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Conformal embedding of a disc with a Lorentz metric into the plane

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1. Introduction

Given an open disc $\mathring{D}^2$ with a Lorentz metric g, we are interested in finding a conformal embedding of $(\mathring{D}^2, g)$ into $E^{1,1}$ where $E^{1,1}$ denotes $\mathbf{R}^2$ with coordinates $\{x, y\}$ and the metric $dx\,dy$. This is equivalent to find a smooth embedding of $(\mathring{D}^2, g)$ into $E^{1,1}$ so that the two transverse foliations of D^2 obtained from the null directions of g are mapped into the horizontal and vertical lines. Our investigation is based on the classification theory of foliations of open discs developed by W. Kaplan (1941), A. Haefliger and G. Reeb (1957). According to this theory, one associates to each foliation of an open disc ($\mathring{D}^2$, one foliation) the quotient space which is a smooth simply-connected (in general non-Hausdorff) 1-manifold. This 1-manifold essentially classifies the once foliated disc up to smooth equivalence. For instance, the 1-manifold associated to the plane with the horizontal foliation is diffeomorphic to the open interval (0,1). Due to the existence of nonstandard smooth structures on simply-connected (non-Hausdorff) 1-manifolds (see Haefliger and Reeb [HR]), there are very few smooth embeddings of the twice foliated disc into the twice foliated plane (see [SW1]). We are therefore led to the problem of finding a conformal (homeomorphic) embedding of an open disc with a Lorentz metric into the plane $E^{1,1}$. By this we mean a topological embedding of $(\mathring{D}^2, g)$ $E^{1,1}$ taking the associated foliations into the horizontal and vertical lines. We will use the language of twice transversely foliated discs instead of the language of conformal classes of Lorentz metrics in discs and all maps are in the C^0 category in this paper. Two such foliations are called isomorphic if there is a homeomorphism between the discs so that the leaves are sent to leaves, i.e., their associated Lorentz metrics are conformally isomorphic. It is not always possible to conformally embed a Lorentz metric on the disc into the plane $E^{1,1}$; the main difficulty is described by the following definition.

Definition 1. *We will say that a leaf L in a twice transversely foliated open disc $(\mathring{D}^2; F_h, F_v)$ is short if there is an arc A in a leaf transverse to L, an end of L and two leaves K and K' of the other foliation on that end of L such that every leaf L' transverse to A other than L meets one of K or K' but L meets neither K nor K' (see Fig. 1). Equivalently, L is short if there is an embedding (as a twice foliated disc but not necessarily a proper embedding) of $\mathbf{R}^2$ with one of the four rays $\{(x,0) : x \geq 0\}$, $\{(x,0) : x \leq 0\}$, $\{(0,y) : y \geq 0\}$, or $\{(0,y) : y \leq 0\}$ removed into $(\mathring{D}^2; F_h, F_v)$ such that the restriction to $\{(x,0) : -1 \leq x < 0\}$, $\{(x,0) : 0 < x \leq 1\}$, $\{(0,y) : -1 \leq y < 0\}$, or $\{0,y) : 0 < y \leq 1\}$ respectively is a proper embedding of a half open interval into L.*

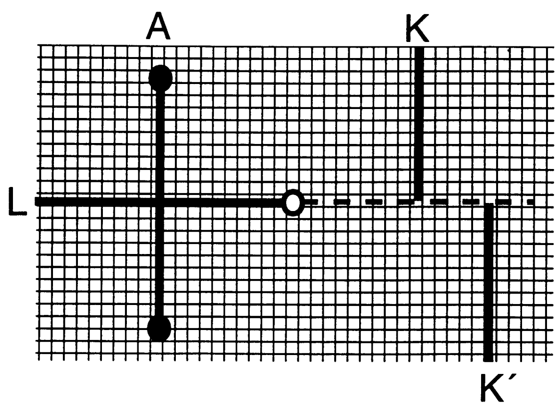


Fig. 1. A short leaf L

Strictly speaking the definition of a short leaf should probably include some notation as to which end of the leaf is short but for simplicity this has been omitted. An example of a twice foliated disc containing a short leaf is given in Fig. 2. The short leaf L ends at the cusp point. This twice-foliated disc also gives an example of a twice-foliated disc which cannot be embedded (as a twice-foliated disc) into the plane. We will show that if a twice-foliated disc has no short leaves or if all the short leaves in $(\mathring{D}^2; F_h, F_v)$ are manageable then an embedding of $(\mathring{D}^2; F_h, F_v)$ in $\mathbf{R}^2$ exists. Thus we have the following result.

Theorem 1. *Let $(\mathring{D}^2; F_h, F_v)$ be an open disc with two transverse foliations. If $(\mathring{D}^2; F_h, F_v)$ has no short leaves, then there is a topological embedding of $\mathring{D}^2$ into the plane sending leaves of F_h (resp. F_v) to horizontal (resp. vertical) lines.*

The related problems in the smooth category are much more difficult. Smooth conformal embedding of Lorentz surfaces was first considered by R. Kulkarni [Ku]. T. Weinstein and her students have done much work in the field of Lorentz

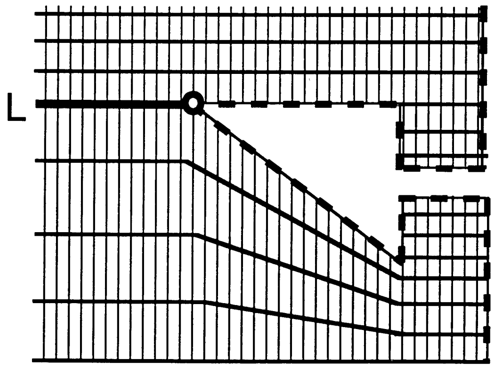


Fig. 2. A twice foliated disc with a short leaf L. (In fact it is a bad-bad short leaf hence nonembeddable

surfaces recently. See [Kr], [Sm], [SW1], [SW2], and [W]. The book [W] is an excellent introduction to this subject.

The organization of the paper is as follows. In Sect. 2, we will recall the basic facts concerning the classification of a foliated disc and we will develop the basic definitions needed to prove the theorem above. We will also define precisely what we mean by manageable short leaves in Sect. 2. In Sect. 3 we prove this stronger version of Theorem 1. Finally in Sect. 4 we give a number of examples of twice-foliated discs which cannot be embedded in the plane. These examples arise naturally when one tries to remove the short leaves of a twice foliated disc. Unfortunately the study of such examples has proved to be quite complicated and tedious. We have therefore contented ourselves with an informal description.

2. Preliminaries

We will use the following notation and conventions. Suppose $\mathring{D}^2$ is an open disc with a codimension one topological foliation F. It is known that $(\mathring{D}^2; F)$ is classified by the directed, simply-connected (non-Hausdorff) orbit manifold M_f, where a point $[L]$ of M_f is a leaf L of F and a Euclidean neighborhood of $[L]$ is the set of leaves intersecting a transverse arc to L, see Kaplan [K1], [K2] and Haefliger and Reeb [HR] for proofs. There are countably many points in M_f where the Hausdorff condition fails. We call these points the branch points of M_f and the corresponding leaves the branch leaves. It is easy to show from the classification theorem that $(\mathring{D}^2; F)$ can be topologically embedded (preserving foliations) in the plane foliated by the horizontal lines if and only if $(\mathring{D}^2; F)$ does not contain a simply-connected subdomain D' so that the restricted foliation $(D'; F|_{D'})$ is isomorphic to horizontal foliation of the “cusp” shown in Fig. 2 above. Suppose that $(\mathring{D}^2; F_h, F_v)$ is an open disc with two transverse foliations. We will call a leaf in F_h (resp. F_v) an h-leaf (resp. v-leaf). Let M_h and M_v be the directed orbit manifolds respectively and let $p_h : \mathring{D}^2 \to M_h$ and $p_v : \mathring{D}^2 \to M_v$

be the quotient maps. It is well known that both p_h and p_v are submersions. By this we mean that for each point $x \in \mathring{D}^2$ there is an open arc containing x which is mapped homeomorphically into M_h (resp. M_v).

Lemma 1. *There exist surjective local homeomorphisms $f_h : M_h \to \mathbf{R}$ and $f_v : M_v \to \mathbf{R}$.*

Proof. For a proof see Haefliger and Reeb 1957. □

An immediate corollary of this lemma is that the product map $f_v \circ p_v \times f_h \circ p_h : \mathring{D}^2 \to \mathbf{R}^2$ gives an immersion, i.e., a local homeomorphism, of the twice foliated disc into the plane sending the leaves to horizontal and vertical lines. The goal of this paper is to study when this immersion can be arranged to be an embedding. The following easy lemma will be the central technical tool needed for the proof.

Lemma 2. *Suppose $(\mathring{D}^2; F_h, F_v)$ is an open disc with two transverse foliations such that either $M_h \cong \mathbf{R}$ or $M_v \cong \mathbf{R}$. Then the product map $f_v \circ p_v \times f_h \circ p_h : \mathring{D}^2 \to \mathbf{R}^2$ is an embedding.*

Proof. Suppose for definitiveness that $M_h = \mathbf{R}$ and suppose that $f_v \circ p_v \times f_h \circ p_h$ is not one-to-one. Then there are two distinct points x, y in $\mathring{D}^2$ which are mapped to the same point in $\mathbf{R}^2$. Hence x and y must lie on the same h-leaf L. Consider the restriction of $f_v \circ p_v$ to L. The restriction of p_v to L is a local homeomorphism, therefore the restriction $f_v \circ p_v$ to L must also be a local homeomorphism. Thus $f_v \circ p_v|_L$ is injective. However since x and y have the same image in $\mathbf{R}^2$, $f_v \circ p_v$ must have the same value at x as at y. This contradicts the assumption that x and y are distinct. □

Suppose we are given a short h-leaf L (whose short end is the right one) in $(\mathring{D}^2; F_h, F_v)$. By definition there is a subset $U \subset \mathring{D}^2$ such that $(U; F_h|_U, F_v|_U)$ is isomorphic (as a twice foliated disc) to $\mathbf{R}^2$ with the ray $\{(x, 0) : x \geq 0\})$ removed and the right end of L is sent to $\{(x, 0) : -1 \leq x < 0\}$. Let $K \subset \mathring{D}^2$ be the v-leaf sent to $\{(-1, y)\}$. Let $V = p_h^{-1}(p_h(K))$. Then by Lemma 2 there is an embedding of V in the plane. Assume this embedding is arranged so that $(U; F_h|_U, F_v|_U)$ is sent to the open square $(-1, 1) \times (-1, 1)$ with the line segment $\{(x, 0) : 0 \leq x < 1\}$ removed and the image V' of V is contained in $(-2, 2) \times (-2, 2)$.

Consider the intersection of $(V')^c$ with the open upper half plane H_+. Let x_+, be the smallest positive number such that $(x_+, 0)$ is a limit point of $(V')^c \cap H_+$. There may be vertical lines in V' which have lower endpoint $(x, 0)$ for some $x > 0$ (necessarily with $x < x_+$) and for which the corresponding v-leaf K' of $(\mathring{D}^2; F_h, F_v)$ would extend below the x-axis. Phrased differently the v-leaf K' meets some h-leaf which lies below every h-leaf of $(V')^c \cap H_+$. If so let w_+ be the infimum of all such x-coordinates if not let $w_+ = \infty$. We will use the parameters x_+ and w_+ to describe the upper side of the short h-leaf L. Corresponding definitions can be made for the intersection of $(V')^c$ with the open lower half plane H_- giving numbers x_- and w_- or for short v-leaves.

Definition 2. *We will say the upper side (resp. lower side) of the short h-leaf L is very bad if $w_+ = 0$ (resp. $w_- = 0$). We will say it is bad if $w_+ < x_+$ (resp. $w_- < x_-$). Otherwise we will say the side is good (and analogously for short v-leaves). We will say a short leaf is of type good-good if both sides are good, good-bad if one side is good and the other is bad, etc..*

This distinction turns out to be central to the problem as the following lemma shows.

Lemma 3. *If any short leaf of $(\mathring{D}^2; F_h, F_v)$ has a very bad side or if both sides are bad, then $(\mathring{D}^2; F_h, F_v)$ cannot be embedded (as a twice foliated disc) in the plane.*

Proof. Any bad side contains a subset homeomorphic to a single "tooth", a very bad side contains a subset homeomorphic to sequence of "teeth" converging down to the origin (see for instance Fig. 7). Therefore it is enough to show that the twice foliated discs built from these models cannot be embedded. This however is easy. □

In contrast, if all short leaves of $(\mathring{D}^2; F_h, F_v)$ are of type good-good then we will show that an embedding of $(\mathring{D}^2; F_h, F_v)$ in the plane always exists. Thus we have the stronger version of Theorem 1 alluded to above.

Theorem 1'. *Let $(\mathring{D}^2; F_h, F_v)$ be an open disc with two transverse foliations. If $(\mathring{D}^2; F_h, F_v)$ has only short leaves of type good-good, then there is an embedding of $\mathring{D}^2$ into the plane sending leaves of F_h (resp. F_v) to horizontal (resp. vertical) lines.*

Before proving this theorem we need to develop a few more useful sets of definitions. First note that the definition of a short leaf and Definition 2 above treat the two sides of a leaf L separately. Thus we will say L is half-short (at a particular end) if there is an arc A in a leaf transverse to L, an end of L, a side of L and a leaf K of the other foliation such that every leaf L' transverse to A on the chosen side of L meets K but L does not meet K. We will further say that this side is good, bad or very bad in the cases described above.

For the remaining definitions fix a v-leaf L, the side of the leaf L under discussion V and an end of the leaf L. We will use the letter L with primes or other marks for v-leaves and the letter K with primes for h-leaves. Interchanging the roles of v- and h-produces the analogous definition for h-leaves. The key idea is that in V near the chosen end of L there is a notion of the angle of the foliations. This angle will be taken to be a (positive) multiple of 45°. Precise definitions will be given below but roughly the angle will be $x°$ if it should be possible to embed V so that near that end of L the embedding lies within an arc of $x°$ about that endpoint of L.

Definition 3. *(1) The angle is 45° if for every v-leaf L' in V there is a neighborhood A (depending on L') of the end of L such that every h-leaf K which meets L inside A misses L' (see Fig. 3).*

(2) The angle is 90° *if there is a* h*-leaf* K *not in* V *with the same endpoint as* L *which is a short or half-short leaf and* L *is on a good short side of* K.

(3) The angle is 135° *if it is not* 45° *or* 90° *and if for any* h*-leaf* K *in* V *above* L *there is a neighborhood* B *of* L *such that* K *does not meet any* v*-leaf* L' *in* B *(see Fig. 4).*

(4) If L *is a short or half-short leaf and* V *is a good short side of* L*, then the angle is* 180°.

(5) If the angle is not one of the above and there is no v*-leaf* $\hat{L}$ *parallel to* L *with a common endpoint, then the angle is* 225°. *If there is such a leaf* $\hat{L}$*, then we define the angle recursively. Let* $\hat{V}$ *be the side of* $\hat{L}$ *not containing* L *and define the angle to be* 180° *more than the angle of* $\hat{V}$ *near* $\hat{L}$ *(see Fig. 5)*

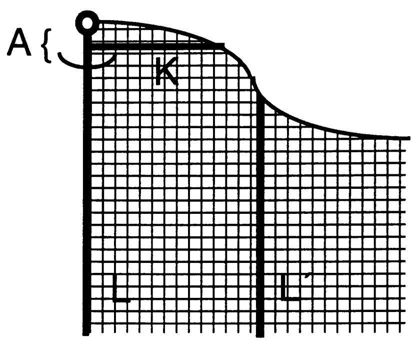


Fig. 3. An end of angle 45°

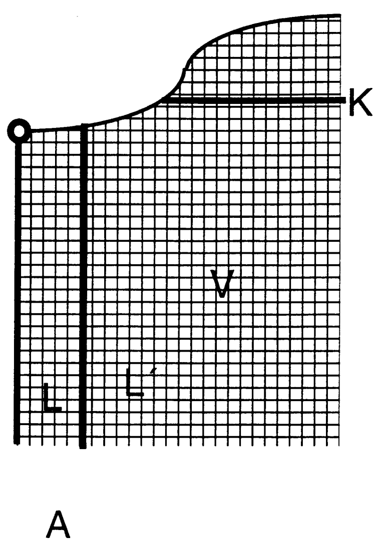


Fig. 4. An end of angle 135°

Notice that for odd multiples of 45° these angles are local in the sense that they only depend on what the two foliations look like in a neighborhood of the end of L. These angles in fact are almost a geometric concept by virtue of Proposition 1 below.

3. Construction of embeddings

With the terminology developed above we can now state and prove a sharper form of Theorem 1' stated above. The outline of the proof is as follows. We break the disc up into blocks each of which has one orbit manifold $\mathbf{R}$. Then we use Lemma 2 to construct embeddings of these blocks into the plane. We wish to combine these embeddings of blocks one at a time to produce an embedding of the entire disc. To make sure that we can do so and keep the union embedded we use the notion of angle defined above. For each block the angles determine how much room we will need to embed it disjoint from previous parts of the construction. We rescale the previous embedding to make sure there is enough room and add the next block. These rescalings are performed in a locally finite manner so we may proceed inductively to embed all of the discs.

Fix a twice foliated disc $(\mathring{D}^2; F_h, F_v)$ with only good-good short leaves. Take a horizontal leaf and let U_0 be the union of all v-leaves intersecting that horizontal leaf. Then U_0 is a twice foliated disc whose vertical leaf space in $\mathbf{R}$. Further U_0 can be embedded as a bounded set in the plane since its vertical 1-manifold is $\mathbf{R}$. Fix such an embedding. The frontier of U_0 in $\mathring{D}^2$ consists of a set of branch v-leaves of $\mathring{D}^2$. Fix such a leaf L and consider extending this embedding to $U_0 \cup L$. The image of L must be a vertical segment. The x-coordinate of this segment is determined since L is a limit of v-leaves in U_0. The range of y-coordinates is also determined; it is the y-coordinates of the h-leaves of U_0 which in $\mathring{D}^2$ meet L. Thus the embedding of U_0 can be extended uniquely to $\bar{U}_0 \subset \mathring{D}^2$. This extension is continuous and 1-1 but need not be a homeomorphism onto its image.

Consider a particular branch v-leaf L and let $V \subset \mathring{D}^2$ be the component of $\mathring{D}^2 - U_0$ containing L. In order to extend the embedding past L we must alter the embedding of U_0 near L to make room for V. The angles defined above will tell us how much room we need. To guarantee that we can get enough room we need the following lemma.

Lemma 4. *Let $(\mathring{D}^2; F_h, F_v)$ be a twice foliated disc with only good-good short leaves. Let L be a leaf and fix an end of L. Then the sum of the angles at that end of L on the two sides of L is at most* $360°$.

Proof. Let V be a side of L. Assume for definitiveness that L is vertical and V is the right side of L. If the angle of V at the end of L is at least $90°$, then V contains a subset homeomorphic to $\{(x, y) : x \geq 0, y < 0\}$ where L is sent to the ray $\{(0, y) : y < 0\}$. Thus if there is a h-leaf K on the side opposite V which shares the endpoint with L, then it is at least half-short. If the angle is more than $90°$, then that side of K must in fact be bad or very bad. If the angle is at least $180°$, then V contains a subset homeomorphic to $\{(x, y) : x > 0$ or $x = 0$ and

$y < 0\}$ where L is sent to the ray $\{(0, y) : y < 0\}$. Thus L is at least half-short. If the angle is more than $180°$, then in fact L must be bad or very bad on that side.

Similar results hold for larger multiples of $90°$. Therefore if the sum exceeds $360°$, then $\mathring{D}^2$ contains a short leaf with at least one side bad or very bad. □

Let V be an open disc $\mathring{D}^2$ together with part of ∂D^2. Suppose we are given two foliations on V $(V; F_h, F_v)$ such that each boundary component is a v-leaf and let L be a boundary component. Fix an open vertical segment L' in the plane (which we will view as the image of L). The angle at an end of L determines a quadrant of the plane based at the corresponding endpoint of L' or a ray parallel to one of the axes and emanating from the endpoint. A model region for V can be built as follows. Choose either monotonic curves emanating from the endpoints of L' lying in the appropriate quadrant and extending to infinity or the coordinate ray whichever is appropriate. These curves together with L' divide the plane into two regions and one region naturally corresponds to V. The model region is this region truncated by intersecting with a rectangle containing L' with sides parallel to the coordinate axes. The model region will be said to be standard if the monotonic curves are chosen to be lines with slope a multiple of $45°$. If U and U' are any two model regions for V, then there is clearly an embedding $h : U \to U'$ which is an isomorphism on the boundary arcs and sends horizontal (resp. vertical) lines to horizontal (resp. vertical) lines. Thus in some sense all model regions for V are equivalent. The following result shows that the angles defined above have the correct geometric meaning.

Proposition 1. *Let $\mathring{D}^2 \subset V \subset D^2$ be such that $V \cap \partial D^2$ is an open subset of ∂D^2. Suppose F_h and F_v are transverse foliations on V such that every component of $V \cap \partial D^2$ is a v-leaf and let L be one such component. If V has only good-good short leaves, V is embedded in the plane and $M_v(int(V)) \cong \mathbf{R}$, then V can be reembedded inside the standard model region determined by the angles at L in such a way that L is sent to the boundary arc of the model.*

Proof. Consider first the top end of L. Assume that the embedding is bounded, that L lies on the y-axis with upper limit point the origin O and that V lies to the right of L. Suppose V meets the positive y-axis. Since $M_v(int(V)) \cong \mathbf{R}$ it must be the case that $int(V)$ never meets the y-axis below L and there must be another branch leaf $\hat{L}$ above L.

If the angle of V at the top of L were $360°$ or more then $\hat{L}$ would be a short leaf and the side containing L would be bad or very bad. Hence the angle of V at the top of L is at most $315°$ and therefore V^c must contain points arbitrarily near O in the third quadrant. Choose points $P_i = (x_i, y_i)$ in V^c in the third quadrant converging to O monotonically in both coordinates. We may choose the P_i so that V misses the vertical rays with upper endpoints the P_i. To see this, note that if V is embedded in the closed right half plane this is obvious. If not we may assume (after possibly omitting the first few terms) that $int(V)$ meets the vertical rays with lower endpoints the P_i. Since $M_v(int(V)) \cong \mathbf{R}$, $int(V)$ must

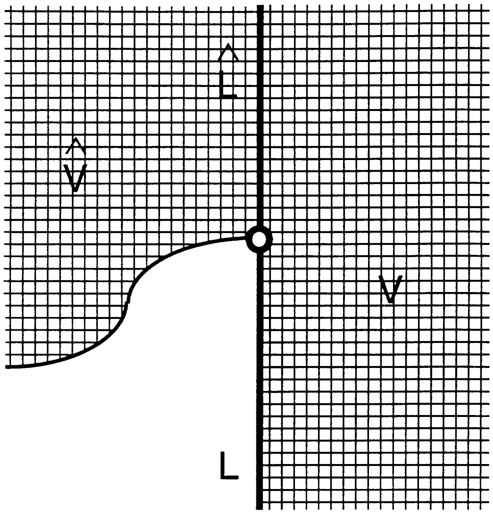


Fig. 5. The recursive definition of angles

miss the vertical ray with upper endpoint P_i. Now consider the v-leaf L' of V sent into the line $\{(x, y_i)\}$. If L' is not a short leaf then moving P_i slightly we may assume $P_i \in int(V)^c$, hence V must also miss the vertical ray with upper endpoint P_i. If L' is a short leaf it must be good-good and hence again V must also miss the vertical ray with upper endpoint P_i. Let U be the complement of these rays in the plane. Then U can be embedded in the plane above the graph of some monotonically nondecreasing function f on $(-\infty, 0]$ with $f(0) = 0$. For later use note that this reembedding can be arranged explicitly as follows. View U as the region $\{(x, y) \in \mathbf{R}^2$: if $x_i \leq x < x_{i+1}$, then $y > y_i\}$ with a collection of strips attached. Then the strip attached to $\{(x, y_i) : x_i < x < x_{i+1}\}$ can be foreshortened to lie above y_{i-1}. This embedding restricted to V embeds it over the graph of some monotonically nondecreasing function and a further rescaling embeds V above the ray $\{(x, x) : x \leq 0\}$ (see Fig. 6).

If the angle of V at the top of L is 315°, then we are done. Suppose the angle is 270°. Let K be the h-leaf which has O as its left endpoint (the leaf sent into the positive x-axis.) Let a be the least real number such that there is a sequence of points $P_i = (x_i, y_i)$ in V^c in the second quadrant converging to $(-a, 0)$ monotonically in both coordinates. Since K is half-short, we must have $a > 0$. Since that side of K is good V must lie above the x-axis for $-a < x < 0$. As above after possibly omitting the first few P_i and moving a few others we may assume V is embedded in the complement of the vertical rays with upper end points the P_i. Let U be the complement of these rays in the plane. Then U can be embedded in the plane above the graph of some monotonically nonincreasing function f on $(-\infty, -a]$ with $f(-a) = 0$. An explicit version can be defined as above. This completes the argument for 270°.

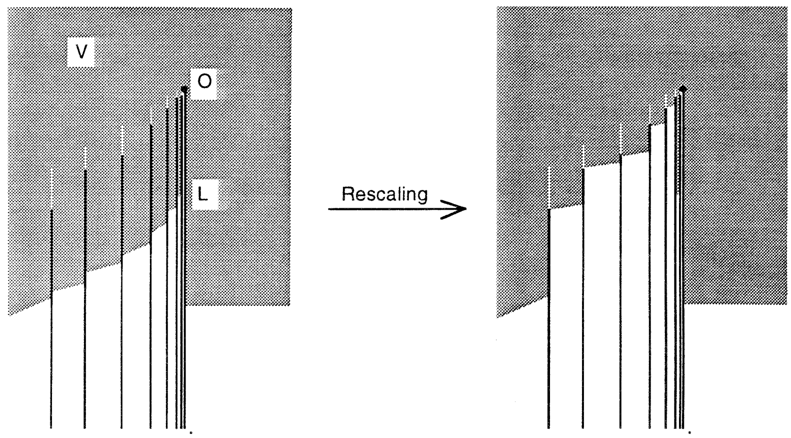


Fig. 6. The rescaling map

Suppose the angle of V at the top of L is at most $225°$. Since the angle of V at the top of L is at most $225°$ V^c must contain points arbitrarily near O in the second quadrant. Now we proceed exactly as in the case of $270°$ except with $a = 0$. Thus V can be embedded in the plane above the graph of some monotonically nonincreasing function f on $(-\infty, 0]$ with $f(0) = 0$. A further rescaling embeds V above the ray $\{(x, -x) : x \leq 0)\}$.

If the angle of V at the top of L is at most $180°$, then V cannot meet the y-axis above L. In the case of $180°$ this completes the argument. For angles of $45°$ through $135°$ the argument now proceeds exactly as above except that we choose points in the first or fourth quadrants and remove the rays above them instead of below. Note that the rescaling done involves only rescaling the y-coordinate and only on horizontal leaves above the leaves through L. Therefore the top and bottom ends can be done independently. Repeating this procedure on the lower end of V puts V inside the desired model. $\square$.

Note that there are really two different types of rescalings. The first one using the removed rays makes a fundamental change in the embedding and puts V inside a (nonstandard) model region. The second is a purely cosmetic rescaling which achieves a standard model. The first rescaling is local. Given a neighborhood $A \times \mathbf{R}$ of L we may alter V only in this neighborhood to produce an embedding whose intersection with some smaller neighborhood $B \times \mathbf{R}$ of L lies in a (nonstandard) model. This extra strength will be used below. A reembedding theorem is actually needed for slightly more complicated regions. Suppose that V' is obtained from V by adjoining model regions to the boundary arcs other than L and that V' also contains only good-good short leaves. The model regions have an interesting property, they retract inside themselves into an arbitrarily small neighborhood of the boundary arc. Therefore they may be arranged to be disjoint from the P_i and from the rays above or below them as required. Thus if V' can be embedded then it can be reembedded inside a model.

Proof of Theorem 1'. We have now assembled all the pieces we need to prove Theorem 1'. As above take a horizontal leaf and let U_0 be the union of all v-leaves intersecting that horizontal leaf. Then U_0 is a twice foliated disc whose vertical leaf space in $\mathbf{R}$. Fix an embedding of U_0 in the plane as guaranteed by Lemma 2 and extend it to an embedding of U_0 as discussed above. View U_0 as a spine to which the rest of $\mathring{D}^2$ is to be attached. Let $\{L_i\}$ be the set of all the branch leaves in the frontier of U_0. Let $\{x_i\}$ be their x-coordinates and choose neighborhoods $\{A_i\}$ of the x_i in $\mathbf{R}$ in such a way that $\mathrm{diam}(A_i) \to 0$ as $i \to \infty$. For L_1, let V_1, be the component of $\mathring{D}^2 - U_0$ containing L_1. The first of the two types of rescalings defined above alters the embedding only in $A_1 \times \mathbf{R}$ and gives room to attach the model region of V_1 to L_1. (Since the second cosmetic rescaling is not used the model region cannot be assumed to be standard but this is unimportant.) By Lemma 4 this procedure can be carried out for all the L_i without producing an overlap with only one proviso. Since there may be countably many of the L_i we must be careful that the rescalings are locally finite.

This would clearly hold if the sets $\{A_i\}$ were locally finite but this condition is too strict. Consider any point $p = (x, y)$ in the image of U_0 and choose a rectangular neighborhood of p in U_0. The rescaling in $A_i \times \mathbf{R}$ consists of finding vertical rays disjoint from U_0 and rescaling the strips between them. If any point in the rectangular neighborhood of p is affected by this rescaling the strip it lies in (and hence A_i) must be at least as wide as the rectangle. Therefore, since $\mathrm{diam}(A_i) \to 0$ only finitely many of the A_i can affect p. Thus we may alter the embedding of U_0.

Now we have $\bar{U}_0$ and the appropriate model regions about the boundary arcs L_j of $\bar{U}_0$ embedded in $\mathbf{R}$. For each L_j let V_j be the component of $\mathring{D}^2 - U_0$ containing L_j. Let $i : [0, \infty) \to p_v(V_j) \subset M_v$ be an embedding such that $i(0) = L_j$ and $f_v \circ i$ is a translation. Then $(U_j = p_v^{-1}(i((0, \infty)))$ is a twice foliated open disc in $V_j \subset \mathring{D}^2$. As above U_j can be embedded as a bounded set in the plane and this embedding extends to an embedding of its closure in V_j with model regions attached to the boundary arcs of the closure in the plane.

Denote this union by W_j. By Proposition 1 W_j reembeds inside a standard (hence any model) region. Hence we can embed the W_j inside the model regions which we created on the embedding of U_0. Iterating this construction and exhausting M_v by rays produces an embedding of all of $\mathring{D}^2$ in the plane as desired. □

4. Nonembeddable examples

Theorem 1' and Lemma 3 above answer a large piece of the embedding question. One need only decide what should be done about good-bad short leaves. Consider a single short h-leaf in $(\mathring{D}^2; F_h, F_v)$ with the upper side bad and the lower side good. Recall that we assign parameters to a short h-leaf L as follows. Embed a neighborhood V of L in the plane so that its image V' contains the open

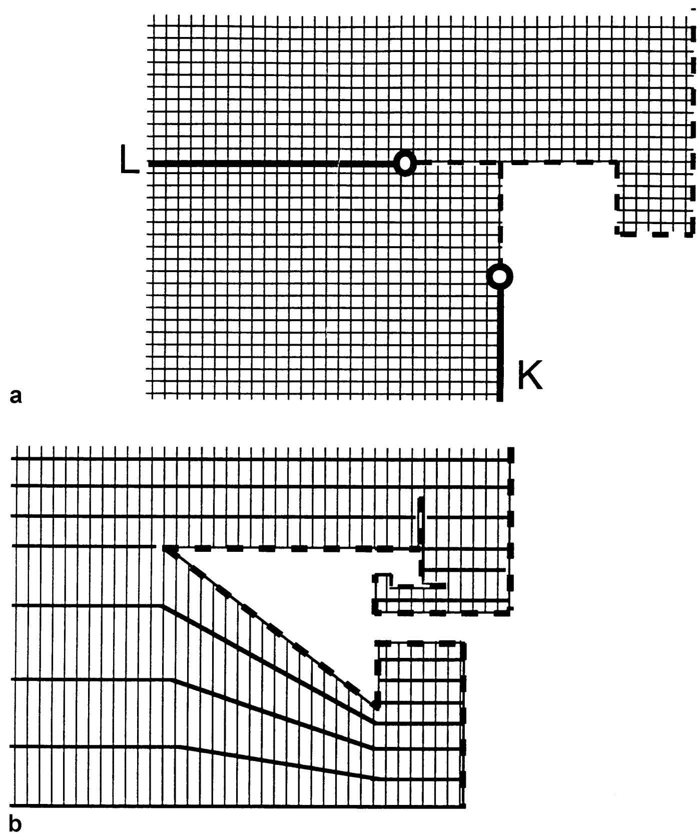


Fig. 7. a A good bad leaf L which if extended creates a bad side to K. **b** A nonembeddable twice foliated disc built from the construction in **a**

square $(-1,1)\times(-1,1)$ with the line segment $\{(x,0) : 1 > x \geq 0\}$ removed, V' is contained in $(-2,2)\times(-2,2)$ and L is sent to $\{(x,0) : -a < x < 0\}$ for some a. Let H_+ be the open upper half plane. Let x_+ be the smallest positive number such that $(x_+,0)$ is a limit point of $(V')^c \cap H_+$. There may be vertical lines in V' which have lower endpoint $(x,0)$ for some $x > 0$ and for which the corresponding v-leaf L of $(\mathring{D}^2; F_h, F_v)$ would extend below the x-axis. If so let w_+ be the infimum of all such x-coordinates. Also recall that x_- is defined similarly (and $w_- = \infty$ since the lower side is good.) Then we can extend (and hence eliminate) the short leaf as follows. Choose some value of $x' < w_+$. By rescaling V' below the x-axis we may assume that $x_- = x'$. Then V' may be enlarged to include $\{(x,0) : 0 \leq x < x'\}$. Gluing this enlarged region to $\mathring{D}^2$ along V produces a new twice foliated disc with this short leaf extended. (That is, since x_- was maximal the lower side of the new leaf is no longer short.)

Unfortunately removing a good-bad short leaf in this fashion may turn a previously good side of a short leaf into a bad side. This will occur if the foliation contains a subset homeomorphic to the one shown in Fig. 7a. If the good-bad short leaf L is extended then the left hand side of the short leaf K goes from being good to being bad. Note that if a twice-foliated disc embeds in the plane then that embedding specifies a way of extending all good-bad short leaves. Thus one gets nonembeddable examples as shown in Fig. 7b from this observation as well. We also get a slight strengthening of Theorem 1'. Suppose $(\mathring{D}^2; F_h, F_v)$ contains only good-good and good-bad short leaves and only finitely many good-bad short leaves. Then unless $(\mathring{D}^2; F_h, F_v)$ contains a subset homeomorphic to one of the examples indicated in Fig. 7b, we may extend all the good-bad short leaves. This produces an embedding of $(\mathring{D}^2; F_h, F_v)$ into a twice foliated disc with only good-good short leaves (this takes easy but nontrivial checking) and hence by Theorem 1' an embedding of $(\mathring{D}^2; F_h, F_v)$ into the plane.

Unfortunately if there are infinitely many good-bad short leaves then the classification of nonembeddable examples becomes intractable. The following construction produces a number of fundamental examples. Every example of a nonembeddable twice foliated disc of which the authors are aware contains one of these as a subset but it is not clear whether they are the only fundamental examples. We get two basic nonembeddable examples from the short leaves which Lemma 3 says cannot be embedded, bad-bad and very bad-good. We omit the nonembeddable very bad-bad and very bad-very bad since either of these contains a subset homeomorphic to a very bad-good short leaf. Taking the sides to be prototypical gives the example shown in Fig. 2 and the example shown in Fig. 8. If a twice foliated disc with some good-bad short leaves can be embedded in the plane then scalings coming from the embedding gives a method of extending all the short leaves. Therefore any example which must produce something containing one of these examples when short leaves are extended must also be nonembeddable. One such construction is to cut into each tooth (but at most finitely often) as shown in Fig. 7. When the resulting short leaves are extended the tooth will be restored. An example of such a side is given in Fig. 7b. This is basically the only example which requires only finitely many cuts. One may also build examples with infinitely many cuts. There are several possible ways of doing this. Suppose for definitivoness that we start with a horizontal short leaf. A second operation is to cut it up with horizontal cuts as shown in Fig. 9a. When the short leaves in this example are extended a good side will be created. Therefore we may perform this construction on any good side to produce another nonembeddable example. A third operation is to use vertical cuts as shown in Fig. 9b. When the short leaves in this example are extended a bad side is produced if the vertical cuts limit to some point to the right of the end of the short leaf and a very bad side will be created if they limit to the end. Therefore we may perform this construction on a bad or very bad side to produce another nonembeddable example. Finally as a fourth operation we may replace any side by a worse one, good by bad or bad by very bad.

We close by giving the following conjecture.

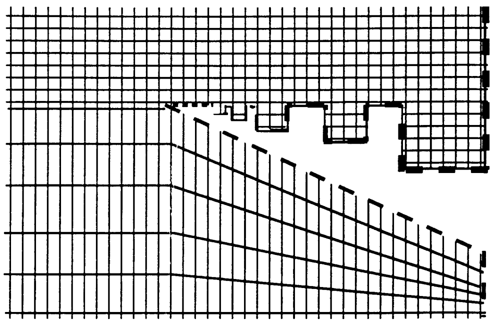
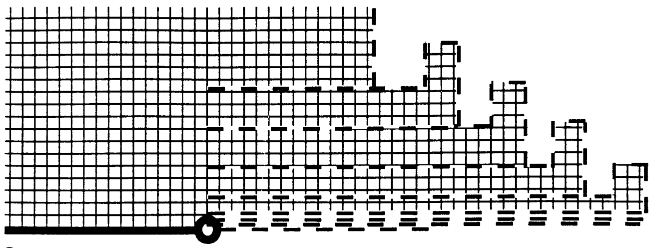
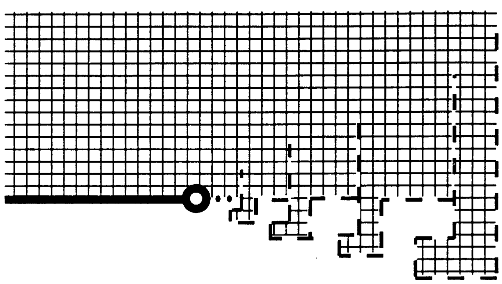


Fig. 8. The other basic nonembeddable example



a



b

Fig. 9a,b. Two of the four operations that produce new nonembeddable examples. **a** Horizontal cuts on a horizontal short leaf. **b** Vertical cuts on a horizontal short leaf

Conjecture. *A twice foliated disc* ($\mathring{D}^2$; F_h, F_v) *embeds in the plane if and only if it has no subdomain* $U \subset \mathring{D}^2$ *such that* $(U; F_h|_U, F_v|_U)$ *is isomorphic to one of the examples described above.*

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