

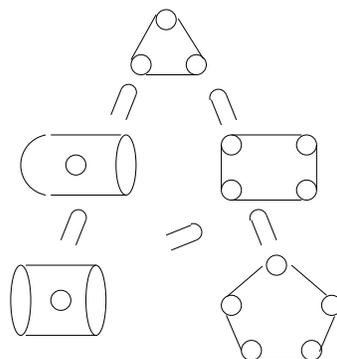
Grothendieck's Reconstruction Principle and 2-dimensional Topology and Geometry

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§0. Introduction

The goal of this paper is an attempt to relate some ideas of Grothendieck in his *Esquisse d'un programme* [Gr1] and some of the recent results on 2-dimensional topology and geometry. Especially, we shall discuss the Teichmüller theory, the mapping class groups, $SL(2, \mathbf{C})$ representation variety of surface groups, and Thurston's theory of measured laminations.

A prominent idea in surface theory is that to study a surface, one should consider all subsurfaces inside it. Indeed, there is a *hierarchy* of compact oriented surfaces of negative Euler number under (essential) inclusion. Each surface in the hierarchy is indexed by its *level* which is the number of disjoint simple loops needed to decompose it into 3-holed spheres (i.e., the complex dimension of the Teichmüller space of complete hyperbolic metrics of finite area). The first three levels in the hierarchy are listed as follows. The level-0 surface is the 3-holed sphere, the level-1 surfaces are 1-holed torus and 4-holed sphere, and the level-2 surfaces are 2-holed torus and 5-holed sphere.



The first three levels of the hierarchy of surfaces

Figure 1

One of the key ideas in [Gr1] which we would like to discuss at length in this paper is Grothendieck's reconstruction principle for the "Teichmüller tower". We quote the relevant paragraph in [Gr1] below. On page 11, line 2-6, Grothendieck wrote (with English translation by L. Schneps) "The a priori interest of a complete knowledge of the first two levels of the tower is to be found in the principle that the entire tower can be reconstructed from these two first levels, in the sense that via the fundamental operation of 'gluing', level-1 gives the complete system of generators, and level-2 a complete system of relations." One may interpret this principle broadly as follows. To study a structure (for instance, hyperbolic structure, complex projective structure, measured lamination, or linear representation of the surface group) and its moduli space on a surface, one should consider the restrictions of the structure to the level-1 subsurfaces and reconstruct the structure from its restrictions. The level-2 surfaces should serve as "relators" in the reconstruction process. For example, one may ask if the reconstruction principle holds

for the characters of representations of the surface groups into general linear group $GL(n, \mathbf{C})$. Namely, suppose f is a complex valued function defined on the fundamental group of the surface so that the restriction of f to the fundamental group of each essential level-1 subsurface is a $GL(n, \mathbf{C})$ -character. Is f the character of some $GL(n, \mathbf{C})$ representation of the surface group? In [Lu3], we show that the answer is affirmative for $SL(2, \mathbf{C})$ representations of the surface groups. It is interesting to note that this principle of reconstruction was taken as one of the basic axioms by physicists in conformal field theory ([MS]).

The main theorems in [Lu1], [Lu2] state that the Teichmüller space and Thurston's measured lamination space for surfaces obey the reconstruction principle. Also using the work of Gervais [Ge], we see that the mapping class group of a surface fits the principle as well [Lu4]. These will be the main topics of this paper. We shall also discuss some open questions arising from reading of [Gr1].

We remark that as far as we know, there is no precise definition of the Teichmüller tower in [Gr1]. See also the books [Gr2], and [Gr3]. What follows is my interpretation of Grothendieck's reconstruction principle and there should be other ways of interpreting it (for instance in algebraic geometry).

One way to illustrate Grothendieck's reconstruction principle is to consider convex planar n -sided polygons. According to the principle, to construct a convex n -sided ($n \geq 5$) polygon, one should consider all convex quadrilaterals inside the polygon (the vertex of the quadrilateral is a vertex of the polygon). The convex polygon is a union of these quadrilaterals by gluing along their overlaps. Now these quadrilaterals overlap in two different ways. An *essential overlap* of two quadrilaterals contains an edge or diagonal. Otherwise, they overlap inessentially (see figure 2). The reconstruction principle states that it suffices to glue quadrilaterals along the essential overlaps. The gluing along the inessential overlaps is a *consequence* of the gluing along essential overlaps (see §1 for more details). As a consequence, to study the geometry of the moduli space of convex polygons, it suffices to understand that of quadrilaterals.

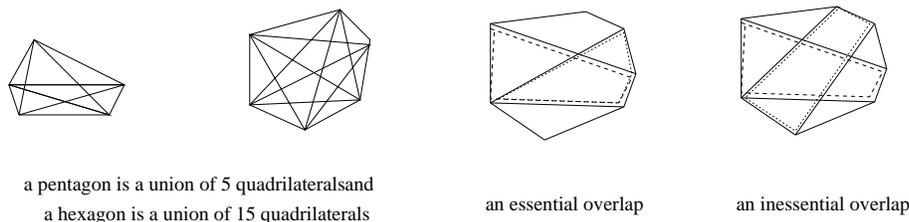


Figure 2

The situation for surfaces is analogous to that of polygons where the 3-holed sphere corresponds to the triangle and 1-holed torus and 4-holed sphere correspond to quadrilaterals. (One should think of the polygons forming an hierarchy under inclusion. And the level of a polygon is the number of disjoint diagonals needed to decompose a polygon into triangles.) Thus according to the reconstruction principle, to construct a hyperbolic metric on a surface of negative Euler number, one should consider all (isotopy classes of) subsurfaces which are homeomorphic to

the 1-holed torus or the 4-holed sphere. These subsurfaces overlap in two different patterns: the overlap is *essential* if there is a homotopically non-trivial loop in the overlap, otherwise it is inessential (see figure 3). The reconstruction principle says that we can glue along essential overlaps to recover the original hyperbolic structure. To be more precise, assign to each level-1 subsurface a hyperbolic structure so that when two level-1 subsurfaces overlap essentially, they overlap geometrically (i.e., the geodesic lengths of all overlapping simple loops are the same in both level-1 surfaces). Then the reconstruction principle states that there exists a hyperbolic metric on the surface whose restrictions to level-1 subsurfaces are (isotopic to) the assigned hyperbolic structures. This is the main result established in [Lu1].

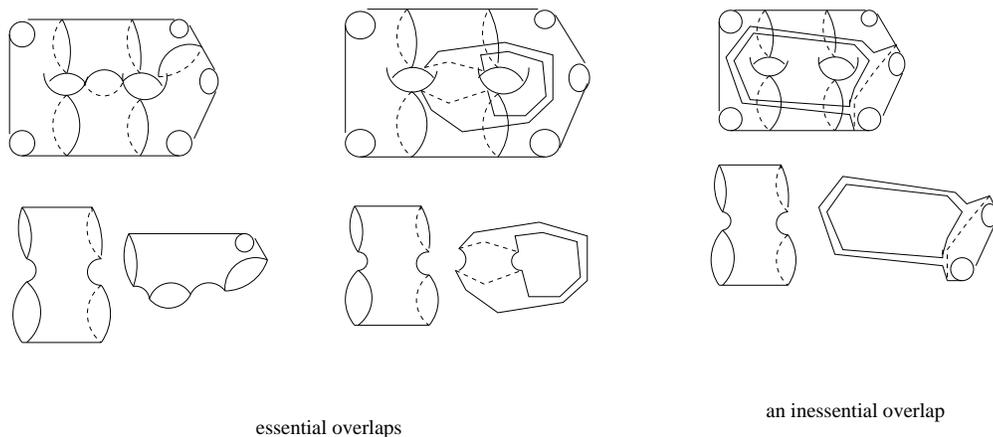


Figure 3

The organization of the paper is as follows. In §1, we study the moduli space of convex polygons in details and use it to illustrate Grothendieck’s principle. We also discuss the ideal triangulations of surfaces. In §2, we recall the Teichmüller spaces, the mapping class groups, and Thurston’s projective measured lamination spaces. It is well known that these three themes represent the geometric, algebraic, and topological aspects of surface theory. In the case of the torus, these three themes correspond to the upper-half space \mathbf{H} , $SL(2, \mathbf{Z})$, and $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ which appears as the boundary of \mathbf{H} . The natural action of the mapping class group corresponds to the action of $SL(2, \mathbf{Z})$ on \mathbf{H} and \mathbf{R} by Möbius transformations. We shall also recall the related topics for level-1 surfaces. In §3, we state the reconstruction theorems for the Teichmüller spaces, the measured lamination spaces and the mapping class groups. In §4, we discuss the key ingredient in the proofs of the reconstruction theorem, namely simple loops on surfaces. We also recall the notion of $SL(2, \mathbf{Z})$ modular structures on a set. The role of the modular structure, or the same, $(\mathbf{Q}P^1, SL(2, \mathbf{Z}))$ is prominent in the reconstruction program as predicted by Grothendieck (see page 248-249 in [Gr1] or §3.8). Topologists have been knowing the role of modular configuration for simple loops on level-1 surfaces since the fundamental work of Max Dehn [De] in 1938. Dehn actually used the structure to give an elegant derivation of the mapping class group of the 4-holed sphere (see §4.1). The special feature of the modular configuration is the huge symmetry built in the configuration. This is, in our view, one reason

why the set of homotopy class of simple loops on the surface is more useful than the fundamental group in establishing the reconstruction principle for many structures (see §3.7, figure 8, and §6.1). In §5 we give a fairly general reason which indicates the special role played by level-2 surfaces in the reconstruction principle. In the last section, we discuss the characters of $SL(2, \mathbf{C})$ representations.

This is not a general survey of recent advances on surface theory. And we have omitted the important contributions of Masur and Minsky and others on 2-dimensional topology and geometry because of our lack of expertise in the areas.

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§1. A Simple Example of Convex Polygons

1.1. We shall illustrate the reconstruction principle and its application by considering the configuration space of convex n -sided polygons. Let us begin with the following problem.

Problem 1. Describe the space $T(n)$ of all convex n -sided polygons up to isometries. Here polygons have marked vertices and isometries preserve markings.

To be more precise, let us distinguish the topological (or combinatorical) and geometric aspects of the problem. By an n -sided polygon we mean a topological disk with n marked points (the vertices) on its boundary. A *convex structure* on a polygon is a metric on the polygon which is isometric to a convex n -sided planar polygon so that the marked points correspond to the vertices. An *edge* in a convex polygon is a line segment joining two vertices, and a *diagonal* is an edge which does not joint adjacent vertices. Given an n -sided polygon P , the space of all convex structures on P modulo isometries preserving the vertices is denoted by $T(P)$ which is essentially $T(n)$.

Having introduced these notations, we may rephrase the problem as follows.

Problem 2. Assign to each edge in a convex n -sided polygon a positive number. When is the assignment corresponds to the edge lengths of an n -sided convex polygon?

The solution for triangles $n = 3$ is well known. Namely, the assignment must satisfy the triangular inequalities that sum of two is larger than the third. For general n , the assignment must satisfy triangular inequalities over three edges forming a triangle and equations over six edges forming a quadrilateral. Grothendieck's reconstruction principle asserts that these are the set of all constrains, i.e., the quadrilaterals (= level-1 polygon) are the "generators" in building convex polygons.

1.2. It is instructive to compare the hierarchies of polygons and surfaces. One first observes that the isometry class of a convex polygon is determined by the lengths of all edges. The corresponding fact in hyperbolic geometry is a result of Fricke-Klein [FK] that the isometry class of a hyperbolic metric on a surface is determined by the lengths of simple geodesic loops. The solution for $T(3)$ is given by $T(3) = \{(a_1, a_2, a_3) \in \mathbf{R}^3 | a_i + a_j > a_k\}$ reflecting the fact

that a triangle is determined up to isometry by its three edge lengths subject to the triangular inequalities. The corresponding fact in hyperbolic geometry is the well known theorem of Fricke-Klein [FK] that a hyperbolic metric on a 3-holed sphere is determined up to isometry by the three lengths of the boundary geodesics and these lengths subject no constraints. For $n \geq 4$, an old way of solving the problem for $T(n)$ is to triangulate the n -sided polygon by $(n-3)$ edges, i.e., one uses the triangle as the basic building block. This corresponds to the Fenchel-Nielsen's decomposition of surfaces into 3-holed spheres (using 3-holed sphere as the basic building block). In this way, one parametrizes the convex polygon by the lengths of the edges in the triangulation. These lengths have to satisfy complicated inequalities due to the convexity. Unlike the Fenchel-Nielsen coordinate for Teichmüller space which can be used to express the Weil-Petersson symplectic form by Wolpert's formula ([Wo]), the length coordinate for convex polygons seems to be less useful in extracting geometric information about $T(n)$ except that it can be used to show that $T(n)$ is a real analytic manifold diffeomorphic to \mathbf{R}^{2n-3} .

1.3. Now Grothendieck's reconstruction principle asserts that quadrilateral be the basic building block. Thus given an n -sided polygon P , one considers all quadrilateral whose edges are edges in P . If the polygon has a convex structure, each quadrilateral in P becomes a convex quadrilateral. These convex quadrilaterals satisfy the obvious consistency condition:

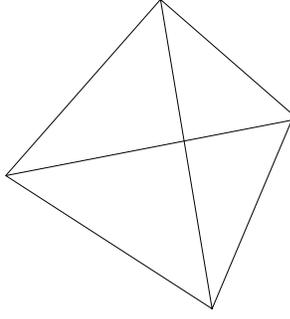
(*) If two quadrilaterals overlap essentially (i.e., there is an edge in the overlap), then the convex quadrilaterals overlap geometrically, i.e., the corresponding lengths of edges in both convex quadrilaterals are the same.

It turns out that the condition (*) is also sufficient to recover the convex polygon for obvious reason.

Reconstruction Principle for Polygons. *To construct a convex n -sided polygon with $n \geq 5$, it suffices to assign to each quadrilateral in the polygon a convex structure so that the assignment satisfies the consistency condition (*).*

1.4. In terms of the reconstruction principle, the solution to problem 2 is simply that the assignment must be realized by a convex quadrilateral for each choice of six edges forming a quadrilateral.

This principle essentially reduces the study of $T(n)$ to that of $T(4)$. To understand $T(4)$, one uses the lengths of the six edges of a quadrilateral. First of all, the lengths satisfy the triangular inequalities that sum of two is larger than the third over each of the four triangles in the quadrilateral. By a simple calculation, one shows that these six lengths satisfy the following constraint:



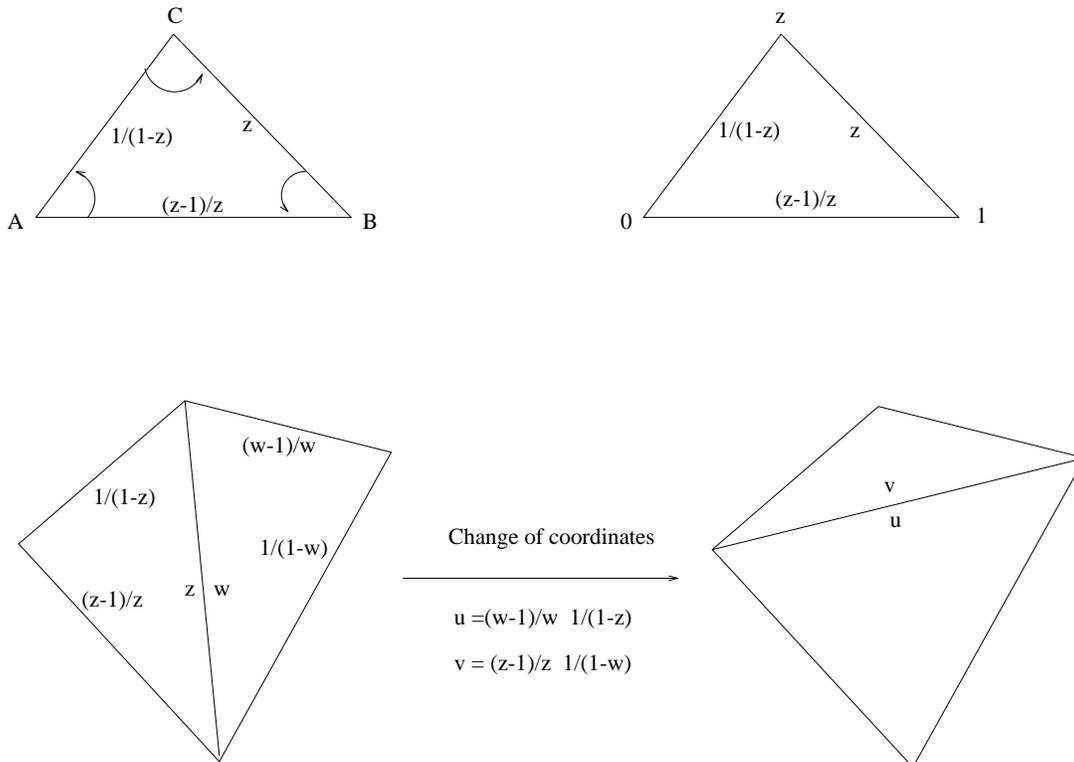
$$(1) \quad \sum_{x \text{ --- } y} (x^2y + xy^2) + \sum_{\begin{array}{c} x \text{ --- } z \\ \diagdown \quad \diagup \\ y \end{array}} xyz = \sum_{\begin{array}{c} x \text{ --- } z \\ \diagdown \quad \diagup \\ y \end{array}} xyz$$

Figure 4

where the sums are over all specified subgraphs (of the complete graph on 4 vertices) whose edges are labeled by their lengths. The convexity condition is equivalent to the *Largest Root Condition* that the lengths of the diagonals are the largest root. (Fix five lengths and think of (1) as a quadratic equation in the length of the remaining diagonal. It has two real roots and the convexity condition says that the largest root is the length.) As a consequence of (1), the Largest Root Condition, the triangular inequalities, and the reconstruction principle, one obtains a complete solution to problem 2.

1.5. Given a convex polygon P , the observable invariants of P seem to be the area of the polygon, the lengths of edges and the angles of intersections of edges. These define the “observable” area, length and angle functions on the configuration space $T(P)$. To be more precise, fix an isotopy class of an edge e in P (resp. a pair of isotopy classes of intersecting edges), one defines a length function (resp. angle function) from $T(P)$ to \mathbf{R} by sending a convex structure to the length (resp. angle) of e in the convex structure. These naturally defined functions seem to play an important role in the geometry of the configuration space $T(P)$. And indeed they are. Here is one way to see it using Thurston’s invariant of oriented triangles. Suppose $\triangle ABC$ is an oriented triangle in the plane so that the cyclic order (A, B, C) is the right-hand orientation in the plane (for simplicity, we shall assume triangles are right-hand oriented in the plane unless stated otherwise). Then the *Thurston invariant* z_A of the triangle at edge BC is defined to be the complex number $\frac{C-A}{B-A}$ in the upper-half plane \mathbf{H} . The edge invariants $z_B = \frac{A-B}{C-B}$ and $z_C = \frac{B-C}{A-C}$ for AC and AB are given by $z_B = 1/(1 - z_A)$ and $z_C = 1/(1 - z_B) = (z_A - 1)/z_A$ respectively. In particular $z_A z_B z_C = -1$. Evidently if two oriented triangles differ by a similarity transformation ($f(w) = aw + b$, $a, b \in \mathbf{C}, a \neq 0$), then their Thurston invariants are the same. For a convex quadrilateral with a marked diagonal, one defines the Thurston invariant to be the pair $(z, w) \in \mathbf{H} \times \mathbf{H}$ where each coordinate is the invariant of a triangle at the marked diagonal. For instance, in terms of Thurston invariants, parallelograms are exactly those convex quadrilaterals with Thurston invariant (z, z) . A simple

calculation shows that if the marked diagonal is changed, then the invariant becomes (u, v) where $u = \frac{w-1}{w(1-z)}$ and $v = \frac{z-1}{(1-w)z}$ as shown in figure 5.



Right -hand orientation in the plane

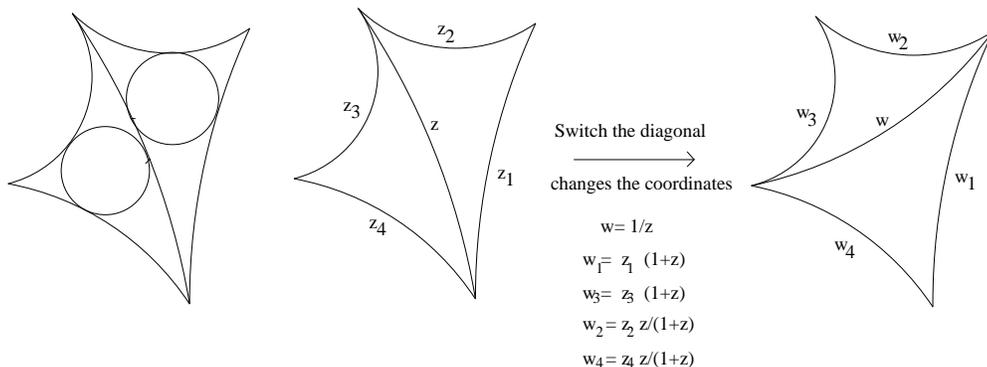
Figure 5

Furthermore the convexity condition is equivalent to both (z, w) and $(u, v) \in \mathbf{H} \times \mathbf{H}$. As a consequence of the transformation formula, one sees that the space of similarity classes of convex quadrilaterals has a natural complex structure. Combining these with the reconstruction principle, one obtains the fact that the projectivized space $T(n)/\mathbf{R}_+$ of similarity classes of convex polygons has a natural complex structure so that angle functions and the logarithm of the ratio of the length functions are pluriharmonic. Furthermore, the space $T(n)/\mathbf{R}_+$ can be explicitly described. This result itself is not surprising since another way of parametrizing $T(n)/\mathbf{R}_+$ is by taking vertices as the coordinates. But the fact that the complex structure is built on that of $T(4)/\mathbf{R}_+$ seems to be interesting. Evidently these “observable” length and angle functions also exist on the Teichmüller spaces. It is natural to ask if these functions are somehow related to the complex structure on the Teichmüller space.

1.6. A close surface of genus g is obtained by identifying the opposite edges of a $4g$ -sided polygon. Suppose that the genus $g > 1$. Now take a $4g$ -sided polygon bounded by line

segments and identify the opposite sides by affine transformations. The quotient surface has a singular affine structure with one singularity coming from the vertices (the monodromy around the singularity is of the form $z \rightarrow kz + (k - 1)a$, $k \in \mathbf{R}_{>0}$). If immersed $4g$ -sided polygons are allowed to be used in the construction, then all complex structures on the surface are obtained in this way. See [Lu7] for more details on relating the Thurston's coordinate, the singular affine structure and the complex structure on the Teichmüller space.

1.7. The reconstruction principle also holds for hyperbolic or spherical convex polygons. Thus the same picture holds in these cases as well. The most interesting one seems to be the ideal polygons in hyperbolic plane where one assigns to each oriented ideal quadrilateral with a marked diagonal the Bonahon-Thurston shearing coordinate ([Bo1], [Th1]). Recall that the shearing coordinate is defined as follows. The *mid-point* of an edge in an ideal triangle is the tangent point of the inscribed circle at the edge. Given an oriented ideal quadrilateral with a marked diagonal, one chooses an orientation on the diagonal. The Bonahon-Thurston coordinate for the marked quadrilateral is the exponential of the signed hyperbolic distance from the left mid-point to the right mid-point of the diagonal. Note that the coordinate is independent of the choice of the orientation on the diagonal. If one changes the diagonal, the coordinate changes to its inverse. The change of coordinate formula for other four edges is given in figure 6.



Right-hand orientation on the plane

Figure 6

Note that the transformation formulas are real algebraic. Since each non-closed surface has an ideal triangulation, this gives an easy way to parametrize the Teichmüller space of the surface using Bonahon-Thurston coordinates. As a consequence, one proves easily that the Teichmüller space is a real analytic manifold diffeomorphic to the Euclidean space. (This seems to be one of the quickest way of showing that the Teichmüller space of a non-closed surface is contractible. The other proof using the Fenchel-Nielsen coordinate seems to be always running into the technical difficulties of showing that Fenchel-Nielsen coordinates associated to different 3-holed sphere decompositions differ by a diffeomorphisms.) These shearing coordinates are closely related to Penner's coordinates for decorated Teichmüller spaces (see [Pe]). See also [Mo] for related material on measured laminations.

§2. The Teichmüller Space, the Mapping Class Group, and the Space of Measured Laminations

2.1. Given a compact orientable surface Σ with or without boundary, there are three themes naturally associated to the surface. Namely, the Teichmüller space $T(\Sigma)$, the mapping class group $\Gamma(\Sigma)$, and the space $\mathcal{S}(\Sigma)$ of isotopy classes of unoriented simple loops not homotopic to a point (or its completion, Thurston's space of measured laminations). These three themes represent the geometric, algebraic, and topological aspects of the surface theory. Recall that the mapping class group $\Gamma(\Sigma) = \text{Homeo}^+(\Sigma, \partial\Sigma)/\text{Iso}$ is the group of orientation preserving self-homeomorphisms modulo isotopies so that the boundary of the surface is fixed pointwise by the homeomorphisms and the isotopies. For surface of negative Euler number, the Teichmüller space $T(\Sigma)$ is the space of all hyperbolic metrics with geodesic boundary on the surface modulo isometries isotopic to the identity. These three themes interact each other in the sense that the mapping class group acts naturally on both $T(\Sigma)$ and $\mathcal{S}(\Sigma)$ by pull back, and the space $\mathcal{S}(\Sigma)$ appeared in Thurston's compactification of the Teichmüller space $T(\Sigma)$.

2.2. One may illustrate these three themes and their interaction by the classical example of the oriented torus $\Sigma_{1,0}$ where the Teichmüller space $T(\Sigma_{1,0})$ is defined to be the space of all flat metrics modulo similarity maps isotopic to the identity. In his doctor thesis in 1913, J. Nielsen proved that two homologous homeomorphism (resp. simple loops) of the torus are isotopic. Thus the space of simple loops $\mathcal{S}(\Sigma_{1,0})$ can be identified naturally with the set of primitive elements in $H_1(\Sigma_{1,0}, \mathbf{Z})$ modulo ± 1 and the mapping class group $\Gamma(\Sigma_{1,0})$ is naturally isomorphic to the automorphism group $\text{Aut}^+(H_1(\Sigma_{1,0}, \mathbf{Z}))$. Define a *marking* on $\Sigma_{1,0}$ to be a pair of oriented simple loops (a, b) intersecting transversely at one point. Fix a marking (a, b) on $\Sigma_{1,0}$. Their homology classes $[a], [b]$ form a basis for the first homology group $H_1(\Sigma_{1,0}, \mathbf{Z})$. In terms of the basis, one can identify $\mathcal{S}(\Sigma_{1,0})$ with the set of rational numbers $\hat{\mathbf{Q}} = \mathbf{Q} \cup \{\infty\}$ by sending each primitive class $\pm(p[a] + q[b])$ to its "slope" p/q . One can also identify the automorphism group $\text{Aut}^+(H_1(\Sigma_{1,0}, \mathbf{Z}))$ with $SL(2, \mathbf{Z})$. The natural action of the mapping class group $\Gamma(\Sigma_{1,0})$ on $\mathcal{S}(\Sigma_{1,0})$ becomes the standard action of $SL(2, \mathbf{Z})$ on the rationals by fractional linear transformations. The marking (a, b) can also be used to parametrize the Teichmüller space $T(\Sigma_{1,0})$ as follows. Fix a flat metric d on $\Sigma_{1,0}$. We isotop a and b into two d -geodesics \hat{a} and \hat{b} which intersect at one point p . Let θ be the angle measured from \hat{a} to \hat{b} at p in the orientation of the surface and let l_a and l_b be the lengths of the geodesics \hat{a} and \hat{b} . Assign to the flat metric d the complex number $z_d = l_a/l_b e^{i\theta}$ in the upper-half plane \mathbf{H} . Evidently the invariant z_d depends only on the similarity class of the flat metric d . Thus one obtains a maps π_m from the Teichmüller space $T(\Sigma_{1,0})$ to \mathbf{H} . This map is a bijection since the inverse can be constructed by sending $z \in \mathbf{H}$ to the torus $\mathbf{C}/(\mathbf{Z} + z\mathbf{Z})$ with marking corresponding to 1 and z . Note that the invariant z_d is independent of the orientations on a and b . Furthermore, the pair (z_d, \bar{z}_d) is the Thurston invariant of the parallelogram obtained by cutting the flat torus $(\Sigma_{1,0}, d)$ open along the geodesics \hat{a}, \hat{b} . Now if we are given a different marking $m' = (a', b')$, there is an $SL(2, \mathbf{Z})$ matrix A which sends $[a]$ to $[a']$ and $[b]$ to $\pm[b']$. A simple calculation shows that two invariants π_m and $\pi_{m'}$ are related by A acting as the fractional linear transformation on \mathbf{H} . Thus the Teichmüller space $T(\Sigma_{1,0})$ can be naturally identified with \mathbf{H} so that the action of the mapping class group becomes the standard action of $SL(2, \mathbf{Z})$ on \mathbf{H} by fractional linear transformations. In short, the three themes $T(\Sigma_{1,0})$, $\Gamma(\Sigma_{1,0})$, and $\mathcal{S}(\Sigma_{1,0})$ for the oriented torus are exactly

$(\mathbf{H}, SL(2, \mathbf{Z}), \hat{\mathbf{Q}})$. It is interesting to note that the complex structure on the Teichmüller space makes both the angle function and the logarithm of ratio of length functions pluriharmonic. Indeed, by fixing a marking $m = (a, b)$ on the torus $\Sigma_{1,0}$, one obtains a fundamental domain map $f_m : T(\Sigma_{1,0}) \rightarrow T(4)/\mathbf{R}_+$ by sending the similarity class $[d]$ to the parallelogram based on \hat{a} and \hat{b} which form a fundamental domain for the flat metric. The complex structure on $T(\Sigma_{1,0})$ makes the map f_m holomorphic, i.e., holomorphic motions in the Teichmüller space corresponds to the homomorphic motions of the fundamental domains. The same phenomenon does not seem to hold for the complex structure on the Teichmüller space of a closed surface of higher genus with hyperbolic metrics ([Ke1], [Wo2]). (A natural choice of fundamental domains for closed surfaces of higher genus seems to be those associated to chains in the surface. See Maskit [Ma] for more information.) However, there are evidences indicating that the complex structure on the Teichmüller space is more closely related to the singular flat metrics on the surface. To be more precise, given a closed Riemann surface Σ of genus g and a fixed point on the surface. There exists a unique singular flat metric of area 1 in the conformal class of Σ so that its singularity is at the fixed point having cone angle $2\pi(2g - 1)$. See [Lu7] for more details on the relationship between these singular flat metrics and the complex structure.

2.3. There is a natural compactification of the upper-half plane \mathbf{H} by the real line $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ where the action of $SL(2, \mathbf{Z})$ extends continuously. Thurston's deep work on surface theory shows that the same compactification also exists for all surfaces. We shall discuss briefly Thurston's work in this section. See [FLP] and [Th2] for more details. To begin with, a proper 1-dimensional submanifold s in a compact surface Σ is called a *curve system* if no component of s is homotopic into $\partial\Sigma$ relative to $\partial\Sigma$. The isotopy class of all curve systems on Σ is denoted by $CS(\Sigma)$ and was introduced by Dehn who called it *the Arithmetic field* of the surface. In the case of a torus, the set $CS(\Sigma)$ is naturally the set of all non-zero lattice points in $H_1(\Sigma_{1,0}, \mathbf{Z})$ modulo ± 1 . There exists a quadratic pairing on $CS(\Sigma)$ given by the geometric intersection number $I(\alpha, \beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}$. For the torus, the pairing is $I((p, q), (p', q')) = |pq' - p'q|$ which is the absolute value of the canonical symplectic form on \mathbf{Z}^2 . This pairing satisfies the homogeneity and non-degenerate property in the sense that $I(k_1\alpha_1, k_2\alpha_2) = k_1k_2I(\alpha_1, \alpha_2)$ ($k_i \in \mathbf{Z}_+$ and $k_i\alpha_i$ means k_i copies of the curve system α_i), and for each α there exists β so that $I(\alpha, \beta) \neq 0$. Thurston's space of measured laminations $ML(\Sigma)$ is the completion of $CS(\Sigma)$ with respect to the pairing I . In linear algebra, given a non-degenerate quadratic form ω on a lattice L of rank r , one can form a completion of (L, ω) by canonically embedding L into R^r so that the form w extends continuously on R^r . If the form is definite, the simplest way to construct the completion is by formally extending ω to $\mathbf{Q}L$ and taking the metric completion of $\mathbf{Q}L$. If the form ω is not definite, one may embed L into the infinite dimensional space \mathbf{R}^L (with the product topology) by sending $x \in L$ to the linear function $\pi(x) = \omega(\cdot, x)$. The canonical completion is given by taking the closure of the set $\mathbf{Q}\pi(L)$. Since the form ω is non-degenerate, the Rieze representation theorem says that the closure is isomorphic to a vector space \mathbf{R}^r and the form ω extends continuously to the closure. Thurston's completion of $(CS(\Sigma), I)$ is the analogous construction. The space $CS(\Sigma)$ is embedded into $\mathbf{R}^{S(\Sigma)}$ by sending α to the *intersection function* $Th(\alpha) = I(\cdot, \alpha)$ and the closure of $\mathbf{Q}_+Th(CS(\Sigma))$ is defined to be the completion, the space of measured lamination $ML(\Sigma)$. Thurston proved a remarkable theorem that the space $ML(\Sigma)$ is homeomorphic to a Euclidean space and the quadratic pairing extends continuously to $ML(\Sigma)$ ([Bo2], [FLP], [Re], [Th1]). Since Thurston's completion is

canonically constructed, the mapping class group $\Gamma(\Sigma)$ acts continuously on $ML(\Sigma)$. In the case of torus, $ML(\Sigma)$ is canonically $H_1(\Sigma_{1,0}, \mathbf{R})/\pm 1$ and the action of the the mapping class group $SL(2, \mathbf{Z})$ is the standard action. The projectivized space $PML = (ML(\Sigma) - 0)/\mathbf{R}_{>0}$ is Thurston's compactification of the Teichmüller space.

§3. The Restriction Maps and the Reconstruction Theorems

3.1. A compact subsurface Σ' in Σ is called *essential* if no component of $\partial\Sigma'$ is null homotopic in Σ . If Σ' is essential with negative Euler number, there exists a natural restriction map from the Teichmüller spaces $T(\Sigma)$ to $T(\Sigma')$ (resp. from $ML(\Sigma)$ to $ML(\Sigma')$). The restriction map is defined as follows. Given a hyperbolic metric d on Σ , we isotop the open surface $int(\Sigma')$ (the interior of Σ') to an open subsurface Σ'' so that Σ'' is bounded by disjoint simple geodesics. The metric completion of $(\Sigma'', d|_{\Sigma''})$ is a hyperbolic metric d' on Σ' with geodesic boundary. The restriction map sends $[d]$ to $[d']$. The restriction map for the measured laminations is defined similarly (see [Lu2]). The key step is to define the restriction map from the space of curve systems $CS(\Sigma)$ to $CS(\Sigma')$. Given $\alpha \in CS(\Sigma)$, choose a representative $a \in \alpha$ so that the number of components of $a \cap \Sigma'$ is minimal. Then the restriction map sends $[a]$ to $[a|_{\Sigma'}]$. The restriction maps are natural in the sense that if we are given two essential subsurfaces $\Sigma_1 \subset \Sigma_2 \subset \Sigma$, then the composition of restrictions is the restriction.

To state the reconstruction theorems, we say that two essential subsurfaces Σ_1 and Σ_2 *overlap essentially* if there is a non-trivial simple loop which is isotopic into both Σ_1 and Σ_2 . If furthermore both Σ_1 and Σ_2 are level-1 subsurfaces, then their possible intersection surfaces are either essential annuli, or an essential 3-holed sphere or they are isotopic.

3.2. With these preparation, we can state the main theorems in [Lu1], [Lu2] and [Lu4] as follows which establish Grothendieck's reconstruction principle for the Teichmüller spaces, the measured lamination spaces and the mapping class groups.

Theorem(Reconstruction of the Teichmüller spaces and the measured lamination spaces). *Each hyperbolic metric (resp. measured lamination) on a surface of level at least 2 is constructed uniquely up to isotopy by assigning a hyperbolic metric (resp. measured lamination) to each essential level-1 subsurface so that when two level-1 subsurfaces overlap essentially, the restrictions of the metrics (resp. measured laminations) to their intersection are isotopic. Furthermore, the restriction of the hyperbolic metric (resp. measured lamination) on the surface to each level-1 essential subsurface is isotopic to the assigned one.*

3.3. For the mapping class group, it was a theorem of Dehn and Lickorish that the mapping class group $\Gamma(\Sigma)$ is finitely generated by Dehn twists along simple loops. The main result in [Lu4] states that

Theorem(Reconstruction of the Mapping Class Groups). *Each orientation preserving self-homeomorphism of a surface which fixes the boundary pointwise is isotopic to a composition of finitely many Dehn twists so that the composition is unique modulo cancellation laws supported in subsurfaces of level-1.*

Earlier work on the subject were Hatcher and Thurston who showed among other things that the subsurfaces can be taken to be genus 2 with 3 holes and Gervais who proved that

subsurface can be taken to be genus 1 with 2 holes. Theorem 2 is just a simple simplification of the work of Gervais.

3.4. Before discussing the related theorems for surfaces of levels 0 or 1, let us recall Thurston's embeddings of the Teichmüller space $T(\Sigma)$ and the measured lamination spaces $ML(\Sigma)$. Given an isotopy class $[d] \in T(\Sigma)$, the *geodesic length function* $l_d : \mathcal{S}(\Sigma) \rightarrow \mathbf{R}_{\geq 0}$ of the metric sends an isotopy class of simple loop to the length of its geodesic representative. For a measured lamination $m \in ML(\Sigma)$, the *geometric intersection number function*, or simply *intersection function* $I_m : \mathcal{S}(\Sigma) \rightarrow \mathbf{R}_{\geq 0}$ is given by $I_m(\alpha) = I(\alpha, m)$. Thurston's embedding $Th : T(\Sigma) \rightarrow \mathbf{R}_{\geq 0}^{\mathcal{S}(\Sigma)}$ (resp. $Th : ML(\Sigma) \rightarrow \mathbf{R}_{\geq 0}^{\mathcal{S}(\Sigma)}$) sends the isotopy class of a metric to its geodesic length function, i.e., $Th([d]) = l_d$ (resp. sends a measured lamination to its intersection function). The fact that the map Th is injective for Teichmüller space was a result of Fricke and Klein. The work of Okumura [Ok1], [Ok2] and Schmultz [Sc] determine the smallest finite set $F \subset \mathcal{S}(\Sigma)$ so that the restriction $l_d|_F$ determines the metric d . The analogous question for measured lamination space seems to be open. A result of Thurston shows $9g - 9$ simple loops suffices to determine an intersection function for closed surface of genus g ([Th1], [FLP]). But the number $9g - 9$ is not the smallest.

3.5. For the level-0 surface, i.e., the 3-holed sphere, the space of simple loops $\mathcal{S}(\Sigma_{0,3})$ consists of isotopy classes of the three boundary components. The Teichmüller space $T(\Sigma_{0,3})$, the measured lamination space $ML(\Sigma_{0,3})$ and the mapping class group $\Gamma(\Sigma_{0,3})$ can be described as follows. By a theorem of Fricke and Klein mentioned before, each isotopy class of a hyperbolic metric on $\Sigma_{0,3}$ is determined by the lengths of three boundary components and these lengths subject to no constraints, i.e., $Th(T(\Sigma_{0,3})) = \mathbf{R}_{>0}^{\mathcal{S}(\Sigma_{0,3})}$. The work of Thurston shows that each measured lamination is determined by its intersection number with three boundary components and these three numbers subject to no constraints, i.e., $Th(ML(\Sigma_{0,3})) = \mathbf{R}_{\geq 0}^{\mathcal{S}(\Sigma_{0,3})}$. Dehn proved in 1938 that $\Gamma(\Sigma_{0,3})$ is isomorphic to free abelian groups on three generators which are the Dehn twists on three boundary components.

With these results on 3-holed sphere, one can restate theorem 3.2 in an equivalent form as follows. Given a surface Σ of level at least 2, a real valued function on the set of simple loops $\mathcal{S}(\Sigma)$ is a geodesic length function (resp. an intersection function) if and only if for each essential surface Σ' of level-1, the restriction of the function to $\mathcal{S}(\Sigma')$ is.

3.6. For level-1 surfaces Σ , i.e., the 1-holed torus $\Sigma_{1,1}$ and 4-holed sphere $\Sigma_{0,4}$, the set of simple loops $\mathcal{S}(\Sigma)$, the Teichmüller space $T(\Sigma)$ and the mapping class group $\Gamma(\Sigma)$ are essentially the same as those of the torus. To be more precise, let us consider the subset $\mathcal{S}'(\Sigma) \subset \mathcal{S}(\Sigma)$ of isotopy classes of simple loops which are not homotopic into the boundary $\partial\Sigma$, i.e., $\mathcal{S}'(\Sigma) = \mathcal{S}(\Sigma) \cap CS(\Sigma)$. There exists a natural bijection i_* between $\mathcal{S}'(\Sigma_{1,1})$ and $\mathcal{S}'(\Sigma_{1,0})$ induced by the inclusion map from $\Sigma_{1,1}$ to $\Sigma_{1,0}$. This isomorphism preserves the intersection pairing. For the 4-holed sphere $\Sigma_{0,4}$, there exists a natural isomorphism P^* between $\mathcal{S}'(\Sigma_{0,4})$ and $\mathcal{S}'(\Sigma_{1,1})$ which satisfies $I(\alpha, \beta) = 2I(P^*(\alpha), P^*(\beta))$. It is defined as follows. Let τ be a hyperelliptic involution on the 1-holed torus $\Sigma_{1,1}$ and let $P : \Sigma_{1,1} \rightarrow \Sigma_{1,1}/\tau$ be the quotient map where $\Sigma_{1,1}/\tau$ is the disc with three cone points of order two (an orbifold). It is well known that the hyperelliptic involution τ preserves the isotopy class of each simple loop and τ commutes with each homeomorphism up to isotopy. Let the 4-holed sphere $\Sigma_{0,4}$ be the subsurface of $\Sigma_{1,1}/\tau$

with three small disc neighborhoods of the cone points removed. Then the isomorphism P^* from $\mathcal{S}'(\Sigma_{0,4})$ to $\mathcal{S}'(\Sigma_{1,1})$ sends the isotopy class $[a]$ to $[b]$ where b is a component of $P^{-1}(a)$. To summarize, for a level-1 surface Σ , there exists a bijection π from $\mathcal{S}'(\Sigma)$ to $\hat{\mathbf{Q}}$ so that $\pi(\alpha) = p/q$ and $\pi(\beta) = p'/q'$ satisfy $pq' - p'q = \pm 1$ if and only if $I(\alpha, \beta) = 1$ for $\Sigma_{1,1}$ and 2 for $\Sigma_{0,4}$. Draw a hyperbolic geodesic in the upper-half plane ending at p/q and p'/q' when $pq' - p'q = \pm 1$. One obtains the so called “modular configuration” (see figure 7). Call three elements in $\mathcal{S}'(\Sigma)$ forming an triangle if they correspond to the vertices of an ideal triangle in the modular configuration and call four elements in $\mathcal{S}'(\Sigma)$ forming a quadrilateral if they correspond to the vertices of an ideal quadrilateral. The modular structure on the space of simple loops $\mathcal{S}'(\Sigma)$ for level-1 surfaces was known to Fricke and Klein ([Ke1]) and to Dehn ([De]) who used the rational numbers to code the set $\mathcal{S}'(\Sigma)$. See also [HT], [MM], [Se], [Th2] and others.

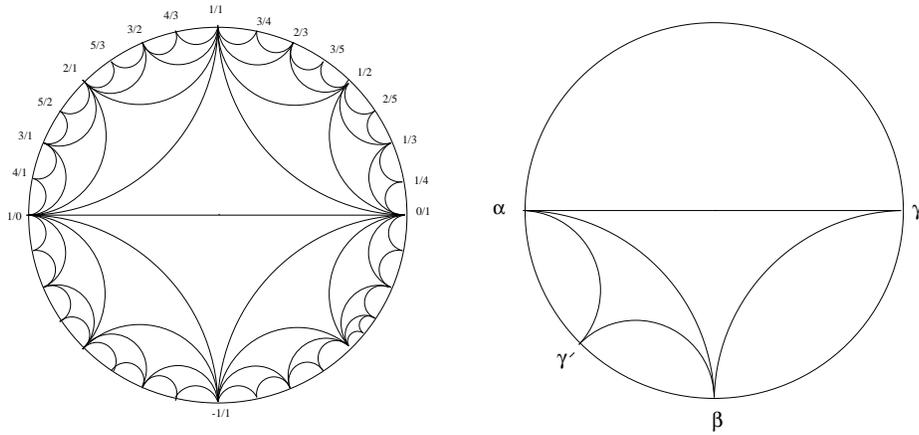
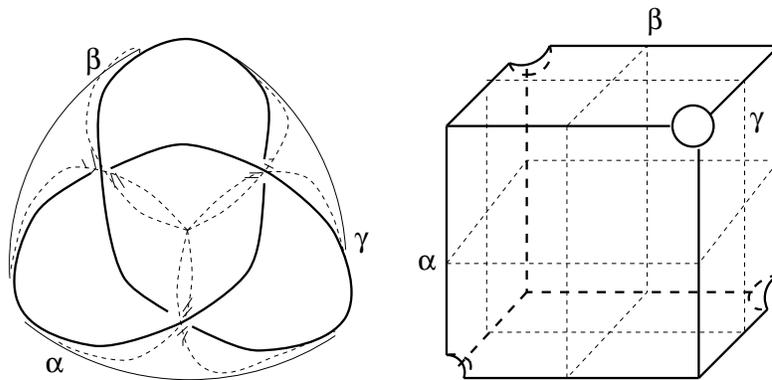


Figure 7

A special feature of the modular configuration is the huge symmetry built in the configuration. This is, in our view, one reason why the set of homotopy class of simple loops on the surface is more useful than the fundamental group in establishing the reconstruction principle for many structures. Suppose we take four vertices $(\alpha, \beta, \gamma; \gamma')$ forming a quadrilateral in $\mathcal{S}'(\Sigma)$ so that both (α, β, γ) and (α, β, γ') are triangles. Then there is an orientation reversing involution of $\mathbf{Q}P^1$ leaving the quadrilateral fixed and interchanging γ and γ' . This involution is realized by an orientation reversing involution of the surface Σ which is the reflection of the figure 9 (where $\gamma = \alpha\beta$) about the yz -plane. On the other hand, given any triangle (α, β, γ) in the modular configuration $\mathcal{S}'(\Sigma)$, there is a \mathbf{Z}_3 action on $\mathbf{Q}P^1$ which permutes the three vertices. Thus there is a \mathbf{Z}_3 action on the surface Σ permuting the isotopy classes. This symmetry is illustrated in figure 8 below where the 1-holed torus is the Seifert surface of the trefoil knot and the 4-holed sphere is the truncated boundary surface of a cube. The symmetry involved in the 4-holed sphere is huge which is difficult to visualize in figure 9. Indeed, as one can see from figure 8, any permutation of the four boundary components is realized by a homeomorphism preserving the set $\{\alpha, \beta, \gamma\}$, i.e., the permutation group on four letters acts on $\Sigma_{0,4}$ preserving set $\{\alpha, \beta, \gamma\}$.



The three-fold symmetry in the modular configuration viewed in the three-space

Figure 8

As an application of this 24-fold symmetry, we consider the trace relations for $SL(2, \mathbf{C})$ matrices. In this case, the analogous question to problem 2 in §1.1 for triangles and quadrilaterals is the following. Given three complex numbers a, b, c , do there exist two matrices $A, B \in SL(2, \mathbf{C})$ so that $tr(A) = a, tr(B) = b$ and $tr(AB) = c$? It is well known that the answer is positive. Next, in analogous to six edge lengths of a quadrilateral, given seven complex numbers $a_1, a_2, a_3, a_{12}, a_{23}, a_{31}$ and a_{123} , under what condition do there exist three $SL(2, \mathbf{C})$ matrices A_1, A_2 , and A_3 so that $tr(A_i) = a_i, tr(A_i A_j) = a_{ij}$ and $tr(A_1 A_2 A_3) = a_{123}$? A solution by Fricke-Klein [FK] and Vogt [Vo] was the following. These three matrices exist if and only if $a_{123}^2 - a_{123}(a_1 a_{23} + a_2 a_{31} + a_3 a_{12} - a_1 a_2 a_3) a_1^2 + a_2^2 a_3^2 + a_{12}^2 + a_{23}^2 + a_{31}^2 + a_{12} a_{23} a_{31} + a_1 a_2 a_3 + a_{12} a_{23} a_{31} - a_1 a_2 a_{12} - a_2 a_3 a_{23} - a_3 a_1 a_{31} - 4 = 0$. This equation, as it stands, is quite complicated. One can easily notice the 3-fold symmetry of the equation under cyclic permutation of $\{a_1, a_2, a_3\}$. In fact, there exists 24-fold symmetry in the equation. Namely, the equation is invariant under any permutation of $\{a_1, a_2, a_3, a_{123}\}$. This can be seen using the modular configuration. Indeed, if we choose the generators of the fundamental group of the 4-holed sphere carefully (see for instance, figure 5 in [Lu1]), then the four boundary components are represented by $x_1, x_2, x_3, x_1 x_2 x_3$ and three simple loops forming a triangle in the modular configuration by $x_1 x_2, x_2 x_3$ and $x_3 x_1$. The first equation in Theorem A4 (b) in the appendix A is a rewriting of the above polynomial equation in terms of the modular configuration.

3.7. The Teichmüller space, measured lamination space and the mapping class groups for level-1 surfaces can be explicitly constructed from the modular configuration on the set of simple loops. To be more precise, the geodesic length functions, the geometric intersection number functions and the Dehn twists satisfy universal relations at the vertices of triangles and quadrilaterals in $\hat{\mathbf{Q}}$. Furthermore, these universal relations form a complete set of relations. See Appendix A for the list of universal relations.

The relations for the Dehn twists were found by Dehn in [De]. D. Johnson [Jo] independently rediscovered the lantern relation (relation (IV) in Theorem A3 in the Appendix A) in 1979. Dehn also proved that these relations are complete for the mapping class group of level-1 surfaces. The relations for the geodesic length function were essentially discovered by Fricke and Klein

[FK] and Vogt [Vo] (thought they were not stated in terms of the modular relations). These are derived from the trace identities for $SL(2, \mathbf{C})$ matrices. That the set of all relations is complete was proved by Keen [Ke2] for the 1-holed torus and was proved in [Lu1] for 4-holed spheres using Maskit combination theorem. The relations for the measured laminations are just the degenerations of that of hyperbolic metrics and they are shown to be complete in [Lu2].

3.8. The relationship between the Teichmüller spaces and the mapping class groups among the level-1 surfaces becomes much clear if one considers the Teichmüller spaces $T_{1,1}$ and $T_{0,4}$ of complete hyperbolic metrics with cups ends (on the open surface), and the reduced mapping class group $\Gamma^*(\Sigma)$ which is the quotient of the mapping class group by the subgroup generated by Dehn twists on boundary components. The key fact is that the hyperelliptic involution τ on $\Sigma_{1,1}$ induces the identity map on both the Teichmüller space $T_{1,1}$ and $\mathcal{S}(\Sigma_{1,1})$ and is in the center of the mapping class group. Indeed, one has a natural biholomorphism between $T_{1,1}$ and $T_{0,4}$ induced by the pull back map $P : \Sigma_{1,1} \rightarrow \Sigma_{1,1}/\tau$. A natural isomorphism from $\Gamma^*(\Sigma_{1,1})$ to $\Gamma(\Sigma_{1,0}) = SL(2, \mathbf{Z})$ is induced by inclusion of $\Sigma_{1,1}$ to $\Sigma_{1,0}$. Since the hyperelliptic involution τ commutes with each homeomorphism, there is an monomorphism from the reduced mapping class group $\Gamma^*(\Sigma_{0,4}) \rightarrow \Gamma^*(\Sigma_{1,1}) / \langle \tau \rangle = PSL(2, \mathbf{Z})$ whose image is the principal congruence subgroup of order 2.

3.9. It is instructive to read [Gr1] on the related topics. We cite the paragraph on page 248-249 in [Gr1] below (with English translation by L. Schneps). “ There is a striking analogy, and I am certain it is not merely formal, between this principle and the analogous principle of Demazure for the structure of reductive algebraic groups, if we replace the term ‘level’ or ‘modular dimension’ with ‘semi-simple rank of the reductive group’. The link becomes even more striking, if we recall that the mapping class group $\Gamma_{1,1}^*$ is no other than $SL(2, \mathbf{Z})$, i.e., the group of integral points of the simple group scheme of ‘absolute’ rank 1 $SL(2)_{\mathbf{Z}}$. Thus, the fundamental building block for the Teichmüller tower is essentially the same as for the tower of reductive groups of all ranks - a group of which, moreover, we may say that it is doubtless in all the essential disciplines of mathematics.”

§4. The Space of Simple Loops on Surfaces and the Modular Structure

4.1. Unlike subsurfaces in a surface, simple loops on surfaces have been the focus of more attention for a long time. Indeed, most of the surface problems can be reduced to ones concerning simple loops and the proofs of theorems 3.2 and 3.3 are no exception. The topological investigation of the set $\mathcal{S}(\Sigma)$ of isotopy classes of essential simple loops began in Dehn’s work [De] on the mapping class groups. As an example of use of simple loops to solve surface problems, let us recall the elegant proof of Dehn that the (reduced) mapping class group $\Gamma^*(\Sigma_{0,4})$ is a free group on two generators generated by Dehn twists on two simple loops intersecting at two points. Dehn first observed that the set $\mathcal{S}'(\Sigma_{0,4})$ of essential simple loops not homotopic into the boundary forms the modular configuration $\hat{\mathbf{Q}}$ and the mapping class group $\Gamma^*(\Sigma_{0,4})$ acts on the modular configuration faithfully preserving both the modular relation and the orientation. Thus $\Gamma^*(\Sigma_{0,4})$ is a subgroup of the modular group $PSL(2, \mathbf{Z})$. Since each boundary component of $\Sigma_{0,4}$ is fixed by the mapping class group elements, $\Gamma^*(\Sigma_{0,4})$ is actually in the principal congruence subgroup of level 2 generated by two matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$. But these two matrices correspond to two Dehn twists mentioned above.

To go from simple loops to subsurfaces, one takes the regular neighborhood of a union of simple loops. In this way, it can be shown for instance that given any two level-1 essential subsurfaces A, B , there is a sequence of level-1 essential subsurfaces starting from A and ending at B so that any two adjacent level-1 subsurfaces overlap in an essential level-0 subsurface.

4.2. The work of Dehn [De] and Lickorish [Li] already suggested strongly that level-1 subsurfaces are fundamental in simplifying the intersections of two simple loops. Indeed, lemma 2 in [Li] states that if two simple loops a, b satisfy either $|a \cap b| \geq 3$ or $|a \cap b| = 2$ with non-zero algebraic intersection number, then there is a Dehn twist which sends b to a new loop having fewer intersection points with a . Thus the only situation which cannot be simplified are: 1) a, b are disjoint, 2) a intersects b at one point and 3) a intersects b at two points of different intersection signs. In these cases, the lowest level connected subsurface which contains both a and b is either a level-0 or a level-1 subsurface. Furthermore, the pair of curves a, b satisfying conditions 2) or 3) corresponds to the basic relation in the modular configuration.

4.3. We have mentioned in several places the notion of modular structure. Here is a formal definition after Thurston's geometric structures on manifolds.

Definition. (a) A $(\hat{\mathbf{Q}}, PSL(2, \mathbf{Z}))$ modular structure on a set X is a maximal collection of charts $\{(U_i, \phi_i) | \phi_i : U_i \rightarrow \hat{\mathbf{Q}} \text{ is injective}\}$ so that the following hold.

(1) $X = \cup_i U_i$.

(2) The transition function $\phi_i \phi_j^{-1}$ is the restriction of an element in $PSL(2, \mathbf{Z})$.

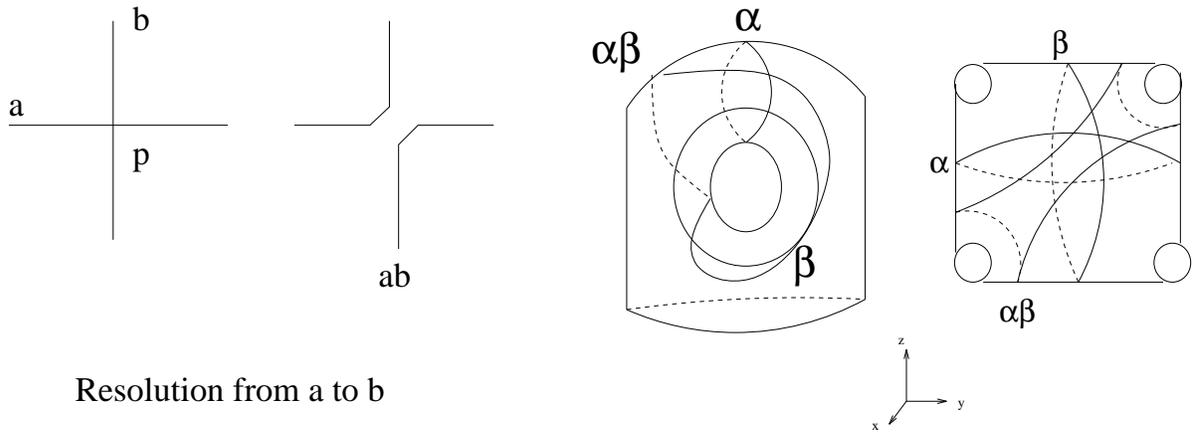
(b) A modular structure on a set X is called *compact* if the group of bijections of X which preserve the modular structure acts on X with finite orbits.

It seems that compactness is essential for developing a useful “function theory” on a set with a modular structure. All interesting examples that we encounter have compact modular structures.

For an oriented surface Σ of level at least 1, the set $\mathcal{S}'(\Sigma)$ of isotopy classes of non-boundary parallel essential simple loops on Σ has a natural compact modular structure invariant under the action of the mapping class group. A special collection of charts for the modular structure is given by $(\mathcal{S}'(\Sigma'), \phi_{\Sigma'})$ where Σ' is an essential level-1 subsurface and $\phi_{\Sigma'} : \mathcal{S}'(\Sigma') \rightarrow \mathcal{S}'(\Sigma_{1,1}) = \hat{\mathbf{Q}}$ is a bijection induced by either an orientation preserving homeomorphism or by an orientation preserving quotient map (see §3.6). To see that the second condition (2) in the definition holds, one simply notes that if two essential level-1 subsurfaces intersect at two non-homotopic simple loops, then they are isotopic. To see the compactness, we note that the mapping class group acts on $\mathcal{S}'(\Sigma)$ preserving the modular structure and the action of the mapping class group has finite orbits. Thus we can talk about triangle and quadrilateral in $\mathcal{S}'(\Sigma)$. Furthermore, since the set of rational numbers $\hat{\mathbf{Q}}$ has the natural orientation invariant under $PSL(2, \mathbf{Z})$, we can talk about oriented triangles in $\mathcal{S}'(\Sigma)$.

Another example of compact modular structure is the set of isotopy classes of 3-holed sphere decompositions of a surface which seems to be related to the Heegaard splittings of 3-manifolds. See Appendix B for more detail.

4.4. One way to see the modular structure on the space of simple loops $\mathcal{S}(\Sigma)$ is to use the notion of resolution of intersection points. Recall that two rational numbers p/q and p'/q' are modular related if $pq' - p'q = \pm 1$ and are denoted by $p/q \perp p'/q'$. Two isotopy classes α and β of curves on surfaces corresponding to a modular related pair are denoted by $\alpha \perp \beta$ or $\alpha \perp_0 \beta$. Here $\alpha \perp \beta$ means that their intersection number $I(\alpha, \beta) = 1$ and $\alpha \perp_0 \beta$ means that $I(\alpha, \beta) = 0$ with zero algebraic intersection number. To find out the vertices of ideal triangles based on $\{\alpha, \beta\}$, we use the resolutions of intersections. Recall that surfaces are oriented. If a, b are two arcs intersecting at one point transversely, then the *resolution of $a \cup b$ at the intersection point from a to b* is defined as follows. Fix any orientation on a and use the orientation on the surface to determine an orientation on b . Then resolve the intersection according to the orientations (see figure 8). The resolution is independent of the orientation chosen on a . If $\alpha \perp \beta$ or $\alpha \perp_0 \beta$, take $a \in \alpha$ and $b \in \beta$ so that $|a \cap b| = I(\alpha, \beta)$. Then the curve obtained by resolving all intersection points in $a \cap b$ from a to b is again an essential non-boundary parallel simple loop. We denote the isotopy class by $\alpha\beta$. One sees easily that positively oriented triangles and quadrilaterals in the modular structure on $\mathcal{S}'(\Sigma)$ are exactly $(\alpha, \beta, \alpha\beta)$ and $(\alpha, \beta, \alpha\beta, \beta\alpha)$. If $\alpha \perp \beta$ or $\alpha \perp_0 \beta$, we use $\partial(\alpha \cup \beta)$ to denote the isotopy class of the boundary of a regular neighborhood of $a \cup b$. In terms of these notations, all universal relations for the geodesic length functions, the intersection functions and the Dehn twists are expressed in terms of $\alpha, \beta, \alpha\beta, \beta\alpha$ and the components of $\partial(\alpha \cup \beta)$. For instance, the relations for the Dehn twists are: 1) if $\alpha \perp \beta$, then $D_\alpha D_\beta = D_\beta D_{\alpha\beta}$ and $(D_\alpha D_\beta D_{\alpha\beta})^4 = D_{\partial(\alpha \cup \beta)}$, and 2) if $\alpha \perp_0 \beta$, then $D_\alpha D_\beta D_{\alpha\beta} = D_{\partial(\alpha \cup \beta)}$ (the lantern relation). Since the modular relation $(\hat{\mathbf{Q}}, \perp)$ has a \mathbf{Z}_3 -symmetry leaving an ideal triangle invariant, we obtain $\alpha(\beta\alpha) = (\alpha\beta)\alpha = \beta$.

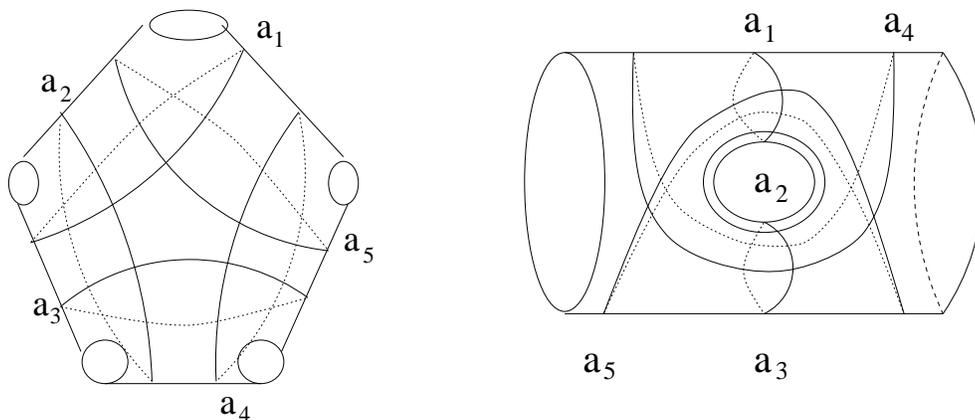


Resolution from a to b

Right-hand orientation on the front faces

Figure 9

4.5. One of the most useful property of the modular structure on $\mathcal{S}'(\Sigma)$ is the following lemma (lemma 7 in [Lu1]) which generalizes Lickorish's lemma 2 in [Li]. It states that given two intersecting elements $\alpha, \beta \in \mathcal{S}'(\Sigma)$ which are not related by the modular relation \perp or \perp_0 , then we can write $\beta = \gamma_1\gamma_2$ with $\gamma_1 \perp \gamma_2$ or $\gamma_1 \perp_0 \gamma_2$ so that (1) $I(\alpha, \gamma_i) < I(\alpha, \beta)$ and $I(\alpha, \gamma_2\gamma_1) < I(\alpha, \beta)$ for $i = 1, 2$ and (2) if $\gamma_1 \perp_0 \gamma_2$, then for each component δ of $\partial(\gamma_1 \cup \gamma_2)$ we have $I(\alpha, \delta) < I(\alpha, \beta)$. As an easily consequence, one shows that the reconstruction principle for the Teichmuller spaces follows from theorem 3.2 for level-2 surfaces. As another consequence, one shows that the space $\mathcal{S}(\Sigma)$ is finitely generated in the following strong sense: There is a finite subset X_0 in $\mathcal{S}(\Sigma)$ so that $\mathcal{S}(\Sigma) = \cup_{n=0}^{\infty} X_n$ where $X_{i+1} = X_i \cup \{\alpha \mid \alpha = \gamma_1\gamma_2 \text{ where either (1) } \gamma_1 \perp \gamma_2, \text{ and } \gamma_1, \gamma_2, \gamma_2\gamma_1 \text{ are in } X_i \text{ or (2) } \gamma_1 \perp_0 \gamma_2 \text{ and } \gamma_1, \gamma_2, \gamma_2\gamma_1 \text{ and each component of } \partial(\gamma_1 \cup \gamma_2) \text{ are in } X_i\}$.



The pentagon relations

Figure 10

4.6. The proof of the reconstruction theorem for level-2 surfaces has always been one of the key steps in establishing the reconstruction principle. In dealing with simple loops on leve-2 surfaces, the following collection of five curves (the pentagon relation) $\{\alpha_1, \dots, \alpha_5$ in $\mathcal{S}'(\Sigma) \mid I(\alpha_i, \alpha_j) = 0$ for indices $|i - j| \neq 1 \pmod{5}\}$ (see figure 10) appears constantly and plays an important role. For 5-holed sphere, one has the following relation $\alpha_i\alpha_{i+1}\alpha_{i+2} = \alpha_{i+3}\alpha_{i+4}$ and for the 2-holed torus, we have $\alpha_1\alpha_2\alpha_3\alpha_4 = \alpha_3\alpha_2\alpha_1$ (see [Lu2]). These five curves for the 5-holed sphere were first observed by Dehn in [De] who showed that the Dehn twists on them generate the reduced mapping class group for 5-holed sphere. Furthermore, these five curves are rigid in the sense that any other collection of five curves with the same disjointness property is the image of $\{\alpha_i\}$ under a homeomorphism ([Lu5]).

4.7. We finish this section with an application of the notion of resolving intersection to a multiplicative structure on the space of curve systems $CS(\Sigma)$. Given two curve systems a, b on an oriented surface with $|a \cap b| = I([a], [b])$, the multiplication ab is defined to be the 1-dimensional submanifold obtained by resolving all intersection points in $a \cap b$ from a to b . It can be shown that ab is again a curve system (see Appendix C for a simple proof when a, b contain no arcs).

This induces a multiplicative structure on $CS(\Sigma)$ by defining $\alpha\beta = [ab]$ where $a \in \alpha$, $b \in \beta$ and $|a \cap b| = I(\alpha, \beta)$. For instance the Dehn twist on a simple loop α applied to β is given by $D_\alpha(\beta) = \alpha^k \beta$ where $k = I(\alpha, \beta)$. This multiplication is natural with respect to the action of the mapping class group and is highly non-commutative. Indeed, if α contains no arc component, then $\alpha\beta = \beta\alpha$ implies $I(\alpha, \beta) = 0$. As a consequence of this, one obtains a new proof of a result of Ivanov ([Iv]) that Dehn twists on two intersecting isotopy classes of simple loops can never be commuting up to isotopy. The most interesting property of the multiplication seems to be the ‘‘cancellation law’’ that if each component of α is not an arc and intersects β , then $\alpha(\beta\alpha) = (\alpha\beta)\alpha = \beta$. This is a generalization of the \mathbf{Z}_3 -symmetry in the modular configuration. As an application of the cancellation law, let us prove a weak form of a result of Thurston [Th2] that if α and β are two surface filling simple loops (i.e., $I(\alpha, \gamma) + I(\beta, \gamma) > 0$ for all $\gamma \in \mathcal{S}'(\Sigma)$), then the self-homeomorphism $f = D_\alpha^{-1}D_\beta$ does not leave any curve system invariant up to isotopy (in fact Thurston proved that f is pseudo-anosov). Indeed, if f leaves an element $\gamma \in CS(\Sigma)$ invariant, then $D_\alpha(\gamma) = D_\beta(\gamma)$, i.e., $\alpha^k \gamma = \beta^l \gamma$. Now multiply the equation by γ from the left and use the cancellation law. One obtains a contradiction to the surface filling property.

§5. Reduction to Level-2 Surfaces

The goal of this section is to establish a fairly general criterion to reducing problems concerning all surfaces to that of level-2 surfaces.

5.1. We shall begin by some abstract definitions. Given a subset X of the set $Y^{\mathcal{S}(\Sigma)}$ of all maps from $\mathcal{S}(\Sigma)$ to Y , we say that the subset X has *property RP* if for any decomposition of the surface Σ as a union of two essential subsurfaces A_1 and A_2 which overlap in a level-0 essential subsurface, then the restriction map from $Y^{\mathcal{S}(\Sigma)}$ to $Y^{\mathcal{S}(A_1) \cup \mathcal{S}(A_2)}$ is injective on X . In another words, if f, g are two elements in X so that $f|_{\mathcal{S}(A_1) \cup \mathcal{S}(A_2)} = g|_{\mathcal{S}(A_1) \cup \mathcal{S}(A_2)}$, then $f = g$. For simplicity, we call the function $f|_{\mathcal{S}(A_i)}$ the *restriction* of f to A_i . We say a subset $X \subset Y^{\mathcal{S}(\Sigma)}$ with property RP is *complete* if for any two elements f_1 and f_2 in the restriction of X to A_1 and A_2 so that their restrictions to the overlap $A_1 \cap A_2$ are the same, then there exists an element $f \in X$ whose restrictions to A_i is f_i for $i = 1, 2$. For instance, the set of all geodesic length functions and the set of all intersection functions have complete property RP. This is equivalent to the following gluing lemma for hyperbolic metrics and measured laminations. Namely, suppose the surface Σ is a union of two essential subsurfaces A_1 and A_2 which overlap in an essential level-0 surface. If we are given two hyperbolic metrics d_i on A_i whose restrictions to the overlap of A_1 with A_2 are isotopic, then there is a hyperbolic metric unique up to isotopy on the surface Σ whose restriction to A_i is isotopic to d_i . The same gluing lemma holds for measured laminations. Note that $SL(2, \mathbf{C})$ characters do not have property RP due to the existence of reducible representations. The mapping class group $\Gamma(\Sigma)$ considered as a subset of $\mathcal{S}(\Sigma)^{\mathcal{S}(\Sigma)}$ does not have property RP either. But if one modifies the definition of $\mathcal{S}(\Sigma)$ by taking the isotopy classes of all *oriented* simple loops, then the mapping class group has complete property RP.

5.2. The main reduction lemma says the following. If X is a subset of $Y^{\mathcal{S}(\Sigma)}$ so that for each level-2 essential subsurface Σ' the restriction of X to $Y^{\mathcal{S}(\Sigma')}$ has property RP, then X has property RP. See Appendix D for a proof of this reduction lemma. As a consequence of this reduction lemma, we have the following fact. Suppose Σ is a surface of level at least three and X and X' are two subsets of $Y^{\mathcal{S}(\Sigma)}$ so that for each level-2 essential subsurface Σ' the restrictions

of X and X' to $Y^{\mathcal{S}(\Sigma')}$ are the same. If furthermore that $X \subset X'$ and X has complete property RP, then $X = X'$. To see this, we use induction on the level of subsurfaces. First of all, by the reduction lemma, both X and X' have property RP. Now to show $X' \subset X$, take an element $x' \in X'$ and decompose Σ into a union of two essential surfaces A_1 and A_2 of smaller levels so that they overlap in a level-0 surface. By the induction hypothesis, we find two elements x_1 and x_2 which are in the restrictions of X to A_1 and A_2 so that x_i is the restriction of x' to A_i . But the restrictions of x_i to the overlap are the same, namely, it is the restriction of x' to the overlap. Thus by the completeness, there is an element x in X whose restrictions to A_i is x_i , Thus $x = x'$ by property RP. This shows $X' \subset X$.

By taking X to be the set of all geodesic length functions and X' to be the set of all real valued functions on $\mathcal{S}(\Sigma)$ which satisfy the universal relations in theorem A1 in Appendix A, we see that the reconstruction principle for all Teichmuller spaces follows from that for level-2 surfaces.

Also, the reduction lemma shows that the problem on the automorphisms of the curve complex of a surface is essentially a problem on level-2 surfaces (see [Lu5]).

5.3. The above gives some hints on the special role played by level-2 surfaces. It also supports Grothendick's principle that in the reconstruction process "relations are supported in level-2 surfaces".

§6. $SL(2, \mathbf{C})$ Representation Variety of Surface Groups

6.1. An $SL(2, \mathbf{C})$ representation of a group is a homomorphism of the group into $SL(2, \mathbf{C})$. The character of the representation sends each group element to the trace of the representation matrix. If the group is the fundamental group of a surface, by using a result of Fricke and Klein [FK] and Vogt [Vo], one shows that the character function is determined by its restriction to the set $\mathcal{S}(\Sigma)$ of homotopy classes of simple loops. The main result in [Lu3] shows that the character function on $\mathcal{S}(\Sigma)$ satisfies the reconstruction principle, i.e., except for finitely many (at least 2^{n-1}) exceptional functions defined on $\mathcal{S}(\Sigma_{0,n})$ for $n \geq 5$, a function on $\mathcal{S}(\Sigma)$ is an $SL(2, \mathbf{C})$ character if and only if for each essential level-1 subsurface Σ' in Σ the restriction of the function to $\mathcal{S}(\Sigma')$ is an $SL(2, \mathbf{C})$ character. An exceptional function $f : \mathcal{S}(\Sigma_{0,n}) \rightarrow \mathbf{C}$ satisfies the following (1) $f(\mathcal{S}(\Sigma_{0,n})) = \{2, -2\}$, (2) for each level-1 subsurface, the restriction of f to the subsurface is a character, (3) there exists a level-2 subsurface Σ' so that $f|_{\mathcal{S}(\Sigma')}$ is exceptional. All exceptional functions are constructed from the basic one defined on $\mathcal{S}(\Sigma_{0,5})$ which sends b_i to 2 and all others to -2 . There are no representations whose characters are these exceptional functions.

Given a surface Σ of level-1, $SL(2, \mathbf{C})$ characters on $\mathcal{S}(\Sigma)$ are characterized by the trace identities on the vertices of triangles and quadrilaterals in the modular relation (see Appendix A for the exact statement). Thus the space of all $SL(2, \mathbf{C})$ characters of a surface group can be explicitly described. The reconstruction theorem also holds for any $SL(2, K)$ characters where the field K is quadratically complete (i.e., each quadratic equation with coefficients in K has roots in K).

6.2. The main difficulty in establishing the reconstruction principle for $SL(2, \mathbf{C})$ characters is due to the existence of reducible representations. Recall that an $SL(2, \mathbf{C})$ representation

is *reducible* if it leaves a 1-dimensional linear subspace in \mathbf{C}^2 invariant. Unlike the discrete faithful subgroups in $SL(2, \mathbf{R})$ which occur in the Teichmuller spaces, there are many irreducible representations of a surface group so that its restriction to a subgroup coming from an essential subsurface of negative Euler number is reducible. Now the gluing lemma (see §5.2) is valid only for representations so that their restrictions to the fundamental group of the intersection surface are irreducible. Thus one should choose the decomposition of a surface Σ as a union of two subsurfaces Σ_1 and Σ_2 carefully. It turns out that the following is true which plays a key role in choosing the decomposition of a surface. Namely, a representation of a surface group into $SL(2, \mathbf{C})$ is irreducible if and only if its restriction to the subgroup of a Euler number -1 subsurface is irreducible ([Lu3]).

As a consequence, we obtain the following result concerning $SL(2, K)$ characters on any group. Suppose K is a field so that each quadratic equation with coefficients in K has roots in K . Given a group G , we are interested in finding all $SL(2, K)$ characters on G . In his work on $SL(2, \mathbf{R})$ characters, Helling [He] introduced the notion of *trace function*. Recall that a K valued function f on G is a *trace function* if any two elements x, y in G , $f(xy) + f(xy^{-1}) = f(x)f(y)$ and $f(id) = 2$. Evidently all $SL(2, K)$ characters on G are trace functions due to the trace identity $tr(AB) + tr(AB^{-1}) = tr(A)tr(B)$. One consequence of the characterization theorem is that each trace function is also a character.

6.3. The role of level-1 surfaces among all surfaces is similar to the role of 2-generator groups among all groups. For instance, by Jorgensen's inequality, a non-elementary subgroup in $SL(2, \mathbf{C})$ is discrete if and only if each of its 2-generator subgroup is discrete. The reconstruction theorem for the Teichmuller space says that a faithful representation of a surface group into $SL(2, \mathbf{R})$ is discrete if and only if its restriction to each subgroup of its level-1 subsurface is discrete and uniformizing a surface of the same type. It is natural to ask if the similar description exists for discrete close surface subgroup of $SL(2, \mathbf{C})$.

6.4. It is interesting to ask if the reconstruction principle holds for representations of the surfaces groups into general linear group $GL(n, \mathbf{C})$. To be more precise, suppose f is a complex valued function defined on the conjugacy classes of the fundamental group of the surface so that the restriction of f to the conjugacy classes of the fundamental group of each essential level-1 subsurface is a $GL(n, \mathbf{C})$ -character. Is f the character of some $GL(n, \mathbf{C})$ representation of the surface group?

Appendix A. The Statement of the Reconstruction Theorems for Level-1 Surfaces

Given a hyperbolic metric d on a surface, the *trace* of the metric d is the function $2\cosh l_d/2$ where l_d is the geodesic length function associated to d .

Theorem A1. (a) For the surface $\Sigma_{1,1}$ with $b = \partial\Sigma_{1,1}$, a function $t : \mathcal{S}(\Sigma_{1,1}) \rightarrow \mathbf{R}_{\geq 2}$ is a trace function of a hyperbolic metric if and only if the following hold.

$$\prod_{i=1}^3 t(\alpha_i) = \sum_{i=1}^3 t^2(\alpha_i) + t(b) - 2 \quad \text{and}$$

$$t(\alpha_3)t(\alpha'_3) = \sum_{i=1}^2 t^2(\alpha_i) + t(b) - 2$$

where $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1, \alpha_2, \alpha'_3)$ are distinct ideal triangles in $\mathcal{S}'(\Sigma_{1,1})$.

(b) For the surface $\Sigma_{0,4}$ with $\partial\Sigma_{0,4} = \cup_{i=1}^4 b_i$, a function $t : \mathcal{S}(\Sigma_{0,4}) \rightarrow \mathbf{R}_{\geq 2}$ is a trace function of a hyperbolic metric if and only if for each ideal triangle $(\alpha_1, \alpha_2, \alpha_3)$ so that (α_i, b_j, b_k) bounds a $\Sigma_{0,3}$ in $\Sigma_{0,4}$ the following hold.

$$\prod_{i=1}^3 t(\alpha_i) = \sum_{i=1}^3 t^2(\alpha_i) + \sum_{j=1}^4 t^2(b_j) + \prod_{j=1}^4 t(b_j) + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^4 t(\alpha_i)t(b_j)t(b_k) - 4 \quad \text{and}$$

$$t(\alpha_3)t(\alpha'_3) = \sum_{i=1}^2 t^2(\alpha_i) + \sum_{j=1}^4 t^2(b_j) + \prod_{j=1}^4 t(b_j) + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^4 t(\alpha_i)t(b_j)t(b_k) - 4$$

where $(\alpha_1, \alpha_2, \alpha'_3)$ and $(\alpha_1, \alpha_2, \alpha_3)$ are two distinct ideal triangles in $\mathcal{S}'(\Sigma_{0,4})$.

Part (a) of theorem A1 was a result of Fricke-Klein [FK] and Keen [Ke2] and part (b) was proved in [Lu1].

Theorem A2. (a) For the surface $\Sigma_{1,1}$, a function $f : \mathcal{S}(\Sigma_{1,1}) \rightarrow \mathbf{R}_{\geq 0}$ is an intersection function if and only if the following hold.

$$f(\alpha_1) + f(\alpha_2) + f(\alpha_3) = \max_{i=1,2,3} (2f(\alpha_i), f([\partial\Sigma_{1,1}]))$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is an ideal triangle, and

$$f(\alpha_3) + f(\alpha'_3) = \max(2f(\alpha_1), 2f(\alpha_2), f([\partial\Sigma_{1,1}]))$$

where $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1, \alpha_2, \alpha'_3)$ are two distinct ideal triangles.

(b) For the surface $\Sigma_{0,4}$ with $\partial\Sigma_{0,4} = b_1 \cup b_2 \cup b_3 \cup b_4$, a function $f : \mathcal{S}(\Sigma_{0,4}) \rightarrow \mathbf{R}_{\geq 0}$ is an intersection function if and only if for each ideal triangle $(\alpha_1, \alpha_2, \alpha_3)$ so that (α_i, b_s, b_r) bounds a $\Sigma_{0,3}$ in $\Sigma_{0,4}$ the following hold.

$$\Sigma_{i=1}^3 f(\alpha_i) = \max_{1 \leq i \leq 3, 1 \leq s \leq 4} (2f(\alpha_i), 2f(b_s), \sum_{s=1}^4 f(b_s), f(\alpha_i) + f(b_s) + f(b_r))$$

$$f(\alpha_3) + f(\alpha'_3) = \max_{1 \leq i \leq 2, 1 \leq s \leq 4} (2f(\alpha_i), 2f(b_s), \sum_{s=1}^4 f(b_s), f(\alpha_i) + f(b_s) + f(b_r))$$

where $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1, \alpha_2, \alpha'_3)$ are two distinct ideal triangles.

Theorem A2 was proved in [Lu2].

Below is the statement of the presentation of the mapping class group for all surfaces of negative Euler number (see [Lu3]).

Theorem A3. For a compact oriented surface Σ of negative Euler number, the mapping class group $\Gamma(\Sigma)$ has the following presentation:

generators: $\{D_\alpha : \alpha \in \mathcal{S}(\Sigma)\}$.

relations: (I) $D_\alpha D_\beta = D_\beta D_\alpha$ if $\alpha \cap \beta = \emptyset$.

(II) $D_{\alpha\beta} = D_\alpha D_\beta D_\alpha^{-1}$ if $\alpha \perp \beta$.

(III) $(D_\alpha D_\beta D_{\alpha\beta})^4 = D_{\partial(\alpha \cup \beta)}$ if $\alpha \perp \beta$.

(IV) $D_\alpha D_\beta D_{\alpha\beta} = D_{\partial(\alpha, \beta)}$ if $\alpha \perp_0 \beta$.

The characterization of the $SL(2, \mathbf{K})$ characters for surface group representations is given by the following. The theorem is proved by Fricke and Klein [FK] and Vogt [Vo], though stated in different terminologies. See for instance [Go] and [Lu3].

Theorem A4. (a) For the surface $\Sigma_{1,1}$ with $b = \partial\Sigma_{1,1}$, a function $t : \mathcal{S}(\Sigma_{1,1}) \rightarrow \mathbf{C}$ is an $SL(2, \mathbf{C})$ trace function if and only if the following hold.

$$\prod_{i=1}^3 t(\alpha_i) = \sum_{i=1}^3 t^2(\alpha_i) - t(b) - 2 \quad \text{and}$$

$$t(\alpha_3) + t(\alpha'_3) = t(\alpha_1)t(\alpha_2)$$

where $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1, \alpha_2, \alpha'_3)$ are distinct ideal triangles in $\mathcal{S}'(\Sigma_{1,1})$.

(b) For the surface $\Sigma_{0,4}$ with $\partial\Sigma_{0,4} = \cup_{i=1}^4 b_i$, a function $t : \mathcal{S}(\Sigma_{0,4}) \rightarrow \mathbf{C}$ is an $SL(2, \mathbf{C})$ trace function if and only if for each ideal triangle $(\alpha_1, \alpha_2, \alpha_3)$ so that (α_i, b_j, b_k) bounds a $\Sigma_{0,3}$ in $\Sigma_{0,4}$ the following hold.

$$\prod_{i=1}^3 t(\alpha_i) = - \sum_{i=1}^3 t^2(\alpha_i) - \sum_{j=1}^4 t^2(b_j) - \prod_{j=1}^4 t(b_j) + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^4 t(\alpha_i)t(b_j)t(b_k) + 4 \quad \text{and}$$

$$t(\alpha_3) + t(\alpha'_3) = -t(\alpha_1)t(\alpha_2) + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^4 t(\alpha_i)t(b_j)t(b_k)$$

where $(\alpha_1, \alpha_2, \alpha'_3)$ and $(\alpha_1, \alpha_2, \alpha_3)$ are two distinct ideal triangles in $\mathcal{S}'(\Sigma_{0,4})$.

Appendix B. The Modular Structure on the Space of 3-Holed Sphere Decompositions

The other natural example of compact modular structure is the set $HD(\Sigma)$ of all isotopy classes of 3-holed sphere decompositions of a surface Σ . The charts are constructed as follows. Suppose (a_1, \dots, a_k) is an element in $HD(\Sigma)$. Take an essential subsurface Σ' of level-1 so that all but one, say a_i of the coordinate, are disjoint from Σ' . Now the chart associate to Σ' is the set of elements $\{(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_k) \in HD(\Sigma) | b_i \in \mathcal{S}'(\Sigma')\}$ with chart map sending the element to the slope of b_i . Again if two charts overlap in two elements, they coincide. A result of Hatcher-Thurston [HT] says that given any two elements in $HD(\Sigma)$ there is a sequence of charts whose union contains these two elements so that any two adjacent charts overlap in at least one element. On the other hand, each element in $HD(\Sigma)$ determines a handlebody structure on the surface Σ obtained by attaching 2-cells to the components of the 3-holed sphere

decomposition and then 3-cells. Evidently if two elements in $HD(\Sigma)$ lie in a chart associated to a 4-holed sphere, then they determine the same handlebody structure. The main result in [Lu5] shows that conversely it is also true. Namely, if two elements in $HD(\Sigma)$ determine the same handlebody structure, then there is a sequence of charts associated to 4-holed spheres whose union contains these two elements so that any two adjacent charts overlap in at least one element.

Appendix C. A Simple Proof That the Multiplication Produces a Curve System

For simplicity, let us assume that the surface is closed (see [Lu2] for general cases). We will give a simple proof of the following fact used in §4. Namely, if a and b are two curve systems so that they intersect minimally in their isotopy classes, then the 1-dimensional submanifold ab obtained by resolving all intersection points in $a \cup b$ from a to b is again a curve system.

Suppose otherwise that the 1-submanifold ab contains a null homotopic component c . We may assume that c is the “inner-most” component, i.e., in the interior of the disc D bounded by c , there are no other components of ab . Let us consider all components of $\Sigma - (a \cup b)$ which are inside D , say A_1, \dots, A_k . Each A_i is an open disc since c is the inner-most. The boundary of A_i consists of arcs in a and b , and the corners of A_i corresponds to the intersection points of a and b . Thus we may call each A_i a polygon bounded by sides in a and b alternatively. Since a intersects b minimally within their isotopy classes, each A_i has at least four sides. Now by the definition of the resolution, the disc D is obtained by resolving corners of A_i 's from a to b . Considering the resolutions at the vertices along the boundary of A_i , one sees that corners open and close alternatively in a cyclic order on the boundary. Form a graph in D by assigning a vertex in each A_i and joining an edge between two vertices if their corresponding polygons A_i and A_j have the same corner which is opened by the resolution. Then, on one hand, the graph is a tree since it is homotopic to the disk. On the other hand, each vertex of the graph has valence at least two since the valence of the vertex is half of the number of sides of the corresponding polygon A_i (by the alternating property). This contradicts the fact that a tree must have a vertex of valence one.

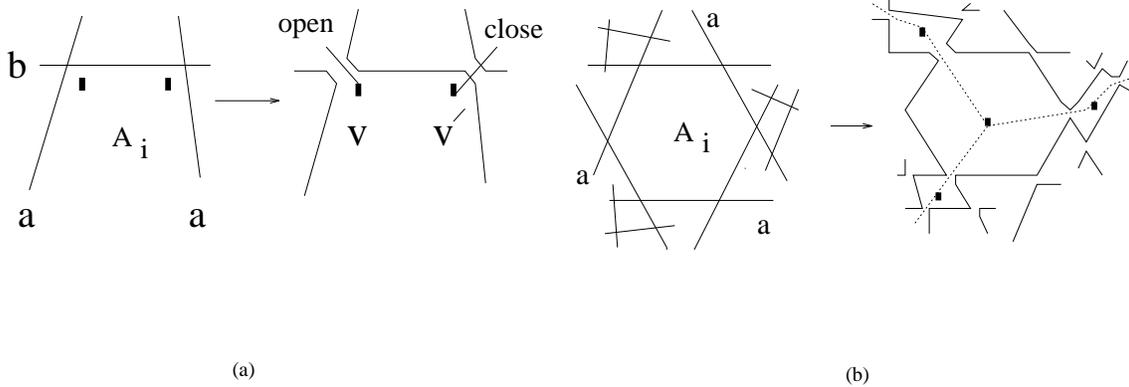


Figure 11

Appendix D. A Proof of the Reduction Lemma in §5

We shall prove the following reduction lemma stated in §5. Suppose Σ is a surface of level at least 3. If X is a subset of $Y^{\mathcal{S}(\Sigma)}$ so that for each level-2 essential subsurface Σ' the restriction of X to $Y^{\mathcal{S}(\Sigma')}$ has property RP, then X has property RP.

To begin the proof, suppose the surface Σ is decomposed into a union of two essential subsurfaces A_1 and A_2 overlapping in an essential level-0 surface and we are given two elements f, g in X whose restrictions to A_i are the same. The goal is to show that $f = g$. To this end, let us construct a 3-holed sphere decomposition (a_1, \dots, a_k) of the surface Σ so that each 3-holed sphere in the decomposition is either in A_1 or in A_2 and $A_1 \cap A_2$ is bounded by a_i 's. Thus if s is an element in $\mathcal{S}(\Sigma)$ which intersects only one element of $\{a_1, \dots, a_k\}$, then s is in $\mathcal{S}(A_i)$ for $i = 1$ or 2 . In particular $f(s) = g(s)$. Now suppose we make an elementary move on $\{a_1, \dots, a_k\}$ to produce a new 3-holed sphere decomposition $\{b_1, \dots, b_k\}$ where all but one of b_i are a_i and the exceptional component, say b_j is modular related to a_j (i.e., $b_j \perp a_j$ or $b_j \perp_0 a_j$) (these moves were introduced in the appendix of [HT]). We claim that if $f(s) = g(s)$ for all elements s which intersects at most one of $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ is obtained from $\{a_1, \dots, a_k\}$ by an elementary move, then $f(s) = g(s)$ for all elements s which intersects at most one of $\{b_1, \dots, b_k\}$. Indeed, by the property RP for level-2 surfaces, we see that $f(s) = g(s)$ for all elements s inside any level-2 subsurface which is bounded by elements in $\{a_1, \dots, a_k\}$. Now if s is an isotopy class which intersects at most one element in $\{b_1, \dots, b_k\}$, then s intersects at most two elements in $\{a_1, \dots, a_k\}$. Thus the isotopy class s is in a level-2 subsurface which is bounded by elements in $\{a_1, \dots, a_k\}$. Thus $f(s) = g(s)$. Now by the result in [HT] that any two 3-holed sphere decompositions of the surface are related by a finite sequence of elementary moves, it follows that $f(s) = g(s)$ for any element s in $\mathcal{S}(\Sigma)$.

In view of the importance of the 3-holed sphere decompositions, it is attempting to make a 2-dimensional cell-complex Z based on 3-holed sphere decompositions of the surfaces as follows. The vertices of Z are the isotopy classes of 3-holed sphere decompositions of the surface and the edges are those pair of vertices related by an elementary move. Now attaching a 2-cell to each 5-gon associated to each pentagon relation (see figure 9), a 2-cell to each 4-gon associated to four elementary moves which are supported in two disjoint level-1 surfaces, and a 2-cell to each 3-gon associated to three elementary moves supported in a level-1 surface. This cell-complex was implicitly introduced in the appendix of [HT]. The simple connectivity of the cell-complex seems to be asserted in [HT]. See also [Ha] for related topics.

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