

3-Dimensional Schlaefli Formula and Its Generalization

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Dedicated to the memory of Xiao-Song Lin

Abstract. Several identities similar to the Schlaefli formula are established for tetrahedra in a space of constant curvature.

Keywords: tetrahedron, dihedral angles, volume, lengths, and the cosine law.

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1 Introduction

One of the most important identities in low-dimensional geometry is the Schlaefli formula. It states that for a tetrahedron in a constant curvature $\lambda = \pm 1$ space, the volume V , the length x_{ij} , and the dihedral angle a_{ij} at the ij -th edge are related by

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$$\frac{\partial V}{\partial a_{ij}} = \frac{\lambda}{2} x_{ij} \quad (1.1)$$

where $V = V(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})$ is a function of the angles. See for instance [5] or [1] for a proof.

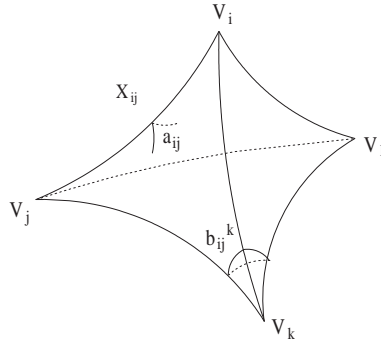


Figure 1.1

In 3-dimensional polyhedral geometry, a space is obtained by isometrically gluing tetrahedra along their codimension-1 faces. The metric is determined by the edge lengths and the curvature at an edge is 2π less the sum of dihedral angles at the edge. From this point of view, the Schlaefli formula relates the most important geometric quantities: the volume, the metric (=edge lengths) and the curvature (= dihedral angles) in a simple elegant identity. The Schlaefli formula plays a vital role in a variational principle for triangulated 3-manifolds. See for instance Regge's work [6] on discrete general relativity.

One consequence of (1.1) is that differential 1-forms

$$\sum x_{ij} da_{ij} \quad \text{and} \quad \sum a_{ij} dx_{ij} \quad (1.2)$$

are closed.

We may recover the volume function V in (1.1) by integrating the 1-form $\sum x_{ij} da_{ij}$. Thus the closeness of the 1-forms in (1.2) essentially captures the Schlaefli formula.

The basic problem in polyhedral geometry is to understand the relationship between the metric and its curvature. In the case of tetrahedra, this prompts us to study the curvature map $K(x) = a$ sending the edge length $x = (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})$ to the dihedral angle $a = (a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})$. The Jacobian matrix $D(K)$ of the curvature map is the 6×6 matrix $[\frac{\partial a_{ij}}{\partial x_{rs}}]_{6 \times 6}$. The closeness of the 1-forms in (1.2) is equivalent to say that the Jacobian matrix $D(K)$ is symmetric. It turns out the Jacobian matrix $[\frac{\partial a_{ij}}{\partial x_{rs}}]_{6 \times 6}$ enjoys many more symmetries. One of the symmetry was discovered by E. Wigner [9] and Taylor-Woodward [8]. The purpose of this paper is to find the com-

plete set of symmetries of the Jacobian matrix $D(K)$ of the curvature map. These symmetries should have applications in 3-dimensional topology and geometry. In particular, the relationships between the Jacobian matrix $D(K)$, the $6j$ symbols, the quantum $6j$ symbols, and the volume conjecture are very attractive problems. See for instance the work of [8] and [7].

The complete set of symmetries was discovered by us a few years ago. We thank Walter Neumann for suggesting us to write it up for publication.

This paper is dedicated to the memory of Xiao-Song Lin who made important contributions to low-dimensional topology. He was a great colleague and friend.

The paper is organized as follows. In §2, we state the main theorem. These theorems are proved in §3. A more general version of it involving complex valued lengths and angles can be found in [4].

2 The main theorem

Let a tetrahedron in \mathbf{S}^3 or \mathbf{H}^3 or \mathbf{E}^3 have vertices v_1, v_2, v_3, v_4 . Let a_{ij} and x_{ij} be the dihedral angle and the edge length at the ij -th edge $v_i v_j$. We consider the angle a_{ij} as a function of the lengths $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$.

Here is our main result.

Theorem 2.1. *Define $P_{rs}^{ij} = \frac{1}{\sin(a_{ij})\sin(a_{rs})} \frac{\partial a_{ij}}{\partial x_{rs}}$. Then these quantities satisfy the following identities for any tetrahedron in any of the three geometries $\mathbf{S}^3, \mathbf{H}^3, \mathbf{E}^3$. The indices i, j, k, l are assumed to be pairwise distinct.*

- (i) (Schlaefli) $P_{rs}^{ij} = P_{ij}^{rs}$.
- (ii) ([9], [8]) $P_{kl}^{ij} = P_{jl}^{ik} = P_{jk}^{il}$.
- (iii) $P_{ik}^{ij} = -P_{kl}^{ij} \cos a_{jk}$.
- (iv) $P_{ij}^{ij} = P_{kl}^{ij} w_{ij}$ where $w_{ij} = (c_{ij}c_{jk}c_{ki} + c_{ij}c_{jl}c_{li} + c_{ik}c_{jl} + c_{il}c_{jk}) / \sin^2(a_{ij})$ and $c_{rs} = \cos(a_{rs})$.
- (v) $P_{rs}^{ij} = P_{r's'}^{i'j'}$ where $\{i, j\} \neq \{r, s\}$ and for a subset $\{a, b\} \subset \{i, j, k, l\}$, the set $\{a', b'\}$ is $\{i, j, k, l\} - \{a, b\}$.

In the spaces \mathbf{S}^3 and \mathbf{H}^3 of constant curvature $\lambda = \pm 1$, a tetrahedron is determined by its dihedral angles a_{ij} . Thus the length x_{ij} can be considered as a function of the angles. The similar theorem is,

Theorem 2.2. *Define $R_{rs}^{ij} = \frac{1}{\sin(\sqrt{\lambda}x_{ij})\sin(\sqrt{\lambda}x_{rs})} \frac{\partial x_{ij}}{\partial a_{rs}}$. Then these quantities satisfy the following identities for any tetrahedron in spherical and hyperbolic geometries. Let the indices i, j, k, l be distinct.*

- (i) (Schlaefli) $R_{rs}^{ij} = R_{ij}^{rs}$.

$$(ii) ([9], [8]) R_{kl}^{ij} = R_{jl}^{ik} = R_{jk}^{il}.$$

$$(iii) R_{ik}^{ij} = R_{kl}^{ij} \cos(\sqrt{\lambda}x_{il}).$$

$$(iv) R_{ij}^{ij} = R_{kl}^{ij} w_{ij} \text{ where}$$

$$w_{ij} = \frac{-c_{ij}c_{ik}c_{il} - c_{ji}c_{jk}c_{jl} + c_{ik}c_{jl} + c_{il}c_{jk}}{\sin^2(\sqrt{\lambda}x_{ij})},$$

and $c_{rs} = \cos(\sqrt{\lambda}x_{rs})$,

(v) $R_{rs}^{ij} = R_{r's'}^{i'j'}$ where $\{i, j\} \neq \{r, s\}$ and for a subset $\{a, b\} \subset \{i, j, k, l\}$, the set $\{a', b'\}$ is $\{i, j, k, l\} - \{a, b\}$.

We remark that the matrices $[\frac{\partial a_{ij}}{\partial x_{rs}}]$ and $[\frac{\partial x_{ij}}{\partial a_{rs}}]$ are inverse of each other when $\lambda \neq 0$. Theorem 2.2 follows from theorem 2.1 by taking the dual. Indeed, in the spherical tetrahedral case, the dual tetrahedron has dihedral angle $\pi - x_{ij}$ and edge length $\pi - a_{ij}$ at the kl -th edge of the dual simplex. Thus, theorem 2.2 follows. The hyperbolic tetrahedra case in theorem 2.2 can be deduced from spherical case by analytical continuation.

Theorem 2.1 suggests that the matrix $MD(K)M = [P_{rs}^{ij}]_{6 \times 6}$ where M is the diagonal matrix whose diagonal entries are $\frac{1}{\sin(a_{ij})}$ exhibits more symmetries than the Jacobian matrix $D(K)$.

Both theorems are special cases of a complex valued edge-length and dihedral angle relation. This will be discussed in [4].

3 A proof of theorem 2.1

We need to recall the cosine law and its derivative form in order to compute the Jacobian matrix $[\frac{\partial a_{ij}}{\partial x_{rs}}]$ effectively.

Let $K^2 = \mathbf{S}^2$, or \mathbf{H}^2 or \mathbf{E}^2 be the space of constant curvature $\lambda = 1, -1$, or 0 . Define a function $S_\lambda(t)$ as follows. $S_0(t) = t$; $S_1(t) = \sin(t)$ and $S_{-1}(t) = \sinh(t)$. The sine law for a triangle of lengths l_1, l_2, l_3 and opposite angles a_1, a_2, a_3 in K^2 can be stated as

$$\frac{S_\lambda(l_i)}{\sin(a_i)} = \frac{S_\lambda(l_j)}{\sin(a_j)} \quad (3.1)$$

A different way to state the sine law is that the expression

$$A_{ijk} = \sin(a_i)S_\lambda(l_j)S_\lambda(l_k)$$

is symmetric in indices i, j, k where $\{i, j, k\} = \{1, 2, 3\}$. For this reason, we call $A_{ijk} = A_{123}$ the A -invariant of the triangle.

Proposition 3.1. ([2], [3]) *Let a triangle in K^2 have inner angles a_1, a_2, a_3 and edge lengths l_1, l_2, l_3 so that l_i -th edge is opposite to the angle a_i . Then*

- (i) $\frac{\partial a_i}{\partial l_j} = -\frac{\partial a_i}{\partial l_i} \cos(a_k)$ where $\{i, j, k\} = \{1, 2, 3\}$,
(ii) $\frac{\partial a_i}{\partial l_i} = \frac{S_\lambda(l_i)}{A_{123}}$

See [2] or [3] for a proof.

Let us introduce some notations before beginning the proof. The indices i, j, k, l are pairwise distinct, i.e., $\{i, j, k, l\} = \{1, 2, 3, 4\}$. The face triangle $\Delta v_i v_j v_k$ will be denoted Δijk . The inner angle at the vertex v_k of the triangle Δijk is denoted by b_{ij}^k . The link at the vertex v_k , denoted by $Lk(v_k)$, is a spherical triangle with edge lengths $b_{ji}^k, b_{il}^k, b_{lj}^k$ and inner angles a_{ki}, a_{kj}, a_{kl} so that a_{ki} is opposite to b_{jl}^k . The A-invariant of the triangle Δijk is denoted by A_{ijk} .

In the calculation below, we consider b_{jk}^i as a function of x_{rs} 's using the cosine law for the triangle Δijk . By the definition, we have,

$$\frac{\partial b_{jk}^i}{\partial x_{rs}} = 0 \quad (3.2)$$

if $\{r, s\}$ is not a subset of $\{i, j, k\}$. The function a_{ij} is considered as a function of b_{st}^r 's by the cosine law applied to either the link $Lk(v_i)$ or $Lk(v_j)$. In this way the dihedral angle a_{ij} , when considered as a function of the lengths x_{rs} 's, is a composition function.

To prove theorem 2.1, note that identity (i) in theorem 2.1 is the Schlaefli formula (1.2). Identity (v) follows from identity (iii). By symmetry, we only need to consider three partial derivatives: $\frac{\partial a_{ij}}{\partial x_{kl}}, \frac{\partial a_{ij}}{\partial x_{ik}}$ and $\frac{\partial a_{ij}}{\partial x_{ij}}$.

3.1 The partial derivatives $\frac{\partial a_{ij}}{\partial x_{ik}}$ and $\frac{\partial a_{ij}}{\partial x_{kl}}$

Consider the link $Lk(v_i)$. Using proposition 3.1(ii), the chain rule and (3.2), we have (see Fig. 2.1(a)),

$$\frac{\partial a_{ij}}{\partial x_{kl}} = \frac{\partial a_{ij}}{\partial b_{kl}^i} \frac{\partial b_{kl}^i}{\partial x_{kl}} = \frac{\partial a_{ij}}{\partial b_{kl}^i} \frac{S_\lambda(x_{kl})}{A_{ikl}}. \quad (3.3)$$

Similarly, using $Lk(v_j)$, we have

$$\frac{\partial a_{ij}}{\partial x_{kl}} = \frac{\partial a_{ij}}{\partial b_{kl}^j} \frac{\partial b_{kl}^j}{\partial x_{kl}} = \frac{\partial a_{ij}}{\partial b_{kl}^j} \frac{S_\lambda(x_{kl})}{A_{jkl}}. \quad (3.4)$$

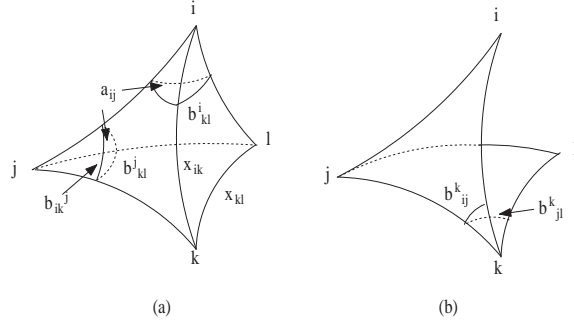


Figure 2.1

Now we use the link $\text{Lk}(v_j)$ to find $\frac{\partial a_{ij}}{\partial x_{ik}}$. By (3.2) and the chain rule, we have

$$\frac{\partial a_{ij}}{\partial x_{ik}} = \frac{\partial a_{ij}}{\partial b_{ik}^j} \frac{\partial b_{ik}^j}{\partial x_{ik}}. \quad (3.5)$$

By proposition 3.1 applied to $\text{Lk}(v_j)$ and Δijk , we see (3.5) is equal to,

$$-\frac{\partial a_{ij}}{\partial b_{kl}^j} \cos(a_{jk}) \frac{S_\lambda(x_{ik})}{A_{ijk}}. \quad (3.6)$$

Using (3.4), we can write (3.6) as,

$$-\frac{\partial a_{ij}}{\partial x_{kl}} \cos(a_{jk}) \frac{A_{jkl} S_\lambda(x_{ik})}{A_{ijk} S_\lambda(x_{kl})}. \quad (3.7)$$

Now by the definition of the A-invariant of triangles Δijk and Δjkl (see Fig. 2.1(b)), we have,

$$A_{jkl} = S_\lambda(x_{jk}) S_\lambda(x_{kl}) \sin(b_{jl}^k) \quad \text{and} \quad A_{ijk} = S_\lambda(x_{jk}) S_\lambda(x_{ik}) \sin(b_{ij}^k). \quad (3.8)$$

Thus (3.7) can be simplified to

$$-\frac{\partial a_{ij}}{\partial x_{kl}} \cos(a_{jk}) \frac{\sin(b_{jl}^k)}{\sin(b_{ij}^k)}. \quad (3.9)$$

By the sine law applied to the spherical triangle $\text{Lk}(v_k)$, we see (3.9) is equal to

$$\frac{\partial a_{ij}}{\partial x_{ik}} = -\frac{\partial a_{ij}}{\partial x_{kl}} \cos(a_{jk}) \frac{\sin(a_{ik})}{\sin(a_{kl})}. \quad (3.10)$$

This is equivalent to identity (iii),

$$P_{ik}^{ij} = -P_{kl}^{ij} \cos a_{jk}. \quad (3.11)$$

Use the Schlaefli formula that $P_{ik}^{ij} = P_{ij}^{ik}$, we obtain from (3.11)

$$-P_{kl}^{ij} \cos(a_{jk}) = -P_{jl}^{ik} \cos(a_{jk}).$$

This shows that

$$P_{kl}^{ij} = P_{jl}^{ik}. \quad (3.12)$$

By symmetry, identity (ii) holds for all indices.

3.2 The partial derivative $\frac{\partial a_{ij}}{\partial x_{ij}}$

By (3.2), the chain rule, we have, in the triangle $\text{Lk}(v_i)$,

$$\frac{\partial a_{ij}}{\partial x_{ij}} = \frac{\partial a_{ij}}{\partial b_{jk}^i} \frac{\partial b_{jk}^i}{\partial x_{ij}} + \frac{\partial a_{ij}}{\partial b_{jl}^i} \frac{\partial b_{jl}^i}{\partial x_{ij}}. \quad (3.13)$$

Using proposition 3.1, we see that (3.13) is equal to

$$\frac{\partial a_{ij}}{\partial b_{kl}^i} \cos(a_{ik}) \cos(b_{ik}^j) \frac{S_\lambda(x_{jk})}{A_{ijk}} + \frac{\partial a_{ij}}{\partial b_{kl}^i} \cos(a_{il}) \cos(b_{il}^j) \frac{S_\lambda(x_{jl})}{A_{ijl}}. \quad (3.14)$$

Using (3.3), we see (3.14) is equal to

$$\frac{\partial a_{ij}}{\partial x_{kl}} [\cos(a_{ik}) \cos(b_{ik}^j) \frac{S_\lambda(x_{jk}) A_{ikl}}{S_\lambda(x_{kl}) A_{ijk}} + \cos(a_{il}) \cos(b_{il}^j) \frac{S_\lambda(x_{jl}) A_{ikl}}{S_\lambda(x_{kl}) A_{ijl}}]. \quad (3.15)$$

Using the sine law for triangles Δikl and $\Delta ij l$ as in (3.8), we can rewrite (3.15) as

$$\frac{\partial a_{ij}}{\partial x_{kl}} [\cos(a_{ik}) \cos(b_{ik}^j) \frac{\sin(b_{il}^k)}{\sin(b_{ij}^k)} + \cos(a_{il}) \cos(b_{il}^j) \frac{\sin(b_{ik}^l)}{\sin(b_{ij}^l)}]. \quad (3.16)$$

Using the sine law in triangles $\text{Lk}(v_k)$ and $\text{Lk}(v_l)$, we see that (3.16) is the same as

$$\frac{\partial a_{ij}}{\partial x_{kl}} [\cos(a_{ik}) \cos(b_{ik}^j) \frac{\sin(a_{kj})}{\sin(a_{kl})} + \cos(a_{il}) \cos(b_{il}^j) \frac{\sin(a_{lj})}{\sin(a_{lk})}]. \quad (3.17)$$

$$= P_{kl}^{ij} [\cos(a_{ik}) \cos(b_{ik}^j) \sin(a_{kj}) \sin(a_{ij}) + \cos(a_{il}) \cos(b_{il}^j) \sin(a_{lj}) \sin(a_{ij})]. \quad (3.18)$$

On the other hand, by the cosine law for the spherical triangle $\text{Lk}(v_j)$, we have

$$\cos(b_{ik}^j) \sin(a_{kj}) \sin(a_{ij}) = \cos a_{kj} \cos a_{ij} + \cos a_{lj}.$$

and

$$\cos(b_{il}^j) \sin(a_{lj}) \sin(a_{ij}) = \cos a_{jl} \cos a_{ij} + \cos a_{jk}.$$

Substitute these into (3.18), we obtain

$$\frac{\partial a_{ij}}{\partial x_{ij}} = P_{kl}^{ij} [c_{ij}c_{jk}c_{ki} + c_{ij}c_{jl}c_{li} + c_{ik}c_{jl} + c_{il}c_{jk}]$$

where $c_{rs} = \cos(a_{rs})$.

This is the identity (iv) since $P_{ij}^{ij} = \frac{1}{\sin^2(a_{ij})} \frac{\partial a_{ij}}{\partial x_{ij}}$.

References

- [1] Alekseevskij, D. V.; Vinberg, . B.; Solodovnikov, A. S. Geometry of spaces of constant curvature. *Geometry, II*, 1–138, *Encyclopaedia Math. Sci.*, 29, Springer, Berlin, 1993.
- [2] Chow Bennett; Luo Feng: Combinatorial ricci flows on surfaces. *Journal Differential Geometry*, 63 (2003), no.1, 197–129.
- [3] Luo, Feng: Rigidity of polyhedral surfaces, arXiv:math.GT/0612714.
- [4] Luo, Feng: Volume and angle structures on 3-manifolds, in preparation.
- [5] Milnor, J. The Schlaefli differential equality, *John Milnor Collected papers*. Vol. 1. *Geometry*. Publish or Perish, Inc., Houston, TX, 1994.
- [6] Regge, T. General relativity without coordinates. *Nuovo Cimento* (10) 19 1961 558–571.
- [7] Roberts, Justin. Classical $6j$ -symbols and the tetrahedron. *Geom. Topol.* 3 (1999), 21–66 (electronic).
- [8] Taylor, Yuka U.; Woodward, Christopher T. $6j$ symbols for $U_q(sl_2)$ and non-Euclidean tetrahedra. *Selecta Math. (N.S.)* 11 (2005), no. 3–4, 539–571.
- [9] Wigner, Eugene P. *Group theory: And its application to the quantum mechanics of atomic spectra*. Pure and Applied Physics. Vol. 5 Academic Press, New York-London 1959 xi+372 pp.