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GLOBAL STABILITY OF WAVE PATTERNS FOR COMPRESSIBLE NAVIER-STOKES SYSTEM WITH FREE BOUNDARY*

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Dedicated to Professor Boling Guo on the occasion of his 80th birthday

Abstract In this article, we investigate the global stability of the wave patterns with the superposition of viscous contact wave and rarefaction wave for the one-dimensional compressible Navier-Stokes equations with a free boundary. It is shown that for the ideal polytropic gas, the superposition of the viscous contact wave with rarefaction wave is nonlinearly stable for the free boundary problem under the large initial perturbations for any $\gamma > 1$ with γ being the adiabatic exponent provided that the wave strength is suitably small.

Key words Compressible Navier-Stokes system; free boundary; combination of viscous contact and rarefaction wave; nonlinear stability

2010 MR Subject Classification 76N10

1 Introduction

The one-dimensional compressible Navier-Stokes equations in the Eulerian coordinate read

$$\begin{cases} \tilde\rho_t + (\tilde\rho\tilde u)_{\tilde x} = 0, \\ (\tilde\rho\tilde u)_t + (\tilde\rho\tilde u^2 + \tilde p)_{\tilde x} = \mu\tilde u_{\tilde x\tilde x}, \\ \left[\tilde\rho\left(\tilde e + \dfrac{\tilde u^2}{2}\right)\right]_t + \left[\tilde\rho\left(\tilde e + \dfrac{\tilde u^2}{2}\right) + \tilde p\tilde u\right]_{\tilde x} = \kappa\tilde\theta_{\tilde x\tilde x} + \mu(\tilde u\tilde u_{\tilde x})_{\tilde x}, \end{cases} \tag{1.1}$$

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where $\tilde{u}(\tilde{x},t)$ is the velocity, $\tilde{\rho}(\tilde{x},t)>0$ is the density, $\tilde{\theta}(\tilde{x},t)>0$ is the absolute temperature, $\tilde{p}=\tilde{p}(\tilde{\rho},\tilde{\theta})$ is the pressure, and $\tilde{e}=\tilde{e}(\tilde{\rho},\tilde{\theta})$ is the internal energy of the gas in gas dynamics, while μ and κ denote the viscosity and the heat-conductivity of the gas, respectively. Here, we study the ideal polytropic gas, that is,

$$\tilde{p}=R\tilde{\rho}\tilde{\theta}=A\tilde{\rho}^{\gamma}e^{\frac{\gamma-1}{R}\tilde{s}},\quad \tilde{e}=c_{\nu}\tilde{\theta},$$

where $\tilde{s}=\tilde{s}(\tilde{\rho},\tilde{\theta})$ is the entropy, $\gamma>1$ is the adiabatic exponent, $c_{\nu}=\frac{R}{\gamma-1}$ is the specific heat, and both A and R are positive constants.

We consider the system (1.1) in the region $\tilde{x}>\tilde{x}(t)$, with the free boundary $\tilde{x}=\tilde{x}(t)$ defined by

$$\begin{cases}\dfrac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t}=\tilde{u}(\tilde{x}(t),t), & t>0,\\ \tilde{x}(0)=0, \end{cases}\tag{1.2}$$

and the free boundary conditions

$$(\tilde{p}-\mu\tilde{u}_{\tilde{x}})\big|_{\tilde{x}=\tilde{x}(t)}=p_0,\qquad \tilde{\theta}\big|_{\tilde{x}=\tilde{x}(t)}=\theta_->0,\tag{1.3}$$

which means that the gas is attached at the boundary $\tilde{x}=\tilde{x}(t)$ with the fixed outer pressure $p_0>0$ and the prescribed temperature $\theta_->0$. The initial data is given by

$$(\tilde{\rho},\tilde{u},\tilde{\theta})\big|_{t=0}=\left(\tilde{\rho}_0,\tilde{u}_0,\tilde{\theta}_0\right)(\tilde{x})\to(\rho_+,u_+,\theta_+)\quad \text{as}\ \ \tilde{x}\to+\infty,\tag{1.4}$$

where $\rho_+>0$, $\theta_+>0$, and u_+ are prescribed constants and we assume $\tilde{\theta}_0\big|_{\tilde{x}=\tilde{x}(t)}=\theta_-$ as the compatibility condition.

As it is convenient to use the Lagrangian coordinate in spatial one-dimensional case, we transform the Eulerian coordinates $(\tilde{x},t)$ to the Lagrangian coordinates (x,t) by

$$x=\int_{\tilde{x}(t)}^{\tilde{x}}\tilde{\rho}(y,t)\mathrm{d}y,\quad t=t,$$

and then the free boundary value problem (1.1)–(1.4) is changed into the half space problem

$$\begin{cases} v_t-u_x=0,\\ u_t+p_x=\mu\left(\dfrac{u_x}{v}\right)_x, & x>0,\ t>0,\\ \left(e+\dfrac{u^2}{2}\right)_t+(pu)_x=\left(\kappa\dfrac{\theta_x}{v}+\mu\dfrac{uu_x}{v}\right)_x, \end{cases}\tag{1.5}$$

with the initial and boundary conditions

$$\begin{cases} \theta|_{x=0}=\theta_-,\\ \left(p(v,\theta)-\mu\dfrac{u_x}{v}\right)(0,t)=p_0,\quad t>0,\\ (v,u,\theta)(x,0)=(v_0,u_0,\theta_0)(x)\to(v_+,u_+,\theta_+)\quad \text{as}\ x\to+\infty, \end{cases}\tag{1.6}$$

where $u(x,t)=\tilde{u}(\tilde{x},t)$, $\theta(x,t)=\tilde{\theta}(\tilde{x},t)$, and $v_+=\rho_+^{-1}$; and $v=v(x,t)=\tilde{\rho}^{-1}(\tilde{x},t)$ denotes the specific volume.

It is well-known that the asymptotic behaviors of (1.5) are characterized by the Riemann solution to the corresponding Euler system:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = 0. \end{cases} \tag{1.7}$$

The Euler system (1.7) is a typical example of the hyperbolic conservation laws. The main feature of the solutions to the hyperbolic conservation laws is the formation of the shock wave no matter how smooth and small the initial values are. Generally speaking, the Riemann solution to Euler system (1.7) contains three basic wave patterns, that is, two nonlinear waves, shock wave and rarefaction wave, and one contact discontinuity in the linearly degenerate field. The above three dilation invariant wave solutions and their linear superpositions in the increasing order of characteristic speed, that is, Riemann solutions, govern both local and large-time behavior of solutions to the Euler system and so govern the large-time behavior of the solutions to the compressible Navier-Stokes equations (1.5).

Indeed, there was great interest and intensive studies on the large time behaviors of the solutions for the Cauchy problem of system (1.5). In the case of the Riemann solution of (1.5) consisting of a single wave pattern, we refer to [2, 17, 22, 26, 34] for the stability of viscous shock wave, [1, 18, 23, 27–29] for rarefaction wave, and [3, 6, 10, 12, 13, 24] for the viscous contact wave. For the superposition wave case, the local stability of the superposition of two viscous shock waves was studied by Huang-Matsumura in [7] without zero mass condition and Huang-Li-Matsumura [6] proved the local stability of the superposition of viscous contact wave and rarefaction waves by introducing a new estimates on the heat kernel.

Recently, the initial boundary value problem of (1.5) attracts increasing interest because it has more physical meanings and of course produces some new mathematical difficulties due to the boundary effect. Some new phenomenon may appear in the initial boundary value problem. Mathematically speaking, how to treat the boundary terms is the main issue. One can refer [25] for the inflow and outflow problems for (1.5) and the references therein. For the initial-boundary value problem (1.5)–(1.6) in this article, there are also some recent results for the large time behavior of the solution. However, most results are concerned with the local stability or the asymptotic stability with"partially" large perturbation to the wave patterns. In fact, Huang-Matsumura-Shi [9] proved the local stability of a viscous contact wave to (1.5) by using the elementary energy methods and a poincare inequality to control the boundary terms. Then, Huang-Zhao [14] proved the asymptotic stability of viscous contact wave and its superposition with rarefaction wave for the system (1.5)–(1.6) with the large perturbation provided that the adiabatic exponent γ is close to 1 enough. Precisely speaking, in [14], the quantity $\gamma-1$ needs to be suitably small, that is, the system is almost isothermal, then the perturbation around wave patterns can be large. Huang-Shi-Wang [11] proved the local stability of viscous shock wave to (1.5) if the viscous shock wave is far away from the boundary. And Hong-Huang [4] proved the local stability of superposition of viscous shock wave and viscous contact wave case to (1.5). For the free boundary problem of corresponding isentropic system with the conservation of energy being neglected, the stability of rarefaction wave and viscous shock wave was proved by

Pan-Liu-Nishihara in [33]. The main aim of this article is to remove the smallness condition of $\gamma-1$ in [14] and to prove the stability of superposition of viscous contact wave and rarefaction wave to the initial and boundary value problem (1.5)–(1.6) under large perturbations for any $\gamma>1$.

Before stating the main results, we first recall the viscous contact wave and approximate rarefaction wave for the compressible Navier-Stokes system (1.5). By the theory of hyperbolic systems of conservation laws (see [32]), we know that there exists a unique point (v_m,u_m,θ_m) such that $p_0=p_m=\frac{R\theta_m}{v_m}$, and (v_m,u_m,θ_m) belongs to the 3-rarefaction wave curve $R(v_+,u_+,\theta_+)$ in the phase plane, where

$$R(v_+,u_+,\theta_+)=\left\{(v,u,\theta)\middle| s=s_+,\quad u=u_+-\int_{v_+}^{v}\lambda(\eta,s_+)\mathrm{d}\eta,\quad v>v_+\right\},$$

$$s=\frac{R}{\gamma-1}\ln\frac{R\theta}{A}+R\ln v\quad\text{and}\quad\lambda(v,s)=\sqrt{A\gamma v^{-\gamma-1}e^{\frac{\gamma-1}{R}s}},$$

that is, (v_m,u_m,θ_m) satisfies

$$p_0=\frac{R\theta_m}{v_m},\quad \theta_m v_m^{\gamma-1}=\theta_+v_+^{\gamma-1},\quad u_m=u_+-\int_{v_+}^{v_m}\lambda(\eta,s_+)\mathrm{d}\eta.$$

We consider the superposition of the viscous contact wave connecting (v_-,u_-,θ_-) with (v_m,u_m,θ_m) and the 3-rarefaction wave connecting (v_m,u_m,θ_m) with (v_+,u_+,θ_+). To state our main results, we first recall the viscous contact wave (V,U,Θ) for the compressible Navier-Stokes system (1.5) defined in [13]. Consider the Euler system (1.7) with the initial Riemann data

$$(v,u,\theta)(x,0)=\begin{cases}(v_-,u_-,\theta_-),\ x<0,\\(v_m,u_m,\theta_m),\ x>0.\end{cases}\tag{1.8}$$

It is known that the contact discontinuity solution takes the form

$$(\tilde V,\tilde U,\tilde\Theta)(x,t)=\begin{cases}(v_-,u_-,\theta_-),\ x<0,t>0,\\(v_m,u_m,\theta_m),\ x>0,t>0,\end{cases}\tag{1.9}$$

provide that

$$u_-=u_m,\quad p_0\triangleq\frac{R\theta_-}{v_-}=p_m\triangleq\frac{R\theta_m}{v_m}.\tag{1.10}$$

In the setting of the compressible Navier-Stokes system (1.5), the inviscid contact discontinuity $(\tilde V,\tilde U,\tilde\Theta)$ becomes smooth to be called "viscous contact wave" and behaves like a diffusion wave due to the dissipation effect. The viscous contact wave $(V^{cd},U^{cd},\Theta^{cd})$ can be constructed as follows. Since the pressure for the profile $(V^{cd},U^{cd},\Theta^{cd})$ is expected to be almost constant as in [10], we set

$$p(V^{cd},\Theta^{cd})\triangleq\frac{R\Theta^{cd}}{V^{cd}}=p_0,$$

which indicates that the leading part of the energy equation $(1.5)_3$ is

$$c_\nu\Theta_t^{cd}+p_0U_x^{cd}=\kappa\left(\frac{\Theta_x^{cd}}{V^{cd}}\right)_x.\tag{1.11}$$

The equation (1.11) and $(1.5)_1$ lead to a nonlinear diffusion equation,

$$\Theta_t^{cd}=a\left(\frac{\Theta_x^{cd}}{\Theta^{cd}}\right)_x,\quad \Theta^{cd}(0,t)=\theta_-,\quad \Theta^{cd}(+\infty,t)=\theta_m,\quad a=\frac{\kappa p_0(\gamma-1)}{\gamma R^2}>0,\tag{1.12}$$

which has a unique self-similar solution $\Theta^{cd}(x,t)=\Theta^{cd}(\xi)$, $\xi=\frac{x}{\sqrt{1+t}}$ due to [5]. Furthermore, on the one hand, $\Theta^{cd}(\xi)$ is a monotone function, increasing if $\theta_m>\theta_-$ and decreasing if $\theta_m<\theta_-$. On the other hand, there exists some positive constant δ^{cd}, such that for $\delta^{cd}=|\theta_m-\theta_-|$, Θ^{cd} satisfies

$$(1+t)|\Theta^{cd}_{xx}|+(1+t)^{\frac{1}{2}}|\Theta^{cd}_x|+|\Theta^{cd}-\theta_m|+|\Theta^{cd}-\theta_-|=O(1)\delta^{cd}e^{-\frac{c_1x^2}{1+t}}\quad \text{as } x\to+\infty, \tag{1.13}$$

where c_1 is positive constant depending only on θ_- and θ_m. Once Θ^{cd} is determined, the viscous contact wave $(V^{cd},U^{cd},\Theta^{cd})(x,t)$ can be defined as follows:

$$V^{cd}=\frac{R}{p_0}\Theta^{cd},\quad U^{cd}=u_m+\frac{\kappa(\gamma-1)}{\gamma R}\frac{\Theta^{cd}_x}{\Theta^{cd}}. \tag{1.14}$$

Then, the viscous contact wave $(V^{cd},U^{cd},\Theta^{cd})(x,t)$ solves the compressible Navier-Stokes system (1.5) time asymptotically, that is,

$$\begin{cases} V^{cd}_t-U^{cd}_x=0,\\ U^{cd}_t+p(V^{cd},\Theta^{cd})_x=\mu\left(\dfrac{U^{cd}_x}{V^{cd}}\right)_x+F^{cd},\\ c_\nu\Theta^{cd}_t+p(V^{cd},\Theta^{cd})U^{cd}_x=\kappa\left(\dfrac{\Theta^{cd}_x}{V^{cd}}\right)_x+\mu\dfrac{(U^{cd})^2_x}{V^{cd}}+G^{cd}, \end{cases} \tag{1.15}$$

where

$$F^{cd}=U^{cd}_t-\mu\left(\frac{U^{cd}_x}{V^{cd}}\right)_x,\quad G^{cd}=-\mu\frac{(U^{cd}_x)^2}{V^{cd}}. \tag{1.16}$$

Now, we turn to rarefaction wave. The 3-rarefaction wave $(v^r,u^r,\theta^r)(x/t)$ connecting (v_m,u_m,θ_m) and (v_+,u_+,θ_+) is the weak solution of the Riemann problem,

$$\begin{cases} v_t-u_x=0,\\ u_t+p_x=0,\qquad x\in\mathbb{R},\quad t>0,\\ \left(e+\dfrac{u^2}{2}\right)_t+(pu)_x=0,\\ (v,u,\theta)(x,0)=\begin{cases}(v_m,u_m,\theta_m),\ x<0\\ (v_+,u_+,\theta_+),\ \ x>0.\end{cases} \end{cases} \tag{1.17}$$

Consider the following Burgers equation:

$$\begin{cases} w_t+ww_x=0,\quad x\in\mathbb{R},\quad t>0,\\ w_0(x):=w(0,t)=\begin{cases} w_-, & x<0,\\ w_-+C_q(w_+-w_-)\displaystyle\int_0^x y^qe^{-y}\mathrm{d}y, & x\ge 0,\end{cases} \end{cases} \tag{1.18}$$

where $w_-=\lambda(v_m,s_+)$, $w_+=\lambda(v_+,s_+)$, and C_q is a constant such that $C_q\int_0^{+\infty}y^qe^{-y}\mathrm{d}y=1$ for large constant $q\ge 8$. Then by the method of characteristic curves, the solution to the above Burgers equation can be expressed explicitly by

$$w(x,t)=w_0(x_0(x,t)),$$

where $x_0(x,t)$ is given by the relation

$$x=x_0(x,t)+w_0(x_0(x,t))t.$$

Correspondingly, the smooth approximate rarefaction wave $(\bar{V}^r, \bar{U}^r, \bar{\Theta}^r)$ can be defined by

$$\begin{cases} \lambda(\bar{V}^r, \bar{\Theta}^r)(x,t) = w(x, 1+t), \\ s(\bar{V}^r, \bar{\Theta}^r)(x,t) = s_+ = s(v_+, \theta_+), \\ \bar{U}^r(x,t) = u_+ - \displaystyle\int_{v_+}^{\bar{V}^r(x,t)} \lambda(\eta, s_+) \mathrm{d}\eta. \end{cases} \tag{1.19}$$

Then, define

$$\left(V^r, U^r, \Theta^r\right)(x,t) := \left(\bar{V}^r, \bar{U}^r, \bar{\Theta}^r\right)(x,t)\Big|_{x \geq 0}. \tag{1.20}$$

And $\left(V^r, U^r, \Theta^r\right)(x,t)$ satisfies

$$\begin{cases} V_t^r - U_x^r = 0, \\ U_t^r + P(V^r, \Theta^r)_x = 0, \\ \left(e(V^r, \Theta^r) + \dfrac{1}{2}(U^r)^2\right)_t + \left(P(V^r, \Theta^r) U^r\right)_x = 0, \\ (V^r, U^r, \Theta^r)|_{x=0} = (v_m, u_m, \theta_m), \\ (V^r, U^r, \Theta^r)|_{t=0} = (V^r, U^r, \Theta^r)(x, 0), \end{cases} \tag{1.21}$$

where $P(V^r, \Theta^r) = R\frac{\Theta^r}{V^r}$.

Denote the wave strength by

$$\delta^{cd} = |\theta_m - \theta_-|, \quad \delta^r = |v_m - v_+| + |u_m - u_+| + |\theta_m - \theta_+|,$$

and $\delta = \max\{\delta^{cd}, \delta^r\}$. Define the superposition wave $(V, U, \Theta)(x,t)$, which is combined by viscous contact wave and rarefaction wave, by

$$\begin{pmatrix} V \\ U \\ \Theta \end{pmatrix}(x,t) = \begin{pmatrix} V^{cd}(x,t) + V^r(x,t) - v_m \\ U^{cd}(x,t) + U^r(x,t) - u_m \\ \Theta^{cd}(x,t) + \Theta^r(x,t) - \theta_m \end{pmatrix}. \tag{1.22}$$

Then, our main result in this article can be stated as follows.

Theorem 1.1 For the initial and boundary value problem (1.5)–(1.6), let (V, U, Θ) be defined in (1.22). Then, there exits a function $m(\delta)$ satisfying $m(\delta) \to +\infty$ as $\delta \to 0$ and a small constant δ_0, such that $|\theta_+ - \theta_-| < \delta_0$ and the initial data satisfies

$$\begin{cases} \|u_0(x) - U(x,0)\|_{H^1(\mathbb{R}_+)} \leq m_0 =: m(\delta), \quad v_0(x), \theta_0(x) \geq m_0^{-1}, \\ \|(v_0(x) - V(x,0), \theta_0(x) - \Theta(x,0))\|_{H_0^1(\mathbb{R}_+)} \leq m_0. \end{cases} \tag{1.23}$$

Then, the problem (1.5)–(1.6) admits a unique global classical solution (v, u, θ) satisfying

$$0 < c_1 \leq v(x,t),\ \theta(x,t) \leq C_1 < +\infty;$$
$$(v - V, u - U, \theta - \Theta)(x,t) \in C\big((0, +\infty); H^1(\mathbb{R}_+)\big);$$
$$(v - V)_x(x,t) \in L^2(0, +\infty; L^2\big(\mathbb{R}_+)\big);$$
$$(u - U, \theta - \Theta)_x(x,t) \in L^2\big(0, +\infty; H^1(\mathbb{R}_+)\big),$$

and the time-asymptotic stability

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}_+} |(v - V, u - U, \theta - \Theta)(x,t)| = 0. \tag{1.24}$$

Remark 1.2 Theorem 1.1 holds for large initial perturbations away from vacuum states and any $\gamma > 1$ provided that the wave strength is suitably small. Thus, we removed the smallness condition of $\gamma - 1$ in [14] and the partial smallness condition of initial perturbation in [3].

Remark 1.3 The result in Theorem 1.1 can be transformed back to the original free boundary problem of (1.1), (1.2)–(1.3), which means that the wave patterns is still nonlinearly stable with suitable shifts in Eulerian coordinate.

We now comment on the proof of this article. It is noted that the smallness of $\gamma - 1$ is crucially used in [14, 29] to control the lower and upper bound of the absolute temperature θ. To remove the smallness condition on $\gamma - 1$, the key issue is how to obtain the uniform lower and upper bound of θ, which is strongly coupled with the uniform bound of the specific volume v. Here, we use some ideas in Huang-Wang [12] for the Cauchy problem. First, it should be noted that the basic energy estimate (see Lemma 3.2 below) is nontrivially derived compared with the case of small initial perturbation. In fact, we essentially use the smallness of wave strength and the underlying structure of wave patterns to control the terms involving the derivative of perturbation around the wave patterns in order to obtain the uniformly basic energy estimate. Second, the specific volume v is shown uniformly bounded from below and above with respect to both time and space through delicate analysis based on a cut-off technique in [21] and the basic energy estimate in Lemma (3.2). Finally, we manipulate some weighted estimates on the perturbation around the wave patterns to derive the uniform bound for the temperature θ. Remark that the underlying structures of viscous contact wave and rarefaction wave are essentially used in the proof. Compared with Cauchy problem case in Huang-Wang [12], one more point is how to control the boundary terms. For the initial and boundary value problems (1.5)–(1.6), we can cope with the boundary terms provided that the initial perturbation $\phi_0(x) \in H_0^1(\mathbf{R}^+)$, which is somehow natural since $\phi(x,t) \in H_0^1(\mathbf{R}^+)$ for any $t > 0$ as in Huang-Zhao [14].

Notations Throughout this article, generic positive constants are denoted by c and C without confusion. For function spaces, $L^p(\Omega), 1 \le p \le \infty$, denotes the usual Lebesgue space on $\Omega \subset \mathbb{R}_+ = (0,\infty)$ with its norm given by

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \quad \| f \|_{L^\infty(\Omega)} := \mathrm{ess.sup}_\Omega |f(x)|.$$

$H^k(\Omega)$ denotes the k^{th} order Sobolev space with its norm

$$\|f\|_{H^k(\Omega)} := \left(\sum_{j=0}^{k} \| \partial_x^j f \|^2 (\Omega) \right)^{\frac{1}{2}}, \quad \text{when } \| \cdot \| = \| \cdot \|_{L^2(\Omega)} .$$

The domain Ω will be often abbreviated without confusion.

2 Preliminaries

We give some preliminary lemmas in this section, whose proof is standard and will be skipped.

Lemma 2.1 Assume that $\delta^{cd} = |\theta_m - \theta_-|$. Then, the viscous contact wave $(V^{cd}, U^{cd}, \Theta^{cd})$ has the following properties:

(1)

$$|V^{cd} - v_m| + |\Theta^{cd} - \theta_m| \le O(1)\delta^{cd} e^{-\frac{cx^2}{1+t}},$$

(2)

$$|\partial_x^k V^{cd}| + |\partial_x^{k-1} U^{cd}| + |\partial_x^k \Theta^{cd}| \le O(1)\delta^{cd}(1+t)^{-\frac{k}{2}} e^{-\frac{cx^2}{1+t}}, \quad k \ge 1.$$

Therefore, we have

$$F^{cd} = O(1)\delta^{cd}(1+t)^{-\frac{3}{2}} e^{-\frac{cx^2}{1+t}}, \quad G^{cd} = O(1)\delta^{cd}(1+t)^{-2} e^{-\frac{cx^2}{1+t}}. \tag{2.1}$$

Lemma 2.2 Let $\delta^r = |(v_m - v_+, u_m - u_+, \theta_+ - \theta_m)|$. The smooth approximation rarefaction wave $(V^r, U^r, \Theta^r)(x,t)$ has the following properties:

(1)

$$U_x^r > 0, \quad (|V_x^r|, |\Theta_x^r|) \le CU_x^r;$$

(2) For any p $(1 \le p \le \infty)$, there exists a constant C_{pq} such that

$$\|(V_x^r, U_x^r, \Theta_x^r)\|_{L^p} \le C_{pq} \min\left(\delta^r, (\delta^r)^{1/p}(1+t)^{-1+1/p}\right),$$

$$\|(V_{xx}^r, U_{xx}^r, \Theta_{xx}^r)\|_{L^p} \le C_{pq} \min\left(\delta^r, ((\delta^r)^{1/p} + (\delta^r)^{1/q})(1+t)^{-1+1/q}\right);$$

(3) There exists some positive constant $C = C(v_m, u_m, \theta_m, s_+)$ such that for $2x \le \lambda(v_m, s_+)$, it holds that

$$|\partial_x^n \{(V^r, U^r, \Theta^r)(x,t) - (v_m, u_m, \theta_m)\}| \le C\delta^r e^{-C(|x|+t)}, \quad n = 0,\ 1,\ 2, \cdots,$$

and for $2x > \lambda(v_m, s_+)$, we have

$$|\partial_x^n \{(V^{cd}, U^{cd}, \Theta^{cd})(x,t) - (v_m, u_m, \theta_m)\}| \le C\delta^{cd} e^{-C(|x|+t)}, \quad n = 0,\ 1,\ 2, \cdots;$$

(4)

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}_+} |(V, U, \Theta)(x,t) - (v^r, u^r, \theta^r)(x/t)| = 0.$$

3 Proof of Theorem 1.1

Put the perturbation $(\phi, \psi, \zeta)(x,t)$ around the superposition wave $(V, U, \Theta)(x,t)$ by

$$(\phi, \psi, \zeta)(x,t) = (v - V, u - U, \theta - \Theta)(x,t). \tag{3.1}$$

Then by (1.5), (1.6), (1.15), and (1.21), the system for the perturbation $(\phi, \psi, \zeta)(x,t)$ becomes

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + (p - P)_x = \mu \left(\dfrac{u_x}{v} - \dfrac{U_x}{V} \right)_x + F, \\ c_\nu \zeta_t + p u_x - P U_x = \kappa \left(\dfrac{\theta_x}{v} - \dfrac{\Theta_x}{V} \right)_x + \mu \left(\dfrac{u_x^2}{v} - \dfrac{U_x^2}{V} \right) + G, \\ \left(\dfrac{R\theta_-}{V + \phi} - \mu \dfrac{U_x + \psi_x}{V + \phi} \right) \Big|_{x=0} = p_0, \\ \zeta(0,t) = 0, \\ (\phi, \psi, \zeta)(x,0) = (v - V, u - U, \theta - \Theta)(x,0) = (\phi_0, \psi_0, \zeta_0)(x), \quad x \in \mathbb{R}_+, \end{cases} \tag{3.2}$$

where $P(x,t)=\frac{R\Theta}{V}(x,t)$,

$$F = (P(V^r,\Theta^r)-P)_x+\left[\mu\left(\frac{U_x}{V}\right)_x-U_t^{cd}\right]:=F_1+F_2, \tag{3.3}$$

$$\begin{aligned} G &= \left[p_0U_x^{cd}+P(V^r,\Theta^r)U_x^r-PU_x\right]+\left[\left(\frac{\kappa\Theta_x}{V}\right)_x+\frac{\mu U_x^2}{V}-\left(\frac{\kappa\Theta_x^{cd}}{V^{cd}}\right)_x\right] \\ &:= G_1+G_2. \end{aligned} \tag{3.4}$$

We shall prove Theorem 1.1 by the local existence and the a priori estimate. We look for the solution (ϕ,ψ,ζ) in the solution space $X([0,+\infty))$,

$$X^M([0,T])=\left\{(\phi,\psi,\zeta)\Big| v,\ \theta\geq M^{-1},\quad \sup_{0\leq t\leq T}\|(\phi,\psi,\zeta)\|_{H^1}\leq M\right\}$$

for some $0<T\leq+\infty$, where the constant M will be determined later. As the local existence of the solution is well known (for example, see [9]), to prove the global existence of Theorem 1.1, we only need to establish the following a priori estimates.

Proposition 3.1 (A priori estimates) Assume that the conditions of Theorem 1.1 hold, then there exists a positive constant δ_0 such that if $\delta<\delta_0$, and $(\phi,\psi,\zeta)\in X^M([0,T])$ for some $M>0$, then it holds that

$$\sup_{0\leq t\leq T}\|(\phi,\psi,\zeta)(t)\|_{H^1}^2+\int_0^T(\|\phi_x\|^2+\|(\psi_x,\zeta_x)\|_{H^1}^2)\leq C_0, \tag{3.5}$$

where C_0 denotes a constant depending only on μ, κ, R, γ, p_0, v_+, u_+, $\theta_\pm$, and m_0 and is independent of M.

Once Proposition 3.1 is proved, we can extend the local solution (u,v,θ) to the global one by the standard continuum process. Moreover, one has

$$\int_0^{+\infty}\left(\|(\phi_x,\psi_x,\zeta_x)(t)\|^2+\left|\frac{\mathrm{d}}{\mathrm{d}t}\|(\phi_x,\psi_x,\zeta_x)(t)\|^2\right|\right)\mathrm{d}t\leq\infty, \tag{3.6}$$

which along with the Sobolev's inequality gives (1.24).

Proposition 3.1 is proved by the following lemmas. First, it should be emphasized that the basic energy estimate is obtained in a nontrivial way, compared with the case of small initial perturbation or uniform constant far-fields case.

Lemma 3.2 There exist some positive constant C_0 and δ_0 such that if $\delta<\delta_0$, it holds that

$$\left\|\left(\psi,\sqrt{\Phi\left(\frac{v}{V}\right)},\sqrt{\Phi\left(\frac{\theta}{\Theta}\right)}\right)(t)\right\|^2+\int_0^t\int\left(\frac{\psi_x^2}{\theta v}+\frac{\zeta_x^2}{\theta^2 v}\right)\mathrm{d}x\mathrm{d}\tau\leq C_0. \tag{3.7}$$

Proof The proof of Lemma 3.2 consists of the following steps.

Step 1 Similar to [9], multiplying $(1.5)_1$ by $-R\Theta(v^{-1}-V^{-1})$, $(1.5)_2$ by ψ, and $(1.5)_3$ by $\zeta\theta^{-1}$, then adding the resulting equations together, we can get

$$\begin{aligned} &\left(\frac{\psi^2}{2}+R\Theta\Phi\left(\frac{v}{V}\right)+c_\nu\Theta\Phi\left(\frac{\theta}{\Theta}\right)\right)_t+\frac{\mu\Theta}{\theta v}\psi_x^2+\frac{\kappa\Theta}{\theta^2 v}\zeta_x^2+Q_1U_x^r+H_x+Q_2 \\ &=F\psi+G\frac{\zeta}{\theta} \end{aligned} \tag{3.8}$$

with

$$\Phi(y)=y-\ln y-1,\quad y>0$$

and

$$H = (p-P)\psi - \mu\left(\frac{u_x}{v} - \frac{U_x}{V}\right)\psi - \frac{\kappa\zeta}{\theta}\left(\frac{\theta_x}{v} - \frac{\Theta_x}{V}\right), \tag{3.9}$$

$$Q_1 = \Phi\left(\frac{\theta V}{v\Theta}\right) + \gamma\Phi\left(\frac{v}{V}\right) > 0, \tag{3.10}$$

$$\begin{aligned} Q_2 = {} & Q_1 U_x^{cd} + \left(\left(\kappa\frac{\Theta_x}{V}\right)_x + \mu\frac{U_x^2}{V} + G\right)\left(\Phi\left(\frac{\Theta}{\theta}\right) - (\gamma-1)\Phi\left(\frac{v}{V}\right)\right) \\ & -\mu\frac{\phi U_x\psi_x}{vV} - \frac{\kappa\Theta_x}{\theta^2 v}\zeta\zeta_x - \frac{\kappa\Theta\Theta_x}{\theta^2 vV}\zeta_x\phi + \frac{\kappa\Theta_x^2}{\theta^2 vV}\zeta\phi - \frac{2\mu U_x}{\theta v}\psi_x\zeta + \frac{\mu U_x^2}{\theta vV}\zeta\phi. \end{aligned} \tag{3.11}$$

As

$$\begin{aligned} |Q_2| \le {} & \frac{\mu\Theta}{4\theta v}\psi_x^2 + \frac{\kappa\Theta}{4\theta^2 v}\zeta_x^2 + C(M)(\phi^2+\zeta^2)(|U_x^{cd}| + (\Theta_x^{cd})^2 + |\Theta_{xx}^{cd}|) \\ & + C(M)(\phi^2+\zeta^2)(|\Theta_{xx}^r| + |U_x^r|^2 + G), \end{aligned} \tag{3.12}$$

where $C(M)$ denotes a constant depending on M, then the boundary condition and $U_x(0,t) = V_t(0,t) = 0$ exactly give

$$\left.\left(\frac{R\theta_-}{V+\phi} - \frac{\mu\phi_t}{V+\phi}\right)\right|_{x=0} = p_0, \quad t > 0. \tag{3.13}$$

Direct computation gives

$$\mu\phi_t(0,t) = -p_0\phi(0,t); \tag{3.14}$$

from which, together with the fact that $\phi_0(x) \in H_0^1(0,+\infty)$, we have

$$\phi(0,t) = \phi_0(0)e^{-\frac{p_0}{\mu}t} = 0. \tag{3.15}$$

Using Pioncaré's type inequality yields

$$|\zeta(x,t)| \le x^{1/2}\|\zeta_x\|, \quad |\phi(x,t)| \le x^{1/2}\|\phi_x\|. \tag{3.16}$$

By Lemma 2.1 and (3.16), we have

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}_+} (\phi^2+\zeta^2)(|U_x^{cd}| + |\Theta_{xx}^{cd}| + |\Theta_x^{cd}|^2)\mathrm{d}x\mathrm{d}\tau \\ \le {} & C\delta\int_0^t \|(\phi_x,\zeta_x)\|^2(1+\tau)^{-1}\int_{\mathbb{R}_+} xe^{-\frac{c_1x^2}{1+\tau}}\mathrm{d}x\mathrm{d}\tau \\ \le {} & C(M)\delta\int_0^t\int_{\mathbb{R}_+}\left(\frac{\theta\phi_x^2}{v^3} + \frac{\kappa\Theta\zeta_x^2}{\theta^2 v}\right)\mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{3.17}$$

As

$$\begin{aligned} |F_1| = {} & \left|R\left(\frac{\Theta^r}{V^r} + \frac{\Theta^{cd}}{V^{cd}} - \frac{\Theta}{V}\right)_x\right| \\ \le {} & C(|V^r - v_m| + |\Theta^r - \theta_m|)|\Theta_x^{cd}| + C(|V^{cd} - v_m| + |\Theta^{cd} - \theta_m|)|\Theta_x^r| \\ \le {} & Ce^{-C(|x|+t)}, \end{aligned} \tag{3.18}$$

and

$$|F_2| \le C\left(|U_t^{cd}| + |U_{xx}^{cd}| + |U_{xx}^r| + |U_x^{cd}V_x^{cd}| + |U_x^{cd}V_x^r| + |U_x^rV_x^{cd}| + |U_x^rV_x^r|\right), \tag{3.19}$$

from Lemma 2.2 (see also Lemma 3.2 in [31]), we have

$$\|F\|_{L^1} \le C\delta^{1/8}(1+t)^{-13/16}. \tag{3.20}$$

Similarly,

$$\|G\|_{L^1} \le C\delta^{1/8}(1+t)^{-13/16}. \tag{3.21}$$

From Lemma 2.2 and Cauchy's inequality, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}_+} (\phi^2+\zeta^2)(|\Theta^r_{xx}| + (U^r_x)^2 + G)\mathrm{d}x\mathrm{d}\tau \\
&\le C\delta^{1/8}\int_0^t (1+\tau)^{-13/16}(\|\phi\|\|\phi_x\| + \|\zeta\|\|\zeta_x\|)\mathrm{d}\tau \\
&\le C(M)\delta^{1/8}\int_0^t\int_{\mathbb{R}_+}\left(\frac{\theta\phi_x^2}{v^3} + \frac{\kappa\Theta\zeta_x^2}{\theta^2 v}\right)\mathrm{d}x\mathrm{d}\tau \\
&\quad + C(M)\delta^{1/8}\int_0^t (1+\tau)^{-13/8}\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)}, \sqrt{\Phi\left(\frac{\theta}{\Theta}\right)}\right)\right\|^2 \mathrm{d}\tau.
\end{aligned} \tag{3.22}$$

By Cauchy's inequality, we obtain

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}_+}\left(F\psi + G\frac{\zeta}{\theta}\right)\mathrm{d}x\mathrm{d}\tau\right| \\
&\le C(M)\int_0^t \|(F,G)\|_{L^1}\|(\psi,\zeta)\|^{1/2}\|(\psi_x,\zeta_x)\|^{1/2}\mathrm{d}\tau \\
&\le C(M)\delta^{1/8}\int_0^t (1+\tau)^{-13/16}\|(\psi,\zeta)\|^{1/2}\|(\psi_x,\zeta_x)\|^{1/2}\mathrm{d}\tau \\
&\le \int_0^t\int_{\mathbb{R}_+}\left(\frac{\mu\Theta\psi_x^2}{4\theta v} + \frac{\kappa\Theta\zeta_x^2}{4\theta^2 v}\right)\mathrm{d}x\mathrm{d}\tau \\
&\quad + C(M)\delta^{1/6}\int_0^t (1+\tau)^{-13/12}\left(1 + \left\|\left(\psi, \sqrt{\Phi\left(\frac{\theta}{\Theta}\right)}\right)\right\|^2\right)\mathrm{d}\tau.
\end{aligned} \tag{3.23}$$

Note that

$$\zeta(0,t) = 0, \quad U_x(0,t) = V_t(0,t) = 0, \quad \text{and} \quad \left(p - \mu\frac{u_x}{v} - P\right)\Big|_{x=0} = p_0 - p_0 = 0,$$

which yields

$$H(0,t) = 0.$$

Then, integrating (3.8) over $\mathbb{R}_+\times(0,t)$, combining (3.17)–(3.23), and using Gronwall's inequality, we have

$$\begin{aligned}
&\left\|\left(\psi, \sqrt{\Phi\left(\frac{v}{V}\right)}, \sqrt{\Phi\left(\frac{\theta}{\Theta}\right)}\right)(t)\right\|^2 + \int_0^t\int_{\mathbb{R}_+}\left(\frac{\psi_x^2}{\theta v} + \frac{\zeta_x^2}{\theta^2 v}\right)\mathrm{d}x\mathrm{d}\tau \\
&\le C_0 + C(M)\delta^{1/8}\int_0^t\int_{\mathbb{R}_+}\frac{\theta\phi_x^2}{v^3}\mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{3.24}$$

Step 2 Following [28], we introduce a new variable $\tilde{v} = \frac{v}{V}$. Then, $(3.2)_2$ can be rewritten by the new variable as

$$\left(\mu\frac{\tilde{v}_x}{\tilde{v}} - \psi\right)_t - (p-P)_x = -F. \tag{3.25}$$

Multiplying (3.25) by $\frac{\tilde{v}_x}{\tilde{v}}$, we have

$$\begin{aligned}&\left(\frac{\mu}{2}\left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2-\psi\frac{\tilde{v}_x}{\tilde{v}}\right)_t+\left(\psi\frac{\tilde{v}_t}{\tilde{v}}\right)_x+\frac{R\theta}{v}\left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2-\frac{R}{v}\zeta_x\frac{\tilde{v}_x}{\tilde{v}}\\&-\frac{\psi_x^2}{v}+\frac{\psi_x\phi U_x}{vV}+R\left(\frac{\theta}{v}-\frac{\Theta}{V}\right)\frac{V_x}{V}\frac{\tilde{v}_x}{\tilde{v}}+\frac{R\phi\Theta_x}{vV}\frac{\tilde{v}_x}{\tilde{v}}=-F\frac{\tilde{v}_x}{\tilde{v}}.\end{aligned}\tag{3.26}$$

The Cauchy's inequality yields that

$$\begin{aligned}&\left|\frac{R}{v}\zeta_x\frac{\tilde{v}_x}{\tilde{v}}\right|+\left|\frac{\psi_x\phi U_x}{vV}\right|+\frac{\psi_x^2}{v}\\&\le\frac{R\theta}{4v}\left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2+C(M)\left(\frac{\zeta_x^2}{\theta^2 v}+\frac{\psi_x^2}{\theta v}\right)+C(M)\phi^2U_x^2,\end{aligned}\tag{3.27}$$

$$\begin{aligned}&\left|R\left(\frac{\theta}{v}-\frac{\Theta}{V}\right)\frac{V_x}{V}\frac{\tilde{v}_x}{\tilde{v}}\right|+\left|\frac{R\phi\Theta_x}{vV}\frac{\tilde{v}_x}{\tilde{v}}\right|+\left|F\frac{\tilde{v}_x}{\tilde{v}}\right|\\&\le\frac{R\theta}{4v}\left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2+C(M)((\phi^2+\zeta^2)(V_x^2+\Theta_x^2)+F^2),\end{aligned}\tag{3.28}$$

and

$$\frac{\phi_x^2}{2v^2}-C(M)\phi^2V_x^2\le\left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2\le\frac{\phi_x^2}{v^2}+C(M)\phi^2V_x^2.\tag{3.29}$$

Note that $\left(\frac{\tilde{v}_t}{\tilde{v}}\right)(0,t)=\left(\frac{\phi_t}{v}\right)(0,t)=0$ and that the right hand sides of (3.27)–(3.29) is already been investigated in Step 1. Integrating (3.26) over $\mathbb{R}_+\times(0,t)$ and combining (3.24), we have

$$\int_{\mathbb{R}_+}\frac{\phi_x^2}{v^2}\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}\frac{\theta\phi_x^2}{v^3}\mathrm{d}x\mathrm{d}\tau\le C_0+C(M)\int_0^t\int_{\mathbb{R}_+}\left(\frac{\psi_x^2}{\theta v}+\frac{\zeta_x^2}{\theta^2 v}\right)\mathrm{d}x\mathrm{d}\tau.\tag{3.30}$$

The proof of Lemma 3.2 is completed by substituting (3.30) into (3.24) and choosing δ suitably small. □

On the basis of Lemma 3.2 and Lemma 5 in [6], one has

Lemma 3.3 There exists a positive constant $C>0$, which may depend on M but is independent of $t>0$, such that if the wave strength $\delta>0$ is suitably small, then it holds that

$$\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{cx^2}{1+\tau}}|(\phi,\psi,\zeta)|^2\mathrm{d}x\mathrm{d}\tau\le C(M).$$

Remark 3.4 The proof of Lemma 3.3 can be found in [6], while it should be noted that there are some boundary terms to be concerned additionally here. In fact, we can change the functions f and g in [6] to be defined by

$$f(x,t)=\int_0^x\omega(y,t)\mathrm{d}y,\qquad g(x,t)=\int_0^x\omega^2(y,t)\mathrm{d}y.$$

Therefore, all the boundary terms to get the estimates in Lemma 3.3 can be controlled by the free boundary conditions (1.6) or vanishes due to the fact that $f(0,t)=g(0,t)\equiv0$. We will skip the details for brevity.

Remark 3.5 For the initial boundary value problem (1.5)–(1.6), Lemma 3.3 is crucially useful for the ψ-component, because for the ϕ and ζ-component, one can use Poincaré inequality as in (3.16)–(3.17) to control the left hand side of the inequality in Lemma 3.3.

Lemma 3.6 Let α_1, α_2 be the two positive roots of the equation $y-\ln y-1=C_0$ and the constant C_0 be the same as in (3.7). Then,

$$\alpha_1\le\int_k^{k+1}\tilde{v}(x,t)\mathrm{d}x,\quad \int_k^{k+1}\tilde{\theta}(x,t)\mathrm{d}x\le\alpha_2,\quad t\ge 0, \tag{3.31}$$

and for each $t\ge 0$, there are points $a_k(t)$, $b_k(t)\in[k,k+1]$ such that

$$\alpha_1\le\tilde{v}(a_k(t),t),\ \tilde{\theta}(b_k(t),t)\le\alpha_2,\quad t\ge 0, \tag{3.32}$$

where $\tilde{v}=\frac{v}{V}$, $\tilde{\theta}=\frac{\theta}{\Theta}$, and $k=0,\ 1,\ 2,\cdots$.

Proof From (3.7), we see that

$$\int_k^{k+1}(\tilde{v}(x,t)-\ln\tilde{v}(x,t)-1)\mathrm{d}x,\quad \int_k^{k+1}(\tilde{\theta}(x,t)-\ln\tilde{\theta}(x,t)-1)\mathrm{d}x\le C_0. \tag{3.33}$$

Applying Jessen's inequality to the convex function $y-\ln y-1=C_0$, we obtain

$$\int_k^{k+1}\tilde{v}(x,t)\mathrm{d}x-\ln\int_k^{k+1}\tilde{v}(x,t)\mathrm{d}x-1,\quad \int_k^{k+1}\tilde{\theta}(x,t)\mathrm{d}x-\ln\int_k^{k+1}\tilde{\theta}(x,t)\mathrm{d}x-1\le C_0,$$

which gives

$$\alpha_1\le\int_k^{k+1}\tilde{v}(x,t)\mathrm{d}x,\quad \int_k^{k+1}\tilde{\theta}(x,t)\mathrm{d}x\le\alpha_2.$$

Moreover, for each $t\ge 0$, by terms of mean value theorem, there are points $a_k(t)$, $b_k(t)\in[k,k+1]$ such that

$$0<\alpha_1\le\tilde{v}(a_k(t),t),\tilde{\theta}(b_k(t),t)\le\alpha_2,\quad t\ge 0. \tag{3.34}$$

□

The following lemma can be found in Jiang [15, Lemma 2.3].

Lemma 3.7 For each $x\in[k,k+1]$, $k=0,1,2,\cdots$, it follows from $(1.5)_2$ that

$$v(x,t)=B(x,t)Y(t)+\frac{R}{\mu}\int_0^t\frac{B(x,t)Y(t)}{B(x,s)Y(s)}\theta(x,s)\mathrm{d}s, \tag{3.35}$$

where

$$B(x,t)=v_0(x)\exp\left(\frac{1}{\mu}\int_x^{+\infty}\big(u_0(y)-u(y,t)\big)\beta(y)\mathrm{d}y\right), \tag{3.36}$$

$$Y(t)=\exp\left(\frac{1}{\mu}\int_0^t\int_{k+1}^{k+2}\sigma(y,s)\mathrm{d}y\mathrm{d}s\right), \tag{3.37}$$

$$\sigma(x,t)=\left(\mu\frac{u_x}{v}-R\frac{\theta}{v}\right)(x,t), \tag{3.38}$$

and

$$\beta(x)=\begin{cases}1, & x\le k+1,\\ k+2-x, & k+1\le x\le k+2,\\ 0, & x\ge k+2.\end{cases} \tag{3.39}$$

By (3.7) and Cauchy's inequality, we have

$$\underline{B}(C_0)\le B(x,t)\le\overline{B}(C_0),\quad \forall x\in[k,k+1],\quad t\ge 0, \tag{3.40}$$

where $\underline{B}(C_0)$ and $\overline{B}(C_0)$ are two constants depending on C_0.

Lemma 3.8 There are two positive constants $\underline{v}(C_0)$ and $\bar{v}(C_0)$ such that

$$\underline{v}(C_0) \leq v(x,t) \leq \bar{v}(C_0), \quad \forall x \in \mathbb{R}_+, \quad t \geq 0, \tag{3.41}$$

where $\underline{v}(C_0)$ and $\bar{v}(C_0)$ are dependent on C_0, independent of x, t.

The proof of Lemma (3.8) can be found in [12] and we skip the details for brevity.

The following lemma is the key to obtain the uniform bound of the absolute temperature.

Lemma 3.9 There exists some positive constants C_0 such that for any given $T > 0$,

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}_+}(\zeta^2+\psi^4)\mathrm{d}x + \int_0^T\int_{\mathbb{R}_+}((\theta+\psi^2)\psi_x^2+\zeta_x^2)\mathrm{d}x\mathrm{d}t \leq C_0. \tag{3.42}$$

Proof The proof of Lemma 3.9 consists of the following steps.

Step 1 First, for $t \geq 0$ and $a > 1$, denote

$$\Omega_a(t) \triangleq \left\{x \in \mathbb{R}\Big|\frac{\theta}{\Theta}(x,t) > a\right\} = \{x \in \mathbb{R}_+ | \zeta(x,t) > (a-1)\Theta(x,t)\}.$$

We derive from (3.7) that Ω_a is bounded, because

$$a|\Omega_a| < \sup_{0\leq t\leq T}\int_{\Omega_a}\frac{\theta}{\Theta}\mathrm{d}x \leq C(a)\sup_{0\leq t\leq T}\int_{\mathbb{R}_+}\Phi\left(\frac{\theta}{\Theta}\right)\mathrm{d}x \leq C(a,C_0). \tag{3.43}$$

Next, multiplying $(3.2)_3$ by $(\zeta-\Theta)_+ = \max\{\zeta-\Theta, 0\}$, noting that $(\zeta-\Theta)_+|_{x=0} = (0-\theta_-)_+ = 0$ and $(\zeta-\Theta)_+|_{x=+\infty} = (0-\theta_+)_+ = 0$, then integrating the resulted equation over $\mathbb{R}_+\times(0,t)$, one has

$$\begin{aligned}
&\frac{c_\nu}{2}\int_{\mathbb{R}_+}(\zeta-\Theta)_+^2\mathrm{d}x + \kappa\int_0^t\int_{\Omega_2}\frac{\zeta_x^2}{v}\mathrm{d}x\mathrm{d}\tau \\
=&\frac{c_\nu}{2}\int_{\mathbb{R}_+}(\zeta_0(x)-\Theta(x,0))_+^2\mathrm{d}x - \int_0^t\int_{\mathbb{R}_+}p\psi_x(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau \\
&-\int_0^t\int_{\mathbb{R}_+}(p-P)U_x(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau + \kappa\int_0^t\int_{\Omega_2}\frac{\zeta_x\Theta_x}{V}\mathrm{d}x\mathrm{d}\tau \\
&-\kappa\int_0^t\int_{\Omega_2}\frac{\phi\Theta_x^2}{vV}\mathrm{d}x\mathrm{d}\tau + \mu\int_0^t\int_{\mathbb{R}_+}\frac{\psi_x^2}{v}(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau \\
&+2\mu\int_0^t\int_{\mathbb{R}_+}\frac{\psi_xU_x}{v}(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau - \mu\int_0^t\int_{\mathbb{R}_+}\frac{\phi U_x^2}{vV}(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau \\
&+\int_0^t\int_{\mathbb{R}_+}G(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau - c_\nu\int_0^t\int\partial_\tau\Theta(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{3.44}$$

Multiplying $(3.2)_2$ by $2\psi(\zeta-\Theta)_+$, and integrating the resulted equation over $\mathbb{R}_+\times[0,t]$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+}\psi^2(\zeta-\Theta)_+\mathrm{d}x + 2\mu\int_0^t\int_{\mathbb{R}_+}\frac{\psi_x^2}{v}(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau \\
=&\int_{\mathbb{R}_+}\psi_0^2(x)(\zeta_0(x)-\Theta(x,0))_+\mathrm{d}x + 2\int_0^t\int_{\mathbb{R}_+}(p-P)\psi_x(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau \\
&+2\int_0^t\int_{\Omega_2}(p-P)\psi\zeta_x\mathrm{d}x\mathrm{d}\tau - 2\int_0^t\int_{\Omega_2}(p-P)\psi\Theta_x\mathrm{d}x\mathrm{d}\tau \\
&+2\mu\int_0^t\int_{\mathbb{R}_+}\frac{\phi U_x}{vV}\psi_x(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau - 2\mu\int_0^t\int_{\Omega_2}\frac{\psi\psi_x\zeta_x}{v}\mathrm{d}x\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+2\mu\int_0^t\int_{\Omega_2}\frac{\phi\psi U_x}{vV}\zeta_x\mathrm{d}x\mathrm{d}\tau+2\mu\int_0^t\int_{\Omega_2}\frac{\psi\psi_x\Theta_x}{v}\mathrm{d}x\mathrm{d}\tau\\
&-2\mu\int_0^t\int_{\Omega_2}\frac{\phi\psi}{vV}U_x\Theta_x\mathrm{d}x\mathrm{d}\tau+2\int_0^t\int_{\mathbb{R}_+}\psi F(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau\\
&+\int_0^t\int_{\Omega_2}\psi^2\partial_\tau\zeta\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\Omega_2}\psi^2\partial_\tau\Theta\mathrm{d}x\mathrm{d}\tau. \qquad (3.45)
\end{aligned}
$$

Adding (3.45) into (3.44) and using $(3.2)_3$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\Big(\frac{c_\nu}{2}(\zeta-\Theta)_+^2+\psi^2(\zeta-\Theta)_+\Big)\mathrm{d}x+\mu\int_0^t\int_{\mathbb{R}_+}\frac{\psi_x^2}{v}(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau+\kappa\int_0^t\int_{\Omega_2}\frac{\zeta_x^2}{v}\mathrm{d}x\mathrm{d}\tau\\
=&\int_{\mathbb{R}_+}\Big(\frac{c_\nu}{2}(\zeta_0(x)-\Theta(x,0))_+^2+\psi_0^2(x)(\zeta_0(x)-\Theta(x,0))_+\Big)\mathrm{d}x\\
&+\int_0^t\int_{\mathbb{R}_+}(p-2P)\psi_x(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\mathbb{R}_+}(p-P)U_x(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau\\
&+\kappa\int_0^t\int_{\Omega_2}\frac{\zeta_x\Theta_x}{V}\mathrm{d}x\mathrm{d}\tau-\kappa\int_0^t\int_{\Omega_2}\frac{\phi\Theta_x^2}{vV}\mathrm{d}x\mathrm{d}\tau+2\mu\int_0^t\int_{\mathbb{R}_+}\frac{\psi_xU_x}{V}(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau\\
&-\mu\int_0^t\int_{\mathbb{R}_+}\frac{\phi U_x^2}{vV}(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau+2\int_0^t\int_{\Omega_2}(p-P)\psi\zeta_x\mathrm{d}x\mathrm{d}\tau\\
&-2\int_0^t\int_{\Omega_2}(p-P)\psi\Theta_x\mathrm{d}x\mathrm{d}\tau-2\mu\int_0^t\int_{\Omega_2}\frac{\psi\psi_x\zeta_x}{v}\mathrm{d}x\mathrm{d}\tau+2\mu\int_0^t\int_{\Omega_2}\frac{\phi\psi U_x}{vV}\zeta_x\mathrm{d}x\mathrm{d}\tau\\
&+2\mu\int_0^t\int_{\Omega_2}\frac{\psi\psi_x\Theta_x}{v}\mathrm{d}x\mathrm{d}\tau-2\mu\int_0^t\int_{\Omega_2}\frac{\phi\psi}{vV}U_x\Theta_x\mathrm{d}x\mathrm{d}\tau+2\int_0^t\int_{\mathbb{R}_+}\psi F(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau\\
&+\int_0^t\int_{\mathbb{R}_+}G(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau-c_\nu\int_0^t\int_{\mathbb{R}_+}\partial_\tau\Theta(\zeta-\Theta)_+\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\Omega_2}\psi^2\partial_\tau\Theta\mathrm{d}x\mathrm{d}\tau\\
&+\frac{\mu}{c_\nu}\int_0^t\int_{\Omega_2}\psi^2\left(\frac{\psi_x^2+2\psi_xU_x}{v}-\frac{\phi U_x^2}{vV}\right)\mathrm{d}x\mathrm{d}\tau+\frac{1}{c_\nu}\int_0^t\int_{\Omega_2}\psi^2G\mathrm{d}x\mathrm{d}\tau\\
&-\frac{1}{c_\nu}\int_0^t\int_{\Omega_2}\psi^2\left(\frac{R\zeta+R\Theta}{v}\psi_x+(p-P)U_x\right)\mathrm{d}x\mathrm{d}\tau+\frac{\kappa}{c_\nu}\int_0^t\int_{\Omega_2}\psi^2\left(\frac{\theta_x}{v}-\frac{\Theta_x}{V}\right)_x\mathrm{d}x\mathrm{d}\tau\\
\triangleq&\int_{\mathbb{R}_+}\Big(\frac{c_\nu}{2}(\zeta_0(x)-\Theta(x,0))_+^2+\psi_0^2(x)(\zeta_0(x)-\Theta(x,0))_+\Big)\mathrm{d}x+\sum_{i=1}^{20}I_i. \qquad (3.46)
\end{aligned}
$$

The estimation of the right hand side of (3.46) can be found in [12], so we skip the details for brevity. Thus, we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+}(\zeta-\Theta)_+^2\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}(\theta\psi_x^2+\zeta_x^2)\mathrm{d}x\mathrm{d}s\\
\le{}& C_0+C_0\int_0^t\left(\max_{x\in\mathbb{R}_+}(\zeta-\frac{1}{2}\Theta)_+^2+\max_{x\in\mathbb{R}}\psi^4\right)\mathrm{d}\tau+C_0\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x^2\mathrm{d}x\mathrm{d}\tau. \qquad (3.47)
\end{aligned}
$$

Step 2 To estimate the last term on the right hand side of (3.47), first, (1.6) and (3.15) leads to

$$
(p-P)|_{x=0}=R\frac{\theta_-}{V(0,t)+\phi(0,t)}-R\frac{\theta_-}{V(0,t)}=R\frac{\theta_-}{V(0,t)}-R\frac{\theta_-}{V(0,t)}=0
$$

and

$$
\psi_x(0,t)=\phi_t(0,t)=0.
$$

Then, multiplying $(3.2)_2$ by ψ^3 and integrating the resulted equation over $\mathbb{R}_+ \times (0,t)$, we have

$$\begin{aligned}
&\frac{1}{4}\int_{\mathbb{R}_+} \psi^4 \mathrm{d}x + 3\mu \int_0^t \int_{\mathbb{R}_+} \frac{\psi^2\psi_x^2}{v} \mathrm{d}x\mathrm{d}\tau \\
= &\frac{1}{4}\int_{\mathbb{R}_+} \psi_0^4 \mathrm{d}x + 3R\int_0^t \int_{\mathbb{R}_+} \frac{\zeta\psi^2\psi_x}{v}\mathrm{d}x\mathrm{d}\tau - 3R\int_0^t\int_{\mathbb{R}_+} \frac{\phi\Theta}{vV}\psi^2\psi_x \mathrm{d}x\mathrm{d}\tau \\
&+3\mu\int_0^t\int_{\mathbb{R}_+} \frac{\phi U_x}{vV}\psi^2\psi_x\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_{\mathbb{R}_+} F\psi^3 \mathrm{d}x\mathrm{d}\tau \\
\triangleq &\frac{1}{4}\int_{\mathbb{R}_+} \psi_0^4\mathrm{d}x + \sum_{i=1}^4 J_i.
\end{aligned} \tag{3.48}$$

We estimate each term on the right hand side of (3.48) as follows:

$$\begin{aligned}
|J_1| &= 3R\int_0^t \int_{\{\zeta>\Theta\}} \frac{\zeta\psi^2\psi_x}{v}\mathrm{d}x\mathrm{d}\tau + 3R\int_0^t\int_{\{\zeta\le\Theta\}} \frac{\zeta\psi^2\psi_x}{v}\mathrm{d}x\mathrm{d}\tau \\
&\le \mu\int_0^t\int_{\{\zeta>\Theta\}} \frac{\psi^2\psi_x^2}{v}\mathrm{d}x\mathrm{d}\tau + C_0\int_0^t\int_{\{\zeta>\Theta\}} \zeta^2\psi^2\mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_0^t\int_{\{\zeta\le\Theta\}} \psi_x^2\mathrm{d}x\mathrm{d}\tau + C_0\int_0^t\int_{\{\zeta\le\Theta\}} \zeta^2\psi^4\mathrm{d}x\mathrm{d}\tau \\
&\le \mu\int_0^t\int \frac{\psi^2\psi_x^2}{v}\mathrm{d}x\mathrm{d}\tau + C_0\int_0^t \max_{x\in\mathbb{R}}\left(\zeta-\frac{1}{2}\Theta\right)_+^2\left(\int\psi^2\mathrm{d}x\right)\mathrm{d}\tau \\
&\quad +C\int_0^t\int_{\{\zeta\le\Theta\}} \frac{\psi_x^2}{\theta}\mathrm{d}x\mathrm{d}\tau + C_0\int_0^t \max_{x\in\mathbb{R}}\psi^4\left(\int_{\{\zeta\le\Theta\}}\zeta^2\mathrm{d}x\right)\mathrm{d}\tau \\
&\le \mu\int_0^t\int\frac{\psi^2\psi_x^2}{v}\mathrm{d}x\mathrm{d}\tau + C_0\int_0^t\left(\max_{x\in\mathbb{R}}\left(\zeta-\frac{1}{2}\Theta\right)_+^2 + \max_{x\in\mathbb{R}}\psi^4\right)\mathrm{d}\tau + C_0.
\end{aligned} \tag{3.49}$$

Recalling (3.7), (3.41) and using Cauchy's inequality, it holds that

$$\begin{aligned}
|J_2| &\le \varepsilon\int_0^t\int_{\mathbb{R}_+}\psi_x^2\mathrm{d}x\mathrm{d}\tau + C(\varepsilon^{-1},C_0)\int_0^t\int_{\mathbb{R}_+}\phi^2\psi^4\mathrm{d}x\mathrm{d}\tau \\
&\le \varepsilon\int_0^t\int_{\mathbb{R}_+}\psi_x^2\mathrm{d}x\mathrm{d}\tau + C(\varepsilon^{-1},C_0)\int_0^t\max_{x\in\mathbb{R}_+}\psi^4\left(\int_{\mathbb{R}_+}\phi^2\mathrm{d}x\right)\mathrm{d}\tau \\
&\le \varepsilon\int_0^t\int_{\mathbb{R}_+}\psi_x^2\mathrm{d}x\mathrm{d}\tau + C(\varepsilon^{-1},C_0)\int_0^t\max_{x\in\mathbb{R}_+}\psi^4\mathrm{d}\tau.
\end{aligned} \tag{3.50}$$

Recalling (3.7), (3.17), (3.22), and (3.30), one has

$$\begin{aligned}
|J_3| &\le \mu\int_0^t\int_{\mathbb{R}_+}\frac{\psi^2\psi_x^2}{v}\mathrm{d}x\mathrm{d}\tau + C_0\int_0^t\int_{\mathbb{R}_+}\phi^2\psi^2U_x^2\mathrm{d}x\mathrm{d}\tau \\
&\le \mu\int_0^t\int_{\mathbb{R}_+}\frac{\psi^2\psi_x^2}{v}\mathrm{d}x\mathrm{d}\tau + C(C_0,M)\int_0^t\int_{\mathbb{R}_+}\phi^2U_x^2\mathrm{d}x\mathrm{d}\tau \\
&\le \mu\int_0^t\int_{\mathbb{R}_+}\frac{\psi^2\psi_x^2}{v}\mathrm{d}x\mathrm{d}\tau + C_0.
\end{aligned} \tag{3.51}$$

It follows from (3.7) and (3.20) that

$$|J_4| \le C\int_0^t \|F\|_{L^1}\|\psi\|_{L^3}^3\mathrm{d}\tau$$

$$
\begin{aligned}
&\le C\delta^{1/8}\int_0^t(1+\tau)^{-7/8}\|\psi\|^{5/2}\|\psi_x\|^{1/2}\mathrm{d}\tau\\
&\le \varepsilon\int_0^t\|\psi_x\|^2\mathrm{d}\tau+C\delta^{1/6}\int_0^t(1+\tau)^{-7/6}\|\psi\|^{10/3}\mathrm{d}\tau\\
&\le \varepsilon\int_0^t\|\psi_x\|^2\mathrm{d}\tau+C_0. \qquad (3.52)
\end{aligned}
$$

Putting the estimates (3.49)–(3.52) into (3.48) gives

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\psi^4\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x^2\mathrm{d}x\mathrm{d}\tau\\
\le{}& C_0+C_0\varepsilon\int_0^t\int_{\mathbb{R}_+}\psi_x^2\mathrm{d}x\mathrm{d}\tau+C(\varepsilon^{-1},C_0)\int_0^t\left(\max_{x\in\mathbb{R}}(\zeta-\frac{1}{2}\Theta)_+^2+\max_{x\in\mathbb{R}}\psi^4\right)\mathrm{d}\tau. \qquad (3.53)
\end{aligned}
$$

Note that

$$
\begin{aligned}
2\int_0^t\int_{\mathbb{R}_+}\psi_x^2\mathrm{d}x\mathrm{d}\tau &\le \int_0^t\int_{\mathbb{R}_+}\frac{\psi_x^2}{\theta}\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}_+}\theta\psi_x^2\mathrm{d}x\mathrm{d}\tau\\
&\le C_0+C_0\int_0^t\int_{\mathbb{R}_+}\theta\psi_x^2\mathrm{d}x\mathrm{d}\tau. \qquad (3.54)
\end{aligned}
$$

Combining (3.47) and (3.53), and choosing ε suitable small, we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+}((\zeta-\Theta)_+^2+\psi^4)\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}((\psi^2+\theta)\psi_x^2+\zeta_x^2)\mathrm{d}x\mathrm{d}\tau\\
\le{}& C_0+C_0\int_0^t\left(\max_{x\in\mathbb{R}_+}\left(\zeta-\frac{1}{2}\Theta\right)_+^2+\max_{x\in\mathbb{R}_+}\psi^4\right)\mathrm{d}\tau. \qquad (3.55)
\end{aligned}
$$

Step 3 It remains to estimate the last term on the right hand side of (3.55). For $x\in\mathbb{R}_+$,

$$
\begin{aligned}
\left(\zeta-\frac{1}{2}\Theta\right)_+^2&=\int_x^\infty 2\left(\zeta-\frac{1}{2}\Theta\right)_+\left(\zeta_x-\frac{1}{2}\Theta_x\right)\mathrm{d}x\\
&\le C\int_{\mathbb{R}_+}\left(\zeta-\frac{1}{2}\Theta\right)_+(|\zeta_x|+|\Theta_x|)\mathrm{d}x\\
&\le \varepsilon\int_{\mathbb{R}_+}\left(\zeta-\frac{1}{2}\Theta\right)_+^2\theta\mathrm{d}x+\frac{C}{\varepsilon}\int_{\{\zeta>\frac{\Theta}{2}\}}\left(\frac{\zeta_x^2}{\theta}+\frac{\Theta_x^2}{\theta}\right)\mathrm{d}x\\
&\le 3\varepsilon\int_{\mathbb{R}_+}\left(\zeta-\frac{1}{2}\Theta\right)_+^2\zeta\mathrm{d}x+\frac{C}{\varepsilon}\int_{\mathbb{R}_+}\frac{\zeta_x^2}{\theta}\mathrm{d}x+\frac{C}{\varepsilon}\int_{\{\zeta>\frac{\Theta}{2}\}}\zeta^2\Theta_x^2\mathrm{d}x\\
&\le 3\varepsilon\max_{x\in\mathbb{R}_+}\left(\zeta-\frac{1}{2}\Theta\right)_+^2\int_{\{\zeta>\frac{\Theta}{2}\}}\zeta\mathrm{d}x+\frac{C}{\varepsilon}\int_{\mathbb{R}_+}\frac{\zeta_x^2}{\theta}\mathrm{d}x+\frac{C}{\varepsilon}\int_{\mathbb{R}_+}\zeta^2\Theta_x^2\mathrm{d}x\\
&\le \varepsilon C_0\max_{x\in\mathbb{R}_+}\left(\zeta-\frac{1}{2}\Theta\right)_+^2+\frac{C}{\varepsilon}\int_{\mathbb{R}_+}\frac{\zeta_x^2}{\theta}\mathrm{d}x+\frac{C}{\varepsilon}\int_{\mathbb{R}_+}\zeta^2\Theta_x^2\mathrm{d}x. \qquad (3.56)
\end{aligned}
$$

This yields that

$$
\max_{x\in\mathbb{R}_+}\left(\zeta-\frac{1}{2}\Theta\right)_+^2\le C_0\int_{\mathbb{R}_+}\frac{\zeta_x^2}{\theta}\mathrm{d}x+C_0\int_{\mathbb{R}_+}\zeta^2\Theta_x^2\mathrm{d}x. \qquad (3.57)
$$

By choosing $\varepsilon>0$ suitably small, we have

$$
\psi^4=\int_x^{+\infty}4\psi^3\psi_x\mathrm{d}x\le 4\int_{\{\zeta>\Theta\}}|\psi|^3|\psi_x|\mathrm{d}x+4\int_{\{\zeta\le\Theta\}}|\psi|^3|\psi_x|\mathrm{d}x
$$

$$
\begin{aligned}
&\leq \varepsilon \int_{\{\zeta>\Theta\}} |\psi|^5\sqrt{\theta}\mathrm{d}x + \frac{C}{\varepsilon}\int_{\{\zeta>\Theta\}} \psi_x^2\frac{|\psi|}{\sqrt{\theta}}\mathrm{d}x + \varepsilon\int_{\{\zeta\leq\Theta\}}\psi^6\theta\mathrm{d}x + \frac{C}{\varepsilon}\int_{\{\zeta\leq\Theta\}}\frac{\psi_x^2}{\theta}\mathrm{d}x \\
&\leq \varepsilon \max_{x\in\mathbb{R}_+}\psi^4\int_{\{\zeta>\Theta\}}(\psi^2+\theta)\mathrm{d}x + C\varepsilon\max_{x\in\mathbb{R}_+}\psi^4\int_{\{\zeta\leq\Theta\}}\psi^2\mathrm{d}x + \frac{C}{\varepsilon}\int_{\mathbb{R}_+}\psi_x^2\left(\frac{|\psi|}{\sqrt{\theta}}+\frac{1}{\theta}\right)\mathrm{d}x \\
&\leq \varepsilon C_0\max_{x\in\mathbb{R}_+}\psi^4 + \frac{C}{\varepsilon}\int_{\mathbb{R}_+}\psi_x^2\left(\frac{|\psi|}{\sqrt{\theta}}+\frac{1}{\theta}\right)\mathrm{d}x,
\end{aligned}\tag{3.58}
$$

which implies that

$$
\max_{x\in\mathbb{R}_+}\psi^4 \leq C_0\int_{\mathbb{R}_+}\psi_x^2\left(\frac{|\psi|}{\sqrt{\theta}}+\frac{1}{\theta}\right)\mathrm{d}x.\tag{3.59}
$$

Substituting (3.57) and (3.59) into (3.55), and recalling (3.7), (3.17), (3.22), and (3.30), it holds that

$$
\begin{aligned}
&\sup_{0\leq t\leq T}\int_{\mathbb{R}_+}((\zeta-\Theta)_+^2+\psi^4)\mathrm{d}x + \int_0^T\int_{\mathbb{R}_+}((\psi^2+\theta)\psi_x^2+\zeta_x^2)\mathrm{d}x\mathrm{d}t \\
&\leq C_0 + C_0\int_0^T\int_{\mathbb{R}_+}\left(\frac{\zeta_x^2}{\theta}+\psi_x^2\left(\frac{|\psi|}{\sqrt{\theta}}+\frac{1}{\theta}\right)\right)\mathrm{d}x\mathrm{d}t \\
&\leq C_0 + \frac{1}{2}\int_0^T\int_{\mathbb{R}_+}(\zeta_x^2+\psi^2\psi_x^2)\mathrm{d}x\mathrm{d}t + C_0\int_0^T\int_{\mathbb{R}_+}\left(\frac{\zeta_x^2}{\theta^2}+\frac{\psi_x^2}{\theta}\right)\mathrm{d}x\mathrm{d}t \\
&\leq C_0 + \frac{1}{2}\int_0^T\int_{\mathbb{R}_+}(\zeta_x^2+\psi^2\psi_x^2)\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.60}
$$

Recalling (3.7) and (3.43), we have

$$
\int_{\{\zeta\leq 2\Theta\}}\zeta^2\mathrm{d}x \leq C\int_{\mathbb{R}_+}\Phi\left(\frac{\theta}{\Theta}\right)\mathrm{d}x \leq C_0,\tag{3.61}
$$

and

$$
\int_{\{\zeta>2\Theta\}}\zeta^2\mathrm{d}x \leq 4\int_{\{\zeta>2\Theta\}}(\zeta-\Theta)^2\mathrm{d}x \leq 4\int_{\mathbb{R}_+}(\zeta-\Theta)_+^2\mathrm{d}x.\tag{3.62}
$$

Thus combining (3.60)–(3.62), the proof of Lemma 3.9 is completed. □

Lemma 3.10 There exist some positive constants C_0 and δ_0 such that if $\delta<\delta_0$, for any $T>0$, we have

$$
\sup_{0\leq t\leq T}\int_{\mathbb{R}_+}(\phi_x^2+\psi_x^2+\zeta_x^2)\mathrm{d}x + \int_0^T\int_{\mathbb{R}_+}(\theta\phi_x^2+\psi_{xx}^2+\zeta_{xx}^2)\mathrm{d}x\mathrm{d}t \leq C_0.\tag{3.63}
$$

Proof Due to (3.7), (3.41), and (3.42), some terms of (3.26) can be estimated again.

$$
\begin{aligned}
&\left|\frac{R}{v}\zeta_x\frac{\tilde{v}_x}{\tilde{v}}\right| + \left|\frac{\psi_x\phi U_x}{vV}\right| + \frac{\psi_x^2}{v} \\
&\leq \frac{R\theta}{4v}\left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2 + C_0\frac{\zeta_x^2}{\theta} + C_0\psi_x^2 + C_0\phi^2U_x^2 \\
&\leq \frac{R\theta}{4v}\left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2 + C_0\left(\zeta_x^2+\frac{\zeta_x^2}{\theta^2}+\theta\psi_x^2+\frac{\psi_x^2}{\theta}\right) + C_0\phi^2U_x^2.
\end{aligned}\tag{3.64}
$$

The other terms in (3.26) can be estimated the same as in Step 2 in Lemma 3.2. Integrating (3.26) over $\mathbb{R}_+\times(0,t)$, then recalling (3.7) and (3.42), we have

$$
\sup_{0\leq t\leq T}\int_{\mathbb{R}_+}\phi_x^2\mathrm{d}x + \int_0^T\int_{\mathbb{R}_+}\theta\phi_x^2\mathrm{d}x\mathrm{d}t \leq C_0.\tag{3.65}
$$

Multiplying $(3.2)_2$ by $-\psi_{xx}$, because $\psi_x(0,t)=\phi_t(0,t)=0$, integrating the resulted equation over $\mathbb{R}_+\times(0,t)$, then we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\frac{\psi_x^2}{2}\mathrm{d}x+\mu\int_0^t\int_{\mathbb{R}_+}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau\\
=&\int_{\mathbb{R}_+}\frac{\psi_{0x}^2}{2}\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}(p-P)_x\psi_{xx}\mathrm{d}x\mathrm{d}\tau-\mu\int_0^t\int_{\mathbb{R}_+}\psi_x\left(\frac{1}{v}\right)_x\psi_{xx}\mathrm{d}x\mathrm{d}\tau\\
&+\mu\int_0^t\int_{\mathbb{R}_+}\left(\frac{U_x}{V}-\frac{U_x}{v}\right)_x\psi_{xx}\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\mathbb{R}_+}F\psi_{xx}\mathrm{d}x\mathrm{d}\tau.
\end{aligned}\tag{3.66}$$

We estimate (3.66) term by term. Recalling (3.42) and (3.65), one has

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}(p-P)_x\psi_{xx}\mathrm{d}x\mathrm{d}\tau\\
=&\int_0^t\int_{\mathbb{R}_+}\left(\frac{R\zeta_x}{v}-\frac{R\theta\phi_x}{v^2}-\frac{R\phi\Theta_x}{vV}-R\left(\frac{\theta}{v^2}-\frac{\Theta}{V^2}\right)V_x\right)\psi_{xx}\mathrm{d}x\mathrm{d}\tau\\
\le&\frac{\mu}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\int_{\mathbb{R}_+}(\zeta_x^2+\theta^2\phi_x^2+(\phi^2+\zeta^2)(V_x^2+\Theta_x^2)+\phi^4V_x^2)\mathrm{d}x\mathrm{d}\tau\\
\le&\frac{\mu}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0+\max_{x,t}\theta\int_0^t\int_{\mathbb{R}_+}\theta\phi_x^2\mathrm{d}x\mathrm{d}\tau\\
\le&\frac{\mu}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0+C_0\max_{x,t}\theta.
\end{aligned}\tag{3.67}$$

By Cauchy's inequality and Sobolev's inequality, and recalling (3.7), (3.42), (3.54), and (3.65), we obtain

$$\begin{aligned}
&-\mu\int_0^t\int_{\mathbb{R}_+}\psi_x\left(\frac{1}{v}\right)_x\psi_{xx}\mathrm{d}x\mathrm{d}\tau\\
=&\int_0^t\int_{\mathbb{R}_+}\left(\frac{\psi_x\phi_x\psi_{xx}}{v^2}+\frac{\psi_xV_x\psi_{xx}}{v^2}\right)\mathrm{d}x\mathrm{d}\tau\\
\le&\frac{\mu}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\int_{\mathbb{R}_+}(\psi_x^2\phi_x^2+\psi_x^2V_x^2)\mathrm{d}x\mathrm{d}\tau\\
\le&\frac{\mu}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\|\psi_x\|_{L^\infty}^2\|\phi_x\|^2\mathrm{d}\tau+C_0\int_0^t\int_{\mathbb{R}_+}\psi_x^2\mathrm{d}x\mathrm{d}\tau\\
\le&\frac{\mu}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\|\psi_x\|\|\psi_{xx}\|\mathrm{d}\tau+C_0\int_0^t\int_{\mathbb{R}_+}\psi_x^2\mathrm{d}x\mathrm{d}\tau\\
\le&\frac{\mu}{4}\int_0^t\int_{\mathbb{R}_+}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\int_{\mathbb{R}_+}\psi_x^2\mathrm{d}x\mathrm{d}\tau\\
\le&\frac{\mu}{4}\int_0^t\int_{\mathbb{R}_+}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0.
\end{aligned}\tag{3.68}$$

Similarly,

$$\begin{aligned}
&\mu\int_0^t\int_{\mathbb{R}_+}\left(\frac{U_x}{V}-\frac{U_x}{v}\right)_x\psi_{xx}\mathrm{d}x\mathrm{d}\tau\\
=&\mu\int_0^t\int_{\mathbb{R}_+}\left(\frac{U_{xx}}{V}-\frac{U_{xx}}{v}-\frac{U_xV_x}{V^2}+\frac{v_xU_x}{v^2}\right)\psi_{xx}\mathrm{d}x\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&= \mu \int_0^t \int_{\mathbb{R}_+} \left(\frac{\phi U_{xx}}{vV} - U_x V_x \frac{\phi(\phi+2V)}{v^2 V^2} + \frac{\phi_x U_x}{v^2} \right) \psi_{xx} \mathrm{d}x\mathrm{d}\tau \\
&\leq \frac{\mu}{8} \int_0^t \int_{\mathbb{R}_+} \frac{\psi_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau + C_0 \int_0^t \int_{\mathbb{R}_+} (\phi^2 (U_{xx}^2 + V_x^2 U_x^2) + \phi^4 U_x^2 V_x^2 + \phi_x^2 U_x^2) \mathrm{d}x\mathrm{d}\tau \\
&\leq \frac{\mu}{8} \int_0^t \int_{\mathbb{R}_+} \frac{\psi_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau + C_0 + C\delta^2 \int_0^t \int_{\mathbb{R}_+} \theta \phi_x^2 \mathrm{d}x\mathrm{d}\tau \\
&\leq \frac{\mu}{8} \int_0^t \int_{\mathbb{R}_+} \frac{\psi_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau + C_0,
\end{aligned} \tag{3.69}
$$

and

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}_+} F \psi_{xx} \mathrm{d}x\mathrm{d}\tau &\leq \frac{\mu}{8} \int_0^t \int_{\mathbb{R}_+} \frac{\psi_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau + C_0 \int_0^t \|F\|^2 \mathrm{d}\tau \\
&\leq \frac{\mu}{8} \int_0^t \int_{\mathbb{R}_+} \frac{\psi_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau + C_0.
\end{aligned} \tag{3.70}
$$

Substituting (3.67)–(3.70) into (3.66) shows

$$
\sup_{0 \leq t \leq T} \int_{\mathbb{R}_+} \psi_x^2 \mathrm{d}x + \int_0^T \int_{\mathbb{R}_+} \psi_{xx}^2 \mathrm{d}x\mathrm{d}t \leq C_0 + C_0 \max_{x,t} \theta. \tag{3.71}
$$

Multiplying $(3.2)_3$ by $-\zeta_{xx}$, noting that $\zeta_t(0,t) = 0$, then integrating the resulted equation over $\mathbb{R}_+ \times (0,t)$, we have

$$
\begin{aligned}
&\frac{c_\nu}{2} \int_{\mathbb{R}_+} \zeta_x^2 \mathrm{d}x + \kappa \int_0^t \int_{\mathbb{R}_+} \frac{\zeta_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau \\
= &\frac{c_\nu}{2} \int_{\mathbb{R}_+} \zeta_{0x}^2 \mathrm{d}x + \int_0^t \int_{\mathbb{R}_+} (pu_x - PU_x) \zeta_{xx} \mathrm{d}x\mathrm{d}\tau \\
&-\kappa \int_0^t \int_{\mathbb{R}_+} \zeta_x \left(\frac{1}{v}\right)_x \zeta_{xx} \mathrm{d}x\mathrm{d}\tau - \kappa \int_0^t \int_{\mathbb{R}_+} \left(\frac{\Theta_x}{v} - \frac{\Theta_x}{V} \right)_x \zeta_{xx} \mathrm{d}x\mathrm{d}\tau \\
&-\mu \int_0^t \int_{\mathbb{R}_+} \left(\frac{u_x^2}{v} - \frac{U_x^2}{V} \right) \zeta_{xx} \mathrm{d}x\mathrm{d}\tau - \int_0^t \int_{\mathbb{R}_+} G \zeta_{xx} \mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{3.72}
$$

Each term on the right hand side of (3.72) will be estimated one by one:

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}_+} (pu_x - PU_x) \zeta_{xx} \mathrm{d}x\mathrm{d}\tau \\
= &\int_0^t \int_{\mathbb{R}_+} \left(\frac{R\theta}{v} \psi_x + \left(\frac{R\zeta}{v} - \frac{R\phi\Theta}{vV} \right) U_x \right) \zeta_{xx} \mathrm{d}x\mathrm{d}\tau \\
\leq &\frac{\kappa}{8} \int_0^t \int_{\mathbb{R}_+} \frac{\zeta_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau + C_0 \int_0^t \int_{\mathbb{R}_+} (\theta^2 \psi_x^2 + (\phi^2 + \zeta^2) U_x^2) \mathrm{d}x\mathrm{d}\tau \\
\leq &\frac{\kappa}{8} \int_0^t \int_{\mathbb{R}_+} \frac{\zeta_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau + C_0 \max_{x,t} \theta \int_0^t \int_{\mathbb{R}_+} \theta \psi_x^2 \mathrm{d}x\mathrm{d}\tau + C_0 \int_0^t \int_{\mathbb{R}_+} (\phi^2 + \zeta^2) U_x^2 \mathrm{d}x\mathrm{d}\tau \\
\leq &\frac{\kappa}{8} \int_0^t \int_{\mathbb{R}_+} \frac{\zeta_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau + C_0 \max_{x,t} \theta + C_0;
\end{aligned} \tag{3.73}
$$

It follows from Cauchy's inequality, (3.42) and (3.65) that

$$
-\kappa \int_0^t \int_{\mathbb{R}_+} \zeta_x \left(\frac{1}{v}\right)_x \zeta_{xx} \mathrm{d}x\mathrm{d}\tau
$$

$$
\begin{aligned}
&\le C\int_0^t\int_{\mathbb{R}_+}\frac{|\zeta_x\phi_x\zeta_{xx}|+|\zeta_xV_x\zeta_{xx}|}{v^2}\mathrm{d}x\mathrm{d}\tau\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\int_{\mathbb{R}_+}(\zeta_x^2\phi_x^2+\zeta_x^2V_x^2)\mathrm{d}x\mathrm{d}\tau\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\sup_{0\le\tau\le t}\|\phi_x\|^2\int_0^t\|\zeta_x\|\|\zeta_{xx}\|\mathrm{d}\tau+C_0\int_0^t\int_{\mathbb{R}_+}\zeta_x^2\mathrm{d}x\mathrm{d}\tau\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\|\zeta_x\|\|\zeta_{xx}\|\mathrm{d}\tau+C_0\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0;
\end{aligned}
\tag{3.74}
$$

Recalling Lemma 2.1 and (3.42), and choosing δ suitable small, we have

$$
\begin{aligned}
&-\kappa\int_0^t\int_{\mathbb{R}_+}\left(\frac{\Theta_x}{v}-\frac{\Theta_x}{V}\right)_x\zeta_{xx}\mathrm{d}x\mathrm{d}\tau\\
&=-\kappa\int_0^t\int_{\mathbb{R}_+}\left(\frac{\Theta_{xx}}{v}-\frac{\Theta_{xx}}{V}-\frac{\Theta_xv_x}{v^2}+\frac{\Theta_xV_x}{V^2}\right)\zeta_{xx}\mathrm{d}x\mathrm{d}\tau\\
&=-\kappa\int_0^t\int_{\mathbb{R}_+}\left(\frac{-\phi\Theta_{xx}}{vV}-\frac{\phi_x\Theta_x}{v^2}+\frac{\phi(\phi+2V)}{v^2V^2}\Theta_xV_x\right)\zeta_{xx}\mathrm{d}x\mathrm{d}\tau\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\int_{\mathbb{R}_+}(\phi^2(\Theta_{xx}^2+\Theta_x^2V_x^2)+\phi_x^2\Theta_x^2+\phi^4\Theta_x^2V_x^2)\mathrm{d}x\mathrm{d}\tau\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0;
\end{aligned}
\tag{3.75}
$$

It follows from (3.71) that

$$
\begin{aligned}
&-\mu\int_0^t\int_{\mathbb{R}_+}\left(\frac{u_x^2}{v}-\frac{U_x^2}{V}\right)\zeta_{xx}\mathrm{d}x\mathrm{d}\tau\\
&=-\mu\int_0^t\int_{\mathbb{R}_+}\left(\frac{\psi_x^2+2\psi_xU_x}{v}-\frac{\phi U_x^2}{vV}\right)\zeta_{xx}\mathrm{d}x\mathrm{d}\tau\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\int_{\mathbb{R}_+}(\psi_x^4+\psi_x^2U_x^2+\phi^2U_x^4)\mathrm{d}x\mathrm{d}\tau\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\|\psi_x\|^3\|\psi_{xx}\|\mathrm{d}\tau+C_0\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}_+}\psi_{xx}^2\mathrm{d}x\mathrm{d}\tau+C_0\sup_{0\le\tau\le t}\|\psi_x\|^4\int_0^t\int_{\mathbb{R}_+}\psi_x^2\mathrm{d}x\mathrm{d}\tau+C_0\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\max_{x,t}\theta^2+C_0;
\end{aligned}
\tag{3.76}
$$

$$
\begin{aligned}
\int_0^t\int G\zeta_{xx}\mathrm{d}x\mathrm{d}\tau&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0\int_0^t\int G^2\mathrm{d}x\mathrm{d}\tau\\
&\le \frac{\kappa}{8}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+C_0.
\end{aligned}
\tag{3.77}
$$

Substituting estimates (3.73)–(3.77) into (3.72) shows

$$\sup_{0\le t\le T}\int_{\mathbb{R}_+}\zeta_x^2\mathrm{d}x+\int_0^T\int_{\mathbb{R}_+}\zeta_{xx}^2\mathrm{d}x\mathrm{d}t\le C_0+C_0\max_{x,t}\theta^2. \tag{3.78}$$

By Sobolev's inequality and (3.42), (3.78), we have

$$\|\zeta\|_{L^\infty}^2\le C\|\zeta\|\|\zeta_x\|\le C_0+C_0\max_{x,t}\theta. \tag{3.79}$$

Noting that

$$\max_{x,t}\theta^2\le 2\max_{x,t}\zeta^2+2\max_{x,t}\Theta^2\le C_0+C_0\max_{x,t}\theta, \tag{3.80}$$

this yields

$$\max_{x,t}\theta\le C_0, \tag{3.81}$$

by which, along with (3.65), (3.71), and (3.78), we complete the proof of Lemma 3.10. □

Proof of Proposition 3.1 The uniform-in-time lower boundedness of the temperature and the proof of Proposition 3.1 can be completed by combining the local existence and the continuation argument, which can be done similarly as in [35, 36]; we omit it for brevity. □

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