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Equivariant isotopy of unknots to round circles

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Abstract

Suppose that γ_0 is an unknotted simple closed curve contained in the 3-sphere which happens to be invariant under a subgroup G of the Möbius group of S^3 = the group (generated by inversions in 2-spheres). It is shown that there is an equivariant isotopy γ_t, $0 \leqslant t \leqslant 1$, from γ_0 to a round circle γ_1.

Keywords: Knot; Unknotting; Möbius group; Möbius energy

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The existence of a Möbius invariant energy functional E : {smooth simple closed curves in the 3-sphere $S^3\} \to R^+$ achieving its minimum precisely on round circles has stimulated renewed interest in finding a "natural" unknotting procedure on the space of topologically unknotted simple closed curves [3,6]. A "natural" flow from unknots to round circles should, at a minimum, satisfy: (1) continuity in initial conditions; (2) conservation of symmetry: a G-invariant loop, $G \subset \text{Möb}(S^3)$, should remain G-invariant as it evolves; and (3) all trajectories converge to a round circle. A flow satisfying (1) and (3) follows from Hatcher's famous paper on the Smale conjecture [4]. In this paper, we prove:

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Theorem. *For every compact subgroup G of the Möbius group* $\text{Möb}(S^3)$, *if $\gamma \subset S^3$ is a smooth unknotted G-invariant simple closed curve, then there exists a smooth family of G-invariant simple closed curves γ_t, $0 \leqslant t \leqslant 1$, with $\gamma_0 = \gamma$ and γ_1 a round circle.*

The Möbius group $\text{Möb}(S^3)$ is the 10-dimensional Lie group of two components generated by inversions in 2-spheres.

This shows that there is no obstruction to combining properties (2) and (3): for all compact subgroups $G \subset \text{Möb}(S^3)$, π_0({round G-invariant circles}) $\to \pi_0$ ({unknotted G-invariant simple closed curves}) is onto.

It is still an open question whether then there could be any topological obstruction to properties (1), (2), and (3) being simultaneously achieved, i.e., to combining parameters and symmetry. There would be no such obstruction if for all compact subgroups $G \subset \text{Möb}(S^3)$:

$$\pi_i(\{\text{round } G\text{-invariant circles}\}) \to \pi_i(\{\text{unknotted } G\text{-invariant simple closed curves}\}) \qquad (*)$$

is an isomorphism.

For example, if G is multiplication by $e^{2\pi i/p}$, then $(*)$ is equivalent to showing $\text{Diff}(L_{p,1}) \cong \text{Normalizer}(G)/G \cong T^2$. This type of generalization of Hatcher's theorem is presently unknown. It is worth noting that $(*)$ probably would have a very interesting formal consequence:

$$\text{pt} = \{\text{round circles}\}/\text{Möb}(S^3) \simeq \{\text{unknotted simple closed curves}\}/\text{Möb}(S^3). \qquad (**)$$

Here is a sketch of the proof that $(*)$ implies $(**)$. Beginning with the largest G (partially ordered by inclusion), piece together deformation retractions of G-principal bundles a strata at a time to obtain a Möbius equivariant deformation retraction of {unknotted simple closed curves} to {round circles}. To make this sketch precise, one would have to enter into the details of the topology of {unknotted simple closed curves} and its G-stratification, a task which we have not considered warranted unless $(*)$ is established.

The organization of this paper is as follows. Suppose G is a compact subgroup of $\text{Möb}(S^3)$ leaving a smooth unknot K invariant. If G is not finite, then K is invariant under a circle action and therefore is a round circle. Thus, it suffices to consider finite groups G. If K lies in a round 2-sphere S^2, then S^2 is G-invariant unless K is a round circle. In the latter case, K is isotopic to a round circle in S^2 G-equivariantly. If K is not in any round 2-sphere, the restriction of G to K gives a faithful representation $G \to \text{Diff}(K)$. Thus, G is a cyclic or a dihedral group. This is the case that we will focus on in this paper. We will recall the list of all cyclic and dihedral subgroups of $\text{Möb}(S^3)$ up to conjugation in Section 1. For the proof, we work our way through the list. The argument varies considerably from case to case and draws from time to time on substantial theorems. If the result derives from some simple unifying topological principle, we have not succeeded in finding it. The theorem in the cyclic group case is in Section 2, and the dihedral group case is in Section 3.

1. Cyclic and dihedral subgroups of $O(4)$

We will list all cyclic and dihedral subgroups of $O(4)$ (Möb(S^3)) up to conjugation in this section. Each subgroup $G \subset O(4)$ will be represented by 4×4 matrices generating the group. To see the group action on S^3, we identify S^3 with $\overline{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\}$ via stereographic projections and represent the action by Möbius transformations in $\mathbb{R}^3$, e.g., the antipodal map is given by $x \mapsto -x/|x|^2$ for $x \in \overline{\mathbb{R}}^3$.

Let $\mathbb{Z}_n$ be the cyclic group of order n generated by α; D_{2n} the dihedral group $\langle \alpha, \beta \mid \alpha^n = 1, \beta^2 = 1, \beta\alpha\beta = \alpha^{-1} \rangle$;

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad I_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

U denotes the unit circle in the xy-plane.

1.1. Cyclic subgroups $\mathbb{Z}_n$

(a) $n = 2$. All $\mathbb{Z}_2$ subgroups of $O(4)$ are conjugate to one of the following four groups. The matrix denotes the generator of $\mathbb{Z}_2$.

$$\alpha = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

α acts as the reflection in the xy-plane in $\overline{\mathbb{R}}^3$ (with a 2-sphere as the fixed point set);

$$\alpha = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

α acts as the π-rotation about the x-axis in $\overline{\mathbb{R}}^3$ (with a circle as the fixed point set);

$$\alpha = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix},$$

α acts as the map $x \mapsto -x$ in $\overline{\mathbb{R}}^3$ (with two points as the fixed point set);

$$\alpha = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix},$$

α is an antipodal map $x \mapsto -x/|x|^2$ in $\overline{\mathbb{R}}^3$ (without fixed point).

(b) $n \geqslant 3$. $\mathbb{Z}_n$ is generated by

$$\alpha = \begin{bmatrix} R_\theta & 0 \\ 0 & R_{\theta'} \end{bmatrix}$$

where $\theta = 2\pi l/n$, $\theta' = 2\pi l'/n$, $0 \leqslant l, l' \leqslant n-1$, $((l, l'), n) = 1$. α acts as the composition of the $2\pi l/n$-rotation about the z-axis with the $2\pi l'/n$-rotation about the unit circle U in the xy-plane.

(c) $n \geqslant 3$. n even. $\mathbb{Z}_n$ is generated by

$$\alpha = \begin{bmatrix} R_\theta & 0 \\ 0 & I_1 \end{bmatrix} \quad \left(I_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

where $\theta = 2\pi l/n$, $(l, n) = 1$, or 2. α acts as the composition of $2\pi l'/n$-rotation about the z-axis with the reflection about the xy-plane.

1.2. Dihedral groups D_{2n}

Each representation of D_{2n} into $O(4)$ leaves a 2-plane invariant. Thus each representation is constructed as the direct sum of two 2-dimensional representations of D_{2n}, see Serre [8].

(a) $n = 2$. $D_{2.2}$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by α and β. There are six mutually nonconjugate subgroups of $O(4)$ isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The generators α, β of the group are listed as follows.

$$\alpha = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix},$$

α is $x \mapsto -x/|x|^2$ and β is the reflection about the xy-plane;

$$\alpha = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

α is $x \mapsto -x/|x|^2$ and β is the π-rotation about the x-axis;

$$\alpha = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

α is the π-rotation about the z-axis and β is the reflection about the xy-plane;

$$\alpha = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix},$$

α is the π-rotation about the z-axis and β is the reflection about the xz-plane;

$$\alpha = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix},$$

α is the π-rotation about the z-axis and β is the π-rotation about the x-axis;

$$\alpha = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix},$$

α is the π-rotation about the unit circle U in the xy-plane and $\beta(x) = -x$.

(b) $n \geqslant 3$. D_{2n} is generated by

$$\alpha = \begin{bmatrix} R_\theta & 0 \\ 0 & R_{\theta'} \end{bmatrix}, \quad \beta = \begin{bmatrix} I_1 & 0 \\ 0 & I_1 \end{bmatrix},$$

where $\theta = 2\pi l/n$, $\theta' = 2\pi l'/n$, $0 \leqslant l,\ l' \leqslant n-1$, $((l,\ l'),\ n) = 1$. α is the composition of the $2\pi l/n$-rotation about the z-axis with the $2\pi l'/n$-rotation about U, and β is the π-rotation about the x-axis.

(c) $n \geqslant 3$. D_{2n} is generated by

$$\alpha = \begin{bmatrix} R_\theta & 0 \\ 0 & \mathrm{id} \end{bmatrix}, \quad \beta = \begin{bmatrix} I_1 & 0 \\ 0 & I_1 \end{bmatrix},$$

where $\theta = 2\pi/n$. α is represented as $2\pi/n$-rotation about the z-axis and β is the π-rotation about the x-axis.

(d) $n \geqslant 3$. D_{2n} is generated by

$$\alpha = \begin{bmatrix} R_\theta & 0 \\ 0 & \mathrm{id} \end{bmatrix}, \quad \beta = \begin{bmatrix} I_1 & 0 \\ 0 & \mathrm{id} \end{bmatrix},$$

where $\theta = 2\pi/n$. α is the $2\pi/n$-rotation about the z-axis and β is the reflection about the xz-plane.

(e) $n \geqslant 3$. D_{2n} is generated by

$$\alpha = \begin{bmatrix} R_\theta & 0 \\ 0 & \mathrm{id} \end{bmatrix}, \quad \beta = \begin{bmatrix} I_1 & 0 \\ 0 & -\mathrm{id} \end{bmatrix},$$

where $\theta = 2\pi/n$. α is the $2\pi/n$-rotation about U and $\beta : x \mapsto -x$ in $\mathbb{R}^3$.

(f) $n \geqslant 3$, *n even*. D_{2n} is generated by

$$\alpha = \begin{bmatrix} R_\theta & 0 \\ 0 & -\mathrm{id} \end{bmatrix}, \quad \beta = \begin{bmatrix} I_1 & 0 \\ 0 & I_1 \end{bmatrix},$$

where $\theta = 2\pi/n$. α is the composition of $2\pi/n$-rotation about the z-axis with the π-rotation about U and β is the π-rotation about the x-axis.

(g) $n \geqslant 3$, *n even*. D_{2n} is generated by

$$\alpha = \begin{bmatrix} R_\theta & 0 \\ 0 & -\mathrm{id} \end{bmatrix}, \quad \beta = \begin{bmatrix} I_1 & 0 \\ 0 & \mathrm{id} \end{bmatrix},$$

$\theta = 2\pi/n$. α is the composition of $2\pi/n$-rotation about the z-axis with the π-rotation about U and β is the reflection about the xz-plane.

(h) *$n \geqslant 3$, n even.* D_{2n} is generated by

$$\alpha = \begin{bmatrix} R_\theta & 0 \\ 0 & I_1 \end{bmatrix}, \qquad \beta = \begin{bmatrix} I_1 & 0 \\ 0 & \mathrm{id} \end{bmatrix},$$

$\theta = 2\pi/n$. α is the composition of $2\pi/n$-rotation about the z-axis with the reflection about the xy-plane and β is the reflection in the xz-plane.

(i) *$n \geqslant 3$, n even.* D_{2n} is generated by

$$\alpha = \begin{bmatrix} R_\theta & 0 \\ 0 & I_1 \end{bmatrix}, \qquad \beta = \begin{bmatrix} I_1 & 0 \\ 0 & -\mathrm{id} \end{bmatrix},$$

$\theta = 2\pi/n$. α is the composition of $2\pi/n$-rotation about the z-axis with the reflection about the xy-plane, and β is the orientation reversing involution of $\mathbb{R}^3$ with two fixed points at $(\pm 1, 0, 0)$.

1.3. Group actions on solid tori

Each isometry of the product metric on $D^2 \times S^1$ is of the form $(z, t) \mapsto (f_1(z), f_2(t))$, where $f_i(z) = \theta_i z$, or $\theta_i \bar{z}$ for some $\theta_i \in S^1$, $D^2 = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$ and $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Proposition 1.1. *Each finite group G acting on $D^2 \times S^1$ is conjugate to an action by isometry on $D^2 \times S^1$ with respect to the product metric.*

The proof is a direct consequence of the equivariant Dehn lemma and the solution of the Smith conjecture. Because many similar cases must be considered, only a sketch is given.

Proof (*sketch*). According to the equivariant Dehn lemma [5] there exists an essential G-invariant disk $D^2 \hookrightarrow D^2 \times S^1$ imbedded in $D^2 \times S^1$ or a G-invariant family $\amalg\ D^2 \hookrightarrow D^2 \times S^1$ of disjointly imbedded essential disks. By elementary 3-manifold topology (chiefly the Schoenflies theorem), cutting along those disks yields a 3-ball or family of 3-balls on which G acts.

It is well known that finite group actions on $S^1 \times S^1 = \partial D^2 \times S^1$ are all conjugate to linear actions, i.e., finite subgroups of the affine group of $Z \oplus Z$ and therefore solvable. The Smith conjecture [1] states that smooth cyclic group actions on B^3 are all conjugate to linear actions and this result has been extended to smooth actions of any solvable group on B^3. Thus the actions of G on the cutting disks and the complementary balls are all linear. By considering cases one may check that these can be reglued to obtain exactly the standard actions of $D^2 \times S^1$. □

Almost certainly, all smooth, finite group actions on B^3 are linear. This is a very special case of the Thurston orbifold theorem. But since the latter has not been published or even fully exposited, we have exploited the fact the most difficult group A_5 acting on B^3 does not arise from actions on a solid torus to avoid relying on Thurston's theorem.

Remark. (1) As a special consequence, we see that there is no self homeomorphism h of $D^2 \times S^1$ so that $h^2(z, t) = (\bar{z}, \bar{t})$.

(2) We will call actions by isometries "standard actions".

2. Cyclic group actions

Suppose K is an unknot which is not lying in a round 2-sphere and is invariant under a cyclic group $\mathbb{Z}_n \subset O(4)$. We will prove that K is equivariantly isotopic to a round circle.

We will use the following notations. $N(K)$ denotes the interior of a small invariant regular neighborhood of K; D^2 denotes $\{z \mid z \in \mathbb{C}, |z| \leqslant 1\}$; S^1 denotes $\{t \mid t \in \mathbb{C}, |t| = 1\}$; $D^2 \times S^1$ denotes a solid torus; I denotes the closed interval [0, 1]. By a core of a solid torus $D^2 \times S^1$ we mean a curve in the interior of $D^2 \times S^1$ isotopic to $\{0\} \times S^1$. Since K is assumed to be trivial, $N(K)^c$ is homeomorphic to $D^2 \times S^1$ and has an induced $\mathbb{Z}_n$-action. If the $\mathbb{Z}_n$-action on $N(K)^c$ has an invariant meridian disc, then K bounds an equivariant disc in S^3. Any finite group action on a disc has a global fixed point. Thus, K is equivariantly isotopic to a round circle centered at the fixed point.

There are five cases which need to be considered in the cyclic group case.

Case 1: *The order* $n = 2$. There are four types of nonconjugate $\mathbb{Z}_2$ actions on $\overline{\mathbb{R}}^3$ according to the dimension of Fix(α) (α is a generator of $\mathbb{Z}_2$).

(a) Fix(α) *is a* 2*-sphere*. We may assume that α is the reflection about the xy-plane. Then $K \cap \text{Fix}(\alpha) =$ 2-point set since K is assumed to be not in a sphere. The action of α on $N(K)^c$ is conjugate to $(z, t) \mapsto (\bar{z}, t)$, $(z, t) \in D^2 \times S^1$. Thus, there is an invariant disc $D \times \{t\}$.

(b) Fix(α) *is a circle*. We may assume that α is the π-rotation about the z-axis. If $K \cap \text{Fix}(\alpha) \neq \emptyset$ it must be two points. In this case α-action on $N(K)^c$ is conjugate to $(z, t) \mapsto (\bar{z}, \bar{t})$, $(z, t) \in D^2 \times S^1$. If $K \cap \text{Fix}(\alpha) = \emptyset$ then α-action on $N(K)^c$ is conjugate to $(z, t) \mapsto (-z, t)$, $(z, t) \in D^2 \times S^1$. In both cases, there are invariant meridian discs.

(c) Fix(α) *is a* 2*-point set*. We may assume that $\alpha(x) = -x$, $x \in \overline{\mathbb{R}}^3$. If $K \cap \text{Fix}(\alpha) = \emptyset$, then α-action in $N(K)^c$ is conjugate to $(z, t) \mapsto (-z, \bar{t})$, $(z, t) \in D^2 \times S^1$. It has an invariant meridian. If $K \cap \text{Fix}(\alpha) \neq \emptyset$, $K \cap \text{Fix}(\alpha) = \{0, \infty\}$. By an equivariant isotopy of K, we may assume that K is a part of a straight line L near 0 and ∞, say outside the region $\{x \mid r \leqslant |x| \leqslant R\}$. We may further equivariantly isotopy K so that K coincides with L near 0 and ∞ respecting the orientation. See

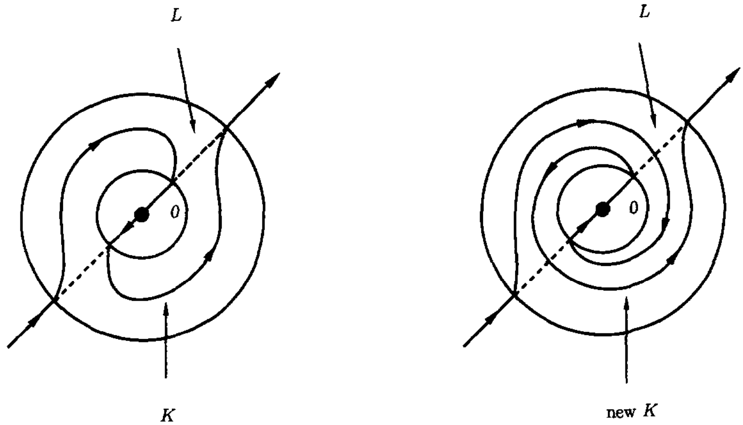

Fig. 1. Untwist K by π-rotation near 0.

Fig. 1. The goal now is to equivariantly isotope K to L in $M=\{x \mid \gamma \leqslant |x| \leqslant R\}$ leaving their endpoints fixed. To this end, we consider the quotient $\{x \mid \gamma \leqslant |x| \leqslant R\}/x \sim \alpha(x)$. It is homeomorphic to $\mathbb{R}P^2 \times I$ where the quotient $L \cap M$ corresponds to $p \times I$. The quotient of $K \cap M$ is an arc $\tilde{K}$ in $\mathbb{R}P^2 \times I$ whose complement is homeomorphic to {Möbius band} $\times I$ since α-action on $N(K)^c$ is standard. Using these product structures, we find an annulus $A = C \times I$ where C is an essential simple closed curve in $\mathbb{R}P^2$ passing through p and another annulus $\tilde{A}$ which is properly embedded containing $\tilde{K}$ so that $\partial\tilde{A} \cap \mathbb{R}P^2 \times \{i\}$ are essential simple closed curves, for $i = 0, 1$.

Since any two essential simple closed curves in $\mathbb{R}P^2$ are isotopic, we may isotope $\tilde{A}$ so that $\partial\tilde{A} \cap \partial A \cap \mathbb{R}P^2 \times \{i\}$ consists of the point $p \times \{i\}$ for $i = 0, 1$, and that $\tilde{A}$ intersects A transversely. Let γ be the intersection arc γ in $A \cap \tilde{A}$ from $p \times \{0\}$ to $p \times \{1\}$. Then, using the annuli A and $\tilde{A}$, we see that $\tilde{K}$ is isotopic to γ in $\tilde{A}$ and γ is isotopic to $p \times I$ in A. Thus $\tilde{K}$ is isotopic to $p \times I$ in $\mathbb{R}P^2 \times I$ (possibly moving the endpoints). On the other hand, by the assumption on the orientations on K and L, the loop $\tilde{K} \cup p \times I$ is null homotopic in $\mathbb{R}P^2 \times I$. Thus $\tilde{K}$ is isotopic to $p \times I$ leaving the endpoint fixed. Lifting this isotopy to $\{x \mid \gamma \leqslant |x| \leqslant R\} = M$, we show that $K \cap M$ is equivariantly isotopic to $L \cap M$ leaving the endpoints fixed.

(d) *α is fixed point free.* This will be covered in Case 2.

Case 2: $n \geqslant 3$, and α generates a free cyclic action. We may assume that α is the composition of the $2\pi/n$-rotation about the z-axis with the $2\pi l/n$-rotation about the unit circle U in the xy-plane, where $(l, n) = 1$. The z-axis and U are both invariant whose quotients in the lens space $L(n, l) = S^3/x \sim \alpha(x)$ are the cores of a genus one Heegaard splitting of $L(n, l)$. The unknot K also descends to a core of a genus one Heegaard splitting of $L(n, l)$ since $\mathbb{Z}_n$-action on $N(K)^c$ is standard. If $n \geqslant 3$, the result that K is equivariantly isotopic to the z-axis or U follows from a theorem of Bonahon [2] on the uniqueness of genus one Heegard splitting of lens space $L(n, l)$. If $n=2$, the result that the genus one Heegaard splitting of $\mathbb{R}P^3$ is unique still holds. Indeed, suppose T is a genus one Heegaard surface in $\mathbb{R}P^3$ in general position with respect to $\mathbb{R}P^2 \subset \mathbb{R}P^3$. Then the intersection $T \cap \mathbb{R}P^2$ consists of several disjoint circles. Since any two essential circles in $\mathbb{R}P^2$ are intersecting, and any two disjoint essential circles in T are parallel, we see that there is exactly one circle C in $T \cap \mathbb{R}P^2$ so that C is essential in both T and $\mathbb{R}P^2$ and that all other circles in $T \cap \mathbb{R}P^2$ are null homotopic in both T and $\mathbb{R}P^2$ respectively. By the standard innermost circle and exchange argument, we may isotopy T so that $T \cap \mathbb{R}P^2$ consists of only one circle (which is automatically essential). Thus all genus one Heegaard splittings of $\mathbb{R}P^3$ are isotopic. This gives a proof of Case 1(d).

Case 3: *$n \geqslant 3$, and α generates a cyclic action with a one-dimensional fixed point set*. We may assume that α is the $2\pi/n$-rotation about the z-axis. Since $n \geqslant 3$, $K \cap \text{Fix}(\alpha) = \emptyset$. The $\mathbb{Z}_n$-action on $N(K)^c$ is conjugate to $(z, t) \mapsto (e^{2\pi i/n}z, t)$ on $D^2 \times S^1$. Thus, there are invariant meridian discs. We would like to state without proof two consequences of this action.

Lemma 2.1. *If an unknot K is invariant under a $2\pi/n$-rotation about an axis L and $K \cap L = \emptyset$, $n \geqslant 2$, then*

(a) *(K, L) is a Hopf link. In particular, the linking number between K and L is one.*

(b) *Suppose K' is another such unknot, $K \cap K' = \emptyset$, then the linking number between K and K' is divisible by n.*

Case 4: *$n \geqslant 3$, and α generates an orientation preserving nonfree action which does not have a global one-dimensional fixed point set*. We may assume that α is the composition of $2\pi/l$-rotation about the z-axis with the $2\pi p/m$-rotation about the unit circle U in the xy-plane, where, $l \neq m$, l, $m \geqslant 2$, $\gcd(p, m)=1$, and $n = \text{lcm}(l, m)$. We divide it into two subcases.

(a) $\gcd(l, m)=1$. Then $n = lm$, and $\mathbb{Z}_m \simeq \mathbb{Z}_l \oplus \mathbb{Z}_m$ where the two generators in the direct sum are the $2\pi/m$-rotation about U denoted by β and the $2\pi/l$-rotation about the z-axis denoted by γ. We claim that the only invariant unknots are $\text{Fix}(\beta)$ and $\text{Fix}(\gamma)$. Indeed, since one of l, m is at least three, K cannot meet both $\text{Fix}(\beta)$ and $\text{Fix}(\gamma)$. $K \cap (\text{Fix}(\beta) \cup \text{Fix}(\gamma)) = \emptyset$ is also impossible. To see this, by Lemma 2.1(b), the linking number $\text{lk}(K, \text{Fix}(\beta))$ is divisible by l since both K and $\text{Fix}(\beta)$ are invariant under γ. On the other hand, by Lemma 2.1(a), $\text{lk}(K, \text{Fix}(\beta))=1$ since K is invariant under β. The other way to see this is to note that the $\mathbb{Z}_l \oplus \mathbb{Z}_m$-action on $N(K)^c$ is nonstandard. Thus, the only case left is that K meets one of $\text{Fix}(\beta)$ or $\text{Fix}(\gamma)$ in two points, say $|K \cap \text{Fix}(\beta)| = 2$. This implies $m=2$. Thus $l \geqslant 3$. However $K \cap \text{Fix}(\beta)$ is γ-invariant. Therefore $|K \cap \text{Fix}(\beta)| \geqslant 3$. This is absurd.

(b) $\gcd(l, m) = a$, *where* $a \leqslant 2$, $l = ab$, $m = ac$ $\gcd(b, c)=1$, $\gcd(p, ac)=1$ *and both* b, $c \geqslant 2$. Then α^{ab} is the $2\pi pb/c$-rotation about U, and α^{ac} is the $2\pi c/b$-rotation about the z-axis. By Case 4(a), $b \geqslant 2$, $c \geqslant 2$ implies that the only invariant unknots are $\text{Fix}(\alpha^{ab})$ and $\text{Fix}(\alpha^{ac})$.

(c) $\gcd(l, m) = a$, $a \geqslant 2$, $l = ab$, $m = ac$, $\gcd(p, ac) = 1$, *and one of b or c is one.* We may assume without loss of generality that $b = 1$. Thus, α is the composition of $2\pi/l$-rotation about the z-axis with the $2\pi p/lc$-rotation about U where $\gcd(p, lm) = 1$. Now α^l is the $2\pi p/c$-rotation about U. Assume first that $c \geqslant 3$, or $c = 2$ together with $U \cap K = \emptyset$. By Lemma 2.1(a), (K, U) forms a Hopf link. Take a small invariant regular neighborhood $N(U)$ of U so that $K \cap N(U) = \emptyset$ and $\{z\text{-axis}\} \cap N(U) = \emptyset$. Then both K and the z-axis are two invariant cores of the solid torus $\overline{N(U)}^c$. Thus they are equivariantly isotopic since $\mathbb{Z}_n$-action on $\overline{N(U)}^c$ is free and conjugate to $(z, t) \mapsto (e^{2\pi i/l}z, e^{2\pi p/n}t)$ in $D^2 \times S^1$. In the rest case, $c = 2$, and $|K \cap U| = 2$. This case does not occur since $n \geqslant 3$ and $K \cap U$ is α-invariant imply that $|K \cap U| \geqslant 3$.

Case 5: The group is generated by an orientation reversing α of order at least three. We may assume that α is the composition of $2\pi/m$-rotation about the z-axis with the reflection about the xy-plane and α has order n. $\text{Fix}(\alpha) = \{0, \infty\}$. Since $n \geqslant 3$, $K \cap \text{Fix}(\alpha) = \emptyset$. If m is odd, $n = 2m$, and α^m is the reflection about the xy-plane. Thus, K is invariant under both $2\pi/m$-rotation about the z-axis and the reflection about the xy-plane. Since $m \geqslant 3$, K is either in the xy-plane (which is excluded by the assumption on K) or K is the z-axis. If m is even, $m = n$. If $m \geqslant 6$, or $m = 4$ together with $K \cap \{z\text{-axis}\} = \emptyset$, then the cyclic group action on $N(K)^c$ is conjugate to the $(z, t) \mapsto (e^{2\pi i/m}\, z, \bar{t})$ action on $D^2 \times S^1$. Thus, there are invariant discs. If $m = 4$ and K intersects the z-axis in two points, then by considering the induced action of $\mathbb{Z}_4$ on $N(K)^c$, we see that the generator α has order 4 and that α^2 is conjugate to $(z, t) \mapsto (\bar{z}, \bar{t})$. This is impossible by the remark following Proposition 1.1.

3. Dihedral group actions

We will continue the use of the notations introduced in Section 2. Thus, K is an invariant unknot which is not in any 2-sphere. The dihedral group D_{2n} is generated by α, β satisfying $\alpha^n = 1$, $\beta^2 = 1$, and $\beta\alpha\beta = \alpha^{-1}$. We will show that K is equivariantly isotopic to a round circle. The cases in the following paragraphs are not in the order that we used in Section 1.2. The most interesting case (Case 1 corresponding to 1.2(b) in the list) is proven first.

Case 1: $n \geqslant 3$ and α is the composition of $2\pi/n$-rotation about the z-axis with $2\pi l/n$-rotation about U, $\gcd(l, n) = 1$ and β is the π-rotation about the x-axis. There are two subclasses.

(a) $K \cap \text{Fix}(\beta) =$ *two points.* Then the orbit space $S^3/\langle\alpha,\beta\rangle$ (as orbifold) has underlying space S^3 and singular set a 2-bridge knot corresponding to $\text{Fix}(\beta)$. The z-axis and the unit circle in the xy-plane U are both D_{2n}-invariant. Their quotients in $S^3/\langle\alpha,\beta\rangle$ form the top and bottom tunnels in the standard presentation of the

2-bridge knot. Since the action D_{2n} on $N(K)^c$ is conjugate to the standard action, there is also an unknot L invariant under D_{2n} which is a core of $N(K)^c$. Thus, the quotients of $\tilde{K}$, $\tilde{L}$ of K and L respectively in $S^3/\langle\alpha,\beta\rangle$ form the top and bottom tunnels of a new presentation of the 2-bridge knot. By a result of Schubert [7] these tunnels as a pair are unique up to isotopy leaving the 2-bridge knot invariant. Thus $\tilde{K}$ is isotopic to one of the quotients of the z-axis or U in $S^3/\langle\alpha,\beta\rangle$ so that during the isotopy the endpoints stay in the 2-bridge knot. This is equivalent to saying that K is equivariantly isotopic to the z-axis or U.

(b) $K \cap \text{Fix}(\beta) = \emptyset$. This case does not occur. Indeed, in the quotient space $S^3/\langle\alpha\rangle$, the descending $\bar{\beta}$ of β has fixed point set $\text{Fix}(\bar{\beta})$ which is disjoint from the quotient $\bar{K}$ of K. Since $\bar{K}$ is $\bar{\beta}$-invariant, $N(\bar{K})^c$ is also $\bar{\beta}$-invariant and contains $\text{Fix}(\bar{\beta})$. By the classification of $\mathbb{Z}_2$-involution on $N(\bar{K})^c$, $\text{Fix}(\bar{\beta})$ must be a core curve of $N(\bar{K})^c$. Thus $\text{Fix}(\bar{\beta})$ is a generator of the lens space $S^3/\langle\alpha\rangle$. This implies $\text{Fix}(\beta)$ is α-invariant in S^3 which contradicts the assumption $n \geqslant 3$.

Case 2: *$n \geqslant 3$ and α is the composition of $2\pi/l$-rotation about the z-axis with the $2\pi q/m$-rotation about U, and β is the π-rotation about the x-axis where $l \neq m$, $l,m \geqslant 2$,* $\gcd(q, m) = 1$, *and* $n = \text{lcm}(l, m)$. By the proof of Case 4 in Section 2, the only case needed to be considered is that $m = lp$. Thus, we may assume that α is the composition of the $2\pi/l$-rotation about the z-axis with the $2\pi q/pl$-rotation about U. By the proof of Case 4 in Section 2, K and U form a Hopf link. Take a small regular invariant neighborhood $N(U)$ of U so that $K \cap N(U) = \emptyset$. Then we have the action of dihedral group D_{2n} on $N(U)^c$ so that both K and the z-axis are invariant cores. The cyclic group $\langle\alpha\rangle$ acts freely on $\overline{N(U)^c}$ and is conjugate to the action $(z, t) \mapsto (e^{2\pi i/l}z, e^{2\pi qi/pl}t)$ on $D^2 \times S^1$. In the quotient space $\overline{N(U)^c}/\langle\alpha\rangle$, K descends to a core curve $\bar{K}$. Furthermore, β descends to an orientation preserving involution $\bar{\beta}$ on $\overline{N(U)^c}/\langle\alpha\rangle$ which is conjugate to $(z, t) \mapsto (\bar{z}, \bar{t})$ in $D^2 \times S^1$. Now, if $K \cap \text{Fix}(\beta) \neq \emptyset$, then $\bar{K}$ and the quotient of the z-axis are $\bar{\beta}$-invariant cores and both intersect $\text{Fix}(\bar{\beta})$ at two points. Thus, they are isotopic by considering a further quotient by $\bar{\beta}$. The case that $K \cap \text{Fix}(\beta) = \emptyset$ does not occur. Indeed, by Lemma 2.1(a), $\text{Fix}(\beta)$ is a core curve of $N(K)^c$. Since $\langle\alpha\rangle$ also acts on $N(K)^c$, by Proposition 1.1, $\langle\alpha\rangle$ will leave $\text{Fix}(\beta)$ invariant which is absurd.

Case 3: $n = 2$. There are six $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ nonconjugate subgroups of $O(4)$. Let α and β be the generators.

(a) *α is the π-rotation about U and β is the map $x \mapsto -x$.* If $K \cap (\text{Fix}(\alpha) \cup \text{Fix}(\beta)) = \emptyset$, then the action on $N(K)^c$ is conjugate to $\alpha : (z, t) \mapsto (-z, t)$ and $\beta : (z, t) \mapsto (-z, \bar{t})$. Thus, there are invariant meridian discs. If $K \cap \text{Fix}(\alpha)$ consists of two points and $K \cap \text{Fix}(\beta) = \emptyset$, then the action on $N(K)^c$ is conjugate to $\alpha : (z, t) \mapsto (\bar{z}, \bar{t})$ and $\beta : (z, t) \mapsto (-z, \bar{t})$ on $D^2 \times S^1$. There still exist invariant meridian discs. The interesting case remaining is when $K \cap \text{Fix}(\beta) = \text{Fix}(\beta) = \{0, \infty\}$. We will distinguish two more cases: $K \cap \text{Fix}(\alpha) = \emptyset$ or $K \cap \text{Fix}(\alpha) \neq \emptyset$. If $K \cap \text{Fix}(\alpha) = \emptyset$, then K and $\text{Fix}(\alpha)$ form a Hopf link. Take a small invariant regular neighborhood N of $N(U)$ so that $N \cap (K \cup \{\text{the } z\text{-axis}\}) = \emptyset$. Then both K and the z-axis are invariant core curves of $\overline{N(U)^c}$. In the quotient solid torus

$\overline{N(U)}^c/\langle\alpha\rangle$, the quotients $\bar{K}$ and $\bar{L}$ of K and the z-axis are still core curves. β descends to $\bar{\beta}$ in $\overline{N(U)}^c/\langle\alpha\rangle$ which is conjugate to $(z, t) \mapsto (-z, \bar{t})$ on $\mathbb{D}^2 \times S^1$. In fact, Fix($\bar{\beta}$) consists of the quotient of Fix(β) and (0, 0, ± 1). Since $\bar{K}$ is $\bar{\beta}$-invariant, $\bar{K} \cap \text{Fix}(\bar{\beta}) = \text{Fix}(\bar{\beta})$. Therefore, the result will follow from the following lemma.

Lemma 3.1. *Suppose K_1, K_2 are two core curves in $D^2 \times S^1$ both invariant under $\bar{\beta}:(z, t) \mapsto (-z, \bar{t})$ and* Fix($\bar{\beta}$) $\subset K_i$ *for $i = 1, 2$. Then K_1 is $\bar{\beta}$-equivariant isotopic to K_2.*

Proof. We may assume for simplicity that $K_2 = \{0\} \times S^1$. Since the pair $(D^2 \times S^1, K_1)$ is $\bar{\beta}$-equivariantly homeomorphic to $(D^2 \times S^1, K_2)$ by Proposition 1.1 in Section 1.3, we find two disjoint meridian discs, D_1 and D_{-1} so that $D_i \cap K_1 = (0, i)$ for $i = 1, -1$ (the fixed points of $\bar{\beta}$). We may also assume that D_i intersects $D \times \{j\}$ transversely for $j = 1, -1$. In particular, $D_i \cap D \times \{i\}$ contains an arc c_i passing through $(0, i)$. The rest of the submanifolds $D_i \cap D \times \{i\}$ are arcs and circles. By the standard innermost disc or half disc argument and the exchange principle, we equivariantly isotope K_1 so that $D_i \cap D \times \{i\}$ consists of only the arc containing $(0, i)$. A further equivariant isotopy sends D_i to $D \times \{i\}$ for $i = 1, -1$. Suppose $1, -1$ decompose S^1 into two arcs A and B. Then isotope $L \cap D \times A$ to $0 \times A$ in $D \times A$ fixing $\partial(0 \times A)$. Using $\bar{\beta}$-invariance, it gives an isotopy of $L \cap D \times B$ to $0 \times B$ in $D \times B$ fixing $\partial(0 \times B)$. Thus, the result follows. □

If $K \cap \text{Fix}(\alpha) \neq \emptyset$, then K is equivariantly isotopic to a line in the xy-plane. To see this, consider the quotient $S^3/\langle\alpha, \beta\rangle$ which is the suspension $\Sigma\mathbb{R}P^2$ of $\mathbb{R}P^2$ with the singular set consisting of the two points and a circle $\bar{U}$ in the middle level of the suspension. Furthermore, $\bar{U}$ is an essential curve in the middle level projective space. The cone points C_1, C_2 correspond to the quotient of Fix(β) and (0, 0, ± 1), and the circle $\bar{U}$ corresponds to the quotient of Fix(α) = U. The quotient of the xy-plane gives a cone over $\bar{U}$ denoted by $C_1(\bar{U})$. The quotient $\bar{K}$ of K in $\Sigma\mathbb{R}P^2$ is an arc joining C_1 to $\bar{U}$. Since the $\langle\alpha, \beta\rangle$-action on $N(K)^c$ is standard, there is an invariant sphere $S^2 \subset \mathbb{R}^3$ containing K and U. The quotient $\tilde{S}^2$ of S^2 in $\Sigma\mathbb{R}P^2$ is a topological cone from C_1 to $\bar{U}$. We may assume that $\tilde{S}^2$ intersects $C_1(\bar{U})$ transversely away from the cone point C_1. Since $\bar{U}$ has a nontrivial normal bundle, the number of intersection points of $\tilde{S}^2$ with $C_1(\bar{U})$ at $\bar{U}$ is odd. This shows there is an intersection arc γ in $\tilde{S}^2 \cap C_1(\bar{U})$ joining C_1 to $\bar{U}$. Therefore $\tilde{K}$ is isotopic to γ in $\tilde{S}^2$ leaving C_1 fixed, and leaving $\bar{U}$ invariant. Similarly, using the cone $C_1(\bar{U})$, we isotope γ to a quotient of a straight line in the xy-plane leaving C_1 fixed and $\bar{U}$ invariant. This gives the equivariant isotopy between K and a straight line in the xy-plane.

(b) *α is the involution $x \mapsto -x/|x|^2$ and β is the π-rotation about the z-axis.* If $K \cap \text{Fix}(\beta) \neq \emptyset$, then the same argument on the tunnels on 2-bridge knots still works. Since $\alpha\beta$ is the π-rotation about U, the same argument works for $K \cap$

$\text{Fix}(\alpha\beta) \neq \emptyset$. In the remaining cases, $K \cap (\text{Fix}(\beta) \cup \text{Fix}(\alpha\beta)) = \emptyset$. Consider the quotient $S^3/\langle\alpha, \beta\rangle$ which is S^3 with singular set consisting of the Hopf link. Each component of the link corresponds to $\text{Fix}(\beta)$ and $\text{Fix}(\alpha\beta)$. The unknot K descends to an unknot $\tilde{K}$ in S^3 so that both pairs ($\tilde{K}$, quotient of $\text{Fix}(\alpha)$) and ($\tilde{K}$, quotient of $\text{Fix}(\beta)$) are Hopf links. Thus $\tilde{K}$ is isotopic to the quotient of $\text{Fix}(\alpha)$ leaving the quotients of $\text{Fix}(\alpha)$ and $\text{Fix}(\beta)$ invariant. The lifting of the isotopy gives the required one in S^3.

(c) *α is the involution $x \mapsto -x/|x|^2$ and β is the reflection about the xy-plane.* Then $\gamma = \alpha\beta$ is the orientation reversing involution fixing $(0, 0, \pm 1)$ only. For any invariant unknot K, $K \cap \text{Fix}(\beta)$ consists of two points. $K \cap \text{Fix}(\gamma) \neq \emptyset$ since otherwise the $\langle\beta, \gamma\rangle$-action on $N(K)^c$ is conjugate to $\beta:(z, t) \mapsto (\bar{z}, t)$ and $\gamma:(z, t) \to (-z, \bar{t})$ on $D^2 \times S^1$ which implies α has fixed point in S^3. Therefore K contains $(0, 0, \pm 1)$. The orbifold $S^3/\langle\alpha, \beta\rangle$ is a cone over $\mathbb{R}P^2$, denoted by $C(\mathbb{R}P^2)$ where the cone point C corresponds to $\text{Fix}(\alpha\beta)$ and $\mathbb{R}P^2$ corresponds to $\text{Fix}(\beta)$. The straight-line segments from the cone point C to $\mathbb{R}P^2$ corresponds to circles in $\overline{\mathbb{R}}^3$ containing $(0, 0, \pm 1)$ (orthogonal to the xy-plane). Our goal is to show that the quotient $\tilde{K}$ of K in $C(\mathbb{R}P^2)$ is isotopic to one of these line segments. Clearly $\tilde{K}$ is an arc joining C to a point in $\mathbb{R}P^2$. Furthermore, the complement of $\tilde{K}$ in $C(\mathbb{R}P^2)$ is homeomorphic to the {open Möbius band} $\times I$ due to the standard action of $\langle\alpha, \beta\rangle$ on $N(K)^c$. By the same argument used in the proof of Case 3(a), $\tilde{K}$ is isotopic to a straight-line segment in $C(\mathbb{R}P^2)$ leaving both C and $\mathbb{R}P^2$ invariant. Thus K is isotopic equivariantly to the z-axis.

(d) *α is the π-rotation about the z-axis and β is the π-rotation about the x-axis.* Then $\alpha\beta$ is the π-rotation about the y-axis. K must intersect one of $\text{Fix}(\alpha)$, $\text{Fix}(\beta)$, or $\text{Fix}(\alpha\beta)$, due to the classification of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action on $N(K)^c$. Similarly, K cannot intersect exactly one of $\text{Fix}(\alpha)$, $\text{Fix}(\beta)$ or $\text{Fix}(\alpha\beta)$ at two points. Thus, K intersects exactly two of $\text{Fix}(\alpha)$, $\text{Fix}(\beta)$ and $\text{Fix}(\alpha\beta)$ each in two points. Then the action on $N(K)^c$ is conjugate to $(z, t) \mapsto (-z, t)$ and $(z, t) \mapsto (\bar{z}, \bar{t})$ on $D^2 \times S^1$. Therefore, there exists an invariant meridian disc.

(e) *α is the π-rotation about the z-axis and β is the reflection about the xy-plane.* Then $K \cap \text{Fix}(\beta)$ consists of two points and $K \cap \text{Fix}(\alpha)$ consists of two points (the other possibility that $K \cap \text{Fix}(\alpha) = \emptyset$ is excluded by the assumption that K is not in a 2-sphere). In this case, the action on $N(K)^c$ is conjugate to $\alpha:(z, t) \mapsto (\bar{z}, \bar{t})$ and $\beta:(z, t) \mapsto (\bar{z}, t)$ on $D^2 \times S^1$. Thus, there exists an invariant meridian disc.

(f) *α is the π-rotation about the z-axis and β is the reflection about the xz-plane.* Since K is not in any 2-sphere, $K \cap \text{Fix}(\beta)$ consists of two points. If $K \cap \text{Fix}(\beta) \subset \text{Fix}(\alpha)$, then the action on $N(K)^c$ is conjugate to $\alpha:(z, t) \mapsto (\bar{z}, \bar{t})$ and $\beta:(z, t) \mapsto (\bar{z}, t)$ in $\mathbb{D}^2 \times S^1$. If $(K \cap \text{Fix}(\beta)) \cap \text{Fix}(\alpha) = \emptyset$, then the action on $N(K)^c$ is conjugate to $\alpha:(z, t) \mapsto (-z, t)$ and $\beta:(z, t) \mapsto (\bar{z}, t)$ in $D^2 \times S^1$. In both cases, there exist invariant meridian discs.

Case 4: *$n \geq 3$ and α is the $2\pi/n$-rotation about the z-axis and β is the π-rotation about the x-axis.* The invariant unknot K does not intersect $\text{Fix}(\alpha)$ since $n \geq 3$. $K \cap \text{Fix}(\beta) \neq \emptyset$ by the classification of D_{2n} on $D^2 \times S^1$. Thus, $K \cap \text{Fix}(\beta)$ consists of two points and the action on $N(K)^c$ is conjugate to $\alpha:(z, t) \mapsto (e^{2\pi i/n}z, t)$ and $\beta:(z, t) \mapsto (\bar{z}, \bar{t})$ on $D^2 \times S^1$. This shows that there exists an invariant meridian disc.

Case 5: *$n \geqslant 3$ and α is the $2\pi/n$-rotation about the z-axis and β is the reflection about the xz-plane.* In this case, $K \cap \text{Fix}(\beta)$ consists of two points and $K \cap \text{Fix}(\alpha) = \emptyset$. Thus, the action on $N(K)^c$ is conjugate to $\alpha:(z, t) \mapsto (e^{2\pi i/n}z, t)$ and $\beta:(z, t) \mapsto (\bar{z}, t)$ on $D^2 \times S^1$. We find again an invariant meridian disc.

Case 6: *$n \geqslant 3$ and α is the $2\pi/n$-rotation about U and β is the map $x \mapsto -x$ in $\overline{\mathbb{R}}^3$.* $K \cap \text{Fix}(\alpha) = \emptyset$ since $n \geqslant 3$. The quotient orbifold $S^3/\langle\alpha\rangle$ is S^3 with singular locus at an unknot $\bar{U}$ of order 2. The involution β descends to $\bar{\beta}: S^3 \to S^3$ which is again the map $x \mapsto -x$. Let the quotient of K be $\bar{K}$ and the quotient of the z-axis be $\bar{L}$. By Lemma 2.1(a), $(\bar{K}, \bar{U})$ and $(\bar{L}, \bar{U})$ are still the Hopf links. Take a $\bar{\beta}$-invariant small neighborhood $N(\bar{U})$ in the quotient. We see that both $\bar{K}$ and $\bar{L}$ are $\bar{\beta}$-invariant cores of $N(\bar{U})^c$. If $K \cap \text{Fix}(\beta) = \{0, \infty\}$, then by Lemma 3.1 $\bar{K}$ is $\bar{\beta}$-equivariantly isotopic to $\bar{L}$ in $N(\bar{U})^c$. Thus, the result follows. In the other case $K \cap \text{Fix}(\beta) = \emptyset$, the action on $N(K)^c$ is conjugate to $\alpha:(z, t) \mapsto (e^{2\pi i/n}z, t)$ and $\beta:(z, t) \mapsto (-z, \bar{t})$ on $\mathbb{D}^2 \times S^1$. Thus, there exists an invariant meridian disc.

Cases 7, 8: *$n \geqslant 3$ and n is even and α is the composition of $2\pi/n$-rotation about the z-axis with the π-rotation about U, and β is either the π-rotation about the x-axis or β is the reflection about the xz-plane.*

Given an invariant unknot K, $K \cap \text{Fix}(\alpha^2) = \emptyset$ by the proof of Case 4(c). Furthermore, K and $\text{Fix}(\alpha^2)$ form a Hopf link. Thus, by taking a small invariant regular neighborhood N of $\text{Fix}(\alpha^2)$ so that $N \cap (K \cup U) = \emptyset$ we see that both K and U are invariant cores of the solid torus $\overline{N^c}$. It remains to show that K is equivariantly isotopic to U in $\overline{N^c}$. To this end, we consider the quotient $\overline{N^c}/\langle\alpha\rangle$ which is a solid torus so that the quotients of K and U are still the core curves of it. Now β descends to an involution $\bar{\beta}$ which is either $(z, t) \mapsto (z, \bar{t})$ or $(z, t) \mapsto (\bar{z}, \bar{t})$ on $D^2 \times S^1$ (note that the case β is the π-rotation about the x-axis and $K \cap \text{Fix}(\beta) = \emptyset$ does not occur). Thus, to finish the proof of this case, it suffices to show that any two $\bar{\beta}$-invariant core curves in $D^2 \times S^1$ are equivariantly isotopic. In both cases, the result follows by considering the further quotient $\overline{N^c}/\langle\alpha, \beta\rangle$ which is an orbifold whose underlying space is a ball.

Case 9. *$n \geqslant 3$, n is even and α is the composition of the $2\pi/n$-rotation about the z-axis with the reflection about the xy-plane and β is the reflection about the xz-plane.* By the proof of Case 5 in Section 2, the invariant unknot K does not intersect $\text{Fix}(\alpha^2)$. Since K is not planar, $K \cap \text{Fix}(\beta)$ consists of two points. Thus, the action on $N(K)^c$ is conjugate to $\alpha:(z, \bar{t}) \mapsto (e^{2\pi i/n}z, \bar{t})$ and $\beta:(z, t) \mapsto (\bar{z}, t)$ on $\mathbb{D}^2 \times S^1$. Thus, there exists an invariant meridian disc.

Case 10: *$n \geqslant 3$, n is even and α is the composition of $2\pi/n$-rotation about the z-axis with the reflection about the xy-plane and β is the orientation reversing Möbius involution of $\overline{\mathbb{R}}^3$ with exactly two fixed points* $(\pm 1, 0, 0)$. If the invariant unknot K does not contain $\text{Fix}(\beta)$, then the dihedral group action on $N(K)^c$ is conjugate to $\alpha:(z, t) \mapsto (e^{2\pi i/n}z, \bar{t})$ and $\beta:(z, t) \mapsto (-z, \bar{t})$. Thus there exists an invariant meridian disc. If $K \cap \mathrm{Fix}(\beta) = \mathrm{Fix}(\beta)$, we consider a small invariant regular neighborhood N of $\mathrm{Fix}(\alpha^2)$ so that $N \cap (K \cup U) = \emptyset$. By Lemma 2.1(a), K and U are invariant core curves of N^c. Our goal is to show that K is isotopic to U inside N^c equivariantly. To see this, we consider the quotient $N^c/\langle \alpha^2 \rangle$. Both α and β descend to $\bar{\alpha}$ and $\bar{\beta}$ in $N^c/\langle \alpha^2 \rangle$ so that $\langle \bar{\alpha}, \bar{\beta} \rangle$ is conjugate to the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action: $\bar{\alpha}:(z, t) \mapsto (\bar{z}, -t)$ and $\bar{\beta}:(z, t) \mapsto (-z, \bar{t})$ on $D^2 \times S^1$. The quotients $\tilde{K}$ and $\tilde{U}$ of K and U are still invariant cores of $N^c/\langle \alpha^2 \rangle$. Furthermore $\tilde{K} \supset \mathrm{Fix}(\bar{\beta})$. Thus, the result follows from the following lemma.

Lemma 3.2. *Suppose L is a core of $D^2 \times S^1$ invariant under $\bar{\alpha}:(z, t) \mapsto (\bar{z}, -t)$ and $\bar{\beta}:(z, t) \mapsto (-z, \bar{t})$ and* $\mathrm{Fix}(\bar{\beta}) \subset L$. *Then L is $\langle \bar{\alpha}, \bar{\beta} \rangle$-equivariantly isotopic to the standard core $0 \times S^1$.*

Proof. Let A be the invariant annulus $\{z \mid z = \bar{z}\} \times S^1$ in $D^2 \times S^1$. Since the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action on $D^2 \times S^1 - N(K)$ is conjugate to the standard action, there exists an invariant annulus A' properly imbedded in $D^2 \times S^1$ so that $L \subset \mathrm{int}(A')$ and $\partial A' = \partial A$. By a slight perturbation, we may assume that A intersects A' transversely in $\mathrm{int}(A')$ and $\mathrm{Fix}(\beta) \subset A \cap A'$. Thus there exists an intersection curve γ in $A \cap A'$ containing (0, 1).

Claim. $\bar{\beta}(\gamma) = \gamma$ *and* $\bar{\alpha}(\gamma) = \gamma$.

The first statement is clear from the invariance of A, A'. If $\bar{\alpha}(\gamma) \neq \gamma$, then $\bar{\alpha}(\gamma) \cap \gamma = \emptyset$. Thus $\bar{\alpha}(\gamma)$ is the other intersection curve in $A \cap A'$ containing $(0, -1)$. Since $\bar{\alpha}$ restricted on A is the π-rotation about the core curve, $\bar{\alpha}(\gamma) \cap \gamma = \emptyset$ implies both γ and $\bar{\alpha}(\gamma)$ are null homotopic. Thus, they bound two disjoint discs in A and they also bound two discs D_1 and D_2 in A'. $D_1 \cup D_2$ is $\bar{\beta}$-invariant by the construction. However, since D_i will be in one side of $D^2 \times S^1 - A$ (at least near ∂D_i), $D_1 \cup D_2$ is not $\bar{\beta}$-invariant.

Therefore $A' \cap A$ contains an $\langle \bar{\alpha}, \bar{\beta} \rangle$-invariant intersection curve γ. Now, L is equivariantly isotopic to γ in A' and γ is equivariantly isotopic to $\{0\} \times S^1$ in A. Thus the result follows. □

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References

[1] H. Bass and J. Morgan, The Smith Conjecture (Academic Press, Orlando, FL, 1984).
[2] F. Bonahon, Diffeotopies des espaces lenticulaires, Topology 22 (1983) 305–314.

[3] M.H. Freedman, Z.-X. He and Z. Wang, Möbius energy of knots and unknots, Ann. of Math., to appear.
[4] A. Hatcher, A proof of the Smale conjecture, Diff(S^3) ≅ $O(4)$, Ann. of Math. 117 (1983) 553–607.
[5] W. Meeks III, L. Simon and S.-T. Yau, The classical plateau problem and the topology of three dimensional manifolds, Topology 21 (1982) 409–442.
[6] J. O'Hara, Energy of a knot, Topology 30 (1991) 24–247.
[7] H. Schubert, Knoten mit zwei brücken, Math. Z. 65 (1956) 133–170.
[8] J.-P. Serre, Linear Representations of Finite Groups (Springer, New York, 1971).