phi\Psi\right)_y, \tag{3.19}$$

where we have used the fact $p'(\bar{V}) < 0$.

By (3.13), it is straightforward to obtain

$$\int_{-\frac{t_0}{\varepsilon}}^{\tau}\int |Q\Psi| dyds \leqslant C(\delta+\eta_0)\int_{-\frac{t_0}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_{yy})(s)\| ds + C\delta\int_{-\frac{t_0}{\varepsilon}}^{\tau}\int(|\bar{U}_{1y}| + |\bar{U}_{2y}|)\Psi^2 dyds. \tag{3.20}$$

On the other hand, from Proposition 2.1, we have

$$\begin{aligned}\bar{R} &= C|V_1 - v_m||V_2 - v_m| + C|U_{1y}||V_2 - v_m| + C|U_{2y}||V_1 - v_m| \\ &\leqslant C\delta^2\exp(-c\delta|y| - c\delta|\tau|).\end{aligned} \tag{3.21}$$

From (3.21), a direct computation gives

$$\begin{aligned}\int_{-\frac{t_0}{\varepsilon}}^{\tau}\int |\bar{R}\Psi| dyds &\leqslant C\delta^2\int_{-\frac{t_0}{\varepsilon}}^{\tau} \|\Psi\|_{L^\infty}\exp(-c\delta|s|)ds \cdot \int \exp(-c\delta|y|)dy \\ &\leqslant C\delta\int_{-\frac{t_0}{\varepsilon}}^{\tau} \|\Psi(s)\|^{\frac{1}{2}}\|\Psi_y(s)\|^{\frac{1}{2}}\exp(-c\delta|s|)ds \\ &\leqslant \sup_{-\frac{t_0}{\varepsilon}\leqslant s\leqslant\tau}\|\Psi(s)\|^2 \cdot \int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\Psi_y(s)\|_{L^2}^2 ds + C\delta^{\frac{4}{3}}\int_{-\frac{t_0}{\varepsilon}}^{\tau}\exp\left(-\frac{4}{3}c\delta|s|\right)ds \\ &\leqslant N(\tau)\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\Psi_y(s)\|_{L^2}^2 ds + C\delta^{\frac{4}{3}}\exp\left(-\frac{c}{2}\delta|\tau|\right)\cdot\int_{-\frac{t_0}{\varepsilon}}^{\tau}\exp\left(-\frac{c}{2}\delta|s|\right)ds \\ &\leqslant N(\tau)\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\Psi_y(s)\|_{L^2}^2 ds + C\delta^{\frac{1}{3}}\exp\left(-\frac{c}{2}\delta|\tau|\right),\end{aligned} \tag{3.22}$$

where in the second inequality we have used the Sobolev inequality

$$\|f\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{2}\|f\|_{L^2}^{\frac{1}{2}}\|f_y\|_{L^2}^{\frac{1}{2}}, \quad \text{for } f \in H^1(\mathbb{R}). \tag{3.23}$$

Using Proposition 2.1 and the Cauchy inequality, we have

$$\begin{aligned}\left|\int_{-\frac{t_0}{\varepsilon}}^{\tau}\int\left(\frac{1}{\bar{V}p'(\bar{V})}\right)_y\Psi_y\Psi dyds\right| &\leqslant C\int_{-\frac{t_0}{\varepsilon}}^{\tau}\int|\bar{V}_y\Psi_y\Psi| dyds \\ &\leqslant C\delta\int_{-\frac{t_0}{\varepsilon}}^{\tau}\int|\bar{U}_y|^{\frac{1}{2}}|\Psi_y\Psi| dyds \\ &\leqslant C\delta\int_{-\frac{t_0}{\varepsilon}}^{\tau}\int\Psi_y^2 dyds + C\delta\int_{-\frac{t_0}{\varepsilon}}^{\tau}\int(|\bar{U}_{1y}| + |\bar{U}_{2y}|)\Psi^2 dyds.\end{aligned} \tag{3.24}$$

Finally, since that

$$-\int_{-\frac{t_0}{\varepsilon}}^{\tau}\int\frac{p''(\bar{V})}{2|p'(\bar{V})|^2}\bar{V}_\tau\Psi^2 dyds \geqslant \int_{-\frac{t_0}{\varepsilon}}^{\tau}\int\frac{p''(\bar{V})}{2|p'(\bar{V})|^2}(|\bar{U}_{1y}| + |\bar{U}_{2y}|)\Psi^2 dyds, \tag{3.25}$$

due to Proposition 2.1, integrating (3.19) and using the smallness of δ, η_0, and (3.20), (3.22), (3.24) and (3.25), we obtain (3.18). The proof of Lemma 3.3 is completed. □

Lemma 3.4. *If $\bar{\delta}_0$ and η_0 are suitably small, for $-\frac{t_0}{\varepsilon} \leqslant \tau \leqslant T \leqslant 0$, it holds for $\delta \leqslant \bar{\delta}_0$ that*

$$\begin{aligned}&\|\Phi(\tau)\|_{H^1}^2 + \|\Psi(\tau)\|_{L^2}^2 + \int_{-\frac{t_0}{\varepsilon}}^{\tau}\int(|\bar{U}_{1y}| + |\bar{U}_{2y}|)\Psi^2 dyds + \int_{-\frac{t_0}{\varepsilon}}^{\tau}\|(\Phi_y, \Psi_y)(s)\|_{L^2}^2 ds \\ &\quad\leqslant C\delta^{\frac{1}{3}}\exp\left(-\frac{c\delta|t-t_0|}{2\varepsilon}\right) + C(\delta+\eta_0)\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\Psi_{yy}(s)\|_{L^2}^2 ds.\end{aligned} \tag{3.26}$$

Proof. From $(3.12)_2$, we have

$$\frac{1}{\bar{V}}\Phi_{y\tau} - p'(\bar{V})\Phi_y - \Psi_\tau = -Q - \bar{R}. \tag{3.27}$$

Multiplying (3.27) by Φ_y implies

$$\left(\frac{1}{2\bar{V}}\Phi_y^2 - \Psi\Phi_y\right)_\tau + |p'(\bar{V})|\Phi_y^2 = \left(\frac{1}{2\bar{V}}\right)_\tau \Phi_y^2 + \Psi_y^2 - (Q+\bar{R})\Phi_y - (\Phi_\tau\Psi)_y. \tag{3.28}$$

Integrating (3.28) over $[-\frac{t_0}{\varepsilon}, \tau] \times \mathbb{R}$ and using the Cauchy inequality, we obtain

$$\begin{aligned}&\int\left(\frac{1}{2\bar{V}}\Phi_y^2 - \Psi\Phi_y\right)(\tau)dy + \frac{1}{2}\int_{-\frac{t_0}{\varepsilon}}^{\tau}\int |p'(\bar{V})|\Phi_y^2 dyds\\ &\quad\leqslant C\delta\exp\left(-\frac{c\delta|t-t_0|}{2\varepsilon}\right) + C\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\Psi_y(s)\|_{L^2}^2 ds + C(\delta+\eta_0)\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\Psi_{yy}(s)\|_{L^2}^2 ds, \end{aligned}\tag{3.29}$$

where we have used that fact that $|\bar{V}_\tau| \leqslant C\delta^2$.

Multiplying (3.18) by a large constant $C_1 > 0$ and by using (3.29) and the smallness of δ, η_0 gives the proof of Lemma 3.4. □

Next, we give the a priori estimates of higher order derivatives.

Lemma 3.5. *If $\bar{\delta}_0$ and η_0 are suitably small, for $-\frac{t_0}{\varepsilon} \leqslant \tau \leqslant T \leqslant 0$, it holds for $\delta \leqslant \bar{\delta}_0$ that*

$$\begin{aligned}&\|\phi(\tau)\|_{H^1}^2 + \|\psi(\tau)\|_{L^2}^2 + \int_{-\frac{t_0}{\varepsilon}}^{\tau}\|(\phi_y,\psi_y)(s)\|_{L^2}^2 ds\\ &\quad\leqslant C\delta^{\frac{1}{3}}\exp\left(-\frac{c\delta|t-t_0|}{2\varepsilon}\right) + C\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|(\Phi_y,\Psi_y)(s)\|_{L^2}^2 ds\\ &\qquad + C(\delta+\eta_0)\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\Psi_{yy}(s)\|_{L^2}^2 ds.\end{aligned}\tag{3.30}$$

Proof. Applying ∂_y to (3.11), we have

$$\begin{cases}\phi_\tau - \psi_y = 0,\\ \psi_\tau + p'(\bar{V})\phi_y = \left(\dfrac{\psi_y}{v}\right)_y + Q_1 + \bar{R}_y,\end{cases}\tag{3.31}$$

where

$$Q_1 = \left(\bar{U}_y\left(\frac{1}{v} - \frac{1}{\bar{V}}\right)\right)_y - (p'(v) - p'(\bar{V}))v_y. \tag{3.32}$$

Multiplying $(3.31)_1$ by ϕ and $(3.31)_2$ by $\frac{\psi}{|p'(\bar{V})|}$, and following the same line of Lemma 3.3, we obtain

$$\begin{aligned}&\|(\phi,\psi)(\tau)\|_{L^2}^2 + \int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\psi_y(s)\|_{L^2}^2 ds\\ &\quad\leqslant C\delta^{\frac{1}{3}}\exp\left(-\frac{c\delta|t-t_0|}{2\varepsilon}\right) + C(\delta+\eta_0)\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|(\Phi_y,\Psi_y,\phi_y)(s)\|_{L^2}^2 ds.\end{aligned}\tag{3.33}$$

On the other hand, substituting $(3.31)_1$ into $(3.31)_2$ gives

$$\frac{\phi_{y\tau}}{\bar{V}} - \psi_\tau - p'(\bar{V})\phi_y = -\left(\frac{1}{\bar{V}}\right)_y\psi_y - \left(\psi_y\left(\frac{1}{v} - \frac{1}{\bar{V}}\right)\right)_y - Q_1 - \bar{R}_y. \tag{3.34}$$

Multiplying (3.34) by ϕ_y, integrating by part and using the Cauchy inequality, we obtain

$$\int\left(\frac{1}{2\bar{V}}\phi_y^2 - \psi\phi_y\right)(\tau)dy + \int_{-\frac{t_0}{\varepsilon}}^{\tau}\int |p'(\bar{V})|\phi_y^2 dyds$$

$$\leqslant C\delta^{\frac{1}{3}}\exp\left(-\frac{c\delta|t-t_0|}{2\varepsilon}\right)+C\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\psi_y(s)\|_{L^2}^2ds$$
$$+C(\delta+\eta_0)\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|(\psi_{yy},\Phi_y)(s)\|_{L^2}^2ds. \tag{3.35}$$

Multiplying (3.33) by a large constant $C_2>0$ and by using the estimate (3.35) and the smallness of δ, η_0, we can complete the proof of Lemma 3.5. □

Lemma 3.6. *If $\bar{\delta}_0$ and η_0 are suitably small, for $-\frac{t_0}{\varepsilon}\leqslant\tau\leqslant T\leqslant 0$, it holds for $\delta\leqslant\bar{\delta}_0$ that*

$$\|\psi_y(\tau)\|_{L^2}^2+\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\psi_{yy}(s)\|_{L^2}^2ds$$
$$\leqslant C\delta^{\frac{1}{3}}\exp\left(-\frac{c\delta|t-t_0|}{2\varepsilon}\right)+C\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|(\phi_y,\psi_y)(s)\|_{L^2}^2ds+C\delta\int_{-\frac{t_0}{\varepsilon}}^{\tau}\|\Phi_y(s)\|_{L^2}^2ds. \tag{3.36}$$

Proof. Multiplying $(3.31)_2$ by $-\psi_{yy}$, and integrating the resulting equation over $[-\frac{t_0}{\varepsilon},\tau]\times\mathbb{R}$, we can prove (3.36). The details are omitted for brevity. □

Proof of Proposition 3.2. Choosing $\eta_0=\delta^{\frac{1}{6}}$ and using the smallness of δ, and combining Lemmas 3.3–3.6, we can close the *a priori* assumption (3.16) and then conclude Proposition 3.2. □

4 Estimates after the interacting time

We continue to study the scaled Navier-Stokes system after the interacting time $\tau>0$, i.e.,

$$\begin{cases}v_\tau-u_y=0,\\ u_\tau+p_y=\left(\dfrac{u_y}{v}\right)_y,\end{cases}\quad y\in\mathbb{R},\quad \tau>0, \tag{4.1}$$

with the initial data $(v,u)(y,0)$ which has already been solved in Section 3. Since the incoming viscous shock profile $(\bar{V},\bar{U})$ can not be used anymore after the interacting time, i.e., $\tau>0$ with $\tau=\frac{t-t_0}{\varepsilon}$, we need to use the outgoing viscous shock profiles $\tilde{S}_1,\tilde{S}_2$ instead. Unlike the previous section where the initial data at $\tau=-\frac{t_0}{\varepsilon}$ is well-chosen such that the perturbation $(\phi,\psi)(y,\tau)|_{\tau=-\frac{t_0}{\varepsilon}}\equiv(0,0)$, here the initial data $(v,u)(y,0)$ for the system (4.1) is determined by the solution obtained before the interacting time and can not be chosen any more. If we choose

$$\begin{pmatrix}\hat{V}\\ \hat{U}\end{pmatrix}(y,\tau):=\begin{pmatrix}\tilde{V}_1(y-\tilde{s}_1\tau)+\tilde{V}_2(y-\tilde{s}_2\tau)-\tilde{v}_m\\ \tilde{U}_1(y-\tilde{s}_1\tau)+\tilde{U}_2(y-\tilde{s}_2\tau)-\tilde{u}_m\end{pmatrix}, \tag{4.2}$$

as the superposition of $\tilde{S}_1,\tilde{S}_2$, then the mass at $\tau=0$, denoted by

$$\vec{I}_0:=\int\begin{pmatrix}v-\hat{V}\\ u-\hat{U}\end{pmatrix}(y,\tau)dy|_{\tau=0} \tag{4.3}$$

may not be zero. Therefore, one can not use the antiderivative technique as in [16, 19]. Usually, it is expected to modify the shock profiles with suitable shifts such that the corresponding integral is zero. For this, let

$$r_1^-=\begin{pmatrix}\tilde{v}_m-v_-\\ \tilde{u}_m-u_-\end{pmatrix},\quad r_2^+=\begin{pmatrix}v_+-\tilde{v}_m\\ u_+-\tilde{u}_m\end{pmatrix}, \tag{4.4}$$

which are linearly independent in $\mathbb{R}^2$ and then there exist two constants $\alpha_i,\ i=1,2$ such that the mass $\vec{I}_0$ is distributed along the two independent directions r_1^+ and r_2^-, i.e.,

$$\vec{I}_0=\alpha_1r_1^-+\alpha_2r_2^+. \tag{4.5}$$

We define

$$\begin{pmatrix} \tilde{V} \\ \tilde{U} \end{pmatrix}(y,\tau) := \begin{pmatrix} \tilde{V}_1(y-\tilde{s}_1\tau+\alpha_1)+\tilde{V}_2(y-\tilde{s}_2\tau+\alpha_2)-\tilde{v}_m \\ \tilde{U}_1(y-\tilde{s}_1\tau+\alpha_1)+\tilde{U}_2(y-\tilde{s}_2\tau+\alpha_2)-\tilde{u}_m \end{pmatrix}. \tag{4.6}$$

From (3.10), one can get

$$\int (v-\bar{V}, u-\bar{U})(y,0)dy = 0,$$

which implies that

$$\int \begin{pmatrix} v-\tilde{V} \\ u-\tilde{U} \end{pmatrix}(y,0)dy = \int \begin{pmatrix} \bar{V}-\tilde{V} \\ \bar{U}-\tilde{U} \end{pmatrix}(y,0)dy =: \vec{I}(\alpha_1,\alpha_2). \tag{4.7}$$

By a direct calculation, one can obtain

$$\frac{\partial \vec{I}(\alpha_1,\alpha_2)}{\partial\alpha_1} = -r_1^- \quad \text{and} \quad \frac{\partial \vec{I}(\alpha_1,\alpha_2)}{\partial\alpha_2} = -r_2^+,$$

which yields that

$$\begin{aligned} \vec{I}(\alpha_1,\alpha_2) &= \vec{I}_0 + \alpha_1\frac{\partial \vec{I}(\alpha_1,\alpha_2)}{\partial\alpha_1} + \alpha_2\frac{\partial \vec{I}(\alpha_1,\alpha_2)}{\partial\alpha_2} \\ &= (\alpha_1 r_1^- + \alpha_2 r_2^+) - \alpha_1 r_1^- - \alpha_2 r_2^+ = 0. \end{aligned} \tag{4.8}$$

Therefore, $(\tilde{V},\tilde{U})$ defined in (4.6) is the desired viscous shock profiles with shifts for the outgoing shocks $\tilde{S}_1, \tilde{S}_2$ and then the antiderivative arguments can be applied. As in Section 3, we also need the smallness of the antiderivative of the perturbation around $(\tilde{V},\tilde{U})$ at $\tau=0$, which seems not be obvious. In fact, let us first estimate the difference between the viscous shock profiles before and after the interacting time, i.e.,

$$\begin{aligned} (\tilde{V}-\bar{V})(y,\tau=0) &= [\tilde{V}_1(y+\alpha_1)+\tilde{V}_2(y+\alpha_2)-\tilde{v}_m] - [V_1(y)+V_2(y)-v_m] \\ &= \begin{cases} ([\tilde{V}_1(y+\alpha_1)-v_-]-[V_1(y)-v_m]) \\ \quad +([\tilde{V}_2(y+\alpha_2)-\tilde{v}_m]-[V_2(y)-v_-]), & \text{for } y\leqslant 0, \\ ([\tilde{V}_1(y+\alpha_1)-\tilde{v}_m]-[V_1(y)-v_+]) \\ \quad +([\tilde{V}_2(y+\alpha_2)-v_+]-[V_2(y)-v_m]), & \text{for } y\geqslant 0. \end{cases} \end{aligned} \tag{4.9}$$

If Proposition 2.1 is simply used, then one has

$$|(\tilde{V}-\bar{V})(y,\tau=0)| = O(1)\delta\exp(-c\delta|y|), \quad \text{for } y\in\mathbb{R}, \tag{4.10}$$

which only gives

$$\left\| \int_{-\infty}^{y} (\tilde{V}-\bar{V})(z,\tau=0)dz \right\|_{L^2}^2 = O(1)\delta^{-1}. \tag{4.11}$$

This means that when the wave strength δ is suitably small, the initial anti-derivative of perturbation in the sense of (4.11) is very large after the interacting time. Therefore, it is hard to study the system (4.1) directly by using this estimate (4.11) which seems too crude. However, from (4.9), it can be observed that there may exist some cancelations so that better estimates might be obtained. Indeed, we have even stronger estimates for (4.9) in the following key lemma.

Lemma 4.1. *There exists a small positive constant $\delta_I>0$, such that if the wave strength $\delta\leqslant\delta_I$, then for any fixed α, it holds that*

$$\begin{aligned} &|([\tilde{V}_1(y+\alpha)-v_-]-[V_1(y)-v_m], [\tilde{U}_1(y+\alpha)-u_-]-[U_1(y)-u_m])| \\ &\quad \leqslant C_\alpha\delta^2\exp\left(-\frac{1}{2}c\delta|y|\right), \quad \textit{for } y\leqslant 0, \end{aligned} \tag{4.12}$$

where the positive constant satisfies $C_\alpha \leqslant C\exp(\mathrm{e}^{C|\alpha|})$.

Proof. Firstly from [21], we have for suitably small δ

$$\tilde{\delta}_1 = \delta_1 + O(1)\delta_1\delta_2 \quad \text{and} \quad \tilde{\delta}_2 = \delta_2 + O(1)\delta_1\delta_2, \tag{4.13}$$

which gives a relation of the wave strength between the incoming and outgoing shock waves. Then it is observed in (2.2) that

$$V_1'(y) = B(v_m, v_+, y)(V_1(y) - v_m), \tag{4.14}$$

where

$$\begin{aligned} B_1(v_m, v_+, y) &= \frac{V_1(y)}{|s_1|}\bigg[s_1^2 + \frac{p(V) - p(v_m)}{V(y) - v_m}\bigg] = \frac{V_1(y)}{|s_1|}\bigg[-\frac{p(v_+) - p(v_m)}{v_+ - v_m} + \frac{p(V) - p(v_m)}{V(y) - v_m}\bigg] \\ &= \frac{V_1(y)}{|s_1|}\bigg[-\frac{1}{2}p''(v_m)(v_+ - v_m) + \frac{1}{2}p''(v_m)(V_1(y) - v_m) + O(1)\delta^2\bigg] \\ &= \frac{v_m}{|s_1|}\bigg[-\frac{1}{2}p''(v_m)(v_+ - v_m) + \frac{1}{2}p''(v_m)(V_1(y) - v_m)\bigg] + O(1)\delta^2. \end{aligned} \tag{4.15}$$

Integrating (4.14) with respect to y over $[y, 0]$ with $y \leqslant 0$ yields that

$$\begin{aligned} v_m - V_1(y) &= (v_m - V_1(0)) \exp\bigg\{-\int_y^0 B_1(v_m, v_+, z)dz\bigg\} \\ &= \frac{1}{2}|v_m - v_+| \exp\bigg\{-\int_y^0 B_1(v_m, v_+, z)dz\bigg\}, \end{aligned} \tag{4.16}$$

where we have used the fact $V_1(0) = \frac{1}{2}(v_+ + v_m)$ in (2.3). Similarly, one can obtain for $y \leqslant 0$,

$$\begin{aligned} v_- - \tilde{V}_1(y + \alpha) &= (v_- - \tilde{V}_1(0)) \exp\bigg\{-\int_{y+\alpha}^0 \tilde{B}_1(v_-, \tilde{v}_m, z)dz\bigg\} \\ &= \frac{1}{2}|\tilde{v}_m - v_-| \exp\bigg\{-\int_{y+\alpha}^0 \tilde{B}_1(v_-, \tilde{v}_m, z)dz\bigg\}, \end{aligned} \tag{4.17}$$

where we have imposed $\tilde{V}_1(0) = \frac{1}{2}(v_- + \tilde{v}_m)$ in Section 2 and

$$\begin{aligned} \tilde{B}_1(v_-, \tilde{v}_m, y) &= \frac{\tilde{V}_1(y)}{|\tilde{s}_1|}\bigg[\tilde{s}_1^2 + \frac{p(\tilde{V}_1) - p(v_-)}{\tilde{V}_1(y) - v_-}\bigg] = \frac{\tilde{V}_1(y)}{|\tilde{s}_1|}\bigg[-\frac{p(\tilde{v}_m) - p(v_-)}{\tilde{v}_m - v_-} + \frac{p(\tilde{V}_1) - p(v_-)}{\tilde{V}_1(y) - v_-}\bigg] \\ &= \frac{\tilde{V}_1(y)}{|\tilde{s}_1|}\bigg[-\frac{1}{2}p''(v_-)(\tilde{v}_m - v_-) + \frac{1}{2}p''(v_-)(\tilde{V}_1(y) - v_-) + O(1)\delta^2\bigg] \\ &= \frac{v_m}{|s_1|}\bigg[-\frac{1}{2}p''(v_-)(\tilde{v}_m - v_-) + \frac{1}{2}p''(v_-)(\tilde{V}_1(y) - v_-)\bigg] + O(1)\delta^2 \\ &= \frac{v_m}{|s_1|}\bigg[-\frac{1}{2}p''(v_m)(v_+ - v_m) + \frac{1}{2}p''(v_m)(\tilde{V}_1(y) - v_-)\bigg] + O(1)\delta^2, \end{aligned} \tag{4.18}$$

where we have used (4.13) and the fact that

$$\frac{\tilde{V}_1(y)}{|\tilde{s}_1|} = \frac{v_m}{|s_1|} + O(1)\delta.$$

By using (4.15), (4.18) and the fact that $p'' > 0$,

$$\begin{aligned} V_1(y) - v_+ &\geqslant c\delta_1, \quad \text{if } \ y \leqslant 0, \\ \tilde{V}_1(y) - \tilde{v}_m &\geqslant c\delta_1, \quad \text{if } \ y \leqslant 0, \end{aligned}$$

it is easy to check that there exist two constants c_3, C_3 only depending on v_- such that

$$C_3\delta_1 \geqslant \tilde{B}_1(v_-, \tilde{v}_m, y) \geqslant c_3\delta_1, \quad \text{and} \quad C_3\delta_1 \geqslant B_1(v_m, v_+, y) \geqslant c_3\delta_1, \quad \text{for } \ y \leqslant 0, \tag{4.19}$$

where we have used the smallness of δ.

By using (4.13), (4.16), (4.17), (4.19), and noting that there has a cancelation in (4.9) due to (4.15) and (4.18), one has for $y \leqslant 0$

$$
\begin{aligned}
&|[\tilde{V}_1(y+\alpha)-v_-]-[V_1(y)-v_m]| \\
&\quad \leqslant \frac{1}{2}\delta_1\left|\exp\left\{-\int_y^0 B_1(v_m,v_+,z)dz\right\}-\exp\left\{-\int_{y+\alpha}^0 \tilde{B}_1(v_-,\tilde{v}_m,z)dz\right\}\right| + C\delta^2 \mathrm{e}^{(-c\delta|y|+C|\alpha|)} \\
&\quad \leqslant C\delta\exp(-c\delta|y|)\left(\int_y^0 \left|B_1(v_m,v_+,z)-\tilde{B}_1(v_-,\tilde{v}_m,z)\right|dz+\delta|\alpha|\right)+C\delta^2\mathrm{e}^{(-c\delta|y|+C|\alpha|)} \\
&\quad \leqslant C\delta^2\exp(-c\delta|y|+C|\alpha|) \\
&\qquad + C\delta\exp(-c\delta|y|+C|\alpha|)\left(\int_y^0 |[\tilde{V}_1(z+\alpha)-v_-]-[V_1(z)-v_m]|dz+\delta^2|y|\right) \\
&\quad \leqslant C\mathrm{e}^{C|\alpha|}\delta^2\exp(-\frac{1}{2}c\delta|y|) \\
&\qquad + C\mathrm{e}^{C|\alpha|}\delta\exp(-c\delta|y|)\int_y^0 |[\tilde{V}_1(z+\alpha)-v_-]-[V_1(z)-v_m]|dz.
\end{aligned} \tag{4.20}
$$

Then Gronwall's inequality implies

$$
\int_y^0 |[\tilde{V}_1(z+\alpha)-v_-]-[V_1(z)-v_m]|dz \leqslant C\exp(\mathrm{e}^{C|\alpha|})\delta, \quad \text{for} \ \ y\leqslant 0. \tag{4.21}
$$

Substituting (4.21) into (4.20), one can obtain

$$
|[\tilde{V}_1(y+\alpha)-v_-]-[V_1(y)-v_m]| \leqslant C\exp(\mathrm{e}^{C|\alpha|})\cdot\delta^2\cdot\exp\left(-\frac{1}{2}c\delta|y|\right), \quad \text{for} \ \ y\leqslant 0. \tag{4.22}
$$

The estimates for $[\tilde{U}_1(y+\alpha)-u_-]-[U_1(y)-u_m]$ can be carried out similarly, and the details will be omitted for brevity. Thus we complete the proof of Lemma 4.1. □

Similarly, one has the following.

Lemma 4.2. *If $\delta\leqslant\delta_I$, then for any fixed α, it holds that*

$$
\begin{aligned}
&|([\tilde{V}_2(y+\alpha)-\tilde{v}_m]-[V_2(y)-v_-],[\tilde{U}_2(y+\alpha)-\tilde{u}_m]-[U_2(y)-u_-])| \\
&\quad \leqslant C_\alpha\delta^2\exp\left(-\frac{1}{2}c\delta|y|\right), \quad \textit{for} \ \ y\leqslant 0,
\end{aligned} \tag{4.23}
$$

$$
\begin{aligned}
&|([\tilde{V}_1(y+\alpha)-\tilde{v}_m]-[V_1(y)-v_-],[\tilde{U}_1(y+\alpha)-\tilde{u}_m]-[U_1(y)-u_-])| \\
&\quad \leqslant C_\alpha\delta^2\exp\left(-\frac{1}{2}c\delta|y|\right), \quad \textit{for} \ \ y\geqslant 0,
\end{aligned} \tag{4.24}
$$

and

$$
\begin{aligned}
&|([\tilde{V}_2(y+\alpha)-v_+]-[V_2(y)-v_m],[\tilde{U}_2(y+\alpha)-u_+]-[U_2(y)-u_m])| \\
&\quad \leqslant C_\alpha\delta^2\exp\left(-\frac{1}{2}c\delta|y|\right), \quad \textit{for} \ \ y\geqslant 0,
\end{aligned} \tag{4.25}
$$

where the positive constant satisfies $C_\alpha\leqslant C\exp(\mathrm{e}^{C|\alpha|})$.

Remark 4.3. Lemmas 4.1 and 4.2 give a better estimate than (4.10) at the interacting time which gives a well control on the connection between the viscous profiles of the incoming and outgoing shock waves.

From Lemmas 4.1 and 4.2 with $\alpha=0$, it holds that

$$
|\vec{I}_0| = \left|\int_{-\infty}^{+\infty}(\hat{V}-\bar{V},\hat{U}-\bar{U})(z,0)dz\right| = |\alpha_1 r_1^- + \alpha_2 r_2^+| \leqslant C\delta. \tag{4.26}
$$

Note that by (4.4), it holds that

$$|r_1^-| \sim \delta, \quad |r_2^+| \sim \delta. \tag{4.27}$$

Therefore, by (4.26), (4.27) and the fact that r_1^- and r_2^+ are linearly independent, one has

$$|(\alpha_1, \alpha_2)| \leqslant C$$

with the positive constant C independent of δ.

Again using Lemmas 4.1 and 4.2, we obtain a better estimate for (4.9) than (4.10), i.e.,

$$|(\tilde{V} - \bar{V}, \tilde{U} - \bar{U})(y, 0)| \leqslant C\delta^2 \exp\left(-\frac{1}{2} c\delta|y| \right), \tag{4.28}$$

with the constant C independent of δ.

By using (4.28) and (2.12), one has the following estimate on the antiderivative of $(\tilde{V} - \bar{V}, \tilde{U} - \bar{U})$ at the interacting time.

Lemma 4.4. *It holds that*

$$\left\| \int_{-\infty}^{y} (\tilde{V} - \bar{V}, \tilde{U} - \bar{U})(z, 0) dz \right\|_{H^2}^2 \leqslant C\delta. \tag{4.29}$$

Remark 4.5. Lemma 4.4 gives a better estimate to (4.9) than (4.11), which is good enough for our analysis.

Based on the above preparation, we continue to study (4.1) after the interacting time. Similar to Section 3, it is easy to know that $(\tilde{V}, \tilde{U})(y, \tau)$ satisfies

$$\begin{cases} \tilde{V}_\tau - \tilde{U}_y = 0, \\ \tilde{U}_\tau + p(\tilde{V})_y = \left(\dfrac{\tilde{U}_y}{\tilde{V}} \right)_y + \tilde{R}_y, \end{cases} \quad \text{for } \tau \geqslant 0, \tag{4.30}$$

where

$$\tilde{R} = (p(\tilde{V}) - p(\tilde{V}_1) - p(\tilde{V}_2) + p(\tilde{v}_m)) - \left(\frac{\tilde{U}_y}{\tilde{V}} - \frac{\tilde{U}_{1y}}{\tilde{V}_1} - \frac{\tilde{U}_{2y}}{\tilde{V}_2} \right). \tag{4.31}$$

Set

$$\tilde{\phi}(y, \tau) = (v - \tilde{V})(y, \tau), \quad \tilde{\psi}(y, \tau) = (u - \tilde{U})(y, \tau), \quad \text{for } \tau \geqslant 0, \tag{4.32}$$

and introduce the anti-derivative variables

$$\tilde{\Phi}(y, \tau) = \int_{-\infty}^{y} \tilde{\phi}(z, \tau) dz, \quad \tilde{\Psi}(y, \tau) = \int_{\infty}^{y} \tilde{\psi}(z, \tau) dz, \quad \text{for } \tau \geqslant 0, \tag{4.33}$$

which can be well-defined in some Sobolev space due to the zero mass condition (4.8). Thus, from (4.1) and (4.30), one can obtain

$$\begin{cases} \tilde{\Phi}_\tau - \tilde{\Psi}_y = 0, \\ \tilde{\Psi}_\tau + (p(v) - p(\tilde{V})) = \left(\dfrac{u_y}{v} - \dfrac{\tilde{U}_y}{\tilde{V}} \right) + \tilde{R}. \end{cases} \tag{4.34}$$

Linearizing the system (4.34) around the profile $(\tilde{V}, \tilde{U})$ yields that

$$\begin{cases} \tilde{\Phi}_\tau - \tilde{\Psi}_y = 0, \\ \tilde{\Psi}_\tau + p'(\tilde{V}) \tilde{\Phi}_y = \dfrac{1}{\tilde{V}} \tilde{\Psi}_{yy} + \tilde{Q} + \tilde{R}, \end{cases} \tag{4.35}$$

where

$$\tilde{Q} = -[p(v) - p(\tilde{V}) - p'(\tilde{V})(v - \tilde{V})] + \left(\frac{1}{v} - \frac{1}{\tilde{V}} \right) (\tilde{\Psi}_{yy} + \tilde{U}_y)$$

$$= O(1)(\tilde{\Phi}_y^2 + |\tilde{\Phi}_y \tilde{\Psi}_{yy}| + |\tilde{U}_y \tilde{\Phi}_y|). \tag{4.36}$$

By using (3.15) and (4.29), we have the estimates of the initial data at the interaction time, i.e.,

$$\|(\tilde{\Phi}, \tilde{\Psi})(\tau)\|_{H^2}^2|_{\tau=0} = \left\| \int_{-\infty}^{y} (u - \tilde{V}, v - \tilde{U})(z, 0) dz \right\|_{H^2}^2 \leqslant C\delta^{\frac{1}{3}}. \tag{4.37}$$

Remark 4.6. The property (4.37) implies that the solution at the interaction time $\tau = 0$ is a small perturbation around the outgoing shock profiles $(\tilde{V}, \tilde{U})$ defined in (4.6).

Now our existence result after the interaction time can be stated as follows.

Theorem 4.7 (Existence result after the interacting time). *There exist positive constants $\tilde{\delta}_0$ $(\leqslant \min(\bar{\delta}_0, \delta_I))$ and C such that, if $\delta \leqslant \tilde{\delta}_0$, then there exists a unique solution $(\tilde{\Phi}, \tilde{\Psi})(\tau) \in X([0, +\infty))$ to* (4.35) *and* (4.37)*. Furthermore, it holds that*

$$\|(\tilde{\Phi}, \tilde{\Psi})(\tau)\|_{H^2}^2 + \int_0^\tau \int (|\tilde{U}_{1y}| + |\tilde{U}_{2y}|) \tilde{\Psi}^2 dy ds + \int_0^\tau \|\tilde{\Phi}_y(s)\|_{H^1}^2 + \|\tilde{\Psi}_y(s)\|_{H^2}^2 ds \leqslant C\delta^{\frac{1}{3}}. \tag{4.38}$$

By [19] and the previous argument of Theorem 3.1 on the existence result before the interacting time, to prove Theorem 4.7, it is sufficient to close the following *a priori* assumption:

$$\tilde{N}(T) =: \sup_{\tau \in [0,T]} \|(\tilde{\Phi}, \tilde{\Psi})(\tau)\|_{H^2}^2 \leqslant \tilde{\eta}_0, \quad \text{for } T \geqslant 0, \tag{4.39}$$

for any time $T > 0$ where the solution is supposed to exist on $[0, T]$, and $\tilde{\eta}_0$ is a positive small constant which is independent of T and will be determined later.

Proposition 4.8 (A priori estimates after the interacting time). *Assume that there exists a solution $(\tilde{\Phi}, \tilde{\Psi}) \in X([0, T])$ to* (4.35) *and* (4.37) *with $T \geqslant 0$, then there exist positive constants $\tilde{\delta}_0$, $\tilde{\eta}_0$ and C independent of ε such that, if $\delta \leqslant \tilde{\delta}_0$ and $\tilde{N}(T) \leqslant \tilde{\eta}_0$, then $(\tilde{\Phi}, \tilde{\Psi})$ satisfies for $\tau \in [0, T]$,*

$$\|(\tilde{\Phi}, \tilde{\Psi})(\tau)\|_{H^2}^2 + \int_0^\tau \int (|\tilde{U}_{1y}| + |\tilde{U}_{2y}|) \tilde{\Psi}^2 dy ds + \int_0^\tau \|\tilde{\Phi}_y(s)\|_{H^1}^2 + \|\tilde{\Psi}_y(s)\|_{H^2}^2 ds \leqslant C\delta^{\frac{1}{3}}. \tag{4.40}$$

Similar to Proposition 3.2, we can prove Proposition 4.8 by the following lemmas. First from (4.39), it is straightforward to imply $\frac{v_-}{8} \leqslant v \leqslant 4v_-$ if $\tilde{\eta}_0 > 0$ is small enough, i.e., v is uniformly bounded from above and below. We have

Lemma 4.9. *If $\tilde{\delta}_0$ and $\tilde{\eta}_0$ are suitably small, for $0 \leqslant \tau \leqslant T < +\infty$, it holds for $\delta \leqslant \tilde{\delta}_0$ that*

$$\begin{aligned}
&\|(\tilde{\Phi}, \tilde{\Psi})(\tau)\|_{L^2}^2 + \int_0^\tau \int (|\tilde{U}_{1y}| + |\tilde{U}_{2y}|) \tilde{\Psi}^2 dy ds + \int_0^\tau \|\tilde{\Psi}_y(s)\|_{L^2}^2 ds \\
&\quad \leqslant C\delta^{\frac{1}{3}} + C(\delta + \tilde{\eta}_0) \int_0^\tau \|(\tilde{\Psi}_{yy}, \tilde{\Phi}_y)(s)\|_{L^2}^2 ds.
\end{aligned} \tag{4.41}$$

Proof. Multiplying $(4.35)_1$ by $\tilde{\Phi}$, $(4.35)_2$ by $-\frac{\tilde{\Psi}}{p'(\tilde{V})}$, we obtain

$$\left(\frac{1}{2}\tilde{\Phi}^2\right)_\tau - \tilde{\Phi}\tilde{\Psi}_y = 0$$

and

$$\begin{aligned}
&\left(-\frac{1}{2p'(\tilde{V})}\tilde{\Psi}^2\right)_\tau - \frac{p''(\tilde{V})}{2|p'(\tilde{V})|^2}\tilde{V}_\tau \tilde{\Psi}^2 - \tilde{\Phi}_y \tilde{\Psi} + \frac{\tilde{\Psi}_y^2}{\tilde{V}|p'(\tilde{V})|} \\
&\quad = \left(\frac{1}{\tilde{V}p'(\tilde{V})}\right)_y \tilde{\Psi}_y \tilde{\Psi} - \frac{\tilde{\Psi}}{p'(\tilde{V})}(\tilde{Q} + \tilde{R}) - \left(\frac{\tilde{\Psi}_y \tilde{\Psi}}{\tilde{V}p'(\tilde{V})}\right)_y.
\end{aligned}$$

Adding the above two equations, we have

$$\left(\frac{1}{2}\tilde{\Phi}^2 + \frac{1}{2|p'(\tilde{V})|}\tilde{\Psi}^2\right)_\tau - \frac{p''(\tilde{V})}{2|p'(\tilde{V})|^2}\tilde{V}_\tau \tilde{\Psi}^2 + \frac{\tilde{\Psi}_y^2}{\tilde{V}|p'(\tilde{V})|}$$

$$= \left(\frac{1}{\tilde{V}p'(\tilde{V})}\right)_y \tilde{\Psi}_y\tilde{\Psi} - \frac{\tilde{\Psi}}{p'(\tilde{V})}(\tilde{Q}+\tilde{R}) + \left(-\frac{\tilde{\Psi}_y\tilde{\Psi}}{\tilde{V}p'(\tilde{V})} + \tilde{\Phi}\tilde{\Psi}\right)_y, \tag{4.42}$$

where we have used the fact $p'(\tilde{V}) < 0$.

By (4.36), it is straightforward to obtain

$$\int_0^\tau \int |\tilde{Q}\tilde{\Psi}| dyds \leqslant C(\delta+\tilde{\eta}_0)\int_0^\tau \|(\tilde{\Phi}_y,\tilde{\Psi}_{yy})(s)\|ds + C\delta\int_0^\tau\int(|\tilde{U}_{1y}|+|\tilde{U}_{2y}|)\tilde{\Psi}^2 dyds. \tag{4.43}$$

On the other hand, from Proposition 2.1, we have

$$\begin{aligned}\tilde{R} &= C|\tilde{V}_1-\tilde{v}_m||\tilde{V}_2-\tilde{v}_m| + C|\tilde{U}_{1y}||\tilde{V}_2-\tilde{v}_m| + C|\tilde{U}_{2y}||\tilde{V}_1-\tilde{v}_m|\\ &\leqslant C\delta^2\exp(-c\delta|y|-c\delta|\tau|),\end{aligned} \tag{4.44}$$

which gives

$$\begin{aligned}\int_0^\tau\int|\tilde{R}\tilde{\Psi}|dyds &\leqslant C\delta^2\int_0^\tau \|\tilde{\Psi}\|_{L^\infty}\exp(-c\delta|s|)ds\cdot\int\exp(-c\delta|y|)dy\\ &\leqslant C\delta\int_0^\tau \|\tilde{\Psi}(s)\|^{\frac12}\|\tilde{\Psi}_y(s)\|^{\frac12}\exp(-c\delta|s|)ds\\ &\leqslant \sup_{0\leqslant s\leqslant\tau}\|\tilde{\Psi}(s)\|^2\cdot\int_{-\frac{t_0}{\varepsilon}}^\tau \|\tilde{\Psi}_y(s)\|_{L^2}^2 ds + C\delta^{\frac43}\int_0^\tau \exp\left(-\frac43 c\delta|s|\right)ds\\ &\leqslant \tilde{N}(\tau)\int_0^\tau\|\tilde{\Psi}_y(s)\|_{L^2}^2 ds + C\delta^{\frac13}.\end{aligned} \tag{4.45}$$

Using Proposition 2.1 and the Cauchy inequality, we have

$$\begin{aligned}\left|\int_0^\tau\int\left(\frac{1}{\tilde{V}p'(\tilde{V})}\right)_y\tilde{\Psi}_y\tilde{\Psi}dyds\right| &\leqslant C\int_0^\tau\int|\tilde{V}_y\tilde{\Psi}_y\tilde{\Psi}|dyds \leqslant C\delta\int_0^\tau\int|\tilde{U}_y|^{\frac12}|\tilde{\Psi}_y\tilde{\Psi}|dyds\\ &\leqslant C\delta\int_0^\tau\int\tilde{\Psi}_y^2 dyds + C\delta\int_0^\tau\int(|\tilde{U}_{1y}|+|\tilde{U}_{2y}|)\tilde{\Psi}^2 dyds.\end{aligned} \tag{4.46}$$

Finally, since that

$$-\int_0^\tau\int\frac{p''(\tilde{V})}{2|p'(\tilde{V})|^2}\tilde{V}_\tau\tilde{\Psi}^2 dyds \geqslant \int_0^\tau\int\frac{p''(\tilde{V})}{2|p'(\tilde{V})|^2}(|\tilde{U}_{1y}|+|\tilde{U}_{2y}|)\tilde{\Psi}^2 dyds, \tag{4.47}$$

due to Proposition 2.1, integrating (4.42) and using the smallness of δ, $\tilde{\eta}_0$, and (4.37), (4.43) and (4.45)–(4.47), we obtain (4.41). The proof of Lemma 4.9 is completed. □

Lemma 4.10. *If $\tilde{\delta}_0$ and $\tilde{\eta}_0$ are suitably small, for $0 \leqslant \tau \leqslant T < +\infty$, it holds for $\delta \leqslant \tilde{\delta}_0$ that*

$$\begin{aligned}&\|\tilde{\Phi}(\tau)\|_{H^1}^2 + \|\tilde{\Psi}(\tau)\|_{L^2}^2 + \int_0^\tau\int(|\tilde{U}_{1y}|+|\tilde{U}_{2y}|)\tilde{\Psi}^2 dyds + \int_0^\tau \|(\tilde{\Phi}_y,\tilde{\Psi}_y)(s)\|_{L^2}^2 ds\\ &\quad\leqslant C\delta^{\frac13} + C(\delta+\tilde{\eta}_0)\int_0^\tau\|\tilde{\Psi}_{yy}(s)\|_{L^2}^2 ds.\end{aligned} \tag{4.48}$$

Proof. From $(4.35)_2$, we have

$$\frac{1}{\tilde{V}}\tilde{\Phi}_{y\tau} - p'(\tilde{V})\tilde{\Phi}_y - \tilde{\Psi}_\tau = -\tilde{Q} - \tilde{R}. \tag{4.49}$$

Multiplying (4.49) by $\tilde{\Phi}_y$ implies

$$\left(\frac{1}{2\tilde{V}}\tilde{\Phi}_y^2 - \tilde{\Psi}\tilde{\Phi}_y\right)_\tau + |p'(\tilde{V})|\tilde{\Phi}_y^2 = \left(\frac{1}{2\tilde{V}}\right)_\tau\tilde{\Phi}_y^2 + \tilde{\Psi}_y^2 - (\tilde{Q}+\tilde{R})\tilde{\Phi}_y - (\tilde{\Phi}_\tau\tilde{\Psi})_y. \tag{4.50}$$

Integrating (4.50) over $[0,\tau]\times\mathbb{R}$ and using (4.37), we obtain

$$\int\bigg(\frac{1}{2\tilde V}\tilde\Phi_y^2-\tilde\Psi\tilde\Phi_y\bigg)(\tau)dy+\frac12\int_0^\tau\int|p'(\tilde V)|\tilde\Phi_y^2dyds \leqslant C\delta^{\frac13}+C\int_0^\tau\|\tilde\Psi_y(s)\|_{L^2}^2ds+C(\delta+\tilde\eta_0)\int_0^\tau\|\tilde\Psi_{yy}(s)\|_{L^2}^2ds, \tag{4.51}$$

where we have used that fact that $|\bar V_\tau|\leqslant C\delta^2$.

Multiplying (4.41) by a large constant $\tilde C_1>0$ and using (4.51) and the smallness of δ, $\tilde\eta_0$, we prove Lemma 4.10. □

The estimates of higher order derivatives are given below.

Lemma 4.11. *If $\tilde\delta_0$ and $\tilde\eta_0$ are suitably small, for $0\leqslant\tau\leqslant T<+\infty$, it holds for $\delta\leqslant\tilde\delta_0$ that*

$$\|\tilde\phi(\tau)\|_{H^1}^2+\|\tilde\psi(\tau)\|_{L^2}^2+\int_0^\tau\|(\tilde\phi_y,\tilde\psi_y)(s)\|_{L^2}^2ds \leqslant C\delta^{\frac13}+C\int_0^\tau\|(\tilde\Phi_y,\tilde\Psi_y)(s)\|_{L^2}^2ds+C(\delta+\tilde\eta_0)\int_0^\tau\|\tilde\Psi_{yy}(s)\|_{L^2}^2ds. \tag{4.52}$$

Proof. Applying ∂_y to system (4.34), we have

$$\begin{cases}\tilde\phi_\tau-\tilde\psi_y=0,\\ \tilde\psi_\tau+p'(\tilde V)\tilde\phi_y=\Big(\dfrac{\tilde\psi_y}{v}\Big)_y+\tilde Q_1+\tilde R_y,\end{cases} \tag{4.53}$$

where

$$\tilde Q_1=\bigg(\tilde U_y\Big(\frac1v-\frac1{\tilde V}\Big)\bigg)_y-(p'(v)-p'(\tilde V))v_y. \tag{4.54}$$

Multiplying $(4.53)_1$ by $\tilde\phi$ and $(4.53)_2$ by $\frac{\tilde\psi}{|p'(\tilde V)|}$, and following the same line of Lemma 4.9, we obtain

$$\|(\tilde\phi,\tilde\psi)(\tau)\|_{L^2}^2+\int_0^\tau\|\tilde\psi_y(s)\|_{L^2}^2ds\leqslant C\delta^{\frac13}+C(\delta+\tilde\eta_0)\int_0^\tau\|(\tilde\Phi_y,\tilde\Psi_y,\tilde\phi_y)(s)\|_{L^2}^2ds. \tag{4.55}$$

On the other hand, substituting $(4.53)_1$ into $(4.53)_2$ gives

$$\frac{\tilde\phi_{y\tau}}{\tilde V}-\psi_\tau-p'(\tilde V)\tilde\phi_y=-\Big(\frac1{\tilde V}\Big)_y\tilde\psi_y-\bigg(\tilde\psi_y\Big(\frac1v-\frac1{\tilde V}\Big)\bigg)_y-\tilde Q_1-\tilde R_y. \tag{4.56}$$

Multiplying (4.56) by $\tilde\phi_y$, integrating by part and using (4.37), we obtain

$$\int\bigg(\frac{1}{2\tilde V}\tilde\phi_y^2-\tilde\psi\tilde\phi_y\bigg)(\tau)dy+\int_0^\tau\int|p'(\tilde V)|\tilde\phi_y^2dyds \leqslant C\delta^{\frac13}+C\int_0^\tau\|\tilde\psi_y(s)\|_{L^2}^2ds+C(\delta+\tilde\eta_0)\int_0^\tau\|(\tilde\psi_{yy},\tilde\Phi_y)(s)\|_{L^2}^2ds. \tag{4.57}$$

Multiplying (4.55) by a large constant $\tilde C_2>0$ and using the estimate (4.57) and the smallness of δ, $\tilde\eta_0$, we can complete the proof of Lemma 4.11. □

Lemma 4.12. *If $\tilde\delta_0$ and $\tilde\eta_0$ are suitably small, for $0\leqslant\tau\leqslant T<+\infty$, it holds for $\delta\leqslant\tilde\delta_0$ that*

$$\|\tilde\psi_y(\tau)\|_{L^2}^2+\int_0^\tau\|\tilde\psi_{yy}(s)\|_{L^2}^2ds\leqslant C\delta^{\frac13}+C\int_0^\tau\|(\tilde\phi_y,\tilde\psi_y)(s)\|_{L^2}^2ds+C\delta\int_0^\tau\|\tilde\Phi_y(s)\|_{L^2}^2ds. \tag{4.58}$$

Proof. Multiplying $(4.53)_2$ by $-\tilde\psi_{yy}$, and integrating the resulting equation over $[0,\tau]\times\mathbb{R}$, we can prove (4.58). The details are omitted for brevity. □

Proof of Proposition 4.8. Choosing $\tilde{\eta}_0 = \delta^{\frac{1}{6}}$, using the smallness of δ and combining Lemmas 4.9–4.12, we can close the *a priori* assumption (4.39) and then conclude Proposition 4.8. □

Proof of Theorem 1.1. Taking $\delta_0 = \min(\bar{\delta}_0, \tilde{\delta}_0, \delta_I)$, from Theorems 3.1 and 4.7, we can obtain a family of smooth solutions $(v^\varepsilon, u^\varepsilon)(t)$ to the Navier-Stokes system (1.2) with the well-prepared initial data (3.2) for all $t > 0$. Thus, (3.15), Proposition 2.1 and Sobolev inequality imply the convergence estimate (1.17) before the interacting time. By using the system (4.35) and the estimate (4.38), we have, after the interacting time,

$$\int_0^\infty \|(\tilde{\Phi}_y, \tilde{\Psi}_y)(\tau)\|_{L^2}^2 d\tau + \int_0^\infty \left| \frac{d}{d\tau} \|(\tilde{\Phi}_y, \tilde{\Psi}_y)(\tau)\|_{L^2}^2 \right| d\tau < \infty. \tag{4.59}$$

From the first term on the left-hand side of (4.59), we know that there exists a sequence $\{\tau_n\}$, $n = 1, 2, 3, \ldots$ with $\tau_n \to +\infty$ as $n \to +\infty$ such that

$$\|(\tilde{\Phi}_y, \tilde{\Psi}_y)(\tau_n)\|^2 \to 0, \quad \text{as} \quad n \to +\infty,$$

and from the second term on the left-hand side of (4.59), we know that $\lim_{\tau \to +\infty} \|(\tilde{\Phi}_y, \tilde{\Psi}_y)(\tau)\|^2$ exists. Therefore, the above two estimates immediately imply that

$$\|(\tilde{\Phi}_y, \tilde{\Psi}_y)(\tau)\|_{L^2}^2 \to 0, \quad \text{as} \quad \tau \to +\infty. \tag{4.60}$$

By using (4.38), (4.60) and Sobolev's inequality (3.23), we obtain the asymptotic behavior of the solution

$$\begin{aligned} \sup_{y \in \mathbb{R}} |(\tilde{\Phi}_y, \tilde{\Psi}_y)(y, \tau)| &\leqslant C \|(\tilde{\Phi}_y, \tilde{\Psi}_y)(\tau)\|^{\frac{1}{2}} \cdot \|(\tilde{\Phi}_{yy}, \tilde{\Psi}_{yy})(\tau)\|^{\frac{1}{2}} \\ &\leqslant C \|(\tilde{\Phi}_y, \tilde{\Psi}_y)(\tau)\|^{\frac{1}{2}} \to 0, \quad \text{as} \quad \tau \to +\infty. \end{aligned} \tag{4.61}$$

Thus, (4.61) and Proposition 2.1 imply the convergence (1.18). The proof is completed. □

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