Automorphisms of Thurston's Space of Measured Laminations

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The purpose of the note is to give a characterization of the action of the mapping class group on Thurston's space of measured laminations.

1. We begin with some abstract definitions. Suppose X is a topological space and $\mathcal{F}$ is a collection of real valued (or complex valued) functions on X. We say that $\mathcal{F}$ defines an $\mathcal{F}$-*structure* $(X, \mathcal{F})$ on X if the topology on X is the weakest topology so that each element in $\mathcal{F}$ is continuous, i.e., the collection $\{f^{-1}(U)|\ U \text{ open in } \mathbf{R},\ f \in \mathcal{F}\}$ forms a subbasis for the topology on X. For instance, take a smooth manifold X and let $\mathcal{F}$ be the set of all smooth functions on X. Then $(X, \mathcal{F})$ is the smooth structure on X. An *automorphism* of a structure $(X, \mathcal{F})$ is a self-homeomorphism ϕ of X so that $\phi^*(\mathcal{F}) = \mathcal{F}$ where $\phi^*(\mathcal{F}) = \{f \circ \phi | f \in \mathcal{F}\}$.

2. Suppose now that $\Sigma = \Sigma_{g,r}$ is a compact orientable surface of genus g with r many boundary components ($r \geq 0$). Let $S(\Sigma)$ be the set of isotopy classes of homotopically non-trivial, not boundary parallel, unoriented simple loops in Σ. Given α and β in Σ, their *geometric intersection number,* denoted by $I(\alpha, \beta)$, is the minimal number of intersection points between their representatives, i.e., $I(\alpha, \beta) = \min\{|a \cap b| | a \in \alpha, b \in \beta\}$. Thurston's space of (compactly supported) measured laminations on Σ, denoted by $ML(\Sigma)$, is defined as follows. Given $\alpha \in S(\Sigma)$, let I_α be the function defined on $S(\Sigma)$ sending β to $I(\alpha, \beta)$. The space $ML(\Sigma)$ is the closure of $\mathbf{Q}_{>0}\{I_\alpha | \alpha \in S(\Sigma)\}$ in $\mathbf{R}^{S(\Sigma)}$ under the product topology. Thurston showed that $ML(\Sigma)$ is homeomorphic to the Euclidean space $\mathbf{R}^{6g-6+2r}$ and the intersection pairing $I : S(\Sigma) \times S(\Sigma) \to \mathbf{R}$ extends to a continuous pairing $I : ML(\Sigma) \times ML(\Sigma) \to \mathbf{R}$ so that $I(k_1 x_1, k_2 x_2) = k_1 k_2 I(x_1, x_2)$ for $k_1, k_2 \in \mathbf{R}_{\geq 0}$. (See [Bo], [FLP], [Re], [Th] and others for a proof.) In particular, for each α in $S(\Sigma)$, the map I_α from $ML(\Sigma)$ to $\mathbf{R}$ sending m to $I(\alpha, m)$ is continuous and the collection $\mathcal{F} = \{I_\alpha\ | \alpha \in S(\Sigma)\}$ forms an $\mathcal{F}$-structure on $ML(\Sigma)$. According to [Th], the structure is called the *piecewise integral linear* structure on $ML(\Sigma)$. See also [Lu1].

Our result is the following.

Theorem 1. *Suppose Σ is a compact surface with or without boundary whose Euler characteristic is negative. Then any automorphism of the piecewise integral linear structure on the space of measured laminations $ML(\Sigma)$ is induced by a self-homeomorphism of the surface.*

3. Proof of theorem 1.

Let ϕ be an automorphism of $(ML(\Sigma), \mathcal{F})$. Then ϕ induces a bijection ψ of $S(\Sigma)$ by the equation $I_\alpha \circ \phi = I_{\psi(\alpha)}$.

We shall first show that ψ is induced by a self-homeomorphism of the surface. To this end, let us recall that two classes α and β in $S(\Sigma)$ are called *disjoint,* denoted by $\alpha \cap \beta = \emptyset$, if $\alpha \neq \beta$ and $I(\alpha, \beta) = 0$. By counting dimension of $I_\alpha^{-1}(0)$, we shall prove that ψ preserves the disjoint relation on $S(\Sigma)$. Now by a result on the automorphism of $(S(\Sigma), \cap)$ (the automorphisms of the curve complex, [Iv], [Ko], [Lu2]), we see that ψ is induced by a self-homeomorphism of the surface.

Given α in $S(\Sigma)$, let $Z_\alpha = I_\alpha^{-1}(0) \subset ML(\Sigma)$. By using the Dehn-Thurston coordinate associated to a 3-holed sphere decomposition of the surface so that α is a decomposing loop

([FLP], [PH]), we see that the dimension $\dim(Z_\alpha)$ is $\dim(ML(\Sigma))$ -1 (only the intersection coordinate with α vanishes).

Lemma 2. *Two elements α and β in $S(\Sigma)$ are disjoint if and only if dim($Z_\alpha \cap Z_\beta$) = dim $ML(\Sigma)$ -2.*

Corollary 3. *The bijection ψ from $S(\Sigma)$ to $S(\Sigma)$ preserves the disjointness.*

Indeed, the equation $I_\alpha \circ \phi = I_{\psi(\alpha)}$ shows that $\phi^{-1}(Z_\alpha) = Z_{\psi(\alpha)}$.

Proof of lemma 2. We may assume that there exist disjoint elements in $S(\Sigma)$, i.e., $\dim(ML(\Sigma)) \geq 4$. Clearly, if α is disjoint from β, then $\dim(Z_\alpha \cap Z_\beta) = \dim(ML(\Sigma)) - 2$. This can be seen by considering the Dehn-Thurston coordinate associated to a 3-holed sphere decomposition so that both α and β are decomposing loops. We now prove that if $\alpha \cap \beta \neq \emptyset$, then $\dim(Z_\alpha \cap Z_\beta) \leq \dim ML(\Sigma) - 3$.

To see this, take $a \in \alpha$ and $b \in \beta$ so that $|a \cap b| = I(\alpha, \beta) > 0$. Let N be a small regular neighborhood of $a \cup b$. If N has null homotopic boundary components in Σ, add the disc bounded by the boundary component to N. As a result, we obtain a connected subsurface Σ' whose boundary components are essential in Σ. Since $\alpha \cap \beta \neq \emptyset$, the Euler characteristic of Σ' is negative and $\Sigma' \neq \Sigma_{0,3}$, i.e., $\dim(ML(\Sigma')) \geq 2$. Furthermore, α and β form a surface filling pair in Σ', i.e., $I(\alpha, m) + I(\beta, m) > 0$ for all $m \in ML(\Sigma')$. This implies that if $m \in ML(\Sigma)$ so that $I(m, \alpha) + I(m, \beta) = 0$, then m is supported in $\Sigma - \Sigma'$, i.e., there exist $m' \in ML(\Sigma - \Sigma')$ and some boundary components of $\alpha_1, ..., \alpha_n$ of Σ' so that m is the disjoint union $m' \alpha_1^{k_1} ... \alpha_n^{k_n}$ where $k_i \in \mathbf{R}_{\geq 0}$. If $\Sigma - \Sigma'$ consists of annuli, then clearly $Z_\alpha \cap Z_\beta = \{0\}$. The result follows. If otherwise, choose a 3-holed sphere decomposition of Σ' and extend it to a 3-holed sphere decomposition of Σ. For each isotopy class γ of a boundary component of Σ', we have $I(m, \gamma) = 0$ for all $m \in Z_\alpha \cap Z_\beta$. Thus, by counting the Dehn-Thurston coordinates associated to the 3-holed sphere decomposition, we obtain $\dim(Z_\alpha \cap Z_\beta) \leq \dim(ML(\Sigma))$ -$\dim(ML(\Sigma')) - 1 \leq \dim(ML(\Sigma)) - 3$. □

Now if $\dim ML(\Sigma) \geq 4$ and $\Sigma \neq \Sigma_{1,2}$, then by theorem 1(a) of [Lu2] (see also [Iv], [Ko]) there exists a self-homeomorphism f of Σ so that $f_*^{-1}(\alpha) = \psi(\alpha)$ for all α in $S(\Sigma)$. In particular, $I_\alpha \circ \phi = I_\alpha \circ f_*$. Since the map from $ML(\Sigma)$ to $\mathbf{R}_{\geq 0}^{S(\Sigma)}$ sending m to $(I_\alpha(m))_{\alpha \in S(\Sigma)}$ is an embedding, we obtain $\phi = f_*$.

It remains to deal with the surfaces $\Sigma = \Sigma_{1,2}$, $\Sigma_{1,1}$ or $\Sigma_{0,4}$. For surface $\Sigma_{1,2}$, we shall prove that $\psi : S(\Sigma_{1,2}) \to S(\Sigma_{1,2})$ preserves the classes represented by separating simple loops. Assume this, then theorem 1(b) of [Lu2] shows that ψ is induced by a self-homeomorphism of the surface. Thus, the above argument goes through.

Suppose otherwise that ψ sends a separating class to a non-separating class. We shall derive a contradiction by relating $ML(\Sigma_{1,2})$ to $ML(\Sigma_{0,5})$. Let τ be an hyper-elliptic involution of $\Sigma_{1,2}$ with four fixed points so that the quotient space $\Sigma_{1,2}/\tau$ is the disc $\mathbf{D}^2$ with four branch points. It is known by the work of Birman [Bi] and Viro [Vi] that $\tau(s)$ is isotopic to s for each simple loop s not homotopic into $\partial\Sigma_{1,2}$. In particular, we obtain $\tau_*(m) = m$ for all $m \in ML(\Sigma_{1,2})$. Let $\pi : \Sigma_{1,2} \to \mathbf{D}^2$ be the quotient map. Consider $\Sigma_{0,5}$ as the disc $\mathbf{D}^2$ with a regular neighborhood $N(B)$ of the the branched point set B removed, i.e., $\Sigma_{0,5} = \mathbf{D} - int(N(B))$. Define $p : S(\Sigma_{0,5}) \to ML(\Sigma_{1,2})$ by sending the isotopy class $[a]$ to the measured lamination

$[\pi^{-1}(a)]$. Note that if $\pi^{-1}(a)$ is connected, then it is a separating loop and if $\pi^{-1}(a)$ is not connected, then it is a union of two parallel copies of a non-separating simple loops. This map p extends to a homeomorphism, still denoted by p, from $ML(\Sigma_{0,5})$ to $ML(\Sigma_{1,2})$ so that $I(p(m_1), p(m_2)) = 2I(m_1, m_2)$ for all $m_1, m_2 \in ML(\Sigma_{0,5})$.

Now consider the homeomorphism $\phi' : ML(\Sigma_{0,5}) \to ML(\Sigma_{0,5})$ given by $p^{-1}\phi p$. Since $I_\alpha \phi = I_{\psi(\alpha)}$ for all $\alpha \in S(\Sigma_{1,2})$, we obtain $\lambda I_\alpha \circ \phi' = I_{\psi'(\alpha)}$ for all $\alpha \in S(\Sigma_{0,5})$ where $\psi' : S(\Sigma_{0,5}) \to S(\Sigma_{0,5})$ is a bijection and $\lambda = 1$ or $1/2$ or 2 depending on the components of $\pi^{-1}(\alpha)$ and $\pi^{-1}(\phi'(\alpha))$ being separating or not. By the assumption that ψ sends some non-separating simple loops to separating ones, the function λ is not a constant. Due to the equation $\lambda I_\alpha \circ \phi' = I_{\psi'(\alpha)}$, the map ϕ' preserves the set $\{Z_\alpha \,|\alpha \in S(\Sigma_{0,5})\}$. By lemma 2, we see that ψ' preserves the disjointness. Thus ψ' is induced by a self-homeomorphism h of $\Sigma_{0,5}$. In particular we obtain $\lambda I_\alpha \circ \phi' = I_\alpha \circ h$ for all α. Since the set of rational multiples of $S(\Sigma_{0,5})$ is dense in $ML(\Sigma_{0,5})$ and both ϕ' and h are are homogeneous, it follows that $\phi' = kh$ for some fixed constant $k = 1$ or $1/2$ or 2. This contradicts the assumption that λ is not a constant.

Finally, we show that any automorphism of $(ML(\Sigma_{1,1}), \mathcal{F})$ and $(ML(\Sigma_{0,4}), \mathcal{F})$ is induced by a surface homeomorphism. Since the structures $(ML(\Sigma_{1,1}), \mathcal{F})$ and $(ML(\Sigma_{0,4}), \mathcal{F})$ are isomorphic, we shall deal with the case $\Sigma_{1,1}$ only.

Let us first identify both $ML(\Sigma_{1,1})$ and $S(\Sigma_{1,1})$ with the first homology groups. Let $i : S(\Sigma_{1,1}) \to H_1(\Sigma_{1,1}, \mathbf{Z})/\pm 1$ be the natural map sending an isotopy class to the corresponding homology classes. It is well known that the map is a bijection from $S(\Sigma_{1,1})$ to $\mathcal{P}/\pm 1$ where $\mathcal{P}$ is the set of primitive elements in $H_1(\Sigma_{1,1}, \mathbf{Z}) \cong \mathbf{Z}^2$. Furthermore, by taking a $\mathbf{Z}$-basis for $H_1(\Sigma_{1,1}, \mathbf{Z})$, each $i(\alpha)$ can be written as $\pm(a, b)$ where a, b are relatively prime integers. Under this identification, the intersection number $I(\alpha_1, \alpha_2) = |a_1 b_2 - a_2 b_1|$ where $\alpha_i = \pm(a_i, b_i)$. In particular, this shows that $ML(\Sigma_{1,1})$ can be naturally identified with $H_1(\Sigma_{1,1}, \mathbf{R})/\pm 1 \cong \mathbf{R}^2/\pm 1$ so that the above intersection number formula still holds.

The action of self-homeomorphisms on $ML(\Sigma_{1,1})$ is induced by the $GL(2, \mathbf{Z})$ action on $\mathbf{R}^2/\pm 1$. Thus, it remains to show that if $\phi = (\phi_1, \phi_2) : \mathbf{R}^2/\pm 1 \to \mathbf{R}^2/\pm 1$ is a self-homeomorphism so that for each pair of relative prime integers $(a, b) \in \mathcal{P}$ there exists a new pair $(a', b') \in \mathcal{P}$ satisfying $|a\phi_1(x, y) - b\phi(x, y)| = |a'x - b'y|$ for all $(x, y) \in \mathbf{R}^2$, then ϕ is induced by an element in $GL(2, \mathbf{Z})$. By taking (a, b) to be $(1, 0)$ and $(0, 1)$, we see that $|\phi_1(x, y)| = |a_1 x + b_1 y|$ and $|\phi_2(x, y)| = |a_2 x + b_2 y|$. Since ϕ is a homeomorphism, $a_1 b_2 - a_2 b_1 \neq 0$. The goal is to show that $a_1 b_2 - a_2 b_1 = \pm 1$. Since both ϕ_1, ϕ_2 are continuous and $|\phi_1(x, y) \pm \phi_2(x, y)|$ is of the form $|ax + by|$, it follows that $\phi_i(x, y) = \pm(a_i x + b_i y)$ for $i = 1, 2$. Now if $|a_1 b_2 - a_2 b_1| \geq 2$, then one can find $(a, b) \in \mathcal{P}$ so that $a\phi_1 - b\phi_2$ is of the form $|cx + dy|$ where c and d have a common non-trivial divisor. This contradicts the assumption. $\square$

4. One consequence of the proof of the theorem 1 is the following characterization of the action of the mapping class group on the projectivized measured lamination space $PML(\Sigma) = ML(\Sigma) - \{0\}/\mathbf{R}_{>0}$.

Theorem 4. (Automorphisms of the projective measured lamination spaces) *Suppose Σ is a compact orientable surface so that $dim(ML(\Sigma)) \geq 2$ and $\Sigma \neq \Sigma_{1,2}$. For each $\alpha \in S(\Sigma)$, let P_α be the image of $\{m \in ML(\Sigma) - \{0\} | I(m, \alpha) = 0\}$ in $PML(\Sigma)$. If ϕ is a self-homeomorphism* *of the projective measured lamination space $PML(\Sigma)$ preserving the collection $\{P_\alpha|\alpha \in S(\Sigma)\}$, then ϕ is induced by a self-homeomorphism of the surface.*

Proof. By lemma 2 and the result on the automorphism of the curve complex, we see that there exists a self-homeomorphism f of the surface so that $f_*\phi^{-1} : PML(\Sigma) \to PML(\Sigma)$ sends each P_α to P_α. The image of $\mathcal{P}(\alpha)$ of $\alpha \in S(\Sigma)$ in $PML(\Sigma)$ can be expressed as a finite intersection $P_{\alpha_1} \cap P_{\alpha_2} \cap ... \cap P_{\alpha_k}$. Thus $f_*\phi^{-1}$ is the identity map on the set $\{\mathcal{P}(\alpha) \ |\alpha \in S(\Sigma)\}$. Since the set $\{ \ \mathcal{P}(\alpha) \ |\alpha \in S(\Sigma)\}$ is dense in $PML(\Sigma)$, it follows that $\phi = f_*$. □

5. *Remark.* The theorem is valid for $\Sigma_{1,2}$ if we assume that the self-homeomorphism ϕ preserves the subset $\{P_\alpha|\alpha$ is a separating class $\}$. Otherwise, it is false. See [Lu2].

6. Similar automorphism results hold for the Teichmuller space and $SL(2, \mathbf{R})$ characters. For simplicity, we state the result for the Teichmuller space. The proof is essentially the same as above and will be omitted. Let $T(\Sigma)$ be the space of isotopy classes of hyperbolic metrics with cusp ends on $int(\Sigma)$. For each $\alpha \in S(\Sigma)$, let $l_\alpha : T(\Sigma) \to \mathbf{R}$ be the geodesic length function sending a metric m to the length of m-geodesic in α. The work of Fricke-Klein [FK] shows that the collection $\{l_\alpha|\alpha \in S(\Sigma)\}$ forms an $\mathcal{F}$-structure on the Teichmuller space.

Theorem 5. *Suppose Σ is a compact surface of negative Euler characteristic. Then any automorphism of $(T(\Sigma), \mathcal{F})$ is induced by a self-homeomorphism of the surface.*

The key step in the proof is to show a result similar to lemma 2. In this case, it is the Margulis lemma that $\alpha \cap \beta = \emptyset$ if and only if $inf\{l_\alpha + l_\beta\} = 0$ on $T(\Sigma)$.

7. Currently, we are unable to solve the automorphism problem for the variety of characters of $SL(2, \mathbf{C})$ representations of a closed surface group with respect to the structure of the trace functions $\{tr_\alpha|\alpha \in S(\Sigma)\}$. Here tr_α sends a character χ to $\chi(\alpha)$. See [CS] for an introduction to the subject. The main difficulty is due to the lacking of intrinsic characterization of disjointness $\alpha \cap \beta = \emptyset$ in terms of the trace functions tr_α and tr_β.

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