SOLVING THURSTON EQUATION IN A COMMUTATIVE RING

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Abstract. We show that solutions of Thurston equation on triangulated 3-manifolds in a commutative ring carry topological information. We also introduce a homogeneous Thurston equation and a commutative ring associated to triangulated 3-manifolds.

1. introduction

1.1. **Statement of results.** Given a triangulated oriented 3-manifold (or pseudo 3-manifold) $(M, \mathcal{T})$, Thurston equation associated to $\mathcal{T}$ is a system of integer coefficient polynomials. W. Thurston [19] introduced his equation in the field $\mathbf{C}$ of complex numbers in order to find hyperbolic structures. Since then, there have been much research on solving Thurston equation in $\mathbf{C}$, [9], [2], [13], [18], [21], [1] and others. Since the equations are integer coefficient polynomials, one could attempt to solve Thurston equation in a ring with identity. The purpose of this paper is to show that interesting topological results about the 3-manifolds can be obtained by solving Thurston equation in a commutative ring with identity. For instance, by solving Thurston equation in the field $\mathbf{Z}/\mathbf{3}Z$ of three elements, one obtains the result which was known to H. Rubinstein and S. Tillmann that a closed 1-vertex triangulated 3-manifold is not simply connected if each edge has even degree.

Theorem 1.1. *Suppose $(M, \mathcal{T})$ is an oriented connected closed 3-manifold with a triangulation $\mathcal{T}$ and R is a commutative ring with identity. If Thurston equation on $(M, \mathcal{T})$ is solvable in R and $\mathcal{T}$ contains an edge which is a loop, then there exists a homomorphism from $\pi_1(M)$ to $PSL(2, R)$ sending the loop to a non-identity element. In particular, M is not simply connected.*

We remark that the existence of an edge which is a loop cannot be dropped in the theorem. Indeed, it was observed in [6], [18], and [20] that for simplicial triangulations $\mathcal{T}$ and any commutative ring R, there are always solutions to Thurston equation. Theorem 1.1 for $R = \mathbf{C}$ was first proved by Segerman-Tillmann [17]. A careful examination of the proof of [17] shows that their method also works for any field R. However, for a commutative ring with zero divisors, the geometric argument breaks down. We prove theorem 1.1 by introducing a homogeneous Thurston equation and studying its solutions. Theorem 1.1 prompts us to introduce the universal construction of a Thurston ring of a triangulated 3-manifold. Theorem 1.1 can be phrased in terms of the universal construction (see theorem 5.2).

Thurston equation can be defined for any ring (not necessary commutative) with identity (see §2). We do not know if theorem 1.1 holds in this case. The

Key words and phrases. triangulation of 3-manifolds, ideal tetrahedra, Thurston equation.
The work is partially supported by NSF grants.

most interesting non-commutative rings for 3-manifolds are probably the algebras of 2×2 matrices with real or complex coefficients. Solving Thurston equation in the algebra $M_{2\times 2}(\mathbf{R})$ has the advantage of linking hyperbolic geometry to $Ad(S^3)$ geometry. See [3] for related topics.

Motivated by theorem 1.1, we propose the following two conjectures.

Conjecture 1. If M is a compact 3-manifold and $\gamma \in \pi_1(M) - \{1\}$, there exists a finite commutative ring R with identity and a homomorphism from $\pi_1(M)$ to $PSL(2, R)$ sending γ to a non-identity element.

Conjecture 2. If $M \neq S^3$ is a closed oriented 3-manifold, then there exists a 1-vertex triangulation $\mathcal{T}$ of M and a commutative ring R with identity so that Thurston equation associated to $\mathcal{T}$ is solvable in R.

Conjecture 2 is supported by the main result in [8]. It states that if M is closed hyperbolic and $\mathcal{T}$ is a 1-vertex triangulation so that all edges are homotopically essential, then Thurston's equation on $\mathcal{T}$ is solvable in $\mathbf{C}$.

1.2. **Organization of the paper.** In §2, we recall briefly triangulations of 3-manifolds and pseudo 3-manifolds and Thurston equation. A homogeneous Thurston equation is introduced. In §3, we recall cross ratio and projective plane in a commutative ring. Theorem 1.1 is proved in §4. In §5, we introduce a universal construction of Thurston ring of a triangulated 3-manifold and investigate the relationship between Thurston ring and Pachner moves. Some examples of solutions of Thurston equation in finite rings are worked out in §6.

1.3. **Acknowledgement.** We thank Sergei Matveev and Carlo Petronio for answering my questions on triangulations of 3-manifolds.

2. Preliminaries on Triangulations and Thurston Equation

All manifolds and tetrahedra are assumed to be oriented in this paper. We assume all rings have the identity element.

2.1. **Triangulations.** A compact oriented triangulated pseudo 3-manifold $(M, \mathcal{T})$ consists of a disjoint union $X = \sqcup_i \sigma_i$ of oriented Euclidean tetrahedra σ_i and a collection of orientation reversing affine homeomorphisms Φ between some pairs of codimension-1 faces in X. The pseudo 3-manifold M is the quotient space X/Φ and the simplices in $\mathcal{T}$ are the quotients of simplices in X. The *boundary* ∂M of M is the quotient of the union of unidentified codimension-1 faces in X. If $\partial M = \emptyset$, we call M *closed*. The sets of all quadrilateral types (to be called *quads* for simplicity) and normal triangle types in $\mathcal{T}$ will be denoted by $\square = \square(\mathcal{T})$ and $\triangle = \triangle(\mathcal{T})$ respectively. See [5], [4] or [7] for more details. The most important combinatorics ingredient in defining Thurston equation is a $\mathbf{Z}/3\mathbf{Z}$ action on $\square$ which we recall now. If edges of an oriented tetrahedron σ are labelled by a, b, c so that opposite edges are labelled by the same letters (see figure 1(a)), then the cyclic order $a \to b \to c \to a$ viewed at vertices is independent of the choice of the outward normal vectors at vertices and depends only on the orientation of σ. Since a quad in σ corresponds to a pair of opposite edges, this shows that there is a $\mathbf{Z}/3\mathbf{Z}$ action on the set of all quads in σ by cyclic permutations. If $q, q' \in \square$, we use $q \to q'$ to indicate that q, q' are in the same tetrahedron so that q is ahead of q' in the cyclic order. The set of all i-simplices in $\mathcal{T}$ will be denoted by $\mathcal{T}^{(i)}$. Given an edge $e \in \mathcal{T}^{(1)}$, a tetrahedron $\sigma \in \mathcal{T}^{(3)}$ and a quad $q \in \square(\mathcal{T})$, we use $e < \sigma$ to indicate that e is adjacent to

σ; $q \subset \sigma$ to indicate that q is inside σ; and $q \sim e$ to indicate that q and e are in the same tetrahedron and q faces e (in the unidentified space X). See [7] for more details. An edge $e \in \mathcal{T}^{(1)}$ is called *interior* if it is not in the boundary ∂M. In particular, if $(M, \mathcal{T})$ is a closed triangulated pseudo 3-manifold, all edges are interior.

2.2. **Thurston equation.**

Definition 2.1. Give a compact triangulated oriented pseudo 3-manifold $(M, \mathcal{T})$ and a ring R (not necessary commutative) with identity, a function $x : \square(\mathcal{T}) \to R$ is called a solution to Thurston equation associated to $\mathcal{T}$ if

(1) whenever $q \to q'$ in $\square$, $x(q')(1 - x(q)) = (1 - x(q))x(q') = 1$,

(2) for each interior edge $e \in \mathcal{T}^{(1)}$ so that $q_1, ..., q_n$ are quads facing e labelled cyclically around e,

$$x(q_1).....x(q_n) = 1, \quad x(q_n)...x(q_1) = 1.$$

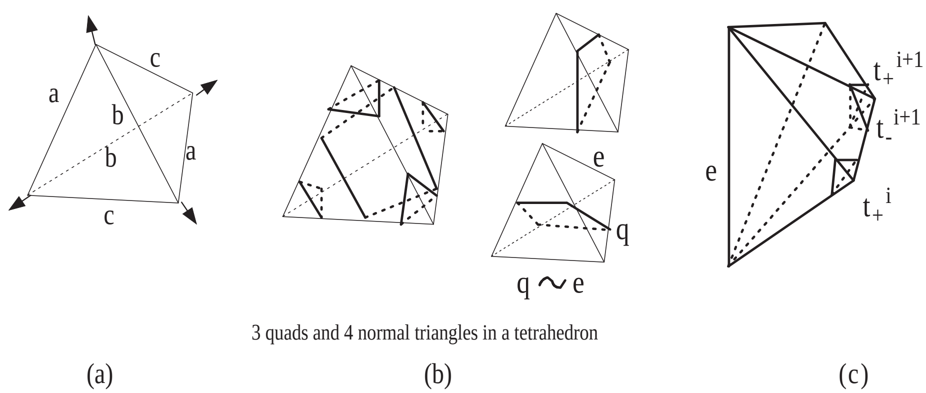

FIGURE 1. cyclic order on three quads in a tetrahedron

Note that condition (1) implies both $x(q)$ and $x(q) - 1$ are invertible elements in R with inverses $1 - x(q'')$ and $x(q')$ where $q'' \to q$. If the ring R is commutative which will be assumed from now on, we only need one equation in each of (1) and (2).

Example 2.2. If $R = \mathbf{Z}/3\mathbf{Z} = \{0, 1, 2\}$ is the field of 3 elements. Then a solution x to Thurston equation must satisfy $x(q) = 2$ for all q. In this case, the first condition (1) holds. The second equation at an edge e becomes $2^k = 1$ where k is the degree of e. Since $2^k = 1$ if and only if k is even, we conclude that Thurston equation has a solution in $\mathbf{Z}/3\mathbf{Z}$ if and only if each interior edge has even degree.

A related homogeneous version of the equation is

Definition 2.3. (Homogeneous Thurston Equation (HTE)) Suppose $(M, \mathcal{T})$ is a compact oriented triangulated pseudo 3-manifold and R is a commutative ring with identity. A function $z : \square \to R$ is called a solution to the homogeneous Thurston equation if

(1) for each tetrahedron $\sigma \in \mathcal{T}$, $\sum_{q \subset \sigma} z(q) = 0$,

(2) for each interior edge e so that the set of all quads facing it is $\{q_1, ..., q_n\}$,

$$\prod_{i=1}^{n} z(q_i) = \prod_{i=1}^{n}(-z(q_i'))$$

or simply

$$\prod_{q\sim e} z(q) = \prod_{q\sim e}(-z(q')),$$

where $q \to q'$.

Note that if z solves HTE and $k : \mathcal{T}^{(3)} \to R$ is any function, then $w(q) = k(\sigma)z(q) : \square \to R$, $q \subset \sigma$, is another solution to HTE. Let R^* be the group of all invertible elements in R. If z solves HTE and $z(q) \in R^*$ for all $q \in \square$, then $x(q) = -z(q)z(q')^{-1}$ for $q \to q'$ solves Thurston equation. Indeed, condition (2) in definition 2.1 follows immediately from condition (2) in definition 2.3 by division. To check condition (1), suppose $q \to q' \to q'' \to q$. Then $z(q'') = -z(q) - z(q')$. Furthermore, $x(q) = -z(q)/z(q')$ and $x(q') = -z(q')/z(q'') = z(q')/(z(q)+z(q')) = 1/(1 - x(q))$. Conversely, we have,

Lemma 2.4. *If R is a commutative ring with identity and $x : \square \to R$ solves Thurston equation, then there exists a solution $z : \square \to R^*$ to HTE so that for all $q \in \square$, $x(q) = -z(q)/z(q')$.*

Proof. By definition, each $x(q)$ is invertible. For each tetrahedron σ containing three quads $q_1 \to q_2 \to q_3$, we have $x(q_2) = 1/(1-x(q_1)), x(q_3) = (x(q_1)-1)/x(q_1)$. Define a map $z : \square \to R^*$ by $z(q_1) = x(q_1), z(q_2) = -1, z(q_3) = 1 - x(q_1)$. Then by definition $x(q) = -z(q)/z(q')$ for all q and $\sum_{q\subset\sigma} z(q) = 0$ for each $\sigma \in \mathcal{T}^{(3)}$. Due to $x(q) = -z(q)/z(q')$ and $\prod_{q\sim e} x(q) = 1$, we see that $\prod_{q\sim e} z(q) = \prod_{q\sim e}(-z(q'))$. $\square$

We remark that the solution z in the lemma depends on the specific choice of the quad q_1 in each tetrahedron.

There is a similar version of homogeneous Thurston equation where we replace the condition $\prod_{q\sim e} z(q) = \prod_{q\sim e}(-z(q'))$ at interior edge e by $\prod_{q\sim e} z(q'') = \prod_{q\sim e}(-z(q'))$ where $q'' \to q \to q'$. In this setting, the transition from HTE to Thurston equation is given by $x(q) = -z(q')/z(q'')$.

3. Cross ratio and projective line in a commutative ring

Let R be a commutative ring with identity and R^* be the group of invertible elements in R. Let $GL(2, R)$ and $PGL(2, R) = GL(2, R)/\sim$ where $M \sim \lambda M$, $\lambda \in R^*$ be the general linear group and its projective group. The group $GL(2, R)$ acts linearly from the left on $R^2 = \{\begin{pmatrix} a \\ b \end{pmatrix} |a, b, \in R\}$. Define the skew symmetric bilinear form $<, >$ on R^2 by $< \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} >= ad - bc$ which is the determinant of $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Note that for $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, its adjacent matrix $adj(X) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ satisfies $X adj(X) = det(X)I$. We also use the transpose to write $\begin{pmatrix} a \\ b \end{pmatrix}$ as $(a, b)^t$.

By the basic properties of the determinant, we have

Lemma 3.1. *Suppose $A, B, A_1, ..., A_n \in R^2$ and $X \in GL(2, R)$. Then*
(1) $< A, B >= - < B, A >$ and $< XA, XB >= det(X) < A, B >$;
(2) If $A = \begin{pmatrix} a \\ b \end{pmatrix}$ and $B = \begin{pmatrix} c \\ d \end{pmatrix}$ so that $< A, B > \in R^*$, then $XA = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $XB = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where $X = \frac{1}{<A,B>} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$;
(3) $< A_1, A_2 > A_3 + < A_2, A_3 > A_1 + < A_3, A_1 > A_2 = 0$;
(4) Let $R_{ijkl} =< A_i, A_j >< A_k, A_l >$. Then $R_{ijkl} = R_{jilk} = R_{klij} = -R_{jikl} = -R_{ijlk}$ and $R_{ijkl} + R_{iklj} + R_{iljk} = 0$.

Indeed, (1) and (2) follow from the properties of determinant. To see (3), let $A_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$, then the left-hand-side of (3) is of the form $\begin{pmatrix} x \\ y \end{pmatrix}$ where $x = det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{bmatrix}$ and $y = det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ by the row expansion formula for determinants. Now these two 3×3 determinants are zero. Thus (3) follows. The first set of identities for R_{ijkl} follow from the definition. The second follows from (3) by applying the bilinear form $<, >$ to it with A_i. We remark that R_{ijkl} enjoys the same symmetries that a Riemannian curvature tensor does.

Definition 3.2. (Cross ratio) Suppose $A_1, ..., A_4 \in R^2$. Then their cross ratio, denoted by $(A_1, A_2; A_3, A_4)$ is defined to be the vector $\begin{pmatrix} R_{1423} \\ R_{1324} \end{pmatrix} = \begin{pmatrix} R_{1423} \\ -R_{1342} \end{pmatrix} \in R^2$ where $R_{ijkl} =< A_i, A_j >< A_k, A_l >$.

For instance,

$$< \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} >= \begin{pmatrix} -ay \\ -bx \end{pmatrix} \tag{1}$$

By lemma 3.1, we obtain,

Corollary 3.3. *Suppose $A_1, ..., A_n \in R^2$. Then*
(1) $(A_1, A_2; A_3, A_4) = (A_3, A_4; A_1, A_2) = (A_2, A_1; A_4, A_3)$,
(2) if $(A_1, A_2; A_3, A_4) = \begin{pmatrix} a \\ b \end{pmatrix}$, then $(A_2, A_1; A_3, A_4) = \begin{pmatrix} b \\ a \end{pmatrix}$,
(3) $(A_1, A_2; A_3, A_4) + (A_1, A_3; A_4, A_2) + (A_1, A_4; A_2, A_3) = 0$,
(4) if $X \in GL(2, R)$, then $(XA_1, XA_2; XA_3, XA_4) = det(X)^2(A_1, A_2; A_3, A_4)$,
(5) if $B, C \in R^2$ and $A_{n+1} = A_1$, then

$$\prod_{i=1}^{n}(< B, A_{i+1} >< C, A_i >) = \prod_{i=1}^{n}(< B, A_i >< C, A_{i+1} >).$$

Corollary 3.3 shows that the cross ratio $(A_1, A_2; A_3, A_4) = (A_i, A_j; A_k, A_l)$ whenever $\{i, j\} = \{1, 2\}, \{k, j\} = \{3, 4\}$ and (i, j, k, l) is a positive permutation of $(1, 2, 3, 4)$, i.e., the cross ratio depends only on the partition $\{i, j\} \sqcup \{k, l\}$ of $\{1, 2, 3, 4\}$ and the orientation of $(1, 2, 3, 4)$. This shows that if σ is an oriented tetrahedron so that its i-th vertex is assigned a vector $A_i \in R^2$, then one can define the cross ratio of a quad $q \subset \sigma$ to be $(A_i, A_j; A_k, A_l)$ where q corresponds to the partition $\{i, j\} \sqcup \{k, l\}$ of the vertex set $\{1, 2, 3, 4\}$ and (i, j, k, l) determines the orientation of σ.

Example 3.4. (Solutions of HTE by cross ratio) Given any compact triangulated pseudo 3-manifold $(M, \mathcal{T})$ and $f : \triangle \to R^2$ so that $f(t) = f(t')$ when two normal triangles t, t' share a common normal arc, we define a map $F : \square(\mathcal{T}) \to R^2$ by $F(q) = (f(t_1), f(t_2); f(t_3), f(t_4)) = \begin{pmatrix} z(q) \\ y(q) \end{pmatrix}$ where $t_1, .., t_4$ are the four normal triangles in a tetrahedron σ containing q so that q separates $\{t_1, t_2\}$ from $\{t_3, t_4\}$ and $t_1 \to t_2 \to t_3 \to t_4$ defines the orientation of σ. Then corollary 3.3 shows that $z : \square \to R$ is a solution to HTE. Note that $y(q) = -z(q')$ with $q \to q'$.

This example serves as a guide for us to solve Thurston equation and HTE. Indeed, the goal is to solve Thurston equation by writing each solution $x : \square \to R$ in terms of cross ratio in the universal cover $(\tilde{M}, \tilde{\mathcal{T}})$.

Let $R^2/\sim$ be the quotient space where $u \sim \lambda u$, $u \in R^2$ and $\lambda \in R^*$. If $x = (a, b)^t \in R^2$, then $[x] = [a, b]^t$ denotes the image of x in $R^2/\sim$.

Definition 3.5. The *projective line* $PR^1 = \{A \in R^2 |$there exists $B \in R^2$ so that $< A, B > \in R^*\}/\sim$ where $A \sim \lambda A$ for $\lambda \in R^*$. A set of elements $\{A_1, ..., A_n\}$ (or $\{[A_1], ..., [A_n]\}$) in R^2 (or in PR^1) is called *admissible* if $< A_i, A_j > \in R^*$ for all $i \neq j$. The *cross ratio* of four points $\alpha_i, i = 1, 2, 3, 4$ in PR^1, denoted by $[\alpha_1, \alpha_2; \alpha_3, \alpha_4]$, is the element $[(A_1, A_2; A_3, A_4)] \in R^2/\sim$ so that $A_i \in \alpha_i$. We also use $[A_1, A_2; A_3, A_4] \in PR^1$ to denote $[(A_1, A_2; A_3, A_4)]$.

Proposition 3.6. *(1) Given an admissible set of three elements $A_1, A_2, A_3 \in R^2$ and $v = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, there exists a unique $A_4 \in R^2$ so that $(A_1, A_2; A_3, A_4) = v$. Furthermore, $A_1, ..., A_4$ form an admissible set if and only if $c_1, c_2, c_1 - c_2 \in R^*$.*

(2) Suppose $A_1, ..., A_4, B_1, ..., B_4 \in R^2$ so that both $\{A_1, ..., A_4\}$ and $\{B_1, ..., B_4\}$ are admissible and $[A_1, A_2; A_3, A_4] = [B_1, B_2; B_3, B_2]$. Then there exists a unique $X \in PGL(2, R)$ so that $[XA_i] = [B_i]$ for all i. Furthermore, if $Y \in GL(2, R)$ so that $[YA_i] = [A_i], i = 1, 2, 3$, then $Y = \lambda I$ for $\lambda \in R^*$.

Proof. To see the existence part of (1), let $A_i = (a_i, b_i)^t$ and consider $X = \frac{1}{<A_1, A_2>} \begin{pmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{pmatrix} \in GL(2, R)$. Then $XA_1 = (1, 0)^t$ and $XA_2 = (0, 1)^t$. By corollary 3.3(4), after replacing A_i by XA_i, we may assume that $A_1 = (1, 0)^t$, $A_2 = (0, 1)^t$. Then by identity (1), $(A_1, A_2; A_3, A_4) = (-a_3 b_4, -a_4 b_3)^t$. By the assumption that $< A_i, A_3 > \in R^*$ for $i = 1, 2$, we see that $a_3, b_3 \in R^*$. It follows that $a_4 = -c_1/b_3$ and $b_4 = -c_2/a_3$. This shows that A_4 exists and is unique.

Now given that $\{A_1, A_2, A_3\}$ is admissible, the set $\{A_1, ..., A_4\}$ is admissible if and only if $< A_4, A_i > \in R^*$, i.e., $c_1, c_2, c_1 - c_2 \in R^*$.

To see part (2), by the proof of part (1) and the assumption $< A_1, A_2 >, < B_1, B_2 > \in R^*$, after replacing A_i by YA_i and B_i by ZB_i for some $Y, Z \in GL(2, R)$, we may assume that $A_1 = B_1 = (1, 0)^t$ and $A_2 = B_2 = (0, 1)^t$. Let $A_3 = (a, b)^t, A_4 = (a', b')^t$, $B_3 = (c, d)^t$ and $B_4 = (c', d')^t$. Then the admissible condition implies that $a, b, c, d, a', b', c', d' \in R^*$. Furthermore, $[A_1, A_2; A_3, A_4] = [B_1, B_2; B_3, B_4]$ implies that there exists $\lambda \in R^*$ so that $a' = \frac{\lambda d d'}{b}$ and $b' = \frac{\lambda c c'}{a}$. This shows that the matrix $X = \begin{bmatrix} \frac{c}{a} & 0 \\ 0 & \frac{d}{b} \end{bmatrix} \in GL(2, R)$ satisfies $XA_1 = \frac{c}{a} B_1$, $XA_2 = \frac{d}{b} B_2$, $XA_3 = B_3$ and $XB_4 = \lambda \frac{cd}{ab} B_4$.

To see the uniqueness, say $YA_i = \lambda_i A_i$ for $\lambda_i \in R^*$. We claim that $\lambda_1 = \lambda_2 = \lambda_3$ and $Y = \lambda_1 I$. Indeed, by definition, $det(Y) < A_i, A_j > = < YA_i, YA_j > = \lambda_i \lambda_j <$

$A_i, A_j >$. Since $< A_i, A_j > \in R^*$, we obtain $\lambda_i \lambda_j = det(Y)$. This implies that $\lambda_i = \lambda_1$ for $i = 2, 3$. We conclude that $Y[A_1, A_2] = \lambda_1[A_1, A_2]$. Since the matrix $[A_1, A_2] \in GL(2, R)$, it follows that $Y = \lambda_1 I$.

□

4. A proof of theorem 1.1

We will prove a slightly general theorem which holds for compact oriented pseudo 3-manifolds $(M, \mathcal{T})$. Let M^* be M with a small regular neighborhood of each vertex removed and let $\mathcal{T}^*$ be the ideal triangulation $\{s \cap M^* | s \in \mathcal{T}\}$ of the compact 3-manifold M^*.

Theorem 4.1. *Suppose $(M, \mathcal{T})$ is a compact triangulated pseudo 3-manifold and R is a commutative ring with identity so that Thurston equation on $\mathcal{T}$ is solvable in R. Then each edge $e \in \mathcal{T}^*$ lifts to an arc in the universal cover $\tilde{M}^*$ of M^* joining different boundary components of $\tilde{M}^*$. Furthermore, if M is a closed connected 3-manifold so that there exists an edge e having the same end points, then there exists a representation of $\pi_1(M)$ into $PSL(2, R)$ sending the loop $[e]$ to a non-identity element.*

The main idea of the proof is based on the methods developed in [9], [17], [18], and [21] which construct pseudo developing map and the holonomy associated to a solution to Thurston equation.

4.1. **Pseudo developing map.** Let $\pi : \tilde{M}^* \to M^*$ be the universal cover and $\tilde{\mathcal{T}}^*$ be the pull back of the ideal triangulation $\mathcal{T}^*$ of M^* to $\tilde{M}^*$. We use $\tilde{\triangle}$ and $\tilde{\square}$ to denote the sets of all normal triangle types and quads in $\tilde{\mathcal{T}}^*$ respectively. The sets of all normal triangle types and quads in $\mathcal{T}^*$ are the same as those of $\mathcal{T}$ and will still be denoted by $\triangle$ and $\square$. The covering map π induces a surjection π_* from $\tilde{\triangle}$ and $\tilde{\square}$ to $\triangle$ and $\square$ respectively so that $\pi_*(d_1) = \pi_*(d_2)$ if and only if d_1 and d_2 differ by a deck transformation element.

Suppose $x : \square \to R$ solves Thurston equation on $\mathcal{T}$ and $z : \square \to R^*$ is an associated solution to HTE constructed by lemma 2.1. Let $w : \square \to PR^1$ be the map $w(q) = [z(q), -z(q')]^t$ where $q \to q'$. Let $\tilde{x} = x\pi_*$, $\tilde{z} = z\pi_*$ and $\tilde{w} = w\pi_*$ be the associated maps defined on $\tilde{\square}$. By the construction, $\tilde{x}$ and $\tilde{z}$ are solutions to Thurston equation and HTE on $\tilde{\mathcal{T}}^*$.

Definition 4.2. (See [18], [17], [21], [9]) Given a solution x to Thurston equation on $(M, \mathcal{T})$, a map $\phi : \tilde{\triangle} \to PR^1$ is called a *pseudo developing map* associated to x if

(1) whenever t_1, t_2 are two normal triangles in $\tilde{\triangle}$ sharing a normal arc, denoted by $t_1 \sim t_2$ in the sequel, then $\phi(t_1) = \phi(t_2)$,

(2) if t_1, t_2, t_3, t_4 are four normal triangles in a tetrahedron σ then $\{\phi(t_1), ..., \phi(t_4)$ $\}$ is admissible and

$$[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = \tilde{w}(q) \quad (2)$$

where $t_1 \to t_2 \to t_3 \to t_4$ determines the orientation of the tetrahedron σ and $q \subset \sigma$ is the quad separating $\{t_1, t_2\}$ from $\{t_3, t_4\}$.

The main result in this section is

Theorem 4.3. *Given any solution x to Thurston equation on a compact pseudo 3-manifold $(M, \mathcal{T})$, there exists a pseudo developing map associated to x.*

Proof. The proof is based on the following result which is an immediate consequence of proposition 3.6.

Lemma 4.4. *Suppose $\{t_1, t_2, t_3, t_4\}$ are four normal triangles in $\sigma \in \tilde{\mathcal{T}}^{(3)}$ so that $t_1 \to t_2 \to t_3 \to t_4$ determines the orientation and q is a quad in σ separating $\{t_1, t_2\}$ from $\{t_3, t_4\}$. If $\phi(t_i) \in PR^1$, $i = 1, 2, 3$, are defined and $\{\phi(t_1), \phi(t_2), \phi(t_3)\}$ is admissible, then there exists a unique $\phi(t_4) \in PR^1$ so that $[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = \tilde{w}(q)$. Furthermore, $\{\phi(t_1), ..., \phi(t_4)\}$ is admissible.*

Indeed, the existence and uniqueness follows from proposition 3.6. The admissibility of $\{\phi(t_1), ..., \phi(t_4)\}$ follows from that fact that $z(q), -z(q'), z(q) - (-z(q')) = -z(q'')$ are in R^* where $\tilde{w}(q) = [z(q), -z(q')]^t$ and $q \to q' \to q''$.

We now use the lemma to construct the pseudo developing map ϕ by the "combinatorial continuation" method. To begin, by working on connected component of M, we may assume that M is connected. Let G be the connected graph dual to the ideal triangulation $\tilde{\mathcal{T}}^*$ of $\tilde{M}^*$, i.e., vertices of G are tetrahedra in $\tilde{\mathcal{T}}^*$ and edges in G are pairs of tetrahedra sharing a codimension-1 face. An edge path $\alpha = [\sigma_1, ..., \sigma_n; \tau_1, ..., \tau_{n-1}]$ in G consists of tetrahedra σ_i and codimension-1 faces τ_i so that $\tau_i \subset \sigma_i \cap \sigma_{i+1}$. If $\sigma_n = \sigma_1$, we say α is an edge loop.

Lemma 4.5. *Suppose α is an edge path from σ_1 to σ_n and t_1, t_2, t_3 are normal triangles in σ_1 adjacent to the codimension-1 face τ_1 so that $\phi(t_1), \phi(t_2), \phi(t_3)$ are defined and admissible. Then there exists an extension ϕ_α of ϕ (depending on α) to all normal triangles t in σ_i's so that conditions (1) and (2) in definition 4.2 hold.*

Proof. Suppose t_4 is the last normal triangle in σ_1. By lemma 4.4, we define $\phi_\alpha(t_4)$ so that identity (2) holds for the quad q separating t_1, t_2 from t_3, t_4 (subject to orientation). Note that this implies that identity (2) holds for all other quads q^* in σ_1 due to the basic property of cross ratio (corollary 3.3) and that x solves Thurston equation. Now we extend ϕ_α to normal triangles in σ_2 as follows. Suppose t'_1, t'_2, t'_3 are the normal triangles in σ_2 so that $t'_i \sim t_i$, i.e., they share a normal arc. Define $\phi_\alpha(t'_i) = \phi_\alpha(t_i)$ and then use lemma 4.4 to extend ϕ_α to the last normal triangle in σ_2. Inductively, we define ϕ_α for all normal triangles t in σ_i. By the construction, both conditions (1) and (2) in definition 4.2 hold for ϕ_α. □

We call ϕ_α the "combinatorial continuation" of ϕ along the edge path α and denote it by $\phi_\alpha^{\sigma_1}$ to indicate the initial value. From the construction, if β is an edge path starting from σ_n to σ_m and $\beta\alpha$ is the multiplication of the edge paths α and β, then

$$\phi_\beta^{\sigma_n}(t) = \phi_{\beta\alpha}^{\sigma_1}(t) \tag{3}$$

for all normal triangles t in σ_m.

Our goal is to show that the extension $\phi_\alpha^{\sigma_1}$ is independent of the choice of edge path α, i.e., $\phi_\alpha^{\sigma_1}(t) = \phi_{\alpha'}^{\sigma_1}(t)$ for two edge paths α and α' from σ_1 to σ_n and $t \subset \sigma_n$. By (3), this is the same as showing $\phi_{\alpha'\alpha^{-1}}^{\sigma_n}(t) = t$. Therefore, it suffices to show

Lemma 4.6. *If α is an edge loop in G from σ_1 to σ_1, then $\phi_\alpha^{\sigma_1}(t) = t$ for all normal triangles t in σ_1.*

Proof. Form the 2-dimensional CW complex W by attaching 2-cells to the graph G as follows. Recall that an edge $e \in \tilde{\mathcal{T}^*}^{(3)}$ is called *interior* if it is not in the boundary $\partial\tilde{M}^*$. For each interior edge e in $\tilde{\mathcal{T}}^*$ adjacent to tetrahedra $\delta_1, ..., \delta_m$, ordered cyclically around e, there corresponds an edge loop $\alpha_e = [\delta_1, ..., \delta_m; \epsilon_1, ..., \epsilon_m]$ where

$\epsilon_i \subset \delta_i \cap \delta_{i+1}$ and $\delta_{m+1} = \delta_1$. We attach a 2-cell to G along α_e for each interior edge e to obtain W. By the construction, the universal cover space $\tilde{M}^*$ is obtained from W by attaching a product space $B \times [0,1)$ along a surface $B \times 0$. Thus W is homotopic to $\tilde{M}^*$. In particular, W is simply connected. This shows that the edge loop α is a product of edge loops of the form α_e, for interior edges e, and loops of the form $\beta\beta^{-1}$ for some edge paths β. By the identity (3), the lemma holds for edge loops of the form $\beta\beta^{-1}$. Therefore, it remains to prove the lemma for edge loops $\alpha = \alpha_e = [\delta_1, ..., \delta_m; \epsilon_1, ..., \epsilon_m]$.

To this end, suppose that ϕ is defined at the normal triangles t_0^1, t_+^1, t_∞^1 in the tetrahedron δ_1 so that the edge e is adjacent to t_0^1, t_∞^1. Let the normal triangles in the tetrahedron δ_i be $t_0^i, t_+^i, t_-^{t+1}, t_\infty^i$ so that t_0^i, t_∞^i are adjacent to the edge e and $t_0^i \to t_\infty^i \to t_+^i \to t_-^{i+1}$ defines the orientation. Then by the construction, $t_0^i \sim t_0^{i+1}$, $t_\infty^i \sim t_\infty^{i+1}$ and $t_+^i \sim t_-^i$ where indices are counted modulo m. See figure 1(c). Let q_i be the quad in δ_i separating $\{t_0^i, t_\infty^i\}$ from $\{t_+^i, t_-^{i+1}\}$ and $\tilde{w}(q_i) = [a_i, b_i]^t$. By the assumption that x solves Thurston equation, $\prod_i a_i = \prod_i b_i$. By the definition of $\phi_{\alpha_e}^{\delta_1}$, denoted by ψ for simplicity, we have $\psi(t_0^i) = \psi(t_0^{i+1}), \psi(t_\infty^i) = \psi(t_\infty^{i+1})$, $\psi(t_+^i) = \psi(t_-^i)$ and

$$[\psi(t_0), \psi(t_\infty); \psi(t_+^i), \psi(t_-^{i+1})] = \tilde{w}(q_i). \tag{4}$$

We claim that $\psi(t_-^{m+1})$ defined by the identity (4) above is equal to $\psi(t_+^1)$. Indeed, by corollary 3.3(5) and $\prod_i a_i = \prod_i b_i$ where all $a_i, b_i \in R^*$ and the admissibility, we see that $[\psi(t_0), \psi(t_\infty); \psi(t_+^m), \psi(t_-^{m+1})] = [\psi(t_0), \psi(t_\infty); \psi(t_+^m), \psi(t_+^1)]$. By the uniqueness of the cross ratio, we conclude that $\psi(t_+^{m+1}) = \psi(t_-^1)$. □

Now to define $\phi : \tilde{\square} \to PR^1$, fix a tetrahedron $\sigma_0 \in \tilde{\mathcal{T}}^*$. Let t_1, t_2, t_3 be three normal triangles in σ_0. Define $\phi(t_1) = [1,0]^t$, $\phi(t_2) = [0,1]^t$ and $\phi(t_3) = [1,1]^t$ and use combinatorial continuation to define ϕ on $\tilde{\triangle}$. □

4.2. **The holonomy representation.** Suppose x is a solution to Thurston equation on $(M, \mathcal{T})$ in a ring R and $\phi : \tilde{\triangle} \to PR^1$ is an associated pseudo developing map. Then there exists a homomorphism $\rho : \pi_1(M^*) \to PSL(2,R)$ so that for all $\gamma \in \pi_1(M^*)$, considered as a deck transformation group for the universal cover $\pi_* : \tilde{M}^* \to M^*$,

$$\phi(\gamma) = \rho(\gamma)\phi. \tag{5}$$

We call ρ a *holonomy representation* of x. It is unique up to conjugation in $PSL(2,R)$. Here is the construction of ρ. Fix an element $\gamma \in \pi_1(M^*)$. By the construction, $\pi_1(M^*)$ acts on $\tilde{M}^*$, $\tilde{\mathcal{T}}^*$, $\tilde{\triangle}$ and $\tilde{\square}$ so that $\pi_*(\gamma) = \pi_*$ for $\gamma \in \pi_1(M^*)$. This implies

$$[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = [\phi(\gamma t_1), \phi(\gamma t_2); \phi(\gamma t_3), \phi(\gamma t_4)]$$

for all normal triangles $t_1, ..., t_4$ in each tetrahedron σ in $\tilde{T}^{(3)}$. By proposition 3.6, there exists an element $\rho_\sigma(\gamma) \in PSL(2,R)$ so that

$$\phi(\gamma t_i) = \rho_\sigma(\gamma)\phi(t_i)$$

where $t_i \subset \sigma$. We claim that $\rho_\sigma(\gamma) = \rho_{\sigma'}(\gamma)$ for any two $\sigma, \sigma' \in \tilde{\mathcal{T}^*}^{(3)}$. Indeed, since any two tetrahedra can be joint by an edge path in the graph G, it suffices to show that $\rho_\sigma(\gamma) = \rho_{\sigma'}(\gamma)$ for two tetrahedra sharing a codimension-1 face τ. Let t_1, t_2, t_3 and t'_1, t'_2, t'_3 be the normal triangles in σ and σ' respectively so that $t_i \sim t'_i$

and t_1, t_2, t_3 are adjacent to τ. Now $\phi(t_i) = \phi(t'_i)$ and $\gamma\phi(t_i) = \gamma\phi(t'_i)$, therefore, $\rho_\sigma(\gamma)\phi(t_i) = \rho_{\sigma'}(\gamma)\phi(t_i)$ for $i = 1, 2, 3$. By the uniqueness part of proposition 3.6(2), it follows that $\rho_\sigma(\gamma) = \rho_{\sigma'}(\gamma)$. The common value is denoted by $\rho(\gamma)$. Given $\gamma_1, \gamma_2 \in \pi_1(M^*)$, by definition, $\rho(\gamma_1\gamma_2)\phi = \phi(\gamma_1\gamma_2) = \rho(\gamma_1)\phi(\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)\phi$ and the uniqueness part of proposition 3.6, we see that $\rho(\gamma_1\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)$, i.e., ρ is a group homomorphism from $\pi_1(M^*)$ to $PSL(2, R)$.

Note that the representation ρ is trivial if and only if $\phi(\gamma t) = \phi(t)$ for all $t \in \tilde{\triangle}$ and $\gamma \in \pi_1(M^*)$. In this case, the pseudo developing map ϕ is defined on $\triangle \to PR^1$ so that $[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = [z(q), -z(q')]^t$. This was the construction in example 3.1. In particular, the holomony representations associated to solutions in example 3.1 are trivial.

4.3. **A proof of theorem 4.1.** Suppose otherwise that there exists an edge $e \in \mathcal{T}^*$ whose lift is an edge e^* in $\tilde{\mathcal{T}}^*$ joining the same boundary component of $\tilde{M}^*$. Take a tetrahedron σ containing e^* as an edge and let t_1, t_2, t_3, t_4 be all normal triangles in σ so that t_1, t_2 are adjacent to e^*. By definition, the pseudo developing map $\phi : \tilde{\triangle} \to PR^1$ satisfies the condition that $\{\phi(t_1), ..., \phi(t_4)\}$ is admissible. In particular, $\phi(t_1) \neq \phi(t_2)$. On the other hand, since e^* ends at the same connected component of $\partial\tilde{M}^*$ which is a union of normal triangles related by sharing common normal arcs, there exists a sequence of normal triangles $s_1 = t_1, s_2, ..., s_n = t_2$ in $\tilde{\triangle}$ so that $s_i \sim s_{i+1}$. In particular, $\phi(s_i) = \phi(s_{i+1})$. This implies that $\phi(t_1) = \phi(t_2)$ contradicting the assumption that $\phi(t_1) \neq \phi(t_2)$.

To prove the second part of theorem 4.1 that M is a closed connected 3-manifold, we first note that $\pi_1(M^*)$ is isomorphic to $\pi_1(M)$ under the homomorphism induced by inclusion. We will identify these two groups and identify $\tilde{M}^*$ as a $\pi_1(M)$ invariant subset of the universal cover $\tilde{M}$ of M. If e is an edge in $\mathcal{T}$ ending at the same vertex v in $\mathcal{T}$, let $\gamma \in \pi_1(M, v)$ be the deck transformation element corresponding to the loop e. We claim that $\rho(\gamma) \neq id$ in $PSL(2, R)$. Indeed, suppose e^* is the lifting of e. Then by the statement just proved, e^* has two distinct vertices u_1 and u_2 in $\tilde{M}$ and $\phi(u_1) \neq \phi(u_2)$. By definition $\gamma(u_1) = u_2$. It follows that $\phi(u_2) = \phi(\gamma u_1) = \rho(\gamma)\phi(u_1)$. Since $\phi(u_1) \neq \phi(u_2)$, we obtain $\rho(\gamma) \neq id$. This ends the proof.

5. A universal construction

Recall that $(M, \mathcal{T})$ is a compact oriented triangulated pseudo 3-manifold. The boundary ∂M of M is triangulated by the subcomplex $\partial\mathcal{T} = \{s \cap \partial M | s \in \mathcal{T}\}$. An edge in $\mathcal{T}$ is called *interior* if it is not in $\partial\mathcal{T}$. The goal of this section is to introduce the *Thurston ring* $\mathcal{R}(\mathcal{T})$ and its homogeneous version $\mathcal{R}_h(\mathcal{T})$. We will study the changes of $\mathcal{R}(\mathcal{T})$ when the triangulations are related by Pachner moves.

We will deal with quotients of the polynomial ring $\mathbf{Z}[\square]$ with $q \in \square$ as variables. As a convention, we will use $p \in \mathbf{Z}[\square]$ to denote its image in the quotient ring $\mathbf{Z}[\square]/\mathcal{I}$.

5.1. **Thurston ring of a triangulation.** The "ground" ring in the construction is the following. Let σ be an oriented tetrahedron so that $q \to q' \to q''$ are the three quads in it. Then the Thurston ring $\mathcal{R}(\sigma)$ is the quotient of the polynomial ring $\mathbf{Z}[q, q', q'']$ modulo the ideal generated by $q'(1-q) - 1$, $q''(1-q') - 1$, and $q(1-q'') - 1$. Note that this implies in $\mathcal{R}(\sigma)$, $q' = 1/(1-q)$ and $q'' = (q-1)/q$ and furthermore, $\mathcal{R}(\sigma) \cong \mathbf{Z}[x, 1/x, 1/(1-x)]$ where x is an independent variable.

Similarly, we defined $\mathcal{R}_h(\sigma)$ to be the quotient ring $\mathbf{Z}[q,q',q'']/(q+q'+q'')$ where $(q+q'+q'')$ is the ideal generated by $q+q'+q''$. Note that $\mathcal{R}_h(\sigma)(\sigma) \cong \mathbf{Z}[x,y]$ the polynomial ring in two independent variables.

Recall that the tensor product $R_1 \otimes R_2$ of two rings R_1 and R_2 is the tensor product of R_1 and R_2 considered as $\mathbf{Z}$ algebras.

Definition 5.1. Suppose $(M, \mathcal{T})$ is a compact oriented pseudo 3-manifold. The *Thurston ring* $\mathcal{R}(\mathcal{T})$ of $\mathcal{T}$ is the quotient of the tensor product $\otimes_{\sigma \in \mathcal{T}^{(3)}} \mathcal{R}(\sigma)$ modulo the ideal generated by elements of the form $W_e - 1$ where $W_e = \prod_{q \sim e} q$ for all interior edges e. The *homogeneous Thurston ring* $\mathcal{R}_h(\mathcal{T})$ is the quotient of $\otimes_{\sigma \in \mathcal{T}^{(3)}} \mathcal{R}_h(\sigma)$ modulo the ideal generated by elements of the form $U_e = \prod_{q\sim e} q - \prod_{q \sim e}(-q')$, $q \to q'$, for all interior edges e. The element $W_e = \prod_{q\sim e} q$ is called the *holonomy* of the edge e.

By the construction, given a commutative ring R with identity, Thurston equation on $\mathcal{T}$ is solvable in R if and only if there exists a non-trivial ring homomorphism from $\mathcal{R}(\mathcal{T})$ to R. Therefore theorem 1.1 can be stated as,

Theorem 5.2. *Suppose $(M, \mathcal{T})$ is a triangulated closed connected 3-manifold so that one edge in $\mathcal{T}$ is a loop. If $\mathcal{R}(\mathcal{T}) \neq \{0\}$, then $\pi_1(M) \neq \{1\}$.*

Note that $\mathcal{R}(\mathcal{T})$ is also the quotient $\mathbf{Z}[\square]/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by $q'(1-q)-1$ for $q \to q'$ and $W_e - 1$ for interior edges e and $\mathcal{R}_h(\mathcal{T})$ is the quotient of $\mathbf{Z}[\square]/\mathcal{I}_h$ where $\mathcal{I}_h$ is the ideal generated by $q+q'+q''$ for $q \to q' \to q''$ and U_e for interior edges e. We remark that if there is $q \in \square$ so that $q = 0$ in $\mathcal{R}(\mathcal{T})$, then $\mathcal{R}(\mathcal{T}) = \{0\}$. Indeed, in this case the identity element $1 = 1 - q(1-q'')$ is in the ideal $\mathcal{I}$, therefore, $\mathcal{R}(\mathcal{T}) = \{0\}$. In particular, if $\mathcal{T}$ contains an interior edge of degree 1 (i.e., adjacent to only one tetrahedron), then $\mathcal{R}(\mathcal{T}) = \{0\}$.

The relationship between $\mathcal{R}(\mathcal{T})$ and $\mathcal{R}_h(\mathcal{T})$ is summarized in the following proposition. To state it, recall that if S is a multiplicatively closed subset of a ring R, then the localization ring R_S of R at S is the quotient $R \times S/\sim$ where $(r_1, s_1) \sim (r_2, s_2)$ if there exists $s \in S$ so that $s(r_1 s_2 - r_2 s_1) = 0$. If $0 \in S$, then $R_S = \{0\}$.

Proposition 5.3. *Let $\mathcal{S} = \{q_1 .... q_m | q_i \in \square\}$ be the multiplicatively closed subset of all monomials in $\square$ in $\mathcal{R}_h(\mathcal{T})$. Then there exist a natural injective ring homomorphism $F : \mathcal{R}(\mathcal{T}) \to \mathcal{R}_h(\mathcal{T})_S$ and a surjective ring homomorphism $G : \mathcal{R}_h(\mathcal{T})_S \to \mathcal{R}(\mathcal{T})$ so that $GF = id$. In particular, $\mathcal{R}(\mathcal{T}) = \{0\}$ if and only if $\mathcal{R}_h(\mathcal{T})_S = \{0\}$.*

Proof. Define a ring homomorphism $F : \mathbf{Z}[\square] \to \mathcal{R}_h(\mathcal{T})_S$ by $F(q) = -q/q'$ where $q \to q'$. We claim that $F(\mathcal{I}) = \{0\}$ and thus F induces a homomorphism, still denoted by F, from $\mathcal{R}(\mathcal{T})$ to $\mathcal{R}_h(\mathcal{T})_S$. The generators of $\mathcal{I}$ are $q'(q-1)-1$ and $W_e - 1$. If $q \to q' \to q''$, then $F(q'(1-q)-1) = -\frac{q'}{q''}(1+\frac{q}{q'}) - 1 = -\frac{q+q'+q''}{q''} = 0$. For an interior edge e, $F(\prod_{q\sim e} q - 1) = \frac{1}{\prod_{q\sim e}(-q')}(\prod_{q\sim e} q - \prod_{q\sim e}(-q')) = 0$. To construct the inverse of F, we define $G : \mathbf{Z}[\square] \to \mathcal{R}(\mathcal{T})$ as follows. For each tetrahedron σ containing $q_1 \to q_2 \to q_3$ where q_1 is specified, define $G(q_1) = q_1, G(q_2) = -1, G(q_3) = 1 - q_1$ in $\mathcal{R}(\mathcal{T})$. By the construction, we have $G(q) = -qG(q')$ for $q \to q'$ in $\square$ and $\sum_{q \subset \sigma} G(q) = 0$ for each tetrahedron σ. We claim that $G(\mathcal{I}_h) = \{0\}$, i.e., G induces a ring homomorphism, still denoted by $G : \mathcal{R}_h(\mathcal{T}) \to \mathcal{R}(\mathcal{T})$. Indeed, we have just verified the first equation associated to each $\sigma \in \mathcal{T}$. For the second type equation, given any interior edge e, due to $G(q) = -qG(q')$, $G(\prod_{q\sim e} q - \prod_{q\sim e}(-q')) = \prod_{q\sim e} G(q) - \prod_{q\sim e}(-G(q')) = [\prod_{q\sim e}(-G(q'))][\prod_{q\sim e} q - 1] = 0$.

Note that by the construction of $\mathcal{R}(\mathcal{T})$, for $q \in \square$, then q and $1-q$ are invertible in $\mathcal{R}(\mathcal{T})$ with inverses $1-q''$ and q' where $q \to q' \to q''$. From the above calculation, we see that G induces a homomorphism from $\mathcal{R}_h(\mathcal{T})_S \to \mathcal{R}(\mathcal{T})$. To check $GF = id$, it suffices to see that $GF(q) = q$ for $q \in \square$. Let $q \to q' \to q''$ where $G(q') = -1$. Then $GF(q) = G(-q/q') = -q/(-1) = q$, $GF(q') = G(-q'/q'') = 1/(1-q) = q'$ and $GF(q'') = G(-q''/q) = (q-1)/q = q''$.

To see the last statement, if $\mathcal{R}_h(\mathcal{T})_S = \{0\}$, then $\mathcal{R}(\mathcal{T}) = \{0\}$ since F is injective. On the other hand, if $\mathcal{R}(\mathcal{T}) = \{0\}$, then we claim that $q = 0$ in $\mathcal{R}_h(\mathcal{T})$ for some $q \in \square$. Indeed, if not, then due to $F(q) = -q/q' \neq 0$, we see that $q \neq 0$ in $\mathcal{R}(\mathcal{T})$. This contradicts the assumption. Therefore the multiplicatively closed set S contains 0. Hence $\mathcal{R}_h(\mathcal{T})_S = \{0\}$.

$\square$

5.2. **Pachner moves.** It is well known that any two triangulations of a closed pseudo 3-manifold are related by a sequence of Pachner moves [14], [15], [11]. There are two types of Pachner moves: $1 \leftrightarrow 4$ move and $2 \leftrightarrow 3$ move. Two more moves of types $0 \leftrightarrow 2_2$ and $0 \leftrightarrow 2_3$ are shown in figure 2. There moves create two new tetrahedra from a triangle and a quadrilateral. The $1 \leftrightarrow 4$ move is a composition of a $0 \leftrightarrow 2_3$ move and a $2 \leftrightarrow 3$ move.

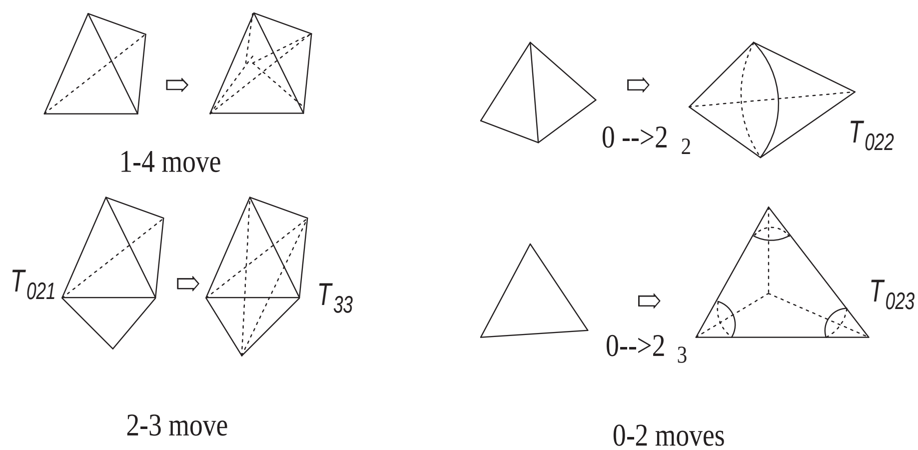

Figure 2. Pachner moves

We will focus on the moves $0 \leftrightarrow 2_i$ $i = 2, 3$ and $2 \leftrightarrow 3$ in the rest of the paper and investigate their effects on the Thurston ring $\mathcal{R}(\mathcal{T})$. For this purpose, we introduce the *directed Pachner moves* $i \to j$ which means the Pachner move change a triangulation of fewer tetrahedra to a triangulation of more tetrahedra, i.e., $0 \to 2_2$, $0 \to 2_3$ and $2 \to 3$. The following problem which improves Pachner's theorem was investigated before. It dual version for special spines was established by Makovetskii [10]. However, we are informed by S. Matveev [12] that the following question is still open.

Problem. Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two triangulations of a closed pseudo 3-manifold M so that $\mathcal{T}_1^{(0)} = \mathcal{T}_2^{(0)}$. Then there exists a third triangulation $\mathcal{T}$ of M so that $\mathcal{T}$ is obtained from both $\mathcal{T}_1$ and $\mathcal{T}_2$ by directed Pachner moves $0 \to 2_2$ and $2 \to 3$.

5.3. **The directed moves** $0 \to 2$ **and** $2 \to 3$. There are four standard triangulations of the 3-ball related to the moves $0 \to 2_2$, $0 \to 2_3$ and $2 \to 3$. Let $\mathcal{T}_{021}, \mathcal{T}_{022}$ and $\mathcal{T}_{023}$ be the standard triangulations of the 3-ball by two tetrahedra σ^+ and σ^-. They are shown in figure 2 where the two tetrahedra in $\mathcal{T}_{02i}$ share i codimension-1 faces. The $2 \to 3$ move replaces $\mathcal{T}_{022}$ by $\mathcal{T}_{33}$. We will calculate the ring $\mathcal{R}(\mathcal{T}_{02i})$ and $\mathcal{R}(\mathcal{T}_{33})$ in this section. The results in subsection are elementary and were known to experts in a less general setting. See [16], [18] and others. We will emphasis the naturality of the associated ring homomorphisms and holonomy preserving properties.

For $\mathcal{T}_{02i}$, take a triangle in $\sigma^+ \cap \sigma^-$ and let its edges be e_1, e_2, e_3 so that $e_3 \to e_2 \to e_1$ in σ^+. Let $q_i^{\pm}$ be the quads in $\sigma^{\pm}$ so that $q_i^{\pm} \sim e_i$. Note that by the construction $q_1^+ \to q_2^+ \to q_3^+$ and $q_3^- \to q_2^- \to q_1^-$.

Lemma 5.4. *For $\mathcal{T}_{021}$, denote q_1^+ and q_1^- by x, y in $\mathcal{R}(\mathcal{T}_{021})$ respectively. Then $\mathcal{R}(\mathcal{T}_{021}) \cong \mathcal{R}(\sigma^+) \otimes \mathcal{R}(\sigma^-)$ and the holonomies W_{e_i} are: $W_{e_1} = xy$, $W_{e_2} = \frac{y-1}{y-xy}$, $W_{e_3} = \frac{x-1}{x-xy}$. The holonomies at all other edges are $x, y, 1/(1-x), 1/(1-y), (x-1)/x, (y-1)/y$. In particular, if $W_{e_1} = 1$, then $W_{e_i} = 1$ for $i = 2, 3$.*

Proof. By definition, $q_2^+ = 1/(1-x), q_3^+ = (x-1)/x$, $q_2^- = (y-1)/y$ and $q_3^- = 1/(1-y)$. Since $W_{e_i} = q_i^+ q_i^-$, the result follows. □

Proposition 5.5. *(1) For the triangulation $\mathcal{T}_{023}$, the inclusion homomorphism $\phi : \mathcal{R}(\sigma^{\pm}) \to \mathcal{R}(\sigma^+) \otimes \mathcal{R}(\sigma^-)$ induces an isomorphism $\Phi : \mathcal{R}(\sigma^{\pm}) \to \mathcal{R}(\mathcal{T}_{023})$. Furthermore, the holonomy W_e of each boundary edge is 1 in $\mathcal{R}(\mathcal{T}_{023})$.*

(2) For the triangulation $\mathcal{T}_{022}$, let $e_0^{\pm} < \sigma^{\pm}$ be the two boundary edges of degree 1 and assume that e_1 is the interior edge. Then the inclusion homomorphism $\phi : \mathcal{R}(\sigma^{\pm}) \to \mathcal{R}(\sigma^+) \otimes \mathcal{R}(\sigma^-)$ induces an isomorphism $\Phi : \mathcal{R}(\sigma^{\pm}) \to \mathcal{R}(\mathcal{T}_{022})$. Furthermore, the holonomies $W_{e_0^{\pm}} = q_1^{\pm}$ and $W_e = 1$ for all other boundary edges e.

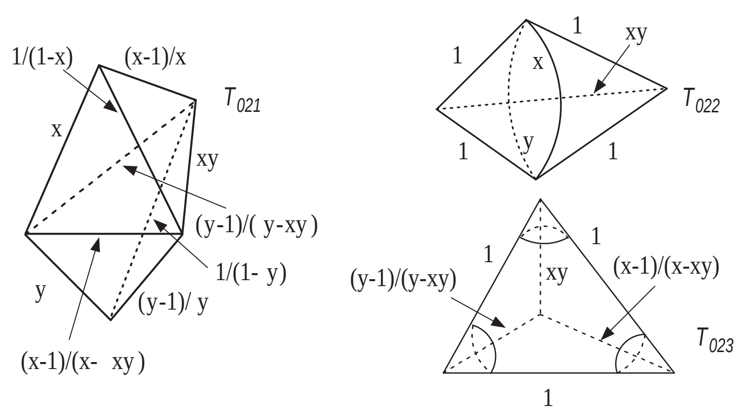

FIGURE 3. Pachner moves and Thurston rings

Proof. To see part (1), let e_1 and e_2 be the interior edges. Since $W_{e_i} = q_i^+ q_i^-$, we have $q_i^- = 1/q_i^+ \in \mathcal{R}(\sigma^+)$ for $i = 1, 2$. By lemma 5.4 and that $q_1^{\pm} q_2^{\pm} q_2^{\pm} = -1$, we have $q_3^- = 1/q_3^+$. It follows that $\Phi : \mathcal{R}(\sigma^+) \to \mathcal{R}(\mathcal{T}_{023})$ is onto. On the other hand, by exactly the same calculation as in lemma 5.4, we see that $W_{e_2} = 1$ and $W_{e_3} = 1$ are consequence of $W_{e_1} = 1$, i.e., the ideal $\mathcal{I}$ in $\mathcal{R}(\sigma^+) \otimes \mathcal{R}(\sigma^-)$ generated by $W_{e_1} - 1$ contains $W_{e_2} - 1$ and $W_{e_3} - 1$. This shows that Φ is injective. Therefore, Φ is a ring isomomorphism. Furthermore, by definition, for each boundary edge e_i^*, the holonomy $W_{e_i^*} = q_i^+ q_i^- = 1$.

To see part (2), let $q^\pm = q_1^\pm$. By definition $\mathcal{R}(\mathcal{T}_{022}) = \mathcal{R}(\sigma^+) \otimes \mathcal{R}(\sigma^-)/(q^+q^- - 1)$. In particular, $q^- = 1/q^+$ and that Φ is an isomorphism. The holonomies $W_e = 1$ follow from the definition and lemma 5.4.

□

Proposition 5.6. *The map $\phi : \square(\mathcal{T}_{021}) \to \mathcal{R}(\mathcal{T}_{33})$ defined by $\phi(\prod_{q \sim e} q) = \prod_{q \sim e} q$ for each degree-1 edge e induces a ring homomorphism $\Phi : \mathcal{R}(\mathcal{T}_{021}) \to \mathcal{R}(\mathcal{T}_{33})$ so that for all edges $e \in \mathcal{T}_{021}$, $\Phi(\prod_{q \sim e} q) = \prod_{q \sim e} q$, i.e., Φ preserves holonomies. Furthermore, let S be the multiplicatively closed set consisting of monomials in $x_i y_i - 1$. Then Φ induces an isomorphism from $\mathcal{R}(\mathcal{T}_{021})_S \to \mathcal{R}(\mathcal{T}_{33})$.*

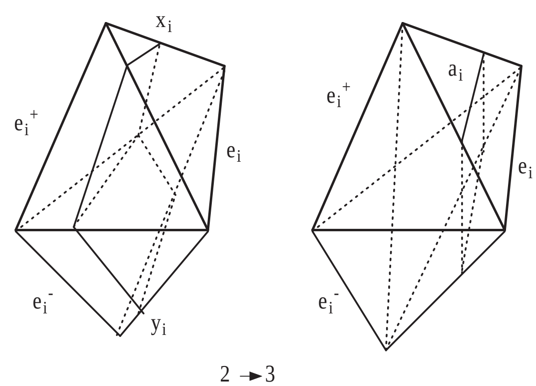

FIGURE 4. 2 → 3 Pachner move

Proof. Let $e_i^\pm$ be the opposite edges of e_i in $\sigma^\pm$ and σ_i be the tetrahedra in $\mathcal{T}_{33}$ so that $e_i < \sigma_i$. The quads in $\sigma^\pm$ are denoted by x_i and y_i so that $x_i \sim e_i$ and $y_i \sim e_i$ respectively. The quads in σ_i facing e_i is denoted by a_i. Let $b_i = a_i'$ and $c_i = b_i'$. Note that by the construction $x_i' = x_{i+1}$ and $y_{i+1}' = y_i$. Furthermore, by definition, $W_{e_i^+}(\mathcal{T}_{021}) = x_i$, $W_{e_i^-}(\mathcal{T}_{021}) = y_i$, $W_{e_i}(\mathcal{T}_{021}) = x_i y_i$, $W_{e_i}(\mathcal{T}_{33}) = a_i$, $W_{e_i^+}(\mathcal{T}_{33}) = b_{i+2}c_{i+1}$, $W_{e_i^-}(\mathcal{T}_{33}) = b_{i+1}c_{i+2}$, and $w_{e_0}(\mathcal{T}_{33}) = a_1 a_2 a_3$ where e_0 is the interior edge in $\mathcal{T}_{33}$. All indices are calculated modulo 3.

By definition, the map $\phi : \square(\mathcal{T}_{021}) \to \mathcal{T}(\mathcal{T}_{33})$ is defined by $\phi(x_i) = b_{i+2}c_{i+1}$, $\phi(y_i) = b_{i+1}c_{i+2}$. To show that ϕ induces a ring homomorphism $\Phi : \mathcal{R}(\mathcal{T}_{021}) \to \mathcal{R}(\mathcal{T}_{33})$, it suffices to show that $\phi(x_{i+1})(1 - \phi(x_i)) = 1$ and $\phi(y_i)(1 - \phi(y_{i+1})) = 1$. Indeed, we have $b_i = 1/(1 - a_i)$, $c_i = (a_i - 1)/a_i$ and $a_i a_{i+1} a_{i+2} = 1$. Thus,

$$\phi(x_{i+1})(1 - \phi(x_i)) = b_i c_{i+2}(1 - b_{i+2}c_{i+1})$$
$$= (\frac{1}{1 - a_i})(\frac{a_{i+2} - 1}{a_{i+2}})(1 - (\frac{1}{1 - a_{i+2}})(\frac{a_{i+1-1}}{a_{i+1}}))$$
$$= (\frac{a_{i+1}a_{i+2}}{a_{i+1}a_{i+2} - 1})(\frac{a_{i+2} - 1}{a_{i+2}})(\frac{1 - a_{i+1}a_{i+2}}{(1 - a_{i+2})a_{i+1}}) = 1$$

The calculation for $\phi(y_i)(1-\phi(y_{i+1})=1$ is very similar and we omit the details.

To establish the identity $\phi(W_e)=W_e$ for the edges $e=e_i$, i.e., we must verify that $\Phi(x_iy_i)=a_i$. Note that $a_ib_ic_i=-1$ and $a_1a_2a_3=1$. Thus $\Phi(x_iy_i)=b_{i+1}b_{i+2}c_{i+1}c_{i+2}=\frac{1}{a_{i+1}a_{i+2}}=a_i$.

To see that Φ induces an isomorphism from $\mathcal{R}(\mathcal{T}_{021})_S$ to $\mathcal{R}(\mathcal{T}_{33})$, consider the map $\psi:\square(\mathcal{T}_{33})\to\mathcal{R}(\mathcal{T}_{021})$ so that $\psi(a_i)=x_iy_i$, $\psi(b_i)=\frac{1}{1-x_iy_i}$ and $\psi(c_i)=\frac{x_iy_i-1}{x_iy_i}$. We claim that ψ induces a ring homomorphism $\Psi:\mathcal{R}(\mathcal{T}_{021})_S\to\mathcal{R}(\mathcal{T}_{33})$ so that for all edges e, $\Psi(\prod_{q\sim e}q)=\prod_{q\sim e}q$ and $\Psi\Phi=id$, $\Phi\Psi=id$.

Indeed, to see that ψ induces a ring homomorphism, we must verify that $\Psi(a_1a_2a_3)=1$. This holds since $\Psi(a_1a_2a_3)=x_1x_2x_3y_2y_2y_3=(-1)(-1)=1$. To check that Ψ preserves the holonomies, it suffices to show $\Psi(W_{e_i^{\pm}})=W_{e_i^{\pm}}$. Since $x_{i+1}=\frac{1}{1-x_i}$ and $y_i=\frac{1}{1-y_{i+1}}$, we have

$$\begin{aligned}\Psi(W_{e_i^+})=\Psi(b_{i+2}c_{i+1})&=(\frac{1}{1-x_{i+2}y_{i+2}})(\frac{x_{i+1}y_{i+1}-1}{x_{i+1}y_{i+1}})\\&=(\frac{1}{1-\frac{y_{i+1}-1}{(1-x_{i+1})y_{i+1}}})(\frac{x_{i+1}y_{i+1}-1}{x_{i+1}y_{i+1}})\\&=\frac{(1-x_{i+1})y_{i+1}(x_{i+1}y_{i+1}-1)}{((y_{i+1}-x_{i+1}y_{i+1})-y_{i+1}+1)x_{i+1}y_{i+1}}\\&=\frac{x_{i+1}-1}{x_{i+1}}=x_i=W_{e_i^+}.\end{aligned}$$

Essentially the same calculation shows $\Psi(W_{e_i^-})=W_{e_i^-}$. Finally, due to holonomy preserving property of $\Phi\Psi$ and $\Psi\Phi$, we have $\Psi\Phi=id$, $\Phi\Psi=id$. $\square$

5.4. **Effects of Pachner moves.** Suppose $(M_i,\mathcal{T}_i)$ $(i=1,2)$ are two compact triangulated oriented pseudo 3-manifolds obtained as the quotients $M_i=X_i/\sim_i$ of disjoint union X_i of tetrahedra. Take $X=X_1\sqcup X_2$ and extend the identifications $\sim_i$ further by identifying pairs of unidentified codimension-1 faces in X by orientation reversing affine homeomorphisms Φ. The quotient $X/\sim=M_1\cup_\Phi M_2$ is called a gluing of M_1 and M_2 along some subsurfaces of ∂M_1 and ∂M_2 by affine homeomorphism Φ. The resulting triangulation will be denoted by $\mathcal{T}_1\cup_\Phi\mathcal{T}_2$. If $M_2=\emptyset$, then Φ is a self-gluing of M_1. We denote the result by $(M_1\cup_\Phi,\mathcal{T}_1\cup_\Phi)$. By definition,

$$\mathcal{R}(\mathcal{T}_1\cup_\Phi\mathcal{T}_2)=(\mathcal{R}(\mathcal{T}_1)\otimes\mathcal{R}(\mathcal{T}_2))/\mathcal{I} \qquad (6)$$

where the ideal $\mathcal{I}$ is generated by elements of the form $W_{e_1}(\mathcal{T}_1)W_{e_2}(\mathcal{T}_2)-1$ with e_1 and e_2 being two boundary edges which are identified to become an interior edge in $\mathcal{T}_1\cup_\Phi\mathcal{T}_2$. Note that there are natural ring homomorphisms induced by the inclusion maps from $\square(\mathcal{T}_i)$ to $\square(\mathcal{T}_1\cup_\Phi\mathcal{T}_2)$.

Using these notations, we can describe the effect of directed Pachner moves $0\to 2_3$, or $0\to 2_2$ and $2\to 3$ on Thurston rings as follows. The moves $0\to 2_3$ and $0\to 2_2$ are of the form of replacing a self-glued $\mathcal{T}\cup_{\Phi_1}$ by $\mathcal{T}\cup_{\Phi_2}\mathcal{T}_{02i}$ for $i=2,3$. The move $2\to 3$ replaces $\mathcal{T}\cup_\Phi\mathcal{T}_{021}$ by $\mathcal{T}\cup_\Phi\mathcal{T}_{33}$.

Combining the definition (6) with the main results in §5.3, we have,

Proposition 5.7. *Suppose $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a directed Pachner move $0\to 2_2$, $0\to 2_3$ or $2\to 3$. Then there exists a holonomy preserving natural ring homomorphism $\mathcal{R}(\mathcal{T})\to\mathcal{R}(\mathcal{T}')$.*

6. Example of solving Thurston equation in finite rings

Suppose $(M, \mathcal{T})$ is a closed oriented pseudo 3-manifold and R is a commutative ring with identity and $x : \square \to R$ solves Thurston equation. If p is a prime number, let F_{p^n} be the finite field of p^n elements.

Example 6.1. For $R = F_3$, then $x : \square \to F_3 - \{0, 1\}$ is the constant map $x(q) = 2$. Thus, as mentioned in §1.1, Thurston equation is solvable if and only if each edge has even degree.

Example 6.2. For $R = F_5 = \{0, 1, 2, 3, 4\}$ and we are looking for $x : \square \to \{2, 3, 4\}$. Due to $1/(1-2) = 4, 1/(1-4) = 3, 4 = 2^2, 3 = 2^3$ so that $2^4 = 1$, we can write $x(q) = 2^{z(q)}$ where $z \in \{1, 2, 3\}$. Thus Thurston equation is solvable if and only if for $q \to q' \to q''$, $(z(q), z(q'), z(q'')) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ so that for each edge e, $\sum_{q \sim e} z(q) = 0 \mod 4$.

Example 6.3. For $R = F_{2^2} = \{0, 1, a, b\}$ where $b = a + 1 = a^2$ and $a^3 = 1$, we have $1/(1-a) = a$ and $1/(1-b) = b$. By writing solution x of Thurston equation as $x(q) = a^{z(q)}$ where $z(q) \in \{1, 2\}$, we see that $z(q) = z(q')$ if $q \to q'$. Therefore, Thurston equation is solvable if and only if there is $z : \mathcal{T}^{(3)} \to \{1, 2\}$ so that for each edge e, $|\{\sigma \in \mathcal{T}^{(3)} | \sigma > e, z(\sigma) = 1\}| + 2|\{\sigma \in \mathcal{T}^{(3)} | \sigma > e, z(\sigma) = 2\}| = 0 \mod 3$.

Example 6.4. For the field F_7, write $x(q) = 3^{z(q)}$. Then Thurston equation is solvable if and only if $z : \square \to \{1, 2, 3, 4, 5\}$ satisfies that $(z(q), z(q'), z(q'')) \in \{(1, 1, 1), (5, 5, 5), (2, 3, 4), (3, 4, 2), (4, 2, 3)\}$ when $q \to q' \to q''$ and for each edge e, $\sum_{q \sim e} z(q) = 0 \mod 6$.

Example 6.5. For the ring $\mathbf{Z}/9\mathbf{Z}$ (not F_{3^2}), since a solution $x(q)$ must satisfy $x(q)$ and $x(q) - 1$ are invertible, we conclude that $x(q) \in \{2, 5, 8\} = \{2, 2^5, 2^3\}$. Write $x(q) = 2^{z(q)}$. Therefore, Thurston equation is solvable if and only if there is $z : \square \to \{1, 3, 5\}$ so that $(z(q), z(q'), z(q'')) \in \{(1, 3, 5), (3, 5, 1), (5, 1, 3)\}$ if $q \to q' \to q''$ and for each edge e, $\sum_{q \sim e} z(q) = 0 \mod 6$. This implies that the degree of each edge must be even.

Example 6.6. For the ring $\mathbf{Z}/15\mathbf{Z}$, the same argument as in example 6.5 shows that Thurston equation is solvable if and only if $x : \square \to \{2, 8, 14\}$ satisfies $(x(q), x(q'), x(q'')) \in \{(2, 14, 8), (14, 8, 2), (8, 2, 14)\}$ if $q \to q' \to q''$ and for each edge e, $\prod_{q \sim e} x(q) = 0 \mod 15$.

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