Some Applications of a Multiplicative Structure on Simple Loops in Surfaces

Feng Luo

To Joan Birman on her 70th birthday

§1. **Introduction**

Given two transversely intersecting unoriented arcs in an oriented surface, it is well known that there exists a natural way of resolving the intersection so that the resolution depends only on the orientation of the surface and the order of the two arcs (see for instance [BS], [De], [Wh]). The analogous "local surgery" operation plays a prominent role in the recent development of knot invariants (see [BL] for instance). This local operation is shown in [Lu] to induce a multiplicative structure on the space of isotopy classes of curves systems in surfaces. The goal of this note is to discuss some simple applications of the multiplicative structure.

The multiplication introduced here is similar to Goldman's Lie bracket on the free module generated by the conjugacy classes in the fundamental group of a surface [Go]. However, the multiplication is highly non-associative and is more geometrically oriented. We are informed by a referee that in some unpublished work of Thurston and Penner on train-tracks, they have also discovered the multiplication.

Suppose $\Sigma = \Sigma_{g,r}$ is a compact oriented surface of genus g with r boundary components. Following Dehn [De] and Thurston [Th2], a *curve system* in Σ is a proper 1-submanifold so that each component of the submanifold is homotopically non-trivial relative to the boundary $\partial\Sigma$. The set $CS(\Sigma)$ of isotopy classes of curve systems in Σ was introduced by Dehn who called it the *arithmetic field* of the surface. For instance, if the surface Σ is a torus, then the space $CS(\Sigma_{1,0})$ is naturally isomorphic to the quotient $H_1(\Sigma_{1,0}, \mathbf{Z}) - \{0\}/\pm 1$. Given a lattice $\mathbf{Z}^2$, there exists a multiplication on $\mathbf{Z}^2 - \{0\}/\pm 1$ which is invariant under the sense preserving linear automorphism group $SL(2, \mathbf{Z})$. Namely, if $\alpha = \pm(x, y)$ and $\alpha' = \pm(x', y')$, then $\alpha * \alpha' = \pm((x, y) + \delta(x', y'))$ where δ is the sign of the number $xy' - x'y$ if $xy' - x'y \neq 0$ and δ is the sign of number k if $(x, y) = k(x', y')$. This multiplication on $CS(\Sigma_{1,0})$ has a topological interpretation in terms of the local resolution. In fact, for all oriented surfaces, there exists a natural multiplication on $CS(\Sigma)$ which is invariant under the action of the orientation preserving homeomorphisms.

Recall that if $\alpha, \beta \in CS(\Sigma)$, then their *geometric intersection number* $I(\alpha, \beta)$ is defined to be the minimal number $\min\{|a \cap b| | a \in \alpha, b \in \beta\}$. Let $CS_0(\Sigma)$ be the subset of $CS(\Sigma)$ consisting of isotopy classes of curve systems with no arc components. The isotopy class of a curve system c is denoted by $[c]$. We use α^k for $k \in \mathbf{Z}_{>0}$ to denote k copies of α.

In [Lu], the following theorem is proved.

Theorem 1. *There exists a multiplication* $CS(\Sigma) \times CS(\Sigma) \to CS(\Sigma)$ *sending* $CS_0(\Sigma) \times CS_0(\Sigma)$ *to* $CS_0(\Sigma)$ *and satisfying the following properties.*

(a) It is preserved by the action of the orientation preserving homeomorphisms.

(b) (Commutative) If $I(\alpha, \beta) = 0$*, then* $\alpha\beta = \beta\alpha$ *and* $I(\alpha\beta, \gamma) = I(\alpha, \gamma) + I(\beta, \gamma)$ *for all* γ*. Conversely, if* $\alpha\beta = \beta\alpha$ *and* $\alpha \in CS_0(\Sigma)$*, then* $I(\alpha, \beta) = 0$*.*

(c) (Associative) If $[c_i] \in CS(\Sigma)$ so that $|c_i \cap c_j| = I([c_i],[c_j])$ for $i,j = 1,2,3$, $i \neq j$, $|c_1 \cap c_2 \cap c_3| = 0$, and there is no contractable region in $\Sigma - (c_1 \cup c_2 \cup c_3)$ bounded either by three arcs in c_1, c_2, c_3 or four arcs one in each of c_1, c_2, c_3 and $\partial\Sigma$, then $[c_1]([c_2][c_3]) = ([c_1][c_2])[c_3]$.

(d) (Cancellation) If $\alpha \in CS_0(\Sigma)$ and each component of α intersects β, then $\alpha(\beta\alpha) = \beta$ and $(\alpha\beta)\alpha = \beta$. Furthermore, $I(\alpha, \alpha\beta) = I(\alpha, \beta\alpha) = I(\alpha, \beta)$.

(e) For any positive integer k, $\alpha^k\beta^k = (\alpha\beta)^k$.

(f) If α is the isotopy class of a simple closed curve, then the positive Dehn twist along α sends β to $\alpha^k\beta$ where $k = I(\alpha,\beta)$.

(g) If α, β are in $CS_0(\Sigma)$, then the sum of any two numbers in $\{I(\alpha,\gamma), I(\beta,\gamma), I(\alpha\beta,\gamma)\}$ is at least the third.

As one consequence, we prove a convexity result concerning the intersection number function. It is a combinatorial analogous to Kerckhoff's theorem that the geodesic length function is convex along earthquake path in the Teichmüller space [Ker]. Indeed, it is conceivable that the multiplication is the extension of Thurston's earthquake to the measured lamination (see [Bo], and [Pa]). If this holds, then theorem 2 is also a consequence of Kerckhoff's theorem ([Bo1]).

Given α and β in $CS_0(\Sigma)$, if $n \in \mathbf{Z}_{<0}$, we define $\alpha^n\beta = \beta\alpha^{-n}$ and $\beta\alpha^n = \alpha^{-n}\beta$.

Theorem 2 (Convexity). *Given $\alpha, \beta \in CS_0(\Sigma)$ and $\gamma \in CS(\Sigma)$, the intersection number function $I(\alpha^n\beta, \gamma)$ is convex in $n \in \mathbf{Z}$.*

Corollary 3. *Suppose α is the isotopy class of a simple loop in a surface, $\beta \in CS_0(\Sigma)$, and D_α is the Dehn twist on α. Then the function $I(D_\alpha^n(\beta), \gamma)$ is convex in $n \in \mathbf{Z}$.*

The paper is organized as follows. In §2, we give a short proof of the main parts of theorem 1. In §3, we prove theorem 2. In §4, we use the multiplication to prove several known results.

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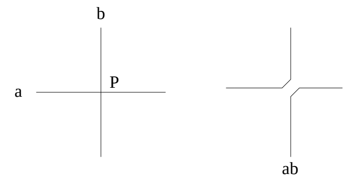
§2. **Definitions and Basic Properties of the Multiplication**

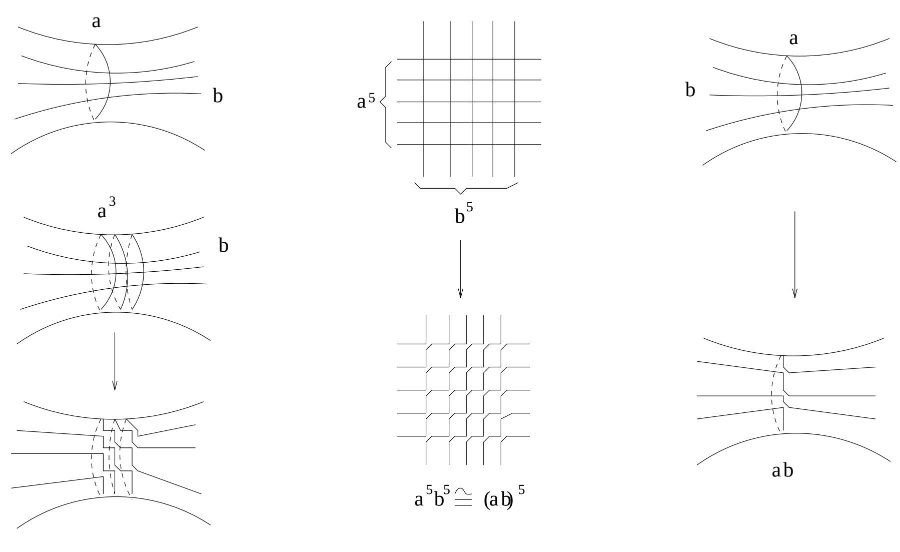
Suppose a and b are two unoriented arcs intersecting transversely at a point p in an oriented surface Σ. Then the *resolution of $a \cup b$ at p from a to b* is defined as follows. Take any orientation on a and use the orientation on the surface to determine an orientation on b. Then resolve the intersection according to the orientations. The resolution is independent of the choice of orientations on a. See figure 1.



The right-hand orientation on the plane

Figure 1

Given two curve systems a, b in Σ which intersect minimally within their isotopy classes, i.e., $I([a],[b]) = |a \cap b|$, the multiplication ab is defined to be the 1-dimensional submanifold obtained by resolving all intersection points in $a \cup b$ from a *to* b. If a and b are disjoint, then ab is the union of a with b. Our first observation (lemma 4) is that ab is again a curve system. Furthermore, the isotopy class of ab depends only on the isotopy classes of a and b. The multiplication of two classes α and β in $CS(\Sigma)$ is defined to be the isotopy class of ab where $a \in \alpha$, $b \in \beta$ and $|a \cap b| = I(\alpha, \beta)$. Note that the positive Dehn twist D_α applied to β is the multiplication $\alpha^k \beta$ where $k = I(\alpha, \beta)$ by the definition. See figure 2.



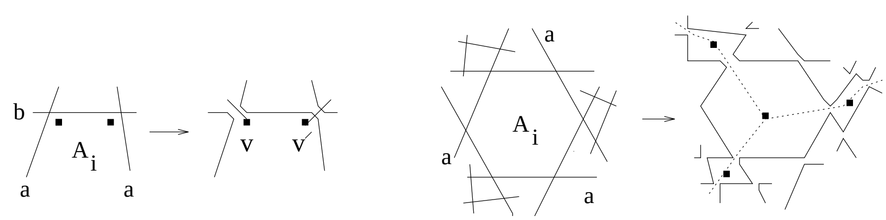
Right-hand orientation on the front faces

Figure 2

Lemma 4. *Suppose a and b are curve systems intersecting minimally within their isotopy classes. Then the 1-dimensional submanifold ab obtained by resolving all intersection points from a to b is again a curve system.*

Proof. We shall prove the lemma in the special case that both a and b contain no arc components in a closed surface. The proof of the general case is essentially the same. See [Lu] for details.

Suppose otherwise that the 1-submanifold ab contains a null homotopic component c. We may assume that c is the "inner-most" component, i.e., in the interior of the disc D bounded by c, there are no other components of ab. Let us consider all components of $\Sigma - (a \cup b)$ which are inside D, say $A_1, ..., A_k$. Each A_i is an open disc since c is the inner-most. The boundary of A_i consists of arcs in a and b, and the *corners* of A_i correspond to the intersection points of a and b in A_i. Thus we may call each A_i a polygon bounded by sides in a and b alternatively. Since a intersects b minimally within their isotopy classes, each A_i has at least four sides. Now by the definition of the resolution, the disc D is obtained by resolving corners of A_i's from a to b. Consider the resolutions at the vertices alone the boundary of A_i. One sees that corners open and closed alternatively in a cyclic order on the boundary. Form a graph in D by assigning a vertex in each A_i and joining an edge between two vertices if their corresponding polygons A_i and A_j have the same corner which is opened by the resolution. Then, on one hand, the graph is a tree since it is homotopic to the disk. On the other hand, each vertex of the graph has valency at least two since the valency of the vertex is half of the number of sides of the corresponding polygon A_i (by the alternating property). This contradicts the fact that a tree must have a vertex of valency one. □



Right-hand orientation on the plane

Figure 3

Now the proof of theorem 1 goes as follows. Properties (a), (e), (f) and the first part of (b) follow from the definition. Property (g) and the second part of (b) follow from property (d). Indeed, by the definition of resolution, we have $I(\alpha\beta, \gamma) \leq I(\alpha, \gamma) + I(\beta, \gamma)$. Now write

$\alpha = \alpha_1\alpha_2$ with $I(\alpha_1, \alpha_2) = 0$ so that $I(\alpha_2, \beta) = 0$ and each component of α_1 intersects β. Then by (d), $\beta = (\alpha_1\beta)\alpha_1$ and $\beta\alpha_2 = \alpha_2\beta$. Thus, $I(\beta, \gamma) = I((\alpha_1\beta)\alpha_1, \gamma) \leq I(\alpha_1\beta, \gamma) + I(\alpha_1, \gamma) \leq I(\alpha\beta, \gamma) + I(\alpha, \gamma)$. To prove part (b), since $\alpha_1\alpha_2\beta = \beta\alpha_1\alpha_2$, we obtain $\alpha_1\beta = \beta\alpha_1$ by dropping the component α_2. By (d), $\beta = \alpha_1(\beta\alpha_1)$. Thus $\beta = \alpha_1^2\beta$. Now by (d) again $0 = I(\beta, \beta) = I(\beta, \alpha_1^2\beta) = I(\beta, \alpha_1^2) = 2I(\beta, \alpha_1)$. This shows that $\alpha_1 = \emptyset$ and $I(\alpha, \beta) = I(\beta, \alpha)$.

It remains to prove parts (c) and (d). The key step in proving (c) is to show that $|c_1c_2 \cap c_3| = I([c_1c_2], [c_3])$ and $|c_1 \cap c_2c_3| = I([c_1], [c_2c_3])$ using the non-existence of triangular and quadrilateral regions in $\Sigma - (\cup_{i=1}^3 c_i)$. The proof of this is essentially the same as the argument used in the proof of lemma 4. We refer the reader to [Lu] for details. Assuming this, then property (c) follows since both $([c_1][c_2])[c_3]$ and $[c_1]([c_2][c_3])$ are obtained from $c_1 \cup c_2 \cup c_3$ by resolving simultaneously the intersection points in $c_i \cap c_j$ from c_i to c_j where $(i,j) = (1,2), (2,3)$ and $(1,3)$. Finally to show property (d), we note that three classes α, β, α satisfy the non-triangular and non-quadrilateral region condition in (c). Thus, by taking $a \in \alpha$, $b \in \beta$ so that $|a \cap b| = I(\alpha, \beta)$ and a' a parallel copy of a, we see that $I(\alpha, \beta) = I(\alpha, \alpha\beta) = I(\alpha, \beta\alpha)$. Furthermorem $\alpha(\beta\alpha)$ is represented by resolving all intersection points in $a \cup b \cup a'$ from a to b and b to a'. A simple calculation shows aba' is isotopic to b. See figure 4.

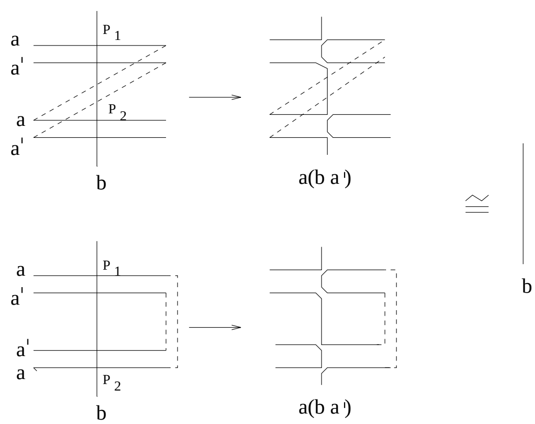


Figure 4

Remark. The non-triangular region condition has already appeared in many papers. The proofs are closely related to the topological cosine formula in [Bo].

§3. **Convexity of Intersection Number Functions**

Recall that if $n \in \mathbf{Z}_{<0}$, we defined $\alpha^n\beta$ to be $\beta\alpha^{-n}$. (Note that under this convention, $\alpha^{-2}\alpha^5 = \alpha^7$.) If each component of α intersects β and α contains no arc component, then by theorem 1(d), we have $\alpha^n(\alpha^m\beta) = \alpha^{n+m}\beta$ for all $m, n \in \mathbf{Z}$.

Theorem 2. *Given $\alpha, \beta \in CS_0(\Sigma)$ and $\gamma \in CS(\Sigma)$, the intersection number function $I(\alpha^n\beta, \gamma)$ is convex in $n \in \mathbf{Z}$.*

Proof. let $f(n) = I(\alpha^n\beta, \gamma)$. Since α^2, β and α satisfy the non-triangular region condition, by theorem 1(c), we have $(\alpha^2\beta)\beta = \alpha^2\beta^2 = (\alpha\beta)^2$. Thus $2f(1) = 2I(\alpha\beta, \gamma) = I((\alpha\beta)^2, \gamma) \leq I(\alpha^2\beta, \gamma) + I(\beta, \gamma) = f(2) + f(0)$.

Now to prove the convexity of f for general classes, it suffices to prove it for the case that each component of α intersects β. Indeed, we may write as before $\alpha = \alpha_1\alpha_2$ with $I(\alpha_1, \alpha_2) = 0$ so that $I(\alpha_1, \beta) = 0$ and each component of α_2 intersects β. Thus $f(n) = I(\alpha_1^{|n|}\alpha_2^n\beta, \gamma) = |n|I(\alpha_1, \gamma) + I(\alpha_2^n\beta, \gamma)$ is the sum of two convex functions.

Assume now that each component of α intersects β. Take $n_1 < n_2$ in $\mathbf{Z}$ so that $n_2 - n_1$ is an even number $2k$. Let $\alpha' = \alpha^k$ and $\beta' = \alpha^{n_1}\beta$. By the assumption on the components of α, we have $\alpha^{\frac{n_1+n_2}{2}}\beta = \alpha'\beta'$, $\alpha^{n_1}\beta = \beta'$, and $\alpha^{n_2}\beta = (\alpha')^2\beta'$. Thus $2f(\frac{n_1+n_2}{2}) \leq f(n_1) + f(n_2)$ follows from the first inequality $2f(1) \leq f(0) + f(2)$ for the pair (α', β'). $\square$

Corollary 3 follows from theorem 2 and theorem 1(f).

Suppose a is a simple loop in an oriented surface. Then the multiplication $a^n b$ is the twisting of b n times around a. In particular, given an annulus containing a as a central curve, if b_1 and b_2 are two curve systems in the annulus so that they have the same end points, then b_1 is isotopic to $a^n b_2$ by an isotopy which is the identity on the boundary of the annulus.

This gives a proof of the following.

Proposition 5. *Suppose $\Sigma_{g,n}$ is a compact oriented surface of genus g with n boundary components $c_1, ..., c_n$. Let $\{\alpha_1, ..., \alpha_{3g+n-3}\}$ be a 3-holed sphere decomposition of the surface. If β_1 and β_2 are two curve systems in $\Sigma_{g,n}$ so that $I(\beta_1, \alpha_i) = I(\beta_2, \alpha_i) > 0$ and $I(\beta_1, [c_j]) = I(\beta_2, [c_j])$, then there exist integers $k_1, ..., k_{3g+n-3}$ so that*

$$(*) \qquad \beta_1 = \alpha_1^{k_1} ... \alpha_{3g+n-3}^{k_{3g+n-3}}\beta_2.$$

Conversely, if β_1 and β_2 are related by the above equation, then they have the same intersection numbers with α_i and $[c_j]$.

Remarks 1. By theorem 1(c), the above expression $(*)$ is well defined.

2. The result still holds when some $I(\beta_1, \alpha_i) = 0$ by appropriately interpreting the equation $(*)$.

Proof. Take $a_i \in \alpha_i$ and $b_j \in \beta_j$ so that $a_i \cap a_k = \emptyset$ and $|b_j \cap a_i| = I(\beta_j, \alpha_j)$. Let $N(a_i)$ be a small regular neighborhood of a_i so that $N(a_i) \cap N(a_j) = \emptyset$. Then each component of $\Sigma_{g,n} - \cup_{i=1}^{3g+n-3} int(N(a_i))$ is a 3-holed sphere. By the classification of the curve systems in the 3-holed sphere, we may assume, after an isotopy of the surface, that $b_1 = b_2$ in $\Sigma_{g,n} - \cup_{i=1}^{3g+n-3} int(N(a_i))$. Now apply the above observation on curve systems $b_1 \cap N(a_i)$ and $b_2 \cap N(a_i)$ in the annulus $N(a_i)$, the result follows. $\square$

In terms of the Dehn-Thurston coordinates of $CS(\Sigma_{g,n})$ with respect to the 3-holed sphere decomposition (see [HP], [FLP]), the integer k_i in the proposition is the difference of the i-th twisting coordinate of the Dehn-Thurston coordinates of b_1 and b_2.

In [LS], the Dehn-Thurston coordinate of the space of measured laminations is reexamed from the point of view of multiplication.

§4. **Some Other Applications**

In this section, we give short proofs of three known results in surface theory using the multiplications.

Theorem 6. *Let α and β be two isotopy classes of simple loops in a surface.*

(a) (Ivanov). *If the Dehn twists on α and β commute, then α is disjoint from β, i.e., $I(\alpha, \beta) = 0$.*

(b) (Thurston). *If α and β fill the surface, i.e., $I(\alpha, \gamma) + I(\beta, \gamma) > 0$ for all $\gamma \in CS(\Sigma)$, then the homeomorphism $D_\alpha^{-1} D_\beta$ has no fixed point in $CS(\Sigma)$.*

Note that Thurston [Th1] proved a much stronger result that the homeomorphism is actually pseudo-anosov.

Proof. For part (a), let $k = I(\alpha, \beta)$. Suppose otherwise that $k > 0$. Consider $D_\alpha D_\beta(\alpha) = D_\beta D_\alpha(\alpha)$. Then we have $\alpha^l(\beta^k \alpha) = \beta^k \alpha$ for some $l > 0$ (indeed, $l = k^2$ by theorem 1(d) that $I(\alpha, \beta^k \alpha) = I(\alpha, \beta^k) = k^2$). By theorem 1(c), we obtain, $\alpha^{l-1}\beta^k = \beta^k \alpha$. Left multiply both sides by α and use the associativity again, we obtain $\alpha^l \beta^k = \beta^k$. By theorem 1(d), $0 = I(\beta^k, \beta^k) = I(\alpha^l \beta^k, \beta^k) = I(\alpha^l, \beta^k) = k^2 l > 0$. This is a contradiction.

For part (b), suppose otherwise that there exists $\gamma \in CS(\Sigma)$ so that $D_\alpha^{-1} D_\beta(\gamma) = \gamma$. Then $\alpha^n \gamma = \beta^m \gamma$ for some $m, n \in \mathbf{Z}_{\geq 0}$. Multiply from the left by γ. By theorem 1(d), we obtain $\alpha^n \gamma' = \beta^m \gamma''$ where $I(\gamma', \alpha) = 0$ and $I(\beta, \gamma'') = 0$. One of the classes γ' and γ'' is non-empty since α, β are surface filling. Say $\gamma' \neq 0$. Then the equation $\alpha^n \gamma' = \beta^m \gamma''$ shows that $I(\gamma', \beta) = 0$ which contradicts the surface filling assumption. □

The following result was obtained in [FLP] (proposition 1 in the appendix of exposé 4) and [MS] (proposition III3.4).

Proposition 7 ([FLP], [MS]). *Suppose $\alpha_1, ..., \alpha_m$ are isotopy classes of simple loops so that $I(\alpha_i, \alpha_j) = 0$ for all i, j. Then for all $\beta \in CS_0(\Sigma)$ and $\gamma \in CS(\Sigma)$, we have*

$$\sum_{i=1}^{m} I(\alpha_i, \beta) I(\alpha_i, \gamma) - I(\beta, \gamma) \leq I(D_{\alpha_1} ... D_{\alpha_m} \beta, \gamma) \leq \sum_{i=1}^{m} I(\alpha_i, \beta) I(\alpha_i, \gamma) + I(\beta, \gamma).$$

Proof. Let $\alpha = \alpha_1^{k_1} .... \alpha_m^{k_m}$ where $k_i = I(\alpha_i, \beta)$. Then $I(\alpha, \gamma) = \sum_{i=1}^m I(\alpha_i, \beta) I(\alpha_i, \gamma)$. The result is a consequence of theorem 1(g). □

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Department of Mathematics

Rutgers University

Piscataway, NJ 08854

email: fluo@math.rutgers.edu