Volume Optimization, Normal Surfaces and Thurston's Equation on Triangulated 3-Manifolds

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We establish a relationship among the normal surface theory, Thurston's algebraic gluing equation for hyperbolic metrics and volume optimization of generalized angle structures on triangulated 3-manifolds. The main result shows that a critical point of the volume on generalized angle structures either produces a solution to Thurston's gluing equation or a branched normal surfaces with at most two quadrilateral types.

Contents

1 Introduction

Given a closed triangulated 3-manifold or pseudo 3-manifold, there are several linear and algebraic systems of equations associated to the triangulation. The most prominent ones are Haken's theory of normal surfaces ([7]), Thurston's algebraic gluing equations for constructing hyperbolic metrics ([22]) using hyperbolic ideal tetrahedra, and the notion of angle structures ([12], [18]). The normal surface theory gives a parametrization of essential surfaces in the manifold and solutions of Thurston's equation produce hyperbolic cone metrics. Thurston used the solutions of the equation to produce a complete hyperbolic metric on the figure-8 knot complement in the earlier stage of formulating his geometrization conjecture. The notion of (Euclidean) angle structures, introduced by Casson, Rivin and Lackenby for 3-manifolds with torus boundary, is a linearized version of Thurston's equation. Recently, Tillmann [21] proved that under a fairly general homotopic assumption on the ideal triangulation of a hyperbolic 3-manifold with torus boundary, there are always solutions to Thurston's equation. The goal of the paper is to investigate the volume optimization of the generalized angle structures and relate it to normal surfaces and Thurston's equations. Using volume optimization of angle structures to find geometric structure was introduced in the field in an important paper by Rivin [19]. It was also known to A. Casson [2]. See also the recent work [6]. This paper follows the same approach in the broader sense. Our main result is the following.

Theorem 1.1. *Suppose* $(M, \mathbf{T})$ *is a triangulated closed pseudo 3-manifold and* M *is orientable. If the triangulation supports no branched normal surfaces with at most two quadrilateral types except the vertex links, then Thurston's algebraic gluing equation associated to the triangulation has a solution.*

This is similar to the main result in [13]. We remark that a simplicial triangulation supports no branched normal surfaces with at most two quadrilateral types except the vertex links. Let us recall briefly Thurston's gluing equation on a triangulated pseudo 3-manifold. Assign each normal quadrilateral in the triangulation **T** a complex number $z \in \mathbf{C}$. The assignment is said to satisfy Thurston's equation if

(a) the three complex numbers assigned to three quadrilaterals in a tetrahedra are z, $\frac{1}{1-z}$ and $\frac{z-1}{z}$ and

(b) for each edge e in the triangulation, if $z_1, ..., z_k$ are the complex numbers assigned to the quadrilaterals facing the edge e, then

$$\prod_{i=1}^{k} z_i = \pm 1 \tag{1}$$

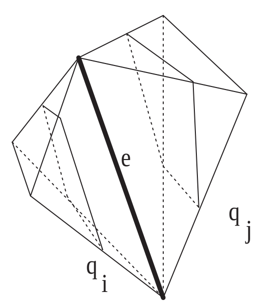

Figure 1: Two quadrilaterals q_i and q_j facing the edge e

We remark that Thurston's equation and its solutions are well defined on any closed, or compact with boundary and more generally any pseudo 3-manifolds with a triangulation. Furthermore, as was first observed by Yoshida [24], those solutions of Thurston's equations where the right-hand-side of (1) is 1 can produce a representation of the fundamental group of the 3-manifold to $PSL(2, \mathbf{C})$. If the right-hand-side of (1) is ± 1, then the solution will produce a representation from a finite index subgroup of $\pi_1(M - V)$ to $PSL(2, \mathbf{C})$.

Theorem 1.1 is a special case of theorem 3.2 in §3 where one shows that under the same assumption as theorem 1.1 there are either solutions to Thurston's equation of the form (a) and for any edge e,

$$\prod_{i=1}^{k} z_i = \pm k(e) \tag{2}$$

where $k(e) \in \mathbf{S}^1$ is a given function satisfying (18) and (19) or there exists a special branched normal surface with at most two quadrilateral types. Theorem 1.1 provides some evidences relating normal surface theory to representations.

The basic idea of the proof of theorem 1.1 is simple. First we introduce the notion of $\mathbf{S}^1$-valued angle structure on triangulated closed pseudo 3-manifold and show that these structures exist on any closed triangulated pseudo 3-manifold. Furthermore, the space of all $\mathbf{S}^1$-valued angle structures is shown to be a closed smooth manifold. The volume of an $\mathbf{S}^1$-valued angle structure can be naturally defined using the Lobachevsky function. In particular, the volume achieves a maximum value at some point. If all $\mathbf{S}^1$-valued angles at the maximum point are not ± 1, then one shows that it produces a solution to Thurston's equation (a) and (1.1). If there is an $\mathbf{S}^1$-valued angle which is ± 1, then by analysis the subderivative of the volume function, one produces a vector with at most two non-zero coordinates which is perpendicular to the tangent space of the $\mathbf{S}^1$-valued angle structures. The work of Tollefsen, Kang-Rubinstein, Tillmann, Thurston and others on the spun normal surface theory, interpreted in term of tangents to angle structures, says that the vector produces a branched normal surface of at most two quadrilateral types.

We remark that all results in the paper can be generalized without difficulties to compact pseudo 3-manifolds with boundary. The simplest way to treat them is by taking the doubling construction. For simplicity, we will not state the corresponding theorems for pseudo manifolds with boundary.

The paper is organized as follows. In §2, we revisit the theory of normal surfaces and spun normal surfaces. In §3, we recall Thurston's work on gluing hyperbolic metrics and the volume of angle structures. Theorem 1.1 is proved in §4.

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2 The theory of normal surfaces revisited

The normal surface theory, developed by Haken in 1950's, is a beautiful chapter in 3-manifold topology. In late 1970's, Thurston introduced the notion of spun normal surfaces and used it to study 3-manifolds. There are works by Tollefson, Kang-Rubinstein, Tillmann, Thurston, Jaco and others which characterize spun normal surfaces using Haken's normal coordinates. It turns out a spun normal surface is most conveniently described in terms of the tangent vectors to angle structures. In fact, the two system of linear equations for the tangential angle structures and the spun normal surfaces are dual to each other. This observation, which is implicit in the work of [23], [10], and [20], is very useful for us in §4 to relate the critical points of the volume functional with the normal surfaces.

We will revisit the normal surface theory and follow the expositions in [8] and [20] closely in this section. Some of the notations used in the section are new.

2.1 Triangulations of closed pseudo 3-manifolds and normal surfaces

Let X be a union of finitely many disjoint oriented Euclidean tetrahedra. The collection of all faces of tetrahedra in X is a simplicial complex $\mathbf{T}^*$ which is a triangulation of X. Identify codimension-1 faces in X in pairs by affine homeomorphisms. The quotient space M is a compact pseudo 3-manifold with a triangulation $\mathbf{T}$ whose simplexes are the quotients of simplexes in $\mathbf{T}^*$. Let V, E, F, T be the sets of all vertices, edges, triangles and tetrahedra in $\mathbf{T}$. If $x, y \in V \cup E \cup F \cup T$, we use $x > y$ to denote that y is a face of x. We use $|Y|$ to denote the cardinality of a set Y.

Note that in this definition of triangulation, we do not assume that simplexes in $\mathbf{T}$ are embedded in M. For instance, it may well be that $|V| = 1$. Furthermore, the non-manifold points in M are either the vertices or the centers of the edges. If we require the affine homeomorphisms used in the gluing for M be orientation reversing, then the pseudo manifold M is oriented and non-manifold points of M are contained in V. Let N be M with a small open regular neighborhood of V removed. Then we call $\{\sigma \cap N | \sigma \in \mathbf{T}\}$ an *ideal triangulation* of N, a compact manifold with boundary.

According to Haken [7], a *normal arc* in X is an embedded arc in a triangle face so that its end points are in different edges and a *norma disk* in X is an embedded disk in a tetrahedron so that its boundary consists of 3 or 4 normal arcs. These are called *normal triangles* and *normal quadrilaterals* respectively.

The projections of normal arcs and normal disks from X to M constitute normal arcs and normal disks in the triangulated space $(M, \mathbf{T})$. A *normal isotopy* is an isotopy of the ambient space X or M which leaves each simplex invariant. Normal arcs and disks will be considered up to normal isotopy. For each tetrahedra, there are four normal triangles and three normal quadrilaterals up to normal isotopy. We use $\triangle$, $\square$ and $\mathbf{A}$ to denote the sets of all normal isotopy classes of normal triangles, quadrilaterals and normal arcs in the triangulation $\mathbf{T}$. There are relationships among various sets

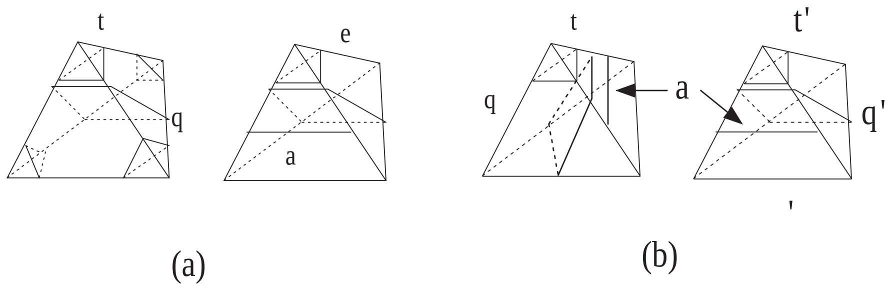

Figure 2: t, t' are normal triangles and q, q' are normal quadrilaterals

$V, E, F, T, \triangle, \square, \mathbf{A}$. These incidence relations, which will be recalled below, are the basic ingredient for defining linear and algebraic equations on $\mathbf{T}$.

Take $t \in \triangle$, $a \in \mathbf{A}$, $q \in \square$, $e \in E$, and $\sigma \in T$. The following notations will be used in the sequel. We use $a < t$ (and $a < q$) if there exist representatives $x \in a$, $y \in t$ (and $z \in q$) so that x is an edge of y (and z). We use $t \subset \sigma$ and $q \subset \sigma$ to denote that representatives of t and q are in the tetrahedron σ. In this case, we say the tetrahedron σ contains t and q. The index $i(q, e)$ is an integer $0, 1$ or 2 defined as follows. $i(q, e) = 0$ if there is no tetrahedron $\sigma > e$ so that $q \subset \sigma$. If there is a tetrahedron $\sigma > e$ so that $q \subset \sigma$, then $i(q, e) = 1$ if e is the only edge in σ facing q and $i(q, e) = 2$ if e are the two edges in σ face q (See figure 3). The index $i(t, e)$ is an integer $0, 1, 2$ or 3 so that it is 0 if there is no tetrahedron $\sigma > e$ which contains t. If there is a tetrahedron $\sigma > e$ and σ contains t, then $i(t, e)$ is the number of vertices of t which lie in e. See figure 3.

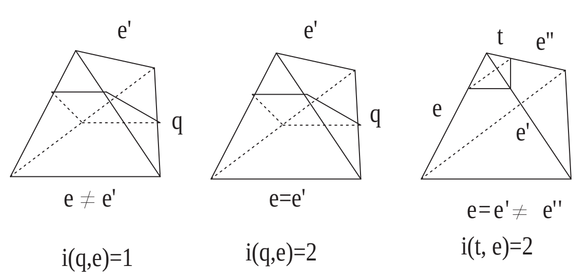

Figure 3: incident indices

We remark that if $\mathbf{T}$ is a simplicial triangulation, then the indices $i(t, e)$ and $i(q, e)$ take only values $0, 1$.

As a convention, in the sequel, we will always use σ, e and q to denote a tetrahedron, an edge and a quadrilateral in the triangulation $\mathbf{T}$ respectively.

2.2 The normal surface equation and the Kang-Rubinstein basis

The normal surface equation is a system of linear equations defined in the space $\mathbf{R}^{\triangle} \times \mathbf{R}^{\square}$, introduced by W. Haken [7]. It is defined as follows. For each normal arc $a \in \mathbf{A}$, suppose σ, σ' are the two tetrahedra adjacent to the triangular face which contains a. Then there is a homogeneous linear equation for $x \in \mathbf{R}^{\triangle} \times \mathbf{R}^{\square}$ associated to a:

$$x(t) + x(q) = x(q') + x(t') \tag{3}$$

where $t, q \subset \sigma$, $t', q' \subset \sigma'$ and $t, t', q, q' > a$. See figure 2(b). Let $\mathbf{S}_{ns}$ be the space of all solutions to (3) as a runs over all normal arcs.

An integer solution to (3) will be called a *branched normal surface* in the sequel. A basis of the solution space $\mathbf{S}_{ns}$ to equations (3) was found by Kang-Rubinstein [10]. To state it, let us introduce one more notation. Given a finite set Z, the *standard basis* of the vector space $\mathbf{R}^Z$ will be denoted by $\{z^* | z \in Z\}$ where $z^*(z) = 1$ and $z^*(z') = 0$ if $z \neq z' \in Z$. We give $\mathbf{R}^Z$ the inner product so that $\{z^* | z \in Z\}$ forms an orthonormal basis. Now for each $\sigma \in T$ and $e \in E$, define the vectors $W_\sigma, W_e \in \mathbf{R}^{\triangle} \times \mathbf{R}^{\square}$ as follows,

$$W_\sigma = \sum_{q \in \square, q \subset \sigma} q^* - \sum_{t \in \triangle, t \subset \sigma} t^* \tag{4}$$

and

$$W_e = \sum_{q \in \square} i(q, e) q^* - \sum_{t \in \triangle} i(t, e) t^*. \tag{5}$$

A basic theorem proved in [10] says,

Theorem 2.1(Kang-Rubinstein). *For any triangulated closed pseudo 3-manifold, the set* $\{W_x | x \in E \cup T\}$ *forms a basis of the solution space* $\mathbf{S}_{ns}$ *of the normal equations.*

For the convenient of the reader, an alternative interpretation of Kang-Rubinstein's proof is given in the appendix.

2.3 Spun normal surfaces and tangential angle structure

Given $x \in \mathbf{R}^{\triangle} \times \mathbf{R}^{\square}$, we will call $x(t)$ and $x(q)$ the t-coordinate and q-coordinate (triangle and quadrilateral coordinates) of x. The spun normal surface theory addresses the following question, first investigated by Thurston [22]. Given a vector $z \in \mathbf{R}^{\square}$, when does there exist a solution $x \in \mathbf{S}_{ns}$ to (3) whose projection to $\mathbf{R}^{\square}$ is z. Geometrically, it asks if a given finite set of normal quadrilaterals can be realized as the set of all normal quadrilaterals in a branched normal surface. The question was completely solved in the work of [23], [10], [20] and [9]. The purpose of this section is to interprete their work in terms of tangential angle structures.

Definition 2.1. A *tangential angle structure* on a triangulated pseudo 3-manifold $(M, \mathbf{T})$ is a vector $x \in \mathbf{R}^{\square}$ so that

for each tetrahedron $\sigma \in T$,

$$\sum_{q \in \square, q \subset \sigma} x(q) = 0, \tag{6}$$

and for each edge $e \in E$,

$$\sum_{q \in \square} i(q, e) x(q) = 0. \tag{7}$$

The linear space of all tangential angle structures on $(M, \mathbf{T})$ is denoted by $TAS(M, \mathbf{T})$ or simply $TAS(\mathbf{T})$.

Recall that a (Euclidean type) angle structure, introduced by Casson, Rivin and Lackenby, is a vector $x \in \mathbf{R}^{\square}_{>0}$ so that for each tetrahedron $\sigma \in T$,

$$\sum_{q \in \square, q \subset \sigma} x(q) = \pi, \tag{8}$$

and for each $e \in E$,

$$\sum_{q \in \square} i(q, e) x(q) = 2\pi. \tag{9}$$

Thus one sees easily that a tangential angle structure is a tangent vector to the space of all angle structures. In [14], a *generalized angle structure* on $(M, \mathbf{T})$ is defined as a vector $x \in \mathbf{R}^{\square}$ so that (8) and (9) hold. It is proved in [14] that a generalized angle structure exists if and only if the euler characteristic of each link $lk(v)$ is zero for $v \in V$. We will consider in this paper those $x \in \mathbf{R}^{\square}$ so that the right-hand-side of (8) is in $\pi + 2\pi\mathbf{Z}$ and the right-hand-side of (9) is in $2\pi\mathbf{Z}$. These will be called $\mathbf{S}^1$*-valued angle structures* on $\mathbf{T}$ and will be shown to exist on any closed pseudo 3-manifold. Evidently $TAS(\mathbf{T})$ is the tangent space of all $\mathbf{S}^1$-valued angle structures.

The following is the result proved by Tollefson (for closed 3-manifold case), Kang-Rubinstein and Tillmann. The result was also known to Jaco.

Theorem 2.2. ([23], [10], [20]) *For a triangulated closed pseudo 3-manifold* $(M,\mathbf{T})$, *let* $Proj_{\square}: \mathbf{R}^{\triangle} \times \mathbf{R}^{\square} \to \mathbf{R}^{\square}$ *be the projection. Then*

$$Proj_{\square}(\mathbf{S}_{ns}) = TAS(\mathbf{T})^{\perp} \tag{10}$$

where $\mathbf{R}^{\square}$ *has the standard inner product so that* $\{q^*|q \in \square\}$ *is an orthonormal basis.*

We remark that theorem 2.2 is not stated in this form in the work of [23], [10], [20]. This interpretation is due to us.

Proof. Suppose $\mathbf{R}^n$ and $\mathbf{R}^m$ are Euclidean spaces with the standard inner product and $A: \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation with transpose $A^t: \mathbf{R}^m \to \mathbf{R}^n$. Then it is well known that $Im(A) = ker(A^t)^{\perp}$. Define a linear map

$$A: \mathbf{R}^E \times \mathbf{R}^T \to \mathbf{R}^{\square} \tag{11}$$

by

$$A(h) = Proj_{\square}(\sum_{e \in E} h(e)W_e + \sum_{\sigma \in T} h(\sigma)W_{\sigma}). \tag{12}$$

By (4) and (5), we have

$$A(h)(q) = \sum_{\sigma \in T, q \subset \sigma} h(\sigma) + \sum_{e \in E} i(q,e)h(e). \tag{13}$$

To understand tangential angle structures, we define a linear map $B: \mathbf{R}^{\square} \to \mathbf{R}^E \times \mathbf{R}^T$ so that

$$B(x)(e) = \sum_{q \in \square} i(q,e)x(q) \tag{14}$$

and

$$B(x)(\sigma) = \sum_{q \in \square, q \subset \sigma} x(q). \tag{15}$$

By definition, we have $TAS(\mathbf{T}) = ker(B)$. We claim that $B = A^t$, i.e., $(B(x),h) = (x, A(h))$ for all $x \in \mathbf{R}^{\square}, h \in \mathbf{R}^E \times \mathbf{R}^T$ where $(,)$ is the standard inner product.

Indeed, by definition, we have

$$\begin{aligned}(B(x),h) &= \sum_{e\in E} h(e)B(x)(e) + \sum_{\sigma\in T} h(\sigma)B(x)(\sigma)\\ &= \sum_{e\in E, q\in\square} i(e,q)x(q)h(e) + \sum_{\sigma\in T, q\in\square, q\subset\sigma} h(\sigma)x(q)\\ &= \sum_{q\in\square} x(q)\sum_{e\in E} i(q,e)h(e) + \sum_{q\in\square} x(q)\sum_{\sigma\in T, q\subset\sigma} h(\sigma)\\ &= (x, A(h)).\end{aligned}$$

This ends the proof.

Corollary 2.3. (Tillmann [Ti]). *(a)* $\dim(TAS(\mathbf{T})) = |V| - |E| + 2|T| = \chi(M) + |T|$.

(b) $\dim(Proj_\square(\mathbf{S}_{ns})) = -\chi(M) + 2|T|$.

Definition 2.2. Suppose $(M,\mathbf{T})$ is a triangulated closed pseudo 3-manifold. We say the triangulation $\mathbf{T}$ is *angle rigid* if there is $q\in\square$ so that $x(q)=0$ for all $x\in TAS(\mathbf{T})$. We say $\mathbf{T}$ is *2-angle rigid* if there exists a non-zero vector $(c_1,c_2)\in\mathbf{R}^2$ and $q_1\neq q_2\in\square$ so that $c_1x(q_1)+c_2x(q_2)=0$ for all $x\in TAS(\mathbf{T})$.

By definition, if $\mathbf{T}$ is angle rigid, then $x(q)$ is a constant for all generalized angle structure x, i.e., the angle at q cannot be deformed. If the triangulation $\mathbf{T}$ has an edge e of degree 1, then $\mathbf{T}$ is angle rigid at the quadrilateral q so that $i(q,e)\neq 0$. If $\mathbf{T}$ has an edge e of degree 2, then $\mathbf{T}$ is 2-angle rigid at the quadrilaterals q_1 and q_2 so that $i(q_j,e)\neq 0$ for $j=1,2$.

One simple consequence of Theorem 2.2 is,

Corollary 2.4. *Under the same assumption as in theorem 2.2,*

(a) $(M,\mathbf{T})$ *is angle-rigid if and only if there exists an embedded normal surface* Σ *in* $\mathbf{T}$ *so that the surface has exactly one normal quadrilateral type.*

(b) (M,T) *is 2-angle rigid if and only if there exists a vector* $v\in\mathbf{S}_{ns}\cap(\mathbf{Z}^{\triangle}\times\mathbf{Z}^{\square})$ *so that* $Proj_\square(v)$ *is non-zero and has at most two non-zero coordinates.*

To see part (a), if there exists a normal surface containing only one quadrilateral type $q\in\square$, then its normal coordinate $x\in\mathbf{R}^{\triangle}\times\mathbf{R}^{\square}$ is a vector so that $Proj_\square(x) =$

$kq^* \in TAS(\mathbf{T})^{\perp}$ for some non-zero scalar k and some $q \in \square$. Thus $z(q) = 0$ for all $z \in TAS(\mathbf{T})$. Conversely, if there exists $q \in \square$ so that $z(q) = 0$ for all $z \in TAS(\mathbf{T})$, then $q^* \in TAS(\mathbf{T})^{\perp}$. By theorem 2.2, $q^* = Proj_{\square}(v)$ for some $v \in \mathbf{S}_{ns}$. We may choose $v \in \mathbf{Q}^{\triangle} \times \mathbf{Q}^{\square}$ since the linear equations (3) are integer coefficient and q^* has integer coordinates. It follows some integer multiple kv has non-negative integer q-coordinates. Now add to the vector kv a positive integer multiples of the normal coordinates of the normal surfaces $lk(v)$, the link of the vertex $v \in V$, so the resulting vector has positive t-coordinates. We obtain a vector $u \in \mathbf{S}_{ns} \cap (\mathbf{Z}^{\triangle}_{\geq 0} \times Z^{\square}_{\geq 0})$ with exactly one non-zero q-coordinate. By the work of Haken, this vector u is the normal coordinate of an embedded normal surface in $(M, \mathbf{T})$. The proof of (b) is similar and will be omitted. However, we are not able to conclude that $v \in \mathbf{Z}^{\triangle}_{\geq 0} \times \mathbf{Z}^{\square}_{\geq 0}$ in this case.

Question. Suppose M is a complete finite volume hyperbolic manifold and $\mathbf{T}$ is geometric triangulation of M so that if $\mathbf{M}$ has cusps, $\mathbf{T}$ is ideal. Is $\mathbf{T}$ 2-angle rigid?

2.4 Existence of S^1-valued angle structures

We begin with a definition which was also known to D. Futer and F. Gueritaud.

Definition 2.2. Suppose $(M, \mathbf{T})$ is a triangulated closed pseudo 3-manifold. Let $k : E \to \mathbf{S}^1$ be given. An $\mathbf{S}^1$-*valued angle structure with curvature* k on $\mathbf{T}$ is a function $x : \square \to \mathbf{S}^1$ so that

for each tetrahedron $\sigma \in \mathbf{T}$,

(16)
$$\prod_{q \in \square, q \subset \sigma} x(q) = -1$$

and for each edge $e \in E$,

(17)
$$\prod_{q \in \square} x(q)^{i(q,e)} = k(e).$$

The set of all $x \in (\mathbf{S}^1)^{\square}$ satisfying (16) and (17) will be denoted by $SAS(\mathbf{T}, k)$. The case that $k(e) = 1$ for all $e \in E$ is the most interesting one.

For a complex number $w \in \mathbf{C}$, we use $\arg(w)$ to denote its argument. If $x \in SAS(\mathbf{T}, k)$, by taking $\arg(x(q))$, we can interpret an $\mathbf{S}^1$-valued angle structure x as a map from $\square \to \mathbf{R}$ satisfying (8) and (9) so that the right-hand side of (8) is in $2\pi\mathbf{Z} + \pi$ and the right-hand-side of (9) is in $2\pi\mathbf{Z} + \arg(k(e))$.

Lemma 2.5. *If $SAS(\mathbf{T},k) \neq \emptyset$, then the function $k : E \to \mathbf{S}^1$ satisfies,*

$$\prod_{e\in E} k(e) = 1 \tag{18}$$

and for each vertex $v \in V$,

$$\prod_{e>v} k(e) = 1. \tag{19}$$

Indeed, to see part (18), using (16) and (17), we can write the left-hand side of (18) as

$$\prod_{q\in\square, e\in E} x(q)^{i(q,e)} = \prod_{\sigma\in T}\prod_{q\in\sigma, e<\sigma} x(q)^{i(q,e)} = \prod_{\sigma\in T}(\prod_{q\subset\sigma} x(q))^2 = 1$$

due to $\sum_{e\in E} i(q,e) = 2$ for each q.

To see part (19), using (16), we can write the left-hand-side of (19) as,

$$\prod_{e>v}\prod_{q\in\square} x(q)^{i(q,e)} = \prod_{\sigma\in T,\sigma>v}(\prod_{q\subset\sigma,e<\sigma,e>v} x(q)^{i(q,e)}) = (-1)^N$$

where N is the number of normal triangles at the vertex v. This number N is the same as the number of triangles in the link $lk(v)$. Since $lk(v)$ is a closed triangulated surface, N is an even number. Thus (19) follows.

Note that (19) is a 2-dimensional fact. One can define the similar notion of $\mathbf{S}^1$-valued angle structure on a closed triangulated surface by assigning each angle of a triangle a complex number of norm 1 so that the product of the complex numbers in each triangle is -1. The curvature at a vertex is the product of all complex numbers assigned to the angles at the vertex. For instance, if $(M^3,\mathbf{T})$ is a triangulated pseudo 3-manifold with an $\mathbf{S}^1$-valued angle structure, then the vertex link $lk(v)$ has the induced $\mathbf{S}^1$-angle structure. The identity (19) says that the product of its curvatures at all vertices is 1.

The main result in this section says that (18) and (19) are also sufficient.

Proposition 2.6. *Given any triangulated closed pseudo 3-manifold $(M,\mathbf{T})$ and $k : E \to \mathbf{S}^1$ satisfying (18) and (19), then $SAS(\mathbf{T},k) \neq \emptyset$. Furthermore, $SAS(\mathbf{T},k)$ is a smooth closed manifold.*

Proof. We may assume without loss of generality that M is connected. Consider the Lie group homomorphism $F : (\mathbf{S}^1)^{\square} \to (\mathbf{S}^1)^E \times (\mathbf{S}^1)^T$ given by

$$F(z)(e) = \prod_{q\in\square} z(q)^{i(q,e)}$$

and

$$F(z)(\sigma) = \prod_{q \in \square, q \subset \sigma} z(q)$$

where $z \in (\mathbf{S}^1)^\square$, $e \in E$ and $\sigma \in T$. The goal is to show that the point $t : E \cup F \to \mathbf{S}^1$ given by $t(e) = k(e)$ for $e \in E$ and $t(\sigma) = -1$ for $\sigma \in T$ is in the image of F.

Suppose otherwise, that t is not in the image of F. Since F is a continuous group homomorphism from a torus to a torus, the image of F is a connected closed subgroup of $(\mathbf{S}^1)^E \times (\mathbf{S}^1)^T$ which misses t. Thus there exists a continuous group homomorphism $h : (\mathbf{S}^1)^E \times (\mathbf{S}^1)^T \to \mathbf{S}^1$ so that $h(t) \neq 1$ and hF is the trivial homomorphism.

Each homomorphism from $(\mathbf{S}^1)^n$ to $\mathbf{S}^1$ is given by a vector $(m_1, ..., m_n) \in \mathbf{Z}^n$, i.e., the homomorphism sends $(x_1, ..., x_n) \in (\mathbf{S}^1)^n$ to $x_1^{m_1} ... x_n^{m_n}$. Thus for the homomorphism h, there exists $\phi \in \mathbf{Z}^E \times \mathbf{Z}^T$ so that for all $x \in (\mathbf{S}^1)^E \times (\mathbf{S}^1)^T$,

$$h(x) = \prod_{e \in E} x(e)^{\phi(e)} \prod_{\sigma \in T} x(\sigma)^{\phi(\sigma)}.$$

By the choice of t, we have $h(t) = \prod_{\sigma \in T} (-1)^{\phi(\sigma)} \prod_{e \in E} k(e)^{\phi(e)}$. Thus $h(t) \neq 1$ says that

$$\prod_{e \in E} k(e)^{\phi(e)} \neq (-1)^{\sum_{\sigma \in T} \phi(\sigma)}. \tag{20}$$

On the other hand, we will show that ϕF being trivial implies that (20) is an equality. The contradiction establishes the proposition.

Since the composition hF is trivial, for any $z \in (\mathbf{S}^1)^\square$,

$$\begin{aligned} h(F(z)) &= \Big(\prod_{e \in E} \prod_{q} z(q)^{i(q,e)\phi(e)}\Big)\Big(\prod_{\sigma \in T} \prod_{q \subset \sigma} z(q)^{\phi(\sigma)}\Big) \\ &= \prod_{q \in \square} z(q)^{\sum_{\sigma, q \subset \sigma} \phi(\sigma) + \sum_{e \in E} \phi(e) i(q,e)}. \end{aligned}$$

By the assumption, $h(F(z)) = 1$ for all choice of $z \in (\mathbf{S}^1)^\square$. Thus we obtain, for each $q \in \square$,

$$\sum_{\sigma, q \subset \sigma} \phi(\sigma) + \sum_{e} i(q,e)\phi(e) = 0. \tag{21}$$

Fix a tetrahedron $\sigma \in \mathbf{T}$, the above equation says that the sum of $\phi(e) + \phi(e')$ of the values of ϕ at two opposite edges e, e' in σ are independent of the choice of e, e'. We will need to use the following lemma.

Lemma 2.7. *Suppose $a_{ij} = a_{ji} \in \mathbf{Z}$ where $i \neq j \in \{1,2,3,4\}$ are six numbers so that*

$$a_{ij} + a_{kl} = c$$

is a constant independent of choice of indices i, j, k, l where $\{i, j, k, l\} = \{1,2,3,4\}$. Then there exist $b_1, .., b_4 \in \mathbf{Z}/2 = \{n/2 | n \in \mathbf{Z}\}$ so that

$$a_{ij} = b_i + b_j$$

for all $i \neq j \in \{1,2,3,4\}$.

Indeed, $b_i = \frac{a_{ij}+a_{ik}-a_{jk}}{2}$ is independent of the choices of $\{i, j, k\}$, $\{i, j, l\}$, or $\{i, k, l\}$ due to the assumption on $a_{ij} + a_{kl} = c$.

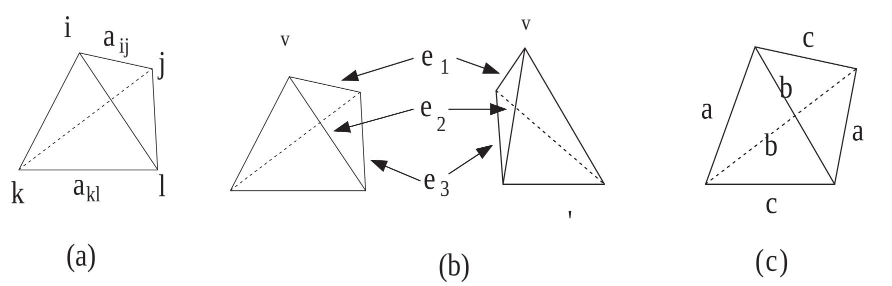

Figure 4: A topological interpretation of lemma 2.7

Thus, by the lemma, there exists a map $w : \{\text{vertice of } \sigma\} \to \mathbf{Z}/2$ so that

(22) $$\phi(e) = w(v, \sigma) + w(v', \sigma)$$

where v, v' are the end points of e. We claim that $w(v, \sigma)$ is independent of the choice of σ. Indeed, consider two tetrahedra σ, σ' sharing a common triangular face f (see figure 4(b)). Then for three edges e_1, e_2, e_3 in f, we solve (22) and obtain

$$w(v, \sigma) = w(v, \sigma') = \frac{\phi(e_1) + \phi(e_2) - \phi(e_3)}{2}$$

where v is the vertex opposite to the edge e_3 in f. It follows that $w(v, \sigma) = w(v, \sigma')$ is independent of the choice of tetrahedra σ and σ' since $(M, \mathbf{T})$ is a pseudo 3-manifold. Let $w : V \to \mathbf{Z}/2$ be the map so that

(23) $$\phi(e) = w(v) + w(v')$$

where v, v' are vertices of e. We claim that either all $w(v)$'s are integers, or all of $w(v)$ are half-integers (i.e., $k+1/2$ for some $k \in \mathbf{Z}$). Indeed, since $\phi(e)$ is an integer, it follows from (23) that either both $w(v), w(v')$ are in $\mathbf{Z}$, or both are in $\mathbf{Z}/2 - \mathbf{Z}$. Since the manifold M is connected, it follows that either $w(v) \in \mathbf{Z}$ for all v, or $w(v) \in \mathbf{Z}/2 - \mathbf{Z}$ for all v.

We now claim that the sum $\sum_{\sigma \in T} \phi(\sigma)$ has to be an even integer. Indeed, by definition $\phi(\sigma) = \sum_{v<\sigma} w(v)$. Thus

$$\sum_{\sigma \in T} \phi(\sigma) = \sum_{v \in V} w(v)(\sum_{\sigma > v} 1)$$
$$= \sum_{v \in V} w(v) |\{\text{triangles in the link lk(v)}\}|$$

For any triangulation of a closed surface, the number of triangles in the triangulation has to be even. Thus if all $w(v) \in \mathbf{Z}$, $\sum_{\sigma \in T} \phi(\sigma)$ is even. In the other case, all $w(v) \in \mathbf{Z}/2 - \mathbf{Z}$. Thus

$$\sum_{\sigma} \phi(\sigma) = \sum_{v \in V} \frac{1}{2} |\{\text{triangles in the link lk(v)}\}| \quad mod(2)$$
$$= \frac{1}{2} |\{\text{ normal triangles in T}\}| \quad mod(2)$$

Now each tetrahedron has 4 normal triangles, thus the total number of normal triangles in T is a divisible by 4. Thus implies again that $\sum_{\sigma \in T} \phi(\sigma)$ is an even number.

This implies that the right-hand-side of (20) is 1. We claim that the left-hand-side of (20) is also equal to 1. There are two cases to be considered. In the first case, all $w(v)$'s are in $\mathbf{Z}$. Then the left-hand-side of (20) becomes

$$\prod_{e \in E} k(e)^{\sum_{v<e} w(v)} = \prod_{v \in V} (\prod_{e>v} k(e))^{w(v)}$$

which is 1 due to (19).

In the second case that $w(v) = W(v) + 1/2$ where $W(v) \in \mathbf{Z}$ for all $v \in V$. We have $\phi(e) = 1 + \sum_{v<e} W(v)$. Thus the left-hand-side of (20) becomes

$$\prod_{e \in E} k(e)^{1+\sum_{v<e} W(v)} = (\prod_{v \in V} \prod_{e>v} k(e)^{W(v)}) \prod_{e \in E} k(e)$$

which is again 1 due to (18) and (19). This contradict shows that $SAS(\mathbf{T}, k) \neq \emptyset$.

Finally, since $SAS(\mathbf{T}, k) = F^{-1}(t)$ where F is a Lie group homomorphism, one concludes that $SAS(\mathbf{T}, k)$ is a closed smooth manifold.

3 Thurston's algebraic gluing equation and volume

In order to define Thurston's equation, we first recall the Neumann-Zagier anti-symmetric bilinear form on $\mathbf{R}^{\square}$. This bilinear form appeared in the important work of Neumann-Zagier [16]. We assume that $(M, \mathbf{T})$ is an oriented closed pseudo 3-manifold in this section so that each tetrahedron in $\mathbf{T}$ has the induced orientation.

3.1 Neumann-Zagier Poisson structure

If σ is an oriented Euclidean tetrahedron with edges from one vertex labelled by a, b, c so that the opposite edges have the same labelling a, b, c (see figure 4(c)), then the cyclic order of edges a, b, c viewed from each vertex depends only on the orientation of the tetrahedron, i.e., is independent of the choice of the vertices. Now each pair of opposite edges in the tetrahedron corresponds to a normal isotopy class of quadrilateral q in σ via the relation $i(q, e) \neq 0$. Let q_1, q_2, q_3 be three quadrilaterals in σ so that $q_1 \to q_2 \to q_3 \to q_1$ is the cyclic order induced by the cyclic order on the opposite edges from a vertex. Let W be the vector space with a basis $\{q_1, q_2, q_3\}$. An anti-symmetric bilinear form $\omega : W \times W \to \mathbf{R}$ is defined by $\omega(q_i, q_j) = 1$ if and only if $(i, j) = (1, 2), (2, 3), (3, 1)$. In particular, $\omega(q_i, q_j) = -\omega(q_j, q_i)$. Given any two quadrilaterals $q, q' \in \square$, set $\omega(q, q')$ to be the value just defined if they are in the same tetrahedron and $\omega(q, q') = 0$ if they are not in the same tetrahedron. In this way, one obtains the Neumann-Zagier anti-symmetric bilinear form

$$\omega : \mathbf{R}(\square) \times \mathbf{R}(\square) \to \mathbf{R}$$

where $\mathbf{R}(\square)$ is the vector space with a basis $\square$. More details of the form can be found in the work of [16], [4] and [20].

The following was proved in [16].

Proposition 3.1(Neumann-Zagier). *Suppose $(M, \mathbf{T})$ is a triangulated, oriented closed pseudo 3-manifold. Then*

(a) for any $q' \in \square$, $\sum_{q \in \square} \omega(q, q') = 0$.

(b) for any pair of edges $e, e' \in E$,

$$\sum_{q, q' \in \square} i(q, e) i(q', e') \omega(q, q') = 0.$$

Indeed, part (a) follows from the anti-symmetric property, i.e., for any $i = 1, 2$ or 3, $\sum_{j=1}^{3} \omega(q_j, q_i) = 0$. Part (b) is more complicated. First, anti-symmetry shows that the identity (b) holds if (1) $e = e'$, or (2) e and e' do not lie in a tetrahedron, or (3) e, e' lie in a tetrahedron and are opposite edges. Now if $e \neq e'$ and e, e' lie in a tetrahedron σ and are not opposite, then e, e' lie a triangular face and there is a second tetrahedron σ' containing e, e'. In this case, due to the orientations on σ and σ', we have

$$\sum_{q,q' \subset \sigma} i(q, e) i(q', e') \omega(q, q') = - \sum_{q,q' \subset \sigma'} i(q, e) i(q', e') \omega(q, q'). \tag{24}$$

Thus part (b) follows. For more details of the proof, see [16], page 316-320.

It is known ([16]) that the restriction of the Neumann-Zagier 2-form to the subspace $\{x = \sum_{q \in \square} a_q q \in \mathbf{R}(\square) |$ for each $\sigma \in \mathbf{T}$, $\sum_{q \subset \sigma} a_q = 0\}$ becomes non-degenerated. The 2-dimensional counter-part of the Neumann-Zagier Poisson structure is the Thurston's anti-symmetric bilinear form on the space of measured laminations. It is very closely related to the Weil-Petersson symplectic form ([17], [1]) on the Teichmuller spaces and plays a vital rule in Kontsevich's work ([11]) on Witten's conjecture. It is expected that Neumann-Zagier Poisson structure should play an equally important role in (2+1) TQFT.

3.2 Thurston's gluing equation

Definition 3.1. Suppose $(M, \mathbf{T})$ is an oriented closed pseudo 3-manifold with a triangulation and $k \in (\mathbf{S}^1)^E$. Thurston's equation (with curvature k) is defined for $z \in \mathbf{C}^\square$ so that

for each $e \in E$,

$$\prod_{q \in \square} z(q)^{i(q,e)} = \pm k(z), \tag{25}$$

and if $q, q' \in \square$ so that $\omega(q, q') = 1$, then

$$z(q')(1 - z(q)) = 1. \tag{26}$$

By (26) and the fact that $f(t) = \frac{1}{1-t}$ satisfies $tf(t)f(f(t)) = -1$, we have, for each tetrahedron $\sigma \in T$,

$$\prod_{q \in \square, q \subset \sigma} z(q) = -1. \tag{27}$$

Note that we do not require that $Im(z(q)) > 0$ which corresponds to the positively oriented ideal tetrahedron ([16]). The work of Yoshida [24] (see also Tillmann [20]) shows that each solution z so that the right-hand-side of (25) is 1 produces a representation of $\pi_1(M-V)$ to $PSL(2,\mathbf{C})$.

The most interesting cases are those where we can specify the right-hand-side of (25) to be 1. However, we are not able to achieve this at the movement. The equation $\prod_{q\in\square} z(q)^{i(q,e)} = k(e)$ deserves a further study.

Note that equation (25) is equivalent to

(28)
$$\prod_{q\in\square} z(q)^{2i(q,e)} = k(e)^2$$

It is the solution to (26) and (28) which is addressed in theorem 1.1.

The main theorem in the paper can be stated as,

Theorem 3.2. *Suppose $(M,\mathbf{T})$ is triangulated closed pseudo 3-manifold and M is orientable. If the triangulation supports no branched normal surfaces with at most two quadrilateral types except the vertex links, then for any curvature function $k \in (\mathbf{S}^1)^E$ satisfying (18) and (19), Thurston's algebraic gluing equation (26) and (28) associated to the triangulation has a solution.*

3.3 Volume of $\mathbf{S}^1$-valued angle structures

Recall that the Lobachevsky function $\Lambda(x) = -\int_0^x \ln|2\sin(u)|du$ is a continuous periodic function of period π defined on $\mathbf{R}$. It is real analytic on $\mathbf{R}-\pi\mathbf{Z}$ so that $\lim_{t\to 0}\Lambda'(t) = +\infty$. For more details, see Milnor [15]. Given $t = e^{\sqrt{-1}a} \in \mathbf{S}^1$, define $\lambda(t) = \Lambda(a)$. This is well defined since $\Lambda(a)$ has π as a period. Furthermore, $\lambda : \mathbf{S}^1 \to \mathbf{R}$ is real analytic on the subset $\mathbf{S}^1 - \{\pm 1\}$. For an $\mathbf{S}^1$-valued angle structure $x : \square \to \mathbf{S}^1$ on $(M,\mathbf{T})$, define the *volume* $\mathbf{V}(x)$ to be

$$\mathbf{V}(x) = \sum_{q\in\square} \lambda(x(q)) = \sum_{q\in\square} \Lambda(\arg(x(q))).$$

By definition $\mathbf{V} : SAS(\mathbf{T},k) \to \mathbf{R}$ is a continuous function and is smooth at those points x where $x(q) \neq \pm 1$ for all q. In particular, it has a maximum and a minimum point. A critical point x of $\mathbf{V}$ is called *non-degenerated* if $x(q) \neq \pm 1$ for all $q \in \square$.

3.4 Non-degenerate critical point of the volume

We will prove,

Lemma 3.3. *Suppose $x \in SAS(\mathbf{T}, k)$ is a non-degenerated critical point of the volume $\mathbf{V} : SAS(\mathbf{T}) \to \mathbf{R}$. Then Thurston's equation (26) and (28) has a solution in $(\mathbf{C}-\mathbf{R})^{\square}$.*

Proof. Suppose q_1, q_2, q_3 are three quadrilaterals in a tetrahedron. Let $x_i = x(q_i)$ be the $\mathbf{S}^1$-valued angle at the quadrilateral. We define the associated complex values $z(q_i)$ by the formula,

$$z(q_i) = \frac{x_j - \overline{x_j}}{x_k - \overline{x_k}} x_i = \frac{\sin(\arg(x_j))}{\sin(\arg(x_k))} x_i$$

where $\omega(q_i, q_j) = 1$ and $\{i, j, k\} = \{1, 2, 3\}$. This is well defined since $x_k - \overline{x_k} \neq 0$ by the assumption. More generally, for $x \in SAS(\mathbf{T})$ and $x(q) \neq \pm 1$ for all q, one defines $z \in \mathbf{C}^{\square}$, by

$$z(q) = x(q) \prod_{r \in \square} (\sin(\arg(x(r)))^{\omega(r,q)}.$$

We claim that z is a solution to Thurston's equation (26) and (28).

First, (26) follows by a direct calculation and the definition. Let us assume that $z_i = z(q_i)$ and that $\omega(q_1, q_2) = 1$. By definition, we have

$$z_1 = \frac{x_2 - \overline{x_2}}{x_3 - \overline{x_3}} x_1$$

and

$$z_2 = \frac{x_3 - \overline{x_3}}{x_1 - \overline{x_1}} x_2.$$

Due to $x_1 x_2 x_3 = -1$ and $x_i \overline{x_i} = 1$, then (26) says that

$$z_2(1 - z_1) = 1.$$

Indeed,

$$z_2(1 - z_1) = (\frac{x_3 - \overline{x_3}}{x_1 - \overline{x_1}} x_2) \frac{x_3 - \overline{x_3} - x_1 x_2 + x_1 \overline{x_2}}{x_3 - \overline{x_3}} = \frac{x_3 + x_1 \overline{x_2}}{x_1 - \overline{x_1}} x_2 = \frac{x_3 x_2 + x_1}{x_1 - \overline{x_1}} = 1$$

To see (28), we need to use the critical point equation for $\mathbf{V}$. By definition, we can identify the tangent space to a point of $SAS(\mathbf{T}, k)$ with $TAS(\mathbf{T})$. Indeed, for any $v \in TAS(\mathbf{T})$ and $x \in SAS(\mathbf{T}, k)$, the path $p(t) = xe^{tv} \in SAS(\mathbf{T}, k)$ given by

$$p(t)(q) = x(q) e^{tv(q)}$$

for $t \in (-\epsilon, \epsilon)$ has tangent vector v at $t = 0$ and all tangent vectors to $SAS(\mathbf{T}, k)$ at x are of this form. Now $\frac{d\mathbf{V}(xe^{tv})}{dt}|_{t=0} = 0$ shows that,

$$\sum_{q \in \square} v(q) \ln |\sin(\arg(x(q)))| = 0. \tag{29}$$

Choose a specific $v \in TAS(\mathbf{T})$ as follows. Fix an edge $e \in E$, by proposition 3.1,

$$v_e = \sum_{q \in \square} \sum_{r \in \square} i(q, e)\omega(r, q) r^* \in TAS(\mathbf{T}). \tag{30}$$

Now substitute v_e for v in (29) and use the fact that $\sum_{q \in \square} i(q, e) v_e(q) = 0$, we obtain, for each $e \in E$,

$$\prod_{q \in \square} z(q)^{i(e,q)} = \prod_{q \in \square} x(q)^{i(e,q)} \prod_{r \in \square} \sin(\arg(x(q)))^{i(e,q)\omega(r,q)}$$
$$= k(e) \prod_{q, r \in \square} \sin(\arg(x(q)))^{i(e,q)\omega(r,q)} = \pm k(e)$$

due to (29) and (30). This verifies (28) and ends the proof.

Corollary 3.4. *Under the same assumption as in lemma 3.3, if $x \in SAS(\mathbf{T}, k)$ is a non-degenerate critical point of the volume $\mathbf{V}$, let $y \in \mathbf{R}^{\square}_{\geq 0}$ be the vector so that $y(q) = -\ln |(\sin(\arg(x(q)))|$. Then $y \in Proj_{\square}(\mathbf{S}_{ns})$.*

Indeed, (29) shows that $y \in TAS(\mathbf{T})^{\perp}$. Thus, by theorem 2.2, $y \in Proj_{\square}(\mathbf{S}_{ns})$.

It will be very interesting to see what the topological information y contains.

4 Volume optimization and normal surfaces

A relationship between the non-degenerated critical points of the volume $\mathbf{V} : SAS(\mathbf{T}, k) \to \mathbf{R}$ and the normal surfaces is established in Corollary 3.4. In this section, we will investigate the case of degenerated critical points of the volume. Since the function $\mathbf{V}$ is not smooth, the definition of the critical points of $\mathbf{V}$ should be specified. First of all, we will show (corollary 4.3) that for any $p \in SAS(\mathbf{T})$ and $u \in TAS(\mathbf{T})$ the limit $\lim_{t \to 0} \frac{d\mathbf{V}(pe^{tu})}{dt}$ always exists as an element in $[-\infty, \infty] = \mathbf{R} \cup \{\infty, -\infty\}$. We say that a point $p \in SAS(\mathbf{T}, k)$ is a *critical point* of the volume $\mathbf{V}$ if for all u in $TAS(\mathbf{T})$,

$$\lim_{t \to 0} \frac{d\mathbf{V}(pe^{tu})}{dt} = 0. \tag{31}$$

Using this definition, one sees easily that the maximum and minimum points of $\mathbf{V}$ are critical points.

A *trivial normal surface* is a normal surface without quadrilateral disks. It is well known that trivial normal surfaces are composed by sum of vertex links. The main theorem in the section, which implies theorem 1.1, is the following,

Theorem 4.1. *Suppose* $(M, \mathbf{T})$ *is an orientable closed triangulated pseudo 3-manifold with* $SAS(M, \mathbf{T}) \neq \emptyset$. *If the volume* $\mathbf{V} : SAS(M, \mathbf{T}, k) \to \mathbf{R}$ *has a degenerated critical, then* $(M, \mathbf{T})$ *contains a non-trivial branched normal surface with at most two quadrilateral types.*

Recall that by proposition 2.6, $SAS(M, \mathbf{T}) \neq \emptyset$ if and only if k satisfies (18) and (19).

4.1 Subderivatives of the volume function

The volume function $\mathbf{V}$ is essentially composed by the function $W : P \to \mathbf{R}$ where $P = \{(x_1, x_2, x_3) \in \mathbf{R}^3 | x_1 + x_2 + x_3 = \pi\}$ and $W(x_1, x_2, x_3) = \Lambda(x_1) + \Lambda(x_2) + \Lambda(x_3)$. The function W is not smooth on the subset defined by some $x_i \in \pi\mathbf{Z}$. However, we can obtain subderivative information of W at these points.

The function $h(t) = t \ln |t|$ can be extended to be a continuous function from $\mathbf{R} \to \mathbf{R}$ by declaring $h(0) = 0$. In the sequel, this extension, still denoted by $t \ln |t|$, will be used.

Lemma 4.2. *Take a point* $a = (a_1, a_2, a_3) \in P$ *and* $b = (b_1, b_2, b_3) \in \mathbf{R}^3$ *so that* $b_1 + b_2 + b_3 = 0$. *Define* $f(t) = \frac{dW(a+tb)}{dt}$. *Then* $\lim_{t \to 0} f(t)$ *exists as an element in* $\mathbf{R} \cup \{\pm\infty\}$ *and*

(a) if $a_i \notin \pi\mathbf{Z}$ *for all* i,

$$\lim_{t \to 0} f(t) = -\sum_{i=1}^{3} b_i \ln |\sin(a_i)|, \tag{32}$$

(b) if $a_i \in \pi\mathbf{Z}$ *for all* i,

$$\lim_{t \to 0} f(t) = -\sum_{i=1}^{3} b_i \ln |b_i| \tag{33}$$

(c) if $a_1 \in \pi\mathbf{Z}$ and $a_2, a_3 \notin \pi\mathbf{Z}$, then

$$\lim_{t\to 0}(f(t) + b_1 \ln|t|) = -b_1 \ln|b_1| - \sum_{i=2}^{3} b_i \ln|\sin(a_i)| \tag{34}$$

Proof. We have $f(t) = -\sum_{i=1}^{3} b_i \ln|2\sin(a_i + tb_i)| = -\sum_{i=1}^{3} b_i \ln|\sin(a_i + tb_i)|$ due to $\sum_{i=1}^{3} b_3 = 0$. Now part (a) follows form the definition.

For part (b), due to $\ln(|\sin(t+\pi)|) = \ln|\sin(t)|$, it follows that $f(t) = -\sum_{i=1}^{3} b_i \ln(|\sin(tb_i)|)$. The result is obvious if $b_i = 0$ for all i. Otherwise, say $b_3 \neq 0$, then $b_3 = -b_1 - b_2$. Substitute it to $f(t)$, we obtain

$$f(t) = -b_1 \ln|\frac{\sin(b_1 t)}{\sin(b_3 t)}| - b_2 \ln|\frac{\sin(b_2 t)}{\sin(b_3 t)}|.$$

By taking the limit as $t \to 0$, we obtain part (b).

For part (c), we write

$$f(t) = -b_1 \ln|\frac{\sin(b_1 t)}{b_1 t}| - b_1 \ln|b_1 t| - \sum_{i=2}^{3} b_i \ln|\sin(a_i + tb_i)|$$

$$= -b_1 \ln|t| - b_1 \ln|b_1| - \sum_{i=2}^{3} b_i \ln|\sin(a_i)| + o(t).$$

where $o(t)$ is the quality so that $\lim_{t\to 0} o(t) = 0$. This establishes part (c) and finishes the proof.

Not that due to $a_1 + a_2 + a_3 = \pi$, cases (a), (b) and (c) are the list of all cases up to symmetry. The limit $\lim_{t\to 0} f(t)$ in cases (b), (c) above is called the subderivative of the function W at the point a. The subderivative, considered as a function of the tangent vector b, is homogeneous of degree 1. However, due to the non-smoothness, the subderivative, as shown in (b), (c), is not a linear function of b.

In the case of the $\mathbf{S}^1$-valued angle structure, consider $X = \{a = (a_1, a_2, a_3) \in (\mathbf{S}^1)^3 | a_1 a_2 a_3 = -1\}$ and the volume $\mathbf{V}(a) = \sum_{i=1}^{3} \lambda(a_i) = \sum_{i=1}^{3} \Lambda(\arg(a_i))$. Consider a tangent vector $b = (b_1, b_2, b_3) \in \mathbf{R}^3$ so that $b_1 + b_2 + b_3 = 0$. Define $f(t) = \frac{d\mathbf{V}(ae^{tb})}{dt}$. Then $\lim_{t\to 0} f(t)$ exists as an element in $\mathbf{R} \cup \{\pm\infty\}$ and the above lemma says,

(a) if $a_i \neq \pm 1$ for all i,

$$\lim_{t\to 0} f(t) = -\sum_{i=1}^{3} b_i \ln|\sin(\arg(a_i))|, \tag{35}$$

(b) if $a_i = \pm 1$ for all i,

$$\lim_{t\to 0} f(t) = -\sum_{i=1}^{3} b_i \ln|b_i| \tag{36}$$

(c) if $a_1 = \pm 1$ and $a_2, a_3 \neq \pm 1$, then

$$\lim_{t\to 0}(f(t) + b_1 \ln|t|) = -b_1 \ln|b_1| - \sum_{i=2}^{3} b_i \ln|\sin(\arg(a_i))| \tag{37}$$

Corollary 4.3. *For any $a \in SAS(\mathbf{T}, k)$ and $b \in TAS(\mathbf{T})$, there exist a linear function $g(b)$ of b and a continuous function $f(b,t)$ of b and $t \in (-\epsilon, \epsilon)$ so that*

$$\frac{d\mathbf{V}(ae^{tb})}{dt} = g(b)\ln|t| + f(b,t).$$

In particular, the limit $\lim_{t\to 0} \frac{d\mathbf{V}(ae^{tb})}{dt}$ always exists as an element in $[-\infty, \infty]$. Furthermore, a local maximum or minimum point of $\mathbf{V}$ is a critical point.

4.2 A proof of theorem 4.1

Suppose $p \in SAS(\mathbf{T}, k)$ is a degenerated critical point of the volume function $\mathbf{V}$. By definition of critical points, we have

$$\lim_{t\to 0} \frac{d\mathbf{V}(pe^{tb})}{dt} = 0$$

for b in $TAS(\mathbf{T})$. By the definition of $\mathbf{V}$, we have

$$\mathbf{V}(x) = \sum_{\sigma\in T} \sum_{q\in\square, q\subset\sigma} \lambda(x(q)).$$

Let $Y = \{q \in \square | p(q) = \pm 1\}$ which is non-empty and $Y' = \{q \in Y|$ there exists $\sigma \in T$ so that $q \subset \sigma$ and for the other two $q', q'' \subset \sigma$ $p(q'), p(q'') \neq \pm 1\}$.

Thus, by (35)-(37), we can write

$$\lim_{t\to 0}\left(\frac{d\mathbf{V}(pe^{tb})}{dt} + \ln|t| \sum_{q\in Y'} b(q)\right) = -\sum_{q\in Y} b(q)\ln|b(q)| - \sum_{q\notin Y} b(q)\ln|\sin(\arg(p(q)))|.$$

By the critical point condition (31), we obtain $\sum_{q\in Y'} b(q) = 0$ and

$$\sum_{q\in Y} b(q)\ln|b(q)| = -\sum_{q\notin Y} b(q)\ln|\sin(\arg(p(q)))| \tag{38}$$

Consider for each $q \in \square$ the linear function $f_q : TAS(\mathbf{T}) \to \mathbf{R}$ defined by $f_q(b) = b(q)$. Then the right-hand-side of (38) is a linear function in b on $TAS(\mathbf{T})$ and the left-hand-side of (38) is a sum of the functions $f_q(b) \ln |f_q(b)|$.

Lemma 4.4. *Suppose W is a finite dimensional vector space over $\mathbf{R}$ and $f_1, ..., f_n$ are non-zero linear functions on W so that there is a linear function g on W satisfying*

$$\sum_{i=1}^{n} f_i(x) \ln |f_i(x)| = g(x). \tag{39}$$

Then for each index i there exists $j \neq i$ and $\lambda_{ij} \in \mathbf{R}$ so that

$$f_i(x) = \lambda_{ij} f_j(x).$$

Proof. We may assume that $W = \mathbf{R}^m$ and $x = (x_1, ..., x_m) \in W$ after a linear change of variables. Write

$$f_i(x) = \sum_{j=1}^{m} a_{ij} x_j.$$

Now suppose the result does not hold, say $f_1(x)$ is not propositional to $f_j(x)'s$ for $j \geq 2$. Then we can find a point $v \in ker(f_1)$ so that $v \notin \cup_{j=2}^{n} ker(f_j)$. Since $f_1 \neq 0$, for simplicity, let us assume that $a_{11} \neq 0$. Now take derivative of (39) with respect to x_1. We obtain an equation of the form

$$\sum_{j=1} a_{1j} \ln |f_j(x)| = h(x) \tag{40}$$

where $h(x)$ is a linear function. Take a sequence of vectors x converging to v in (40), we obtain a contradiction since $a_{11} \neq 0$. This ends the proof.

If one of the linear function f_q in (38) is the zero map, then by definition $q^* \in (TAS(\mathbf{T}))^{\perp}$. By theorem 2.2 and corollary 2.4(a), there exists an embedded normal surface with one quadrilateral type. If otherwise that all f_q are not zero, using lemma 4.4 for (39) where the linear subspace W is $TAS(\mathbf{T})$ and linear functions are f_q with $q \in Y$, we conclude that there exists $q_1 \neq q_2 \in \square$ and $\lambda \in \mathbf{R}$ so that $f_{q_1}(b) = \lambda f_{q_2}(b)$ for all b in $TAS(\mathbf{T})$. This shows for all $b \in TAS(\mathbf{T})$, the inner product $(b, q_1^* - \lambda q_2^*) = 0$. By theorem 2.2, $q_1^* - \lambda q_2^*$ is in $Proj_{\square}(\mathbf{S}_{ns})$. Thus theorem 4.1 follows from corollary 2.4(b) and theorem 2.2. This ends the proof.

Note that we have not used the equation $\sum_{q \in Y'} b(q) = 0$ in the proof. By the definition of Y', any two quadrilaterals q, q' in Y' are in different tetrahedra. Thus, by theorem

2.2, the vector $\sum_{q\in Y'} q^*$ is the q-coordinate of a branched normal surface. By the work of Haken, we can make the branched normal surface embedded in this case. It is an interesting question to study the geometric information coded by this embedded normal surface.

5 Some questions

It is very interesting to know if one can improve theorem 1.1 so that the right-hand-side of (1.1) is 1. If this holds, then by the work of Yoshida [24], one can produce a representation of the fundamental group of $M - V$ to $PSL(2, \mathbf{C})$. It is also very interesting to know when solutions to Thurston's equation produces irreducible representations of the fundamental group.

Solving Thurston's equation over the real numbers, i.e., $z \in \mathbf{R}^{\square}$ is also very interesting.

Finally, it is interesting to know which triangulations support normal surface or branch normal surfaces with at most two quadrilateral types. For instance, can one list those compact 3-manifold which have a 0-efficient triangulation so that the triangulation contains a normal surface with one quadrilateral type?

6 Appendix

We give a new proof of Kang-Rubinstein theorem. First, one checks easily that both W_σ and W_e are in $\mathbf{S}_{ns}$. Next, by a simple dimension counting, one sees that $\dim(\mathbf{S}_{ns}) \leq |E| + |T|$. Thus, it suffices to prove that $\{W_\sigma, W_e | \sigma \in T, e \in E\}$ is an independent set. To this end, suppose otherwise that there exists $h \in \mathbf{R}^E \times \mathbf{R}^T$ so that

$$\sum_{e\in E} h(e)W_e + \sum_{\sigma\in T} h(\sigma)W_\sigma = 0.$$

We can write it as,

$$\sum_{t\in\triangle}(-\sum_{e>t} h(e) - \sum_{\sigma>t} h(\sigma))t^* + \sum_{q\in\square}(\sum_{e\in E} h(e)i(q,e) + \sum_{\sigma\in T, q\subset\sigma} h(\sigma))q^* = 0$$

Since $\{t^*, q^*\}$ form a basis, we obtain for each $t \in \triangle$

$$\sum_{e>t} h(e) - \sum_{\sigma>t} h(\sigma) = 0 \tag{41}$$

and for each $q \in \square$

$$\sum_{e\in E} h(e)i(q,e) + \sum_{\sigma\in T, q\subset\sigma} h(\sigma) = 0 \tag{42}$$

Consider a fixed tetrahedron $\sigma \in T$. We claim that the system of linear equations (41) and (42) for the six edges of σ has only the trivial solution, i.e., $h(e) = h(\sigma) = 0$. In particular, this shows that $\{W_e, W_\sigma\}$ is independent.

To see the claim, let us label the vertices of σ by $1, 2, 3, 4$ and the six edges by e_{ij} were $i \neq j \in \{1, 2, 3, 4\}$. Let $h_{ij} = h(e_{ij})$ and $f = h(\sigma)$. Then (41) and (42) say: at the i-th vertex

$$h_{ij} + h_{ik} + h_{il} = f \tag{43}$$

and

$$h_{ij} + h_{kl} = f \tag{44}$$

for $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Consider the sum of two equations (43) at the i-th and j-th vertices subtracting the sum of the two equations (43) at the k-th and l-th vertices. We obtain, $h_{ij} = h_{kl}$, i.e., $h(e) = h(e')$ when e, e' are opposite edges. Now by (44), we see that $h_{ij} = f/2$ for all $i \neq j$. Now substitute back to (43), we obtain $3f/2 = f$. Thus $f = 0$ and $h_{ij} = 0$, i.e., $h(e) = h(\sigma) = 0$.

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