A Characterization of Spherical Cone Metrics on Surfaces

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Abstract We study the spherical cone metrics on surfaces from the point of view of inner angles. A rigidity result is obtained. The existence of spherical cone metric of Delaunay type is also established.

§1. **Introduction**

1.1. In an attempt to understand the geometric triangulations of closed 3-manifolds with constant sectional curvature metrics, we are led to the study of spherical cone metrics on the 2-sphere. These cone metrics appear as the link of a vertex in the 3-dimensional geometric triangulation. Naturally, ones asks if we can understand the spherical cone metrics in terms of the inner angles of the spherical triangles. The main results of the paper give a characterization of the spherical cone metrics in terms of the inner angles and a characterizations of the spherical Delaunay triangulations of the 2-spheres. Similar results for Delaunay triangluations of the surfaces in the Euclidean or hyperbolic cone metrics have been worked out beautifully by Rivin [Ri] and Leibon [Le]. One of the interesting consequence of our work is the following rigidity result. Namely, a totally geodesic triangulation of the 2-sphere (in the standard metric) is uniquely determined by its *edge invariants.* Here the edge invariant of the triangulation at each edge is the sum of the two inner angles facing the edge.

1.2. We now set up the frame work. Suppose S is a closed surface and T is a triangulation of the surface. Here by a triangulation we mean the following: take a finite collection of triangles and identify their edges in pairs by homeomorphisms. Let V, E, F be the sets of all vertices, edges and triangles in the triangulation T respectively. If a, b are two simplices in the triangulation T, we use $a < b$ to denote that a is a face of b. The set of *corners* of T is $\{(e,f) | e \in E, f \in F$ so that $e < f\}$ and is denoted by $C(T)$. By a *linear spherical structure* on the triangulated surface (S,T) we mean a map $x : C(T) \to (0,\pi)$ so that for each $f \in T$ and the three edges e_1, e_2, e_3 of f, the numbers $x_i = x(e_i, f)$, $i = 1,2,3$, form the inner angles of a spherical triangle. A *spherical cone metric* on the triangulated surface (S,T) is a map $l : E \to (0,\pi)$ so that for each triangle f and its three edges e_1, e_2, e_3, the three numbers $l_i = l(e_i)$, $i = 1,2,3$, form the edge lengths of a spherical triangle. Evidently, given any spherical cone metric, there is a natural linear spherical structure associated to it by measuring its inner angles. One of the goal in the paper is to characterize the set of all spherical cone metrics inside the space of all linear spherical structures. To this end, we introduce the notion of the *edge invariant* D_x of the linear spherical structure x. The edge invariant D_x is the map defined on the set of all edges E so that the its value at an edge is the sum of the two inner angles facing the edge, i.e., $D_x(e) = x(e,f) + x(e,f')$ where $f, f' \in F$ and $e < f, e < f'$ (it may occur that $f = f'$).

Theorem 1.1. *Given any triangulated closed surface and a real valued function D defined on the set of all edges of the triangulation, there is at most one spherical cone metric having D as the edge invariant.*

Theorem 1.2. *Given any triangulated closed surface and any edge invariant function*

$D : E \to (0, \pi)$ so that there are two linear spherical structures having D as the edge invariant, then there exists a spherical cone metric having D as the edge invariant function.

Similar results for Euclidean background geometry have been established by Igor Rivin [Ri]. Similar questions for hyperbolic cone metrics is still open. The work of Leibon [Le] established the analogous results for functions defined on E other than the edge invariant functions. The functional used in Rivin's approach is the volume of the ideal hyperbolic 3-simplex associated to the Euclidean triangle. The geometric meaning of the functional used in the paper is not clear to us.

Given any function $D : E \to (0, \infty)$, whether there exists a linear spherical structure having D as the edge invariant is a linear programming problem. We defer the study of this problem in a future paper.

The space of all spherical cone metrics on (S, T), denoted by $CM(S, T)$ is an open convex polyhedron of dimension $|E|$, the number of edges. The space of all edge invariant functions on (S, T), denoted by $EI(S, T)$ is also an open convex polyhedron of dimension $|E|$. The map $\Pi : CM(S, T) \to EI(S, T)$ sending a cone metric to its edge invariant is evidently a smooth map between two open cells of the same dimension. Theorem 1.1 shows that the map is injective and theorem 1.2 shows that its image contains a convex polytope subset of edge invariants strictly bounded above by π. It is interesting to know what the image of Π is. Is it possible that the image of Π is an open convex polyhedron in $EI(S, T)$? The situation is a bit similar to Thurston's proof of his circle packing theorem for triangulated surface of negative Euler characteristic ([Th]).

We remark that a slightly stronger verson of theorem 1.2 can also be established for edge invariants $D(E) \subset (0, \pi]$. See theorem 2.1.

The strategy of proving theorems 1.1 and 1.2 goes as follows. For each linear spherical structure, we introduce the concept of *capacity* of the structure. The capacity defines a strictly convex function on the space $LS(S, T)$ of all linear spherical structures on (S, T). For a given edge invariant $D : E \to (0, \infty)$, we consider the subset $LS(S, T; D)$ of $LS(S, T)$ consisting of all linear spherical structures with D as the edge invariant. We prove that the critical points of the capacity function restricted to the subspace $LS(S, T; D)$ are exactly the spherical cone metrics on (S, T). Since a strictly convex function cannot have more than one critical points, theorem 1.1 follows. To show theorem 1.2, we show that the capacity function which has a natural continuous extension to the compact closure of $LS(S, T; D)$ cannot achive its minimal points in the boundary.

1.3. The paper is organized as follows. In section 2, we recall some known facts about the derivatives of the cosine laws. We also introduce the capacity function. Some of the basic properties of the capacity function are established. In particular, we prove theorems 1.1 and 1.2 in section 2 assuming two important properties of the capacity function. These two properties are established in sections 3 and 4. In section 3, we show that the capacity function has a continuous extension to the degenerated spherical triangles by relating it to the Lobachevsky function. In section 4, we study the behavior of the derivative of the capacity function at the degenerated spherical triangles.

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§2. **Spherical Triangles and Proofs of Theorems 1.1 and 1.2**

We prove theorems 1.1 and 1.2 assuming several technical properties on spherical triangles in this section. For simplicity, we assume that the indices i, j, k are pairwise distinct in this section.

2.1. Given a spherical triangle with inner angles x_1, x_2, x_3, let y_1, y_2, y_3 be the edge lengths so that y_i-th edge is facing the angle x_i. The cosine law states that,

$$\cos y_i = \frac{\cos x_i + \cos x_j \cos x_k}{\sin x_j \sin x_k}, \tag{2.1}$$

where $\{i, j, k\} = \{1, 2, 3\}$.

Furthermore, the partial derivatives of y_i as a function of $x = (x_1, x_2, x_3)$ is given by the following lemma.

Lemma 2.1. *For any spherical triangle of inner angles x_i, x_j, x_k and the corresponding edge lengths y_i, y_j, y_k, where $\{i, j, k\} = \{1, 2, 3\}$, we have,*

(a) $\partial y_i/\partial x_i = \sin(x_i)/A_{ijk}$ where $A_{ijk} = \sin y_i \sin x_j \sin x_k$ satisfies $A_{ijk} = A_{jki}$,

(b) $\partial y_i/\partial x_j = \partial y_i/\partial x_i \cos y_k$.

The proof is a simple exercise in calculus. We omit the details.

The space of all spherical triangles parametrized by its inner angles x_1, x_2, x_3, denoted by M_3, is the open tetrahedron $\{x = (x_1, x_2, x_3) \in (0, \pi)^3 | x_i^* > 0, \sum_{i=1}^3 x_i > \pi\}$ where $x_i^* = 1/2(\pi + x_i - x_j - x_k)$, $\{i, j, k\} = \{1, 2, 3\}$. To see that these inequalities are necessary, we first note that the sum of inner angles of a spherical triangle is larger than π. To see $x_1^* > 0$, we note that if x_1, x_2, x_3 are the inner angles of a spherical triangle A, then $x_1, \pi - x_2, \pi - x_3$ also form the inner angles of a spherical triangle B so that $A \cup B$ forms a region bounded by two great circles intersecting at an angle x_1. It follows that the sum $x_1 + \pi - x_2 + \pi - x_3 > \pi$. This shows $x_1^* > 0$ is necessary. It is not difficult to show that these four inequalities are also sufficient.

Corollary 2.2. *(a) The differential 1-form $w = \sum_{i=1}^3 \ln \tan(y_i/2) dx_i$ is closed in the open set M_3.*

(b) The function $\theta(x) = \int_{(\pi/2,\pi/2,\pi/2)}^x w$ is well defined on M_3 and is strictly convex.

We remark that corollary 2.2(a) also holds for hyperbolic triangles. Namely one can define the similar $\theta(x)$ function for hyperbolic triangles using partial derivatives of the cosine law. The only difference is that the $\theta(x)$ function for hyperbolic triangles is no longer convex.

Proof. To show part (a), it suffices to prove $\partial(\ln \tan(y_i/2))/\partial x_j$ is symmetric in i, j. By lemma 2.1, the partial derivative is found to be

$$1/\sin(y_i) \partial y_i/\partial x_j = \cos(y_k)[\sin(x_i)/\sin(y_i)]/A_{ijk}. \tag{2.2}$$

By the sine law, one sees clearly that the partial derivative is symmetric in i, j. Note also that

$$\partial(\ln \tan(y_i/2))/\partial x_i = [\sin(x_i)/\sin(y_i)]/A_{ijk}. \tag{2.3}$$

Since the space M_3 is simply connected, we see that the function $\theta(x)$ is well defined on M_3. To show that the function θ is strictly convex, let us calculate its Hessian matrix $H = [h_{rs}]_{3\times 3}$. By definition, we have $h_{rs} = \partial(\ln\tan(y_r/2))/\partial x_s$. By (2.2) and (2.3), we have $h_{ij} = h_{ii}\cos y_k$ and $h_{11} = h_{22} = h_{33} > 0$ by the sine law. Thus the matrix H is a positive multiplication of the matrix $[a_{rs}]$ where $a_{ij} = \cos y_k$ and $a_{ii} = 1$. For a spherical triangle of edge lengths y_1, y_2, y_3, the matrix $[a_{rs}]$ is always positive definite. Indeed, let v_1, v_2, v_3 be the three unit vectors in the 3-space forming the vertices of the spherical triange, then by definition, a_{rs} is the inner product of v_r with v_s. Thus the matrix $[a_{rs}]$ is positive definite since it is the Gram matrix of three independent vectors. QED

2.3. The closure of M_3 in $\mathbf{R}^3$ is given by $\bar{M}_3 = \{x \in [0,\pi]^3 | x_i^* \geq 0, x_1 + x_2 + x_3 \geq \pi\}$. In sections 3 and 4, we will establish the following two properties concerning the function θ.

Proposition 3.1. *The function* $\theta(x) = \int_{(\pi/2,\pi/2,\pi/2)}^{x} \sum_{i=1}^{3} \ln\tan(y_i/2) dx_i$ *is given by the following,*

$$\theta(x_1, x_2, x_3) = -\sum_{i=1}^{3} \Lambda(x_i^*) - \Lambda(\pi/2 + (x_1 + x_2 + x_3)/2)$$

where $\Lambda(t) = -\int_0^t \ln|2\sin u| du$ *is the Lobachevsky function. In particular,* θ *has a continuous extension to the closure* $\bar{M}_3$ *of the moduli space of spherical triangles* M_3.

Proposition 4.1. *For any point* $a \in \bar{M}_3 - M_3$ *and a point* $p \in M_3$*, let* $f(t)$ *be the function* $\theta((1-t)a + tp)$ *where* $t \in (0,1)$. *If* a *is not one of* $(0,0,\pi), (0,\pi,0), (\pi,0,0), (\pi,\pi,\pi)$*, then*

$$\lim_{t\to 0^+} f'(t) = -\infty.$$

If $a \in \{(0,0,\pi), (0,\pi,0), (\pi,0,0), (\pi,\pi,\pi)\}$*, then the limit* $\lim_{t\to 0^+} f'(t)$ *exists and is a finite number.*

In the rest of the section, we prove theorems 1.1 and 1.2 assuming propositions 3.1 and 4.1.

2.4. Given a spherical triangle of inner angles x_1, x_2, x_3, we define its *capacity* to be $\theta(x_1, x_2, x_3)$ where θ is the function introduced in corollary 2.2. For a linear spherical structure, we define its capacity to be the sum of the capacities of its spherical triangles. To write down the capacity function explicitly, let us fix some notations. First, let us label the set of all corners in (S,T) by integers $\{1, ..., n\}$. If three corners labeled by a, b, c are of the form $(e_1, f), (e_2, f), (e_3, f)$, we denote it by $\{a, b, c\} \in \Delta$ and call $\{a, b, c\}$ forms a triangle. For a linear spherical structure $x : C(S,T) \to (0,\pi)$, we use x_r to denote the value of x at the r-th corner and consider $x = (x_1, ..., x_n)$ as a vector in $\mathbf{R}^n$. Under this identification, the space of all linear spherical structures $LS(S,T) = \{x \in (0,\pi)^n |$ whenever r, s, t form a triangle, $(x_r, x_s, x_t) \in M_3\}$ becomes an open convex polyhedron of dimension n. The capacity of the linear spherical structure x, denoted by $\Theta(x)$, is given by,

$$\Theta(x) = \sum_{\{r,s,t\}\in\Delta} \theta(x_r, x_s, x_t).$$

Since $\theta(x_1, x_2, x_3)$ is strictly convex, we have,

Lemma 2.2. *The capacity function Θ defined on $LS(S,T)$ is a strictly convex function.*

2.5. For any map $D : E \to (0, \infty)$, we define the subspace of all linear spherical structures with edge invariant functions equal to D, denoted by $LS(S,T;D)$ the set consisting of $x \in LS(S,T)$ so that $D_x = D$.

Lemma 2.3. *If $LS(S,T;D)$ contains at least two points, then the critical points of $\theta|_{LS(S,T;D)}$ are exactly those linear spherical structures derived from spherical cone metrics.*

Proof. For simplicity, let us set $G = \Theta|_{LS(S,T;D)}$. Applying the Lagrangian multipliers to Θ on $LS(S,T)$ subject to the set of linear constraints $D_x(e) = D(e)$ for $e \in E$, we see that at a critical point of G, there is a map $C : E \to \mathbf{R}$ (the multipliers) so that, for all indices i,

$$\partial\Theta/\partial x_i = C_e \tag{2.4}$$

where the i-th corner is of the form (e, f), i.e., the i-th corner is facing the edge e. Let the three corners of the triangle f be labeled by i, j, k. Then $\partial\Theta/\partial x_i = \ln\tan(y_i/2)$ where y_i is given by the cosine law (2.1). This shows, by (2.4), that the edge length of e in the spherical triangle of inner angles x_i, x_j, x_k depends only on C_e. In particular, if f' is the second triangle in F having e as an edge, then the length of e calculated in f' in the linear spherical structure is the same as the length of e calculated using f. In summary, we see that there is a well defined assignment of edge lengths $l : E \to (0, \pi)$ so that the assignment on the three edges of each triangle forms the lengths of a spherical triangle and the inner angles induced by l is x.

To see the result in the other direction, suppose we have a point in $LS(S,T;D)$ which is induced from a spherical cone metric $l : E \to (0, \infty)$. We want to show that the point is a critical point of G. Since the constraints $D_x = D$ are linear, the critical points p of G on $LS(S,T;D)$ are the same as those points $q \in LS(S,T;D)$ so that there is a map $C : E \to \mathbf{R}$ satisfying (2.4) at q. Evidently at a linear spherical structure derived from a spherical cone metric $l : E \to \mathbf{R}_{>0}$, we define C_e to be $\ln\tan(l(e)/2)$. Then (2.4) follows. QED

As a consequence of lemma 2.2 and lemma 2.3, we see theorem 1.1 follows.

2.6. To prove theorem 1.2, by proposition 3.1, the function Θ on the space of all linear spherical structure $LS(S,T;D)$ has a continuous extension to the closure $\bar{L}S(S,T;D)$ of $LS(S,T;D)$ in $\mathbf{R}^n$. The closure is evidently compact since it is contained in $[0,\pi]^n$. Take a minimal point a of Θ in the closure $\bar{L}S(S,T;D)$. If the point a is in $LS(S,T;D)$, we are done. If $a \in \partial LS(S,T;D)$, there is a triple of indices $\{u', v', w'\}$ so that $(a_{u'}, a_{v'}, a_{w'})$ is in the boundary of M_3. Take a point $p \in LS(S,T;D)$ and consider the smooth path $\gamma(t) = (1-t)a + tp$ for $t \in (0,1]$ in $LS(S,T;D)$. Let $g(t) = \theta(\gamma(t))$. We have $g(t) \geq g(0)$ for all $t > 0$ by the choice of the point a. Thus, $\liminf_{t\to 0^+} dg/dt \geq 0$. But, by proposition 4.1, we have

$$\lim_{t\to 0^+} dg/dt = -\infty. \tag{2.5}$$

This produces a contradiction. Here is the more detailed argument to see (2.5).

Let Δ_1 be the set of all triples of indices $\{\{u,v,w\}|$ so that $\{u,v,w\} \in \Delta$ and $(a_u, a_v, a_w) \in \partial M_3\}$ and $\Delta_2 = \Delta - \Delta_1$. Then the function g can be written as,

$$g(t) = \sum_{\{u,v,w\}\in\Delta_1} \theta(x_u(t), x_v(t), x_w(t)) + \sum_{\{u,v,w\}\in\Delta_2} \theta(x_u(t), x_v(t), x_w(t))$$

where $x(t) = x(\gamma(t))$. The derivative $g'(t)$ can be expressed as,

$$g'(t) = \sum_{\{u,v,w\}\in\Delta_1} d/dt[\theta(x_u(t), x_v(t), x_w(t))] + \sum_{\{u,v,w\}\in\Delta_2} d/dt[\theta(x_u(t), x_v(t), x_w(t))].$$

Note that since the edge invariant D is assumed to be strictly less than π, if $\{u,v,w\}$ is in Δ_1, then the triple (a_u, a_v, a_w) is in $\partial M_3 - \{(0,0,\pi),(0,\pi,0),(\pi,0,0),(\pi,\pi,\pi)\}$. Thus by proposition 4.1, as t tends to 0, each terms in the first sum tends to $-\infty$. Each term in the second sum tends to a finite number as t tends to 0. Thus we see (2.5) holds.

2.7. The above proof in fact shows the following stronger result. *A cycle* in the triangulated surface (S,T) is an ordered collection of edges and triangles $\{e_1, f_1, e_2, f_2, ..., e_n, f_n\}$ so that e_i and e_{i+1} are edges in f_i and e_1, e_n are edges of f_n. An edge invariant assignment D is said to contain a {0,0,π}-cycle if there is a cycle of edges and a point $a \in \partial LS(S,T;D)$ so that $D_a(e_i) = \pi$ and the inner angles of each f_i in a are $0, 0, \pi$.

Theorem 2.1. *Given any triangulated surface and any edge invariant function $D : E \to (0,\pi]$ which contains no $\{0,0,\pi\}$-cycles, if there are two linear spherical structures having D as the edge invariant, then there exists a spherical cone metric having D as the edge invariant function.*

The proof is evident.

§3. **Continuous Extension of the Capacity Function**

We show that the capacity of spherical triangles extends continuously to the degenerated triangles. For the rest of the section, we take a spherical triangle of inner angles x_1, x_2, x_3 and edge lengths y_1, y_2, y_3 so that y_i-th edge is facing the x_i-th inner angle. We use $x = (x_1, x_2, x_3)$ and $x_i^* = 1/2(\pi + x_i - x_j - x_k)$. As a convention, we assume the indices $\{i,j,k\} = \{1,2,3\}$. Recall that the Lobachevsky function $\Lambda(t) = -\int_0^t \ln|2\sin u| du$. The function is continuous on the real line $\mathbf{R}$ and is an odd periodic function of period π. See Milnor [Mi] for more details. The main result of the section is the following.

Proposition 3.1. *The capacity function* $\theta(x) = \int_{(\pi/2,\pi/2,\pi/2)}^{x} \sum_{i=1}^{3} \ln\tan(y_i/2) dx_i$ *is given by the following,*

(3.1) $$\theta(x_1,x_2,x_3) = -\sum_{i=1}^{3} \Lambda(x_i^*) - \Lambda(\pi/2 + (x_1+x_2+x_3)/2)$$

In particular, θ has a continuous extension to the closure $\bar{M}_3$ of the moduli space of spherical triangles $M_3 = \{(x_1, x_2, x_3) \in (0, \pi)^3 | x_1 + x_2 + x_3 > \pi$ and $x_i^* > 0,\ i = 1, 2, 3\}$.

Proof. The proof is a straight forward computation using the cosine law. Recall that the cosine law (2.1) says

$$\cos y_i = \frac{\cos x_i + \cos x_j \cos x_k}{\sin x_j \sin x_k}.$$

Use the summation formulas for cosine function that

$$\cos(a + b) = \cos a \cos b - \sin a \sin b,$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b,$$

$$\cos a + \cos b = 2 \cos((a + b)/2) \cos((a - b)/2),$$

$$\cos a - \cos b = 2 \sin((a + b)/2) \sin((b - a)/2),$$

we can rewrite the cosine law as one of the following,

$$\cos y_i - 1 = 2\frac{\sin x_i^* \cos((x_i + x_j + x_k)/2)}{\sin x_j \sin x_k}, \tag{3.2}$$

and

$$\cos y_i + 1 = 2\frac{\sin x_j^* \sin x_k^*}{\sin x_j \sin x_k}. \tag{3.3}$$

In particular,

$$\frac{1 - \cos y_i}{1 + \cos y_i} = -\frac{\sin x_i^* \cos((x_i + x_j + x_k)/2)}{\sin x_j^* \sin x_k^*}. \tag{3.4}$$

However, we also have the trigonometric identity,

$$\tan^2(u/2) = \frac{1 - \cos u}{1 + \cos u}. \tag{3.5}$$

Since by definition, $\partial\theta/\partial x_i = \ln \tan(y_i/2)$, by (3.2)-(3.5), we have

$$\partial\theta/\partial x_i = 1/2[\ln \sin x_i^* - \ln \sin x_j^* - \ln \sin x_k^* + \ln(|\sin((x_1 + x_2 + x_3)/2 + \pi/2)|)] \tag{3.6}$$

Since the function $F(x_1, x_2, x_3)$ given by the right hand side of the (3.1) has the partial derivative,

$$\partial F/\partial x_i = 1/2[\ln(2 \sin x_i^*) - \ln(2 \sin x_j^*) - \ln(2 \sin x_k^*) + \ln(2|\sin((x_1 + x_2 + x_3)/2 + \pi/2)|)]$$

$$= 1/2[\ln\sin x_i^* - \ln\sin x_j^* - \ln\sin x_k^* + \ln(|\sin((x_1+x_2+x_3)/2+\pi/2)|)],$$

we see that $\partial F/\partial x_i = \partial\theta/\partial x_i$. In particular, these two functions differ by a constant on M_3. Since $\theta(\pi/2,\pi/2,\pi/2) = 0 = F(\pi/2,\pi/2,\pi/2)$, the result follows. In particular, we see that θ has a continuous extension to the 3-space $\mathbf{R}^3$. QED

Remark 3.1. Proposition 3.1 shows that the function $\theta(x_1, x_2, x_3)$ is closely related to the function

$$V(x_1, x_2, x_3) = \sum_{i=1}^{3}(\Lambda(x_i) + \Lambda(x_i^*)) + \Lambda((x_1+x_2+x_3)/2+\pi/2),$$

namely $V(x) = \sum_{i=1}^{3}\Lambda(x_i) - \theta(x)$. The function $V(x)$ is well known to be twice the hyperbolic volume of a hyperbolic tetrahedron with three vertices at the sphere at infinite so that the link at the finite vertex is the spherical triangle of inner angles x_1, x_2, x_3 (see [Vi], also [Le]). For simplicity, we call $V(x)$ the *volume function.* Note that this volume function is not concave on M_3. This fact was first noticed by Peter Doyle [Le]. But $V(x)$ is concave in the set $\{(x_1,x_2,x_3) \in [0,\pi]^3 | x_1+x_2+x_3 \le \pi\}$. The restrictions of the function $V(x)$ to the subsets $\{(x_1,x_2,x_3) \in [0,\pi]^3 | x_1+x_2+x_3 = \pi\}$ and $\{(x_1,x_2,x_3) \in [0,\pi]^3 | x_1+x_2+x_3 < \pi\}$ are the ones used by both Rivin [Ri] and Leibon [Le] in the study of the Delaunay triangulations on surfaces with Euclidean and hyperbolic cone metrics. The other related works are [CV] and [BS].

§4. **Degeneration of Spherical Triangles**

The goal of this section is to understand how a sequence of spherical triangles degenerates and to understand the behavior of the derivatives of the capacity on the sequence of degenerated spherical triangles. Recall that the moduli space M_3 of spherical triangles is an open regular tetrahedron in the 3-space. The closure $\bar{M}_3$ of M_3 is the closed tetrahedron. We call a point in the boundary $\partial M_3 = \bar{M}_3 - M_3$ a *degenerated* spherical triangle (with respect to inner angles). To be more precise, a degenerated triangle is the limit of a sequence of spherical triangles so that the inner angles converge. The goal of the section is to prove,

Proposition 4.1. *For any point $a \in \bar{M}_3 - M_3$ and a point $p \in M_3$, let $f(t) = \theta((1-t)a + tp)$ where $t \in [0,1]$. If a is not equal to any of the points $(0,0,\pi)$, $(0,\pi,0)$, $(\pi,0,0)$, (π,π,π), then*

$$\lim_{t\to 0^+} f'(t) = -\infty. \tag{4.1}$$

If $a \in \{(0,0,\pi), (0,\pi,0), (\pi,0,0), (\pi,\pi,\pi)\}$, then the limit $\lim_{t\to 0^+} f'(t)$ exists and is a finite number.

4.1. The moduli space M_3 of spherical triangles is given by $\{x \in (0,\pi)^3 | x_i^* > 0, x_1 + x_2 + x_3 > \pi\}$ which is the open regular tetrahedron inscribed in the standard cube $[0,\pi]^3$. The four vertices of the tetrahedron are $v_1 = (\pi,0,0), v_2 = (0,\pi,0), v_3 = (0,0,\pi)$ and $v_4 = (\pi,\pi,\pi)$ and its four triangular faces lie in the planes given by the linear equations

$x_i^* = 0$, i=1,2,3, and $x_1 + x_2 + x_3 = \pi$ respectively. We now decompose the boundary ∂M_3 into a disjoint union of six parts, denoted by I, II, III, IV, V and VI, as follows. Here I is the open triangle $\Delta v_1 v_2 v_3$. Part II is the union of the three open triangles $\Delta v_4 v_i v_j$ where $\{i,j\} \subset \{1,2,3\}$. Part III is the union of three open edges of the triangle I, i.e., III is the union of open intervals $v_i v_j$ where $\{i,j\} \subset \{1,2,3\}$. Part IV is the union of the three open intervals $v_4 v_i$. Part V is $\{(\pi,\pi,\pi)\}$. Part VI is $\{(0,0,\pi),(0,\pi,0),(\pi,0,0)\}$. The algebraic description of them is as follows.

$$I = \{a \in (0,\pi)^3 | a_1 + a_2 + a_3 = \pi, a_i^* \in (0,\pi)\},$$

$$II = \cup_{i=1}^3 \{a \in (0,\pi)^3 | a_i^* = 0, a_j^*, a_k^* \in (0,\pi), a_1 + a_2 + a_3 > \pi\},$$

$$III = \cup_{i=1}^3 \{a \in [0,\pi)^3 | a_i^* = 0, a_j^*, a_k^* \in (0,\pi), a_1 + a_2 + a_3 = \pi\},$$

$$IV = \cup_{i=1}^3 \{a \in (0,\pi]^3 | a_i = \pi, a_j^* = a_k^* = 0, a_i^* \in (0,\pi), a_1 + a_2 + a_3 > \pi\},$$

$$V = \{a \in [0,\pi]^3 | a_i^* = 0, i = 1,2,3, a_1 + a_2 + a_3 = 3\pi\},$$

$$VI = \cup_{i=1}^3 | a_j^* = a_k^* = 0, a_i^* = 2\pi, a_1 + a_2 + a_3 = \pi\}.$$

As usual, we have used the convention that $\{i,j,k\} = \{1,2,3\}$ above.

4.2. We now prove proposition 4.1 by considering the limit $\lim_{t \to 0^+} f'(t)$ according to the type of the degenerated spherical triangle a. Let $a = (a_1, a_2, a_3), p = (p_1, p_2, p_3)$ and let $x_i = x_i(t) = (1-t)a_i + tp_i$. We use $y_i = y_i(t)$ to denote the corresponding edge lengths of the triangle $x = (x_1, x_2, x_3)$. Note that, $x_i \to a_i$ and $x_i^* \to a_i^*$ as time t tends to 0, also $dx_i/dt = p_i - a_i$. By definition,

$$f'(t) = \sum_{i=1}^3 \ln \tan(y_i(t)/2)(p_i - a_i). \tag{4.2}$$

By (3.6), we write,

$$\ln \tan(y_i/2) = S(x_i^*) - S(x_j^*) - S(x_k^*) + C(x) \tag{4.3}$$

where $S(u) = 1/2 \ln \sin(u)$ and $C(x) = 1/2 \ln |\cos((x_1 + x_2 + x_3)/2)|$. Assume in the following computation that (i,j,k) is a cyclic permutation of $(1,2,3)$, or more precisely, we take $j = i+1, k = i+2$ where indices are counted modulo 3. Substitute (4.3) into (4.2), we obtain,

$$f'(t) = \sum_{i=1}^3 (S(x_i^*) - S(x_{i+1}^*) - S(x_{i+2}^*) + C(x))(p_i - a_i)$$

$$= \sum_{i=1}^3 (S(x_i^*) - S(x_{i+1}^*) - S(x_{i+2}^*))(p_i - a_i) + C(x)(\sum_{i=1}^3 p_i - \sum_{i=1}^3 a_i)$$

(4.4) $$= 2\sum_{i=1}^{3} S(x_i^*)(p_i^* - a_i^*) + C(x)(\sum_{i=1}^{3} p_i - \sum_{i=1}^{3} a_i)$$

We now discuss the limit of $f'(t)$ as t tends to 0 according to the type of the degenerated triangle a.

4.3. Case 1, the triangle a has type I, i.e., $a_1+a_2+a_3 = \pi$ and $a_i, a_i^* \in (0, \pi)$. In particular, $\lim_{t\to 0^+} S(x_i^*) = S(a_i^*)$ exists in $\mathbf{R}$. Thus the unbounded term in (4.4) is the last term $C(x)(\sum_{i=1}^3 p_i - \sum_{i=1}^3 a_i)$ which tends to $-\infty$ due to $a_1 + a_2 + a_3 = \pi$, $p_1 + p_2 + p_3 > \pi$ and $\lim_{t\to 0^+} C(x) = -\infty$. This shows the proposition for case 1.

4.4. Case 2, the triangle a has type II. For simplicity, we may assume that $\pi+a_1 = a_2+a_3$, i.e., $a_1^* = 0$, $a_i, a_2^*, a_3^* \in (0, \pi)$, and $a_1 + a_2 + a_3 \in (\pi, 3\pi)$. Then the unbounded term in (4.4) is $2S(x_1^*)(p_1^* - a_1^*)$. All other terms are bounded since the $\lim_{t\to 0^+} S(x_i^*(t)) = S(a_i^*)$ is finite for $i = 2, 3$ and $\lim_{t\to 0^+} C(x) = 1/2 \ln |\cos(a_1 + a_2 + a_3)/2)|$ is also finite. On the other hand, $p_1^* > 0$, $a_1^* = 0$ and $\lim_{t\to 0^+} S(x_1^*) = -\infty$, we see that $\lim_{t\to 0^+} f'(t) = -\infty$.

4.5. Cases 3,4, the triangle a has type III or IV. In these cases, exactly two of the four equations $a_1^* = 0, a_2^* = 0, a_3^* = 0$, or $a_1 + a_2 + a_3 = \pi$ hold. To be more precise, in the case III, we may assume without loss of generality that $a_1^* = 0$, $\sum_{i=1}^3 a_i = \pi$, $a_2^*, a_3^* \in (0, \pi)$. Thus, in (4.4), exactly two terms, $2S(x_1^*)(p_1^* - a_1^*)$ and $C(x)(\sum_{i=1}^3 p_i - \pi)$ tend to $-\infty$ as t approaches 0. The other two terms remain bounded. Thus the result follows.

In the case IV, we may assume for simplicity that $a_1^* = a_2^* = 0$ and $\sum_{i=1}^3 a_i > \pi$ and $a_3^* > 0$. Then due to $0 < \sum_{i=1}^3 a_i^* = 3\pi - \sum_{i=1}^3 a_i$, we have $\sum_{i=1}^3 a_i < 3\pi$. This shows that $\lim_{t\to 0} C(x) = C(a)$ is finite. Thus in (4.4), there are again exactly two terms, namely $2S(x_1^*)(p_1^* - a_1^*)$ and $2S(x_2^*)(p_2^* - a_2^*)$ tend to $-\infty$ as t approaches 0. The other two terms remain bounded. Thus the result follows again.

4.6. Case 5, the triangle a is an equator (π, π, π). In this case $a_i^* = 0$ and $a_1+a_2+a_3 = 3\pi$. Using (4.2) and (4.3), we have,

$$f'(t) = \sum_{i=1}^{3} (S(x_i^*) - S(x_j^*) - S(x_k^*) + C(x))(p_i - \pi)$$

(4.5) $$= \sum_{i=1}^{3} [(S(x_i^*) - S(x_j^*)) + (C(x) - S(x_k^*))](p_i - \pi)$$

We note that both limits $\lim_{t\to 0^+}(S(x_i^*) - S(x_j^*))$ and $\lim_{t\to 0^+}(C(x) - S(x_k^*))$ exist in $\mathbf{R}$. Indeed, by definition,

$$x_i^* = 1/2[\pi + x_i - x_j - x_k] = 1/2[\pi + (1 - t)(a_i - a_j - a_k) + t(p_i - p_j - p_k)]$$

$$= 1/2[\pi + (1 - t)(-\pi) + t(p_i - p_j - p_k)$$

$$= 1/2(t(p_i - p_j - p_k + \pi)) = tp_i^*.$$

$$x_1 + x_2 + x_3 = t(p_1 + p_2 + p_3) + (1-t)3\pi = 3\pi + t(p_1 + p_2 + p_3 - 3\pi)$$

Thus, $S(x_i^*) - S(x_j^*) = 1/2(\ln\sin(tp_i^*) - \ln\sin(tp_j^*))$ which tends to $1/2(\ln\sin p_i^* - \ln\sin p_j^*)$ as t tends to 0. Similarly, $C(x) - S(x_k^*)$ tends to the finite number $1/2(\ln|\sin((p_1 + p_2 + p_3 - 3\pi)/2)| - \ln(\sin(p_k^*)))$.

4.7. Case 6, the triangle a is of type VI. For simplicity, we assume that $a = (\pi, 0, 0)$. Thus $a_1 + a_2 + a_3 = \pi$, $a_1^* = 2\pi$, $a_2^* = a_3^* = 0$. We use (4.5) to calculate the limit $\lim_{t\to 0} f'(t)$. The calculation is exactly the same as that of case 5. Indeed, each of the four terms $S(x_i^*)$ and $C(x)$ tends to $-\infty$ as t approaches zero. On the other hand, by the same arguement as in 4.6, both of the limits $\lim_{t\to 0^+} S(x_i^*)/S(x_k^*)$ and $\lim_{t\to 0^+} S(x_i^*)/C(x)$ are finite. Thus the result follows.

This ends the proof of proposition 4.1.

4.8. **Remark.** We give a geometric interpretation of the stratification I, II, ..., VI of the degenerated triangles. The type I boundary point $x \in \{x \in (0,\pi)^3 | x_1 + x_2 + x_3 = \pi\}$ corresponds to the "Euclidean triangle". Geometrically, it represents a point which is the limit of spherical triangles shrinking to a point so that its inner angles tend to three numbers in $(0,\pi)$. In particular, if one defines the edge length $y_i = 0$ for these triangle, the cosine law (2.1) still makes sense in terms of taking limit. The type II points in $\{x \in (0,\pi)^3 | x_1 + x_2 + x_3 > \pi, x_i^* = 0, x_j^* > 0, x_k^* > 0\}$ correspond to the other codimension-1 faces. They represent the "exceptional Euclidean triangles". Geometrically, it is the limit of sequence of spherical triangles expanding to a union of two geodesics from a point to its antipodal point so that the inner angles tend to three numbers in $(0,\pi)$. In particular, the edge lengths are $y_i = 0, y_j = y_k = \pi$ and a type II triangle has two vertices. Note that the edge length function y_i extends continously on the set $M_3 \cup I \cup II$. There are two types of codimension-2 faces. The first type, denoted by III, consists of three open edges of the form $\{x = (x_1, x_2, x_k) \in [0,\pi)^3 | \ x_i = 0,\ x_j, x_k > 0$ and $x_j + x_k = \pi\}$. This is a further degeneration of "Euclidean triangles". The second type of codimension-2 face, denoted by IV, consists of the three open edges of the form $\{x = (x_1, x_2, x_3) \in (0,\pi]^3 | x_i = \pi, x_j = x_k \in (0,\pi)\}$. Geometrically, it corresponds to a degenerated spherical triangle so that two of its three distinct vertices are antipodal points. Due to the location of the third vertex (of inner angle π), the length functions y_r does not extend continuously from M_3 to $M_3 \cup IV$. Finally, there are two types of vertices. The first type, denoted by V, is the point (π,π,π) corresponding to the equator and the second type, denoted by VI, consists of $(0,0,\pi), (0,\pi,0), (\pi,0,0)$ corresponding to a degenerated triangle whose three distinct vertices lie in a great circular arc of length at most π.

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