SIAM REVIEW
Vol. 42, No. 2, pp. 161–230

Front Propagation in Heterogeneous Media*

Jack Xin†

Abstract. A review is presented of recent results on front propagation in reaction-diffusion-advection equations in homogeneous and heterogeneous media. Formal asymptotic expansions and heuristic ideas are used to motivate the results wherever possible. The fronts include constant-speed monotone traveling fronts in homogeneous media, periodically varying traveling fronts in periodic media, and fluctuating and fractal fronts in random media. These fronts arise in a wide range of applications such as chemical kinetics, combustion, biology, transport in porous media, and industrial deposition processes. Open problems are briefly discussed along the way.

Key words. fronts, heterogeneous media, homogenization, statistical analysis, modeling

AMS subject classifications. 35K55, 35K57, 41A60, 60H30, 65C20

PII. S0036144599364296

1. Introduction.

Front propagation and interface motion occur in many scientific areas such as chemical kinetics, combustion, biology, transport in porous media, and industrial deposition processes. In spite of these different applications, the basic phenomena can all be modeled using nonlinear parabolic partial differential equations (PDEs) or systems of such equations. Since the pioneering work of Kolmogorov, Petrovsky, and Piskunov (KPP) [100] and Fisher [62] in 1937 on traveling fronts in reaction-diffusion (R-D) equations, the field has gone through enormous growth and development. However, studies of fronts in heterogeneous media have been more recent. Heterogeneities are always present in natural environments, such as fluid convection effects in combustion, inhomogeneous porous structures in transport of solutes, noise effects in biology, and deposition processes. It is a fundamental problem to understand how heterogeneities influence the characteristics of front propagation such as front speeds, front profiles, and front locations. Our goal here is to give a review of recent results on front propagation in heterogeneous media in a coherent and motivating manner. It is not our intention to give a complete survey, and so our references will cover only a portion of the literature.

We begin in section 2 with the well-studied scalar homogeneous R-D equations and explain the basic properties of front solutions, such as front existence and stability, front speed selection, and variational characterization. Many of these properties carry over to heterogeneous fronts.

*Received by the editors January 20, 1999; accepted for publication (in revised form) November 25, 1999; published electronically April 24, 2000. This work was partially supported by NSF grant DMS-9625680.

http://www.siam.org/journals/sirev/42-2/36429.html

†Department of Mathematics, Texas Institute of Computational and Applied Mathematics, University of Texas at Austin, Austin, TX 78712 (jxin@fireant.ma.utexas.edu).

A new theme associated with fronts in heterogeneous media is the understanding of multiple scales and their interaction. In section 3 we illustrate how to apply homogenization ideas to front problems for the case of periodic media (Figure 1). Basic ideas of homogenization theory explained through concrete examples serve as useful guides. Periodically varying traveling waves then come up naturally as robust solutions. Depending on the form of the nonlinearity, the homogenization procedure for the heterogeneous medium varies. Several regimes also arise according to whether the front width is smaller than, comparable to, or larger than the wavelength of the oscillating medium. In the large space and time limit, fronts move with averaged speeds depending on the structures of the medium. The front location evolves according to Hamilton–Jacobi (H-J) equations under suitable conditions. It turns out that the homogenization of H-J equations is intimately related to the large-time front speed of heterogeneous KPP equations.

In section 4 we turn to fronts in random media (Figure 2), which are are much less well understood, partially because homogenization is more difficult to carry out in the random setting. Handling both nonlinearity and randomness is a great challenge, and mathematical results are far fewer. We present recent rigorous results on two noisy Burgers equations and three noisy KPP equations. A new phenomenon is that in addition to the averaged front speed provided by a successful extension of homogenization into the random setting, front locations are random and undergo diffusion about the mean positions. Another new phenomenon is that of front acceleration through a rough (on the large scale) turbulent velocity field and the resulting anomalous scaling limits. We describe the related modeling activities in studying premixed turbulent flames as well as other noisy dynamics involving fronts.

Most of the results that we discuss in detail concern scalar equations. In the final subsection of each main section, we mention briefly (1) open problems, (2) extensions of results discussed, for example, to systems of equations, and (3) modeling activities in the physics literature. Modeling results are selected only to complement the existing rigorous works and to introduce physical phenomena. We do not discuss numerical methods.

2. Fronts in Homogeneous Media.

We review the classical existence and stability results for traveling fronts for R-D equations of the KPP, Zeldovich, bistable, and combustion types. We then discuss front stability, front selection, and variational principles for front speeds.

2.1. Traveling Fronts in Scalar R-D Equations.

One-dimensional scalar R-D equations of the form

$$u_t = u_{xx} + f(u), \tag{2.1.1}$$

with $x \in \mathbb{R}$, arise in many scientific areas, such as chemical kinetics, population genetics, and combustion. The unknown u may accordingly stand for concentration of a chemical reactant, population density of a biological species, or temperature of a reacting mixture. The functional form of f also varies. To be specific for the study of traveling fronts, we will be concerned throughout this article with the following five types of nonlinearity:

1. $f(u) = u(1-u)$: the *KPP* [100] or *Fisher nonlinearity* [62];
2. $f(u) = u^m(1-u)$, m an integer ≥ 2: the *mth-order Fisher nonlinearity* (called the *Zeldovich nonlinearity* if $m = 2$);
3. $f(u) = u(1-u)(u-\mu)$, $\mu \in (0,1)$: the *bistable nonlinearity*;

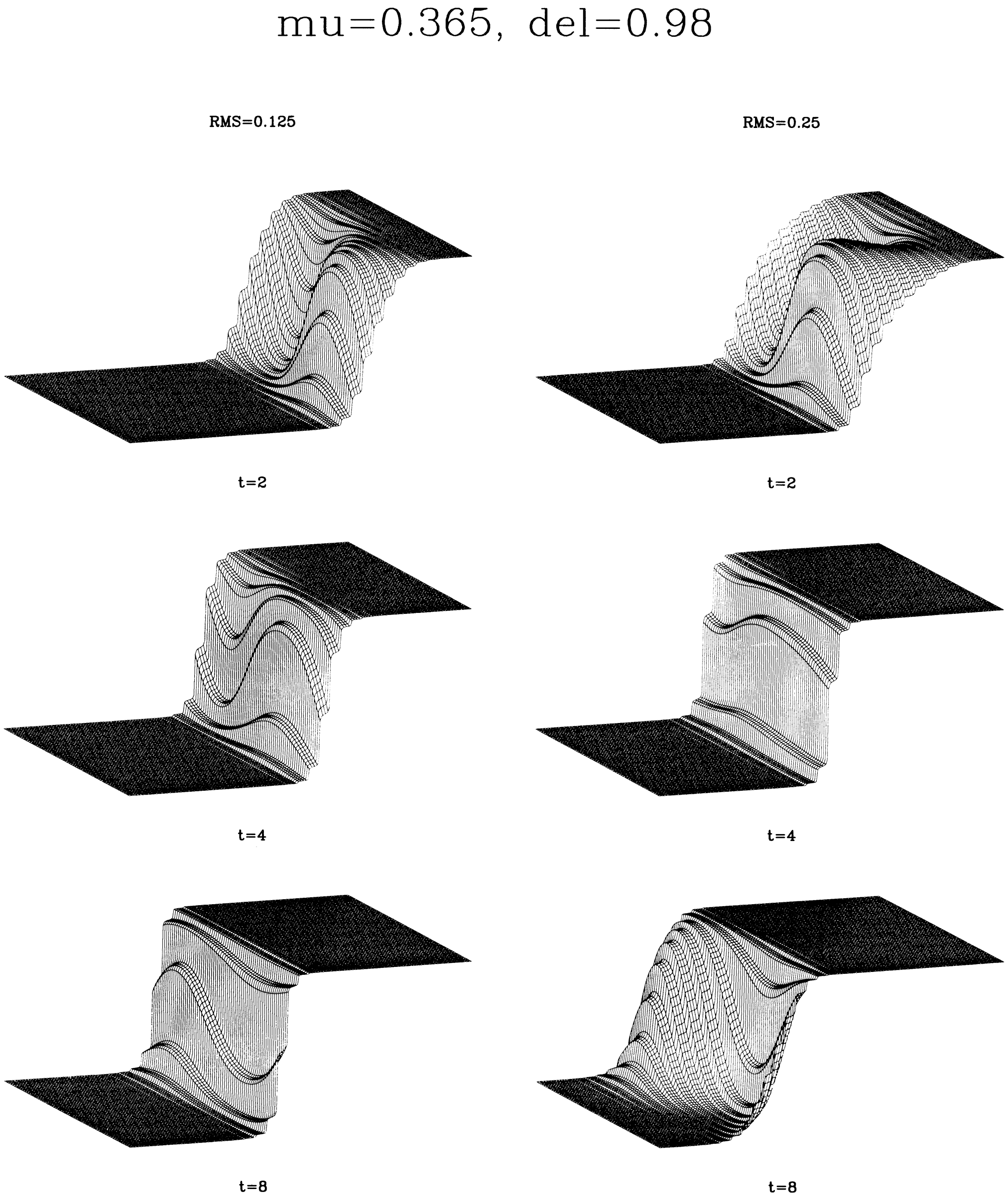

Fig. 1 *Example of fronts in periodic media. Numerically simulated propagating fronts of the reaction-advection-diffusion equation* $u_t + \vec{v}(x,y)\cdot\nabla_{x,y}u = \epsilon(1+\delta\sin(100x))\Delta_{x,y}u + \epsilon^{-1}u(1-u)(u-\mu)$*, where* $\epsilon = 0.05$*,* $\delta = 0.98$*,* $\mu = 0.365$*. The advection velocity* $\vec{v}$ *is a mean-zero divergence-free field generated by a periodic array of counter-rotating vortices with vortex length scale* $\lambda = 0.3873$*. The strength of the advection is measured by the rms velocity* $\langle |v|^2 \rangle^{1/2}$*, where the bracket denotes the period average. The values are* $\langle |v|^2 \rangle^{1/2} = 0.125$ *for the left column and* $\langle |v|^2 \rangle^{1/2} = 0.25$ *for the right column. The spatial domain is* $(x,y) \in [-2,2] \times [0,1]$*, with* 512×128 *uniform grid points. Simulations are performed with a second-order upwind method. We observe front wrinkling due to spatially periodic diffusion and advection. Comparing the two columns, we also see the speedup of the front with increasing values of* $\langle |v|^2 \rangle^{1/2}$*. Intuitively, a wrinkled front region increases the effective area for the reaction and results in a higher front speed.* Reprinted from Physica D, 81, J. X. Xin and J. Zhu, Quenching and Propagation of Bistable Reaction-Diffusion Fronts in Multidimensional Periodic Media, pp. 94–110, 1995, with permission from Elsevier Science.

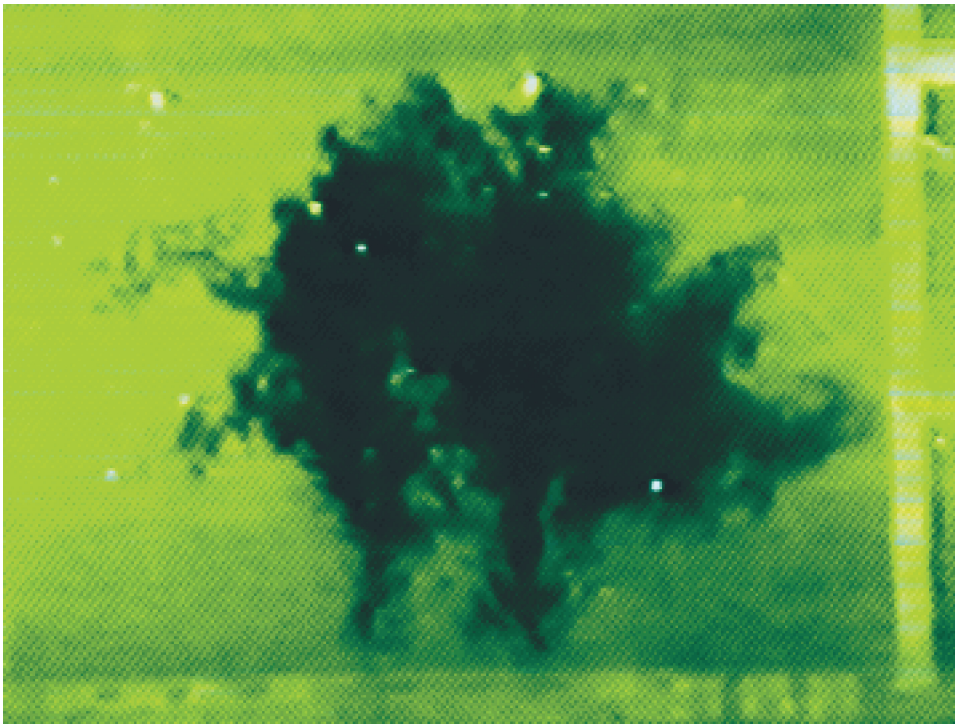

Fig. 2 *Example of front in random medium. Planar laser-induced fluorescence light sheet image of an experimental arsenous-acid/iodate autocatalytic front in random capillary wave flow of broad scales. Field view* 14 *cm* × 14 *cm, with the ratio U of rms flow velocity u' and laminar front speed S_L (e.g., front speed when no flow is present) equal to* 650*. The arsenous-acid/iodate reaction takes place in an aqueous solution and has two advantages over gas reactions:* (1) *it allows small density changes across the reaction front,* (2) *it permits large rms values of u' where front normal velocity still depends on local flow and curvature. Both properties are helpful for comparing experimental findings with predictions from the H-J-type front models such as the G-equation (see section* 4.4*). The capillary wave flow is achieved in a thin layer of liquid in a vertically vibrated tray (*20 *cm each side). At large amplitude, the flow field becomes random and develops a broad range of spatial and temporal scales. In the experiment, the flow has zero ensemble mean and is isotropic and quasi-two-dimensional. For details of the experimental set-up, see* [82]*. We observe the resulting fractal nature of the front due to the rough flow velocity.* Image used with the permission of Paul Ronney.

4. $f(u) = e^{-E/u}(1-u)$, $E > 0$: the *Arrhenius combustion nonlinearity* or combustion nonlinearity with activation energy E but no ignition temperature cutoff;

5. $f(u) = 0$ $\forall u \in [0,\theta] \cup \{1\}$, $f(u) > 0$ $\forall u \in (\theta, 1)$, $f(u)$ Lipschitz continuous: the *combustion nonlinearity with ignition temperature* θ.

Types 1 and 2 come from chemical kinetics (for example, from autocatalytic reactions, as we shall discuss later), with type 2 being the high-order generalization of type 1. Type 3 comes from biological applications (such as FitzHugh–Nagumo (FHN) systems) and also more recently from phase field models of solidification. Types 4 and 5 appear in the study of premixed flames in combustion science, and type 4 can be regarded as a limit of type 5 as θ tends to zero.

If we look at the graphs of $f(u)$ for the five types of nonlinearity, sketched in Figure 3, we see that they differ near $u = 0$ and behave similarly near $u = 1$. The type 1 nonlinearity has a positive slope at $u = 0$. The type 2 nonlinearity has

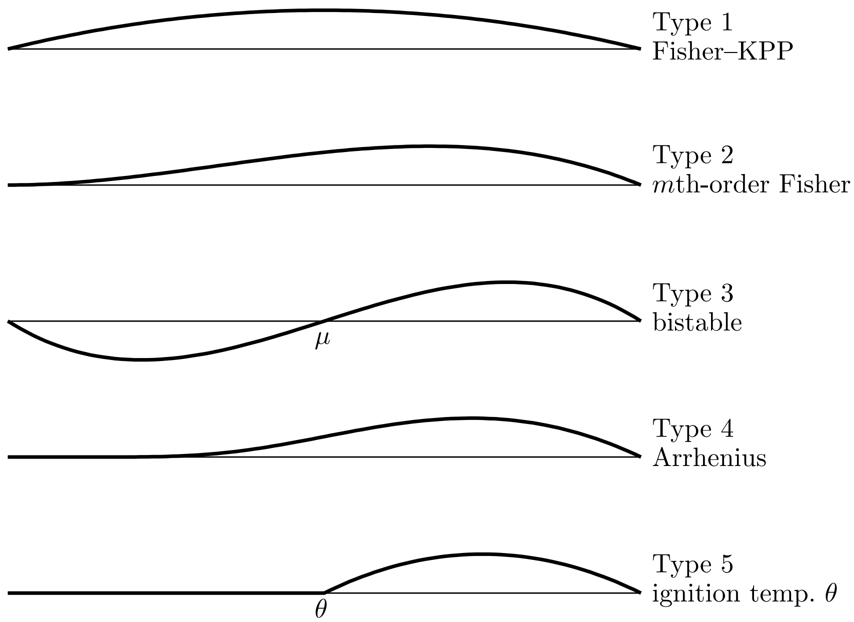

Fig. 3 *Five types of nonlinearities $f(u)$ considered throughout this article. If we gradually deform the curve of $f = f(u)$ near $u = 0$ from above the u-axis (type 1) to below it (type 3), we can experience all five types of nonlinearities.*

zero slope (and derivatives up to order $m-1$ for $m \geq 2$). The type 4 nonlinearity has an exponentially small tail near zero, so all derivatives at zero vanish. Type 5 is identically zero for an interval $u \in [0, \theta]$, i.e., there is no reaction below ignition temperature. Type 3 has a negative slope at $u = 0$, goes down to a negative minimum, goes up and through an intermediate zero μ, goes up to its positive maximum, and finally comes back to its third zero at $u = 1$. Type 3 is the only one that changes sign. Its total area $\int_0^1 f(u)du$ is positive if $\mu \in (0, \frac{1}{2})$, zero if $\mu = \frac{1}{2}$, and negative if $\mu > \frac{1}{2}$.

The simplest nontrivial solution of (2.1.1), which models the steady motion of a flame or a transition layer between two chemical species, is the traveling front solution of the form $u = U(x - ct) \equiv U(\xi)$, where c is the wave speed and U is the wave profile that connects 0 and 1. Substituting this form into (2.1.1), we obtain

(2.1.2) $$U_{\xi\xi} + cU_\xi + f(U) = 0,$$

with boundary conditions $\lim_{\xi\to-\infty} U(\xi) = 0$ and $\lim_{\xi\to+\infty} U(\xi) = 1$. Since u is a concentration or a temperature, we also impose the physical conditions $U(\xi) \geq 0$. The above problem can be thought of as a nonlinear eigenvalue problem with eigenvalue c and eigenfunction U. We note that as soon as we have a traveling front solution, we get another solution moving in the opposite direction at the same speed by the transform $\xi \to -\xi$, $c \to -c$. However, U will take the values 1 at $\xi = -\infty$ and 0 at $\xi = +\infty$. We can also get another front moving in the same direction by translating ξ to $\xi + \text{const}$.

It is convenient to perform a phase plane analysis by writing (2.1.2) as a first-order system of ODEs,

(2.1.3) $$U_\xi = V, \qquad V_\xi = -cV - f(U).$$

Now we are looking for a trajectory in the phase plane that goes from $(0,0)$ to $(1,0)$. Multiplying both sides of (2.1.2) by U_ξ and integrating over $\xi \in \mathbb{R}$, we obtain (assuming $\mu \in (0, \frac{1}{2})$ in the case of a nonlinearity of type 3)

$$c = -\frac{\int_0^1 f(U)dU}{\int_{\mathbb{R}} U_\xi^2 d\xi} < 0.$$

The linearized system about $U = 0$ is

$$\frac{d}{d\xi}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -f'(0) & -c \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}.$$

The eigenvalues of this 2×2 matrix are given by

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4f'(0)}}{2}.$$

In the case of a type 1 nonlinearity, if $c^2 \geq 4f'(0)$ or $c \leq c_1^* \equiv -2\sqrt{f'(0)}$, $(0,0)$ is an unstable node. In the type 3 case, since $f'(0) < 0$, $(0,0)$ is a saddle. In either case, a similar linearization at $(1,0)$ shows that $(1,0)$ is always a saddle, thanks to $f'(1) < 0$. Since there is a family of unstable directions going out of an unstable node, and only one direction in or out of a saddle, one can show by isolating the flows in a triangular region in the first quadrant of the U-V plane that there is a connecting trajectory for each $c \leq c_1^*$ for type 1 and a unique connecting trajectory for type 3. Moreover, $U_\xi > 0$ always holds, thanks to the trajectory being in the first quadrant. Since the ODE system (2.1.3) is autonomous, U is unique only up to a constant translation of ξ.

We shall call the front solution corresponding to $c = c_1^*$ the *critical front*. The critical front moves at the slowest speed in absolute value, and its asymptotic behavior as $|\xi| \to \infty$ is (see Aronson and Weinberger [1])

$$U(\xi) = 1 - Ce^{-\beta\xi} + O(e^{-2\beta\xi}), \quad \xi \to +\infty,$$

$$U(\xi) = (A - B\xi)e^{-c_1^*\xi/2} + O(\xi^2 e^{-c_1^*\xi}), \quad \xi \to -\infty, \tag{2.1.4}$$

where A, B, and C are positive constants and $2\beta = -c_1^* - \sqrt{(c_1^*)^2 - 4f'(1)} > 0$. In contrast, the faster fronts with $c < c_1^*$ have exponential decay $O(e^{\text{const.}\,\xi})$ as $|\xi| \to \infty$, because the two roots $\lambda_{1,2}$ at $(0,0)$ are both simple. The faster fronts decay more slowly than the critical fronts as $U \to 0$.

For the type 3 cubic polynomial, Huxley (see [141]) solved (2.1.2)–(2.1.3) exactly:

$$U(\xi) = \frac{1}{1 + e^{-\xi/\sqrt{2}}}, \quad c = \sqrt{2}\left(\mu - \frac{1}{2}\right) \tag{2.1.5}$$

for $\mu \in (0, \frac{1}{2}]$. If $\mu \in [\frac{1}{2}, 1]$, one simply switches c to $-c$ and ξ to $-\xi$.

For the remaining three types, $f'(0) = 0$, and so $(0,0)$ has an unstable and a neutral direction. More delicate analysis is required. In the type 2 case, one can show that there is a center manifold near $(0,0)$ and a connecting trajectory from the center manifold to the saddle at $(1,0)$ for each $c < c_m^* < 0$. If $c = c_m^*$, the connection goes from the unstable manifold at $(0,0)$ to the saddle. For a type 2 nonlinearity with $m = 2$, the critical front profile approaches zero at the rate $O(e^{-c_2^*\xi})$ as $\xi \to -\infty$, while the profiles of faster fronts approach zero at an algebraic rate $O(\xi^{-1})$ [21]. This is different from the KPP case (2.1.4). The absolute values of c_m^* decrease with increasing m.

For type 4 and type 5 nonlinearities, a different method using degree theory on finite intervals to construct approximate solutions, then taking their infinite line limit, is much more expedient and robust; see Berestycki, Nicolaenko, and Scheurer [17] and Marion [111]. We will explain this method in detail in the coming sections on fronts in periodic media. The results of [17] and [111] show in particular that in the case of a type 4 nonlinearity, a continuum family of traveling front solutions exist, one for each $c \le c_0^* < 0$, just as for type 1 and type 2. However, type 5 is different from type 4 in that for each ignition temperature $\theta > 0$, there is a unique c_θ^* so that a corresponding front profile U exists and is unique up to a constant translation in ξ. We see that type 5 is just like type 3. See Fife [59] or Uchiyama [147] for a phase plane justification of the result.

The isotropic scalar R-D equation in several dimensions is

$$u_t = \Delta_x u + f(u), \tag{2.1.6}$$

where $x \in \mathbb{R}^n$ and Δ_x is the n-dimensional Laplacian. For each unit vector $\vec{k}$, the traveling front in direction $\vec{k}$ is $U(\vec{k} \cdot x - ct)$, with c and U corresponding to exactly the same one-dimensional traveling front solution as before. The wave speed c and profile U are both isotropic (independent of $\vec{k}$).

2.2. Asymptotic Stability and Selection of Fronts. The next step after we know the existence of traveling fronts is to ask about their dynamic asymptotic stability in the large time limit. This is important because the fronts should be stable if they are to be experimentally observed. Stability means that if initial data are prescribed in the form $u_0(x) = U_c(x) + \tilde{u}(x)$, where $U_c(x)$ is a front profile corresponding to the speed c and $\tilde{u}(x)$ is a smooth and spatially decaying perturbation, then $u(x,t)$ converges to $U_c(x - ct + \xi_0)$ in a suitably weighted Banach space as $t \to \infty$ for some constant ξ_0.

The reason we have a constant translation in the definition can be seen as follows. Due to spatial translation-invariance of the original equation, we have a family of traveling fronts $U_c(x - ct + x_0)$ for each allowable wave speed c. Let us take $\tilde{u}(x) = U_c(x + x_0) - U_c(x)$, which is a perturbation with spatial decay. Now for initial data $u(x,0) = U_c(x) + \tilde{u}(x) = U_c(x + x_0)$, the solution for later time is just $U_c(x - ct + x_0)$, which does not converge to $U_c(x - ct)$ unless $x_0 = 0$. In the case of a continuum of speeds, we can also take $\tilde{u}(x) = U_{c'}(x) - U_c(x)$, $c' \neq c$, and the later time solution is $U_{c'}(x - c't)$, again not converging to $U_c(x - ct)$ as $t \to \infty$. Even the wave speed is different.

These simple examples show that it is a subtle problem to establish asymptotic stability, especially in the case of multiple speeds. A great deal turns out to depend on the rate of decay of the initial perturbations as $|\xi| \to \infty$. Intuitively, the tiny amount of perturbation in the far field takes a long time to crawl into a front from its tails; however, its effect is crucial since asymptotic stability concerns large-time behavior. Let us show that the bistable (type 3) R-D front under a small initial perturbation $\tilde{u}$ is asymptotically stable. We linearize the problem by writing $u = U(x-ct)+\tilde{u}(x-ct,t)$. In the moving frame $\xi = x - ct$, $\tilde{u} = \tilde{u}(\xi,t)$ satisfies

$$\tilde{u}_t \;=\; \tilde{u}_{\xi\xi} + c\tilde{u}_\xi + f'(U)\tilde{u} + N(\tilde{u}) \;\equiv\; L\tilde{u} + N(\tilde{u}), \tag{2.2.1}$$

where $N(\tilde{u})$ contains quadratic or higher order nonlinear terms. The location of the spectrum of L, denoted by $\sigma(L)$, carries information on the decay of $\tilde{u}$. According to general theory, $\sigma(L)$ consists of an essential spectrum and an isolated point spectrum if L is considered in the space $L^2(\mathbb{R})$ of square integrable functions. The essential spectrum is included in the union of the essential spectra of the limiting operators (Henry [86])

$$L_{\pm} = \lim_{\xi \to \pm\infty} L = \tilde{u}_{\xi\xi} + c\tilde{u}_{\xi} + f'(U(\pm\infty))\tilde{u}.$$

Since $f'(U(\pm\infty)) < 0$, the essential spectrum of $L_{\pm}$ is strictly in the left half-plane, with a gap from the imaginary axis (by Fourier transform). Now we consider the operator e^L, which is a positive bounded linear operator from $L^2(\mathbb{R})$ into itself. The essential spectrum of e^L is contained in $B(0,r)$, a disc about 0 of radius $r < 1$. The remaining spectrum of e^L consists of isolated eigenvalues of finite multiplicities. The restriction of e^L to the space spanned by the eigenfunctions of these isolated eigenvalues is a finite-dimensional positive linear operator. On the other hand, we know that $LU_{\xi} = 0$ by differentiating the U equation with respect to ξ, and we know that $0 < U_{\xi} \in L^2(\mathbb{R})$. Thus U_{ξ} is a positive eigenfunction of e^L with eigenvalue 1. By the Krein–Rutman or Perron–Frobenius theorems [146], [52], 1 is a simple eigenvalue, and the rest of the point spectrum of L lies strictly inside the unit disc. It follows that if we decompose the perturbation $\tilde{u}(\xi,t)$ into the sum of a part along U_{ξ} and a part orthogonal to it, the orthogonal part decays to zero and the neutral U_{ξ} part leads to the translation by ξ_0. Moreover, since the decay rate is exponential in time (due to the spectral gap), the nonlinear term $N(\tilde{u})$ is slaved to the linear part.

In case of KPP (type 1) or type 5 nonlinearities, however, the above proof fails. The limiting operator L_{-} has a continuous spectrum with positive real part (if $f'(0) > 0$) or a continuous spectrum touching the origin (if $f'(0) = 0$). The spectral gap disappears. The cure proposed in Sattinger [141] is to restrict attention to perturbations in a weighted space that specifies a rate of spatial decay. To see how this works, let us change variables: $\tilde{u} = e^{\alpha\xi}v$, $v \in L^2$, with a constant $\alpha > 0$, for $\xi \leq 0$. Then v satisfies the equation

$$v_t = v_{\xi\xi} + (c + 2\alpha)v_{\xi} + (\alpha^2 + c\alpha + f'(U))v, \tag{2.2.2}$$

whose advantage is that the coefficient of v at $\xi = -\infty$, $\alpha^2 + c\alpha + f'(0)$, can be negative. In the KPP case, $c \leq c_1^* = -2\sqrt{f'(0)}$. Hence if $c < c_1^*$, then there is a positive α such that $\alpha^2 + c\alpha + f'(0) = \alpha^2 + c\alpha + (c_1^*)^2/4 < 0$. One choice is $\alpha = -c/2$. In this way, we can still apply the argument of the bistable case. For type 5, with $f'(0) = 0$, any small positive number (less than $-c_{\theta}^*$) will do. Technically, one chooses a weight function $w(\xi) = e^{-\alpha\xi} + 1$ for all $\xi \in \mathbb{R}$. The weighted L^2 space is $\{\tilde{u} \in L^2 : w\tilde{u} \in L^2\}$. It was noted in [141] that this kind of weight causes the solution to converge to the same U_c with translation $\xi_0 = 0$, since the neighboring solutions are infinitely far away in the weighted norm. The critical case $c = c^*$, however, remains uncured by even this remedy, one reason being that there is no exponential convergence.

The study of the asymptotic stability of critical fronts has been much more recent; see Kirchgässner [99] for KPP nonlinearity, Bricmont and Kupiainen [31] for Ginzburg–Landau (G-L, that is, $f(u) = u(1-u^2)$) fronts and Gallay [72] for both G-L and KPP, and Eckmann and Wayne [45] for more general parabolic equations. The key step of [99] is to write the solution in the moving frame as $u(\xi,t) = U(\xi) + U'(\xi)v(\xi,t)$, where U is a shorthand for $U_{c^*}(\xi)$ and $c^* = c_1^*$. The equation satisfied by v becomes

$$v_t = v_{\xi\xi} + \left(c^* + 2\frac{U''}{U'}\right)v_{\xi} - U'v^2, \tag{2.2.3}$$

or in self-adjoint form,

$$v_t = b(\xi)^{-2}(b(\xi)^2 v_{\xi})_{\xi} - U'v^2, \tag{2.2.4}$$

where $b^2(\xi) = (U')^2 e^{c^*\xi}$. The stability bound of [99] is based on (2.2.4):

$$||bv(\xi,t)||_{L^\infty(\mathbb{R})} \le \frac{C}{1+t^{1/4}} ||bv(\xi,0)||_{L^2(\mathbb{R})}. \tag{2.2.5}$$

Note that by (2.1.4), b decays to zero exponentially as $\xi \to +\infty$ and grows like $O(\xi^2)$ as $\xi \to -\infty$, so the convergence of v to zero is not uniform in ξ. Moreover, the spatial shift ξ_0 is again zero.

Using the renormalization method (see [32]), Gallay [72] gives the leading self-similar asymptotics of v:

$$v(\xi,t) \sim t^{-3/2} A^* \psi^*(\xi t^{-1/2}), \tag{2.2.6}$$

where the constant A^* depends on the initial condition and

$$\psi^*(\xi) = \frac{1}{\sqrt{4\pi}} \begin{cases} 1, & \xi > 0, \\ e^{-\xi^2/4}, & \xi \le 0 \end{cases}$$

is universal. To see the decay factor $t^{-3/2}$, note that the variable coefficient $(c^* + 2U''/U')$ of v_ξ behaves like $2\xi^{-1}$ for $\xi \to -\infty$ and tends to $\sqrt{(c^*)^2 - 4f'(1)}$ as $\xi \to +\infty$. So, near $\xi = -\infty$, the linear part of (2.2.3) is

$$v_t = v_{\xi\xi} + \frac{2}{\xi} v_\xi,$$

as in the three-dimensional heat equation in radial coordinates, and hence the decay factor is $t^{-3/2}$. The other regime, near $\xi = \infty$, leads to exponential decay in time. Combining the two contributions and the coupling in the intermediate regime eventually produces $t^{-3/2}$. For details of how to establish self-similarity and the precise statement of the result, we refer to [72].

We note that all the stability results so far are for small perturbations. To extend them to order 1 or arbitrary perturbations, a global tool such as a maximum principle is needed. For the global asymptotic analysis and critical front selection, see Aronson and Weinberger [1], [2]; Fife [59], [60]; the original paper by KPP [100], with the initial data being the indicator function of the negative line; the probabilistic analysis of the scalar KPP equation in Bramson [29], Freidlin [66], McKean [115], and references therein; and Kanel [93] for the type 5 nonlinearity.

The type 2 and type 4 cases can be handled by the weighted norm idea of [141] outlined above, thanks to $f'(0) = 0$, for local stability. In the KPP (type 1) case, it is known [1], [66] that if the initial data is of compact support, then the edges of support develop into a pair of outgoing fronts moving at the critical (minimum) speed $\pm c_1^*$. It appears to be unknown whether the critical speed is also selected in this way for the type 2 or type 4 nonlinearities. Intuitively, such a result is expected for type 2, since the faster fronts of type 2 are only generated by perturbations behaving like $O(x^{-1})$ as $|x| \to \infty$, and they are not as easy to initiate as in the KPP case.

The asymptotic stability of traveling fronts subject to perturbations in several dimensions is an interesting problem, not yet explored as much as in the one-dimensional case. Let us consider a front moving in the x_1 direction and write $y = (x_2, \ldots, x_n)$. The linearized equation for the perturbation $\tilde{u}$ in the moving frame $\xi = x_1 - ct$ is

$$\tilde{u}_t = \tilde{u}_{\xi\xi} + c\tilde{u}_\xi + \Delta_y \tilde{u} + f'(U)\tilde{u}. \tag{2.2.7}$$

If the transverse variable y is finite (say, if $\tilde{u}$ is periodic in y), then decomposing into Fourier modes in y reveals that the term $\Delta_y \tilde{u}$ contributes a nonpositive spectrum.

Thus the problem can be reduced to one dimension in a straightforward way. The more interesting case is when the transverse variable y is unbounded. Then the term $\Delta_y \tilde{u}$ introduces a continuous spectrum, and the gap near the origin disappears. Just as in the case of critical KPP with $c = c^*$, the continuous spectrum cannot be shifted by introducing weights. The decay of perturbations coming from the far field of the transverse direction is essentially governed by the heat equation $\tilde{u}_t = \Delta_y \tilde{u}$, and so the rate of decay is at best algebraic. The higher the dimension of y, the better the decay of the transverse component of the perturbation. In Xin [158] and Levermore and Xin [103], it was shown that an initially localized perturbation v of a front for the bistable (type 3) nonlinearity decays to zero like $O(t^{-(n-1)/4})$ in a Sobolev space of high enough order if $n \geq 4$ and decays to zero uniformly on any compact set in (ξ, y) if $n \geq 2$. The analysis is based on a Lyapunov functional, the maximum principle, and the known one-dimensional results. It remains to study the other types of nonlinearity, in particular, the critical KPP front in several space dimensions.

2.3. Variational Principles for Front Speeds. Since the wave speeds of traveling fronts are in general unknown in closed form, their variational characterization is an invaluable way of estimating them. Hadeler and Rothe [81] considered the general continuously differentiable nonlinearity $f(u)$ such that

$$f(0) = f(1) = 0, \quad f(u) > 0, \quad u \in (0,1), \quad f'(0) > 0, \quad f'(1) < 0. \tag{2.3.1}$$

They were the first to establish a min-max variational principle for the minimum wave speed:

$$|c^*| = \inf_{\rho} \sup_{u\in(0,1)} \left\{ \rho'(u) + \frac{f(u)}{\rho(u)} \right\}, \tag{2.3.2}$$

where ρ is any continuously differentiable function on $[0,1]$ such that

$$\rho(u) > 0, \quad u \in (0,1), \quad \rho(0) = 0, \quad \rho'(0) > 0. \tag{2.3.3}$$

The formula (2.3.2) is based on the phase plane construction of the fronts. Under (2.3.1), for each allowable c, there is a connection from an unstable node to a saddle, and the front profile is strictly monotone. Let $u = u(x-ct) = u(\xi)$ connect $u = 1$ and $u = 0$ from left to right, so that $c \geq c^* > 0$. Then $u_{\xi\xi} + cu_\xi + f(u) = 0$, $u(-\infty) = 1$, and $u(\infty) = 0$. Now regard u_ξ as a function of u by defining $p = p(u) = -u_\xi > 0$ at $u = u(\xi)$. The function $p(u)$ is a solution of

$$p(u)p'(u) - cp(u) + f(u) = 0, \tag{2.3.4}$$

with $p(0) = 0$, $p(1) = 0$, and $p(u) > 0$ on $(0,1)$. The expression inside the supremum of (2.3.2) is just what we find from (2.3.4) on writing c in terms of u, p, and $p'(u)$. If $p(u)$ is not a solution of (2.3.4), then $(u, p(u))$ can represent a curve connecting the node and the saddle in the phase plane, but with the flow field at the curve pointing toward the solution curve of (2.3.4). This geometric information translates into the inequality that c is no less than the supremum in (2.3.2) for some ρ satisfying (2.3.3). It follows that any allowable c, in particular c^*, is no less than the min-max of (2.3.2). The equality is attained by $\rho = p_*(u)$ corresponding to the speed c^*.

It follows from (2.3.2) that

$$2\sqrt{f'(0)} \leq |c^*| \leq 2\sqrt{L}, \quad L = \sup_{u\in(0,1)} \frac{f(u)}{u}, \tag{2.3.5}$$

which gives the well-known KPP minimal speed $2\sqrt{f'(0)}$ if L is achieved at $u = 0$. To see (2.3.5), we take $\rho(u) = au$, $a > 0$. Then $|c^*| \leq a + L/a$. Minimizing over a establishes the upper bound. The lower bound is easily deduced by restricting the supremum to those functions u in a small neighborhood of zero. Hadeler and Rothe [81] further used (2.3.2) to find the exact minimal speed for $f(u) = u(1-u)(1+\nu u)$: $|c^*| = 2$ if $-1 \leq \nu \leq 2$, $|c^*| = (\nu+2)/\sqrt{2\nu}$ if $\nu \geq 2$.

Recently, Benguria and Depassier [12] obtained a general variational wave speed formula for any f such that $f(0) = f(1) = 0$.

THEOREM 2.1. *Let f be any of the five types of nonlinearity, and assume that a monotone front exists. Then the minimum (or unique) speed c^* is given by*

$$(c^*)^2 = \sup \left(2 \frac{\int_0^1 fg\,du}{\int_0^1 (-g^2/g')du} \right), \tag{2.3.6}$$

where the supremum is over all positive decreasing functions $g \in (0,1)$ for which the integrals exist. Moreover, the maximizer exists if $|c^*| > 2\sqrt{f'(0)}$.

The above theorem appears to be the first variational result in such generality. The formula holds for f changing signs in $(0,1)$. The Huxley formula (2.1.5) is recovered by putting $g(u) = ((1-u)/u)^{1-2\mu}$. A similar variational formula for f in (2.3.1) without the constraint $f'(1) < 0$ is established in [11].

The proof of (2.3.6) is elementary, and it uses (2.3.4) again. Let $g = g(u)$ be any positive function on $(0,1)$ such that $h = -g'(u) > 0$. Multiplying (2.3.4) by $g(u)$ and integrating over $u \in [0,1]$, we have after integration by parts the equality

$$\int_0^1 fg\,du \;=\; c\int_0^1 pg\,du - \int_0^1 \frac{1}{2}hp^2\,du. \tag{2.3.7}$$

For positive c, g, and h, the function $\varphi(p) = cpg - \frac{1}{2}hp^2$ has its maximum at $p = cg/h$, and so $\varphi(p) \leq c^2g^2/2h$. It follows that

$$c^2 \geq 2\frac{\int_0^1 fg\,du}{\int_0^1 (g^2/h)du} \equiv I(g), \tag{2.3.8}$$

which implies (setting $c = c^*$ if c is nonunique) that $(c^*)^2$ is no less than the supremum of (2.3.6). Next, we show that the equality holds for a function $\hat{g}$. Notice that the condition $p = cg/h$ is solvable in g and gives an expression for the maximizer $\hat{g}$,

$$\hat{g} = \exp\left(-\int_{u_0}^{u} cp^{-1}du \right), \tag{2.3.9}$$

with $u_0 \in (0,1)$. Clearly, $\hat{g}$ is positive and decreasing, with $\hat{g}(1) = 0$ since $p \sim O((1-u))$ for $u \sim 1$. At $u = 0$, however, $\hat{g}$ diverges since the exponent goes to $+\infty$. A natural choice for $\hat{g}$ now is $p = p_*(u)$, if we verify that the two integrals are finite in $I(\hat{g})$.

For nonlinearities of types 2, 3, 4, and 5, p_* approaches zero exponentially and

$$p_* \sim \frac{c + \sqrt{c^2 - 4f'(0)}}{2}u \;\equiv\; mu.$$

Thus, near $u = 0$, $\hat{g} \sim u^{-c/m}$ and $f\hat{g}$ and $\hat{g}^2/\hat{h}$ diverge at most like $u^{1-c/m}$. The integrals of $I(\hat{g})$ are finite if $m/c > 1/2$. This condition holds if $f'(0) \leq 0$, which is indeed true for types 2, 3, 4, and 5, and also for f in (2.3.1) if $(c^*)^2 > 4f'(0)$. If $(c^*)^2 =$

$4f'(0)$, which is the case for type 1, the maximizer does not exist. However, choosing the test function $g(u) = a(2-a)u^{a-2}$ with $a \in (0,1)$, we calculate $\int_0^1 (g^2/h)du = 1$. Integration by parts twice shows that as $a \to 0$, $I(g) = 2(2-a)a\int_0^1 fu^{a-2}du \to 4f'(0)$. The proof is complete.

2.4. Further Remarks. Let us comment on a few systems of R-D equations from which the various types of scalar equation of this section came and briefly explain the new phenomena that arise with systems.

Scalar R-D equations with nonlinearities of types 1, 2, 4, and 5 come from systems of the form

$$\begin{aligned} u_t &= d\Delta_x u + vf(u), \quad x \in \mathbb{R}^n, \\ v_t &= \Delta_x v - vf(u), \end{aligned} \tag{2.4.1}$$

where $d > 0$. If (2.4.1) models premixed flame fronts in a one-step exothermic chemical reaction of the form $A \to B$, then u is the temperature of the reacting mixture, v is the mass fraction of the reactant A, and d is called the *Lewis number*. The function f takes the Arrhenius form $e^{-E/T}$ with activation energy constant $E > 0$.

If $d = 1$, then adding the two equations shows that $u+v$ satisfies the heat equation and hence is forever equal to 1 if this is so arranged at $t = 0$. Replacing v by $1-u$ in the first equation of (2.4.1), we find a scalar R-D equation of type 4, and type 5 then arises as we introduce a temperature cutoff θ. For existence and uniqueness of fronts, see [17], [26], and [111]. It is well known that if d is much larger or smaller than 1, fronts are unstable; see [7], [14], [91], [144], and references therein. Intuitively, the very distinct diffusion constants cause the front to develop spatial-temporal scales as a way of keeping balance. With $d > 1$, fronts oscillate in time, and with $d < 1$, they generate transverse spatial oscillations in two or three dimensions. The scales continue to grow with d, and eventually the solutions are chaotic.

When (2.4.1) models isothermal autocatalytic reactions of the form $A + mB \to (m+1)B$, $m \geq 1$, with rate law proportional to vu^m, v and u are the concentrations of the reactant A and the catalyst B. The function f is now $f(u) = vu^m$. Again, when $d = 1$, we recover a scalar R-D equation of type 1 if $m = 1$ and of type 2 if $m \geq 2$. Existence and dynamics of fronts are discussed in [22], [23], [24], and [64]. Similarly, if d is sufficiently far from 1, fronts are unstable and can be chaotic; see [87], [108], and [116].

For (2.4.1) in either application above and general bounded continuous initial data, the maximum of u is uniformly bounded in time if $d \leq 1$ [114] and can grow at most like $O(\log\log t)$ if $d > 1$ [37]. Due to the lack of a maximum principle for the u equation, the upper bound of the maximum of u is delicate to obtain. In general, the maximum of u will go above 1 for large t even if the initial data of u is bounded between 0 and 1. This is in clear contrast with scalar equations. Although a neat argument of comparison of heat kernels leads to a uniform maximum norm bound in time when $d \leq 1$ [114], in general one has to use nonlinear functionals (generalized Lyapunov functionals for a nongradient system like (2.4.1)) to control the growth of u [37]. Mathematically speaking, the absence of a maximum principle and a classical Lyapunov functional (nonincreasing in time) is responsible for the appearance of complex dynamics such as chaos.

On the other hand, there is still order and pattern in (2.4.1). For spatially decaying initial data, solutions will eventually decay to zero. This can be seen as follows. Intuitively, u tends to grow and v tends to decay when they meet each other because of the plus and minus signs in front of the nonlinear terms. One can think of u as a predator and v as a prey. If the initial amount of prey v is finite (the case of decaying data), sooner or later the predator will consume the prey, then die of hunger. What is interesting is that when $vf(u) = vu^2$, the decay of v is $O(t^{-p})$ with $p > 1/2$ in one space dimension. This anomalous scaling has been proved for initial data of any size [33] and is a consequence of competition between plus and minus nonlinear terms, with the cubic nonlinearity being critical. The nonlinearity in the u equation drops out to leading order as $t \to \infty$, and the reduced system (that is, (2.4.1) with $u_t = du_{xx}$ in place of the u equation) has a family of self-similar solutions giving the desired anomalous scaling.

Scalar R-D equations of type 3 come from the FHN system in mathematical biology,

$$\begin{aligned} &u_t = u_{xx} + u - u^3 - v, \quad x \in \mathbb{R}, \\ &v_t = \epsilon(u - \gamma v), \end{aligned} \tag{2.4.2}$$

where $\gamma > 0$ and $\epsilon > 0$ is a small parameter. In the limit $\epsilon \to 0$, (2.4.2) reduces to a bistable scalar R-D equation. See [59], [90], [119], [121], and [140].

A type 1 (KPP) or type 3 (bistable) scalar equation can also be treated as a special case of a vector equation of the same type with vector field $\vec{f}$. For type 1, see [8], where $\vec{f}$ is a vector field pointing inside the first quadrant with flow trajectories all converging to a stable equilibrium point. Such a system has a maximum principle (or an invariance region of solutions), and so the maximum norm is bounded in time. For type 3 [139], one can simply take $\vec{f} = -\nabla_u V(u)$, where V is a potential with multiple minima and $u = \vec{u} = (u_1, \ldots, u_n)$. Such a system is a gradient system, and the natural Lyapunov functional exists and is nonincreasing in time. Due to the presence of either the maximum principle or variational structure, front solutions to these two types of systems are stable [138], [153], [154].

The fact that the minimal front speeds of KPP equations can be determined by linearization at the unstable steady state is also known as the marginal stability criterion (MSC). The validity of MSC in models different from the classical KPP equations (see [2]) has been a central theme of many studies in the physics literature. See [10], [40], [78], [34], [35], and [124], among others. It is an interesting problem to prove or disprove MSC for non-KPP equations [78].

3. Fronts in Periodic and Slowly Varying Media. We discuss a classical homogenization result for a scalar elliptic equation and use that to motivate the analytical form of traveling fronts in periodic media. Existence, stability, and propagation of fronts come next, with explanations of the PDE techniques involved. We also describe the large deviation approach and viscosity solution method for the KPP-type R-D equation. The large deviation approach extends to analyzing the speeds of the one-dimensional KPP fronts in random media. Open problems are mentioned.

3.1. Periodic Media and Homogenization. Multiscale problems are common in applications, such as finding the effective conductivity of a composite material or the effective permeability for flows in porous media, where one has at least two scales, the large scale of the sample and the small scale of the imbedded inclusions or pores. These two scales normally differ significantly and render the full resolution of the problem difficult. Therefore, it is of great theoretical and practical interest to find out how to upscale the collective effect of the small scale into the large scale, and so simplify the problem. When the small scale possesses a periodic structure, the upscale problem has a well-developed theory called homogenization. See Bensoussan, Lions, and Papanicolaou [13] for a systematic account of the foundational works.

We give here an example of homogenization and use formal asymptotic analysis to illustrate the ideas. Consider a two-point boundary value problem for a second-order ODE with rapidly oscillating periodic coefficients,

$$(a(\epsilon^{-1}x)u^\epsilon_x)_x = f(x), \quad x \in [0,1], \tag{3.1.1}$$

with boundary condition $u^\epsilon(0) = u^\epsilon(1) = 0$. Here a is a positive smooth function with period 1 in $y \equiv \epsilon^{-1}x$ and $f(x)$ is a bounded continuous function in x. We are going to examine the limit of u^ϵ as $\epsilon \to 0$, where the large scale x and the small scale $\epsilon^{-1}x$ are separated. Since there are two separate scales in the problem, it is natural to search for a two-scale expansion of the solution in the form

$$u^\epsilon \;\sim\; u_0(\epsilon^{-1}x) + \epsilon u_1(x, \epsilon^{-1}x) + \epsilon^2 u_2(x, \epsilon^{-1}x) + \cdots, \tag{3.1.2}$$

where the $y = \epsilon^{-1}x$ dependence has period 1 also. Substituting the ansatz (3.1.2) into (3.1.1) and regarding x and y as independent variables, we have (noting that the x derivative is replaced by the operator $\partial_x + \epsilon^{-1}\partial_y$)

$$(\partial_x + \epsilon^{-1}\partial_y)(a(y)(\partial_x + \epsilon^{-1}\partial_y)(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots)) \;=\; f. \tag{3.1.3}$$

At the highest order $O(\epsilon^{-2})$, we have

$$\partial_y(a(y)\partial_y u_0) = 0, \tag{3.1.4}$$

which has only a y-independent periodic solution. Thus $u_0 = u_0(x)$. At the next highest order $O(\epsilon^{-1})$, we have

$$\partial_y(a(y)(\partial_x u_0 + \partial_y u_1)) = 0, \tag{3.1.5}$$

which implies

$$a(y)(\partial_x u_0 + \partial_y u_1) = c(x) \tag{3.1.6}$$

for some function $c(x)$. Dividing (3.1.6) by a and integrating the resulting equation over $y \in [0,1]$ yields

$$\frac{d}{dx}u_0 = c(x)\langle a^{-1}\rangle, \tag{3.1.7}$$

where $\langle\cdot\rangle$ denotes the integral or average over $y \in [0,1]$. At the next order $O(1)$, we have

$$\partial_x(a(y)(\partial_x u_0 + \partial_y u_1)) + \partial_y(a(y)(\partial_x u_1 + \partial_y u_2)) \;=\; f. \tag{3.1.8}$$

Averaging (3.1.8) over $y \in [0,1]$ gives

$$\partial_x\langle\, a(y)(\partial_x u_0 + \partial_y u_1)\,\rangle = f,$$

which in view of (3.1.6) is just $dc/dx = f$. This then becomes, as we insert (3.1.7),

$$\frac{d}{dx}\left(a^*\frac{d}{dx}u_0\right) = f, \tag{3.1.9}$$

where $a^* = \langle a^{-1}\rangle^{-1}$, the harmonic mean of a. Equation (3.1.9) is the homogenized equation and is the same type of equation as that from which we started; however, its coefficient has been changed to the harmonic mean of the original one in the fast oscillating variable $y = \epsilon^{-1}x$. Now we only need to solve the large-scale equation (3.1.9) subject to the same boundary condition, and the small-scale effect has been built in already.

Rigorous justifications of the above formal asymptotics in any number of dimensions were presented in [13] using the energy method and in Evans [49] using the weak convergence method. For the first homogenization result in random media, see Papanicolaou and Varadhan [125]. The equation (3.1.5) is posed on the periodic domain in terms of the y variable and is called the cell problem. Only in one dimension can it be solved in closed form; as a result, we know the homogenized coefficient explicitly. In several dimensions, the corresponding elliptic boundary value problem can be homogenized, but the homogenized coefficients are not known explicitly in general. The formulation of the cell problem and the formula for a^* in several dimensions are given in subsection 3.5.

3.2. Traveling R-D Fronts in Periodic Media. Now let us consider what happens if we let the R-D fronts discussed in section 2 pass through a medium with periodic structure. If we model the medium with a periodic coefficient as we did above, then a model equation for R-D fronts is

$$u_t = (a(x)u_x)_x + f(u), \tag{3.2.1}$$

where $a(x)$ is a positive 1-periodic smooth function and $f(u)$ is a nonlinear function of one of the five types. Since we expect solutions to behave like fronts, we should look at them in the large-space and large-time scaling limit. That is, let us consider (3.2.1) under the change of variables $x \to \epsilon^{-1}x$, $t \to \epsilon^{-1}t$ for ϵ small. The rescaled equation is

$$u_t^\epsilon = \epsilon(a(\epsilon^{-1}x)u_x^\epsilon)_x + \epsilon^{-1}f(u^\epsilon), \tag{3.2.2}$$

which resembles a homogenization problem except that there is also a singular prefactor ϵ^{-1} in front of the nonlinear term. We realize that there are two scales present in this problem. One is the width of the front, and the other is the wavelength of the periodic medium. The first is easy to capture if we look at the rescaled form of a traveling front in a homogeneous medium, or $U(\epsilon^{-1}(x-ct))$. The second can be built in as in the homogenization ansatz (3.1.2). Combining the two ideas, we come up with the following two-scale ansatz for R-D fronts in periodic media:

$$u^\epsilon \sim U(\epsilon^{-1}(x-c^*t), \epsilon^{-1}x) + \cdots, \tag{3.2.3}$$

where c^*, the average wave speed, plays the role of a^* in the homogenization example shown before. Certainly, we impose periodicity in $y = \epsilon^{-1}x$ and a 0 or 1 far-field boundary condition in $s = (x - c^*t)/\epsilon$.

Substituting (3.2.3) into (3.2.2), we find that $U(s, y)$ satisfies the PDE

$$(\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)U) + c^*U_s + f(U) = 0. \tag{3.2.4}$$

If (3.2.4) has a solution under the boundary conditions

$$U(s, \cdot) \text{ has period } 1, \quad U(+\infty, y) = 1, \quad U(-\infty, y) = 0, \tag{3.2.5}$$

the leading term of (3.2.3) is actually an exact solution! Recalling that the scaling was just to motivate ourselves, we see that we could have worked with the original equation (3.2.1) to begin with. The exact traveling front then has the functional form

$U(x-ct,x)$, which was first found and constructed by Xin [157]. It is also called a periodically varying wavefront in [88].

Comparing (3.1.2) and (3.2.3), we see that the two scales of (3.2.3) are not necessarily separate. In fact, they can be arbitrary, while in (3.1.2) the two scales are vastly separate. In this sense, (3.2.3) is a general two-scale representation. Also for this reason, we end up with a PDE cell problem to solve instead of an ODE cell problem. We will see that what makes (3.2.3) possible is the nonlinearity $f(U)$ and that the extreme cases when the front width is either much larger or much smaller than the wavelength of the medium are simpler.

It is easy to generalize the above form of traveling front to several space dimensions. Let us consider an R-D equation of the form

$$\begin{aligned} u_t &= \nabla_x \cdot (a(x)\nabla_x u) + b(x)\cdot \nabla_x u + f(u), \\ u|_{t=0} &= u_0(x), \end{aligned} \tag{3.2.6}$$

where

(A1) $a(x) = (a_{ij}(x))$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is a smooth positive-definite matrix on $\mathbb{R}^n$, 1-periodic in each coordinate x_i;

(A2) $b(x) = (b_j(x))$ is a smooth divergence-free vector field, 1-periodic in each coordinate x_i, with mean zero.

Equations of the form (3.2.6) appear in the study of premixed flame propagation through turbulent media [36], where u is the temperature of the combustible gas, $b(x)$ is the prescribed turbulent incompressible fluid velocity field with zero ensemble mean, $f(u)$ is the Arrhenius reaction term, and $a(x)$ is taken to be a constant matrix. Since the fluid velocity b is given as we solve for the temperature u, the above problem is called passive, and the traveling fronts are called passive fronts. In [36], based on their formal asymptotic analysis in the large activation temperature limit, Clavin and Williams found that the temperature profile of u propagates with the effective turbulent flame speed. To make progress and to avoid the difficulty of dealing with complex flows involving a wide range of spatial and temporal scales such as turbulence, we consider here the special periodic case, hoping to achieve a better understanding of the concept of effective flame speed. It appears to be unknown under what general conditions such a speed is well defined in turbulent media, but interesting findings recently have given an indication of what is going on, as we shall see.

Let us fix a unit vector $\vec{k}$ and look for a traveling front moving in this direction with speed $c = c(\vec{k})$. The traveling front is of the form $u(x,t) = U(\vec{k}\cdot x - ct, x)$, where the wave speed c is an unknown constant depending on $\vec{k}$, and U, the wave profile, as a function of $s = \vec{k}\cdot x - ct$ and $y = x$, satisfies the boundary conditions $U(-\infty, y) = 0$ and $U(+\infty, y) = 1$, and $U(s,\cdot)$ has period 1. Upon substitution of $u(x,t)$ into (3.2.6), we obtain the following traveling front equation for $U = U(s,y)$ and c:

$$(\vec{k}\partial_s + \nabla_y)(a(y)(\vec{k}\partial_s + \nabla_y)U) + b(y)\cdot(\vec{k}\partial_s + \nabla_y)U + cU_s + f(U) = 0. \tag{3.2.7}$$

Solutions of (3.2.7) have been systematically studied by Xin in a series of works [157], [159], [160], [161], [162]. The main results on traveling fronts are summarized in the following theorem.

THEOREM 3.1 (existence and uniqueness). *Let T^n be the n-dimensional unit torus, define $\overline{a} = \int_{T^n} a(x)dx$, and assume that conditions* (A1) *and* (A2) *hold.*

(1) *If the nonlinearity $f(U)$ is of type* 3 *(the bistable nonlinearity) with $\mu \in (0, \frac{1}{2})$, there is a positive number δ_{cr} such that if $||a(x) - \overline{a}||_{H^s(T^n)} < \delta_{cr}$ and $||b(x)||_{H^s(T^n)} <$*

δ_{cr}, $s = s(n) > n+1$, (3.2.7) *has a unique classical front* (U, c) *such that* $0 < U < 1$, $U_s > 0$ *for all* $(s, y) \in \mathbb{R} \times T^n$, *and* $c < 0$.

(2) *If the nonlinearity* $f(U)$ *is of type 5 (the combustion nonlinearity with ignition temperature cutoff), then for all* a *and* b, (3.2.7) *has a unique classical front* (U, c) *satisfying the same properties.*

Here uniqueness means that c is uniquely determined by the coefficients (a, b) and the nonlinearity $f(U)$, and U is unique up to a constant translation in s due to the translation-invariance of (3.2.7). We see that there is a threshold phenomenon in the bistable case. Intuitively, this is because the unequal potential wells of the antiderivative of $f(u)$ (which are essentially the driving force behind front motion) can have effectively the same depth due to the effects of periodic media. As a result, front propagation is suppressed or quenched. A similar situation occurs in the homogeneous case when the intermediate zero of $f(u)$ is equal to $\frac{1}{2}$. In other words, traveling fronts of the form we seek may cease to exist in the bistable case when the spatial variation of the coefficients is large enough. Standing waves with zero wave speed exist instead. We will illustrate this point further.

For the other types of nonlinearity (types 1, 2, and 4), $f(u) \geq 0$, and so fronts always move and there is no quenching issue as long as b has mean zero and thus has no bias towards positive or negative speeds. However, as we saw in section 1, fronts are no longer unique and, in particular, there is a continuum of wave speeds above a minimal number. Such a wave speed spectrum is expected in the periodic case. Historically, the type 1 (KPP or Fisher) case was studied first [75], [66], using the probabilistic large deviation method, which deals with compactly supported data and the resulting minimal speed. In such an approach, one does not need to worry about other wave speeds and wave profiles. We will come back to this KPP property later. In the theoretical biology literature, Shigesada, Kawasaki, and Teramoto [142] studied the KPP critical fronts in one-dimensional periodic media, using formal arguments and linearization to find the approximate speeds.

The mathematical study of periodically varying traveling waves in equations of the three types 1, 2, and 4 is limited to one space dimension, but it already reveals the picture of a continuum of wave speeds and the appearance of a minimal wave speed. Hudson and Zinner [88] considered a one-dimensional generalized Fisher (KPP) R-D equation

$$u_t = u_{xx} + f(u, x), \tag{3.2.8}$$

where f is jointly continuous in u and x, Lipschitz continuous in u, and periodic in x with period 1. Moreover, f satisfies the property that there exists a 1-periodic continuous function $\bar{u}(x) > 0$ such that $f(u, x) > 0$ for $u \in (0, \bar{u}(x))$ and $f(u, x) \leq 0$ for $u > \bar{u}(x)$. They found that there is a minimal speed $\tilde{c}^*$ given by

$$\tilde{c}^* = \inf_{r,y} \sup_{x \in [0,1]} \frac{y''(x) + \mu(x) y(x)}{r y(x)}, \tag{3.2.9}$$

where the infimum is taken over all $r > 0$ and $y(x) \in C^2(\mathbb{R})$ for which $y(x) > 0$ for all x, $y(x)e^{-rx}$ is periodic with period 1, and

$$\mu(x) = \sup_{u \in (0, \bar{u}(x))} \frac{f(u, x)}{u}.$$

Their result can be stated as follows.

Theorem 3.2. *For all* $c \geq \tilde{c}^*$, *there exists a function* $u(t, x)$ *that is a solution of* (3.2.8) *in the distribution sense. The solution can be written in terms of a wave profile* U, *namely,* $u(t, x) = U(x + ct, x)$, $(t, x) \in \mathbb{R}^2$, *where* $U(x + ct, \cdot)$ *is periodic with period* 1. *Furthermore,* $U(\cdot, x)$ *is nondecreasing for each* x *and* $u(t, \cdot)$ *is uniformly Lipschitz continuous with* $\lim_{x \to -\infty} u(t, x) = 0$, $\liminf_{x \to +\infty} u(t, x) \geq \min \bar{u}(x)$, *and* $\limsup_{x \to +\infty} u(t, x) \leq \max \bar{u}(x)$ *for all* $x \in \mathbb{R}$.

In the KPP case (type 1), (3.2.9) agrees with the variational formula of [75], $\tilde{c}^* = |c^*|$, as we shall see later. In general, (3.2.9) is only an upper bound on the minimal wave speed; in fact, $\tilde{c}^* > |c^*|$ in the case $f(u, x) = u^2(1 - u)$.

Let us mention that [157] and [159], [160], [161], [162] work directly with the nonlinear eigenvalue problem (3.2.7), while [88] works with a time-dependent spatially discretized system first, then takes a continuum limit to recover the traveling front from the limiting solutions. The two different approaches yield the same functional form of the traveling fronts that we motivated using homogenization at the beginning of this subsection.

3.3. Constructing Traveling Fronts in Periodic Media. When constructing one-dimensional traveling fronts, one often relies on the dynamical systems approach (as in section 2) by establishing a desired connection orbit in phase space; see Smoller [146]. Although in [73] Gardner successfully extended the approach to analyzing multidimensional traveling fronts in a model problem, this is, in general, a difficult task, especially for variable coefficient equations. Here we would like to introduce two other methods that work in any number of dimensions and rely more on functional analytical tools. Since one works directly with the boundary value problem of the traveling wave equations and PDE tools can be applied, the results are much more general and robust.

The idea of the first method (the degree approach) typically divides into two steps. In step 1, we pose the same boundary value problem on a truncated finite domain. If the original equation is elliptic, then existence of solutions reduces to finding a fixed point of a nonlinear map, which follows from the standard Leray–Schauder degree theory [77], [166]. This is a more or less abstract step. In step 2, we derive bounds on the solutions of step 1 independently of the size of the truncated domain and pass to the (original) infinite domain limit. In addition, we also estimate the asymptotic behavior of the limiting solution at $\pm\infty$ to make sure that the limit is the desired traveling front. The second step is concrete, and specifics of the problem (such as types of nonlinearity) play a key role. Such a degree approach was first developed in Berestycki, Nicolaenko, and Scheurer [16] for systems of one-dimensional fronts and was later extended to multidimensional fronts [15].

Now let us see how the method works for fronts in periodic media. Consider (3.2.7) with a type 5 nonlinearity. Our first observation is that the three linear terms do not form a strongly elliptic operator (such as the Laplacian $\Delta_{s,y}$), since the second derivatives are along directions $(k_i, 0, \ldots, 0, y_i, 0, \ldots, 0) \in \mathbb{R}^{n+1}$, $i = 1, \ldots, n$, that do not cover all $n+1$ directions. The other derivative along direction $(1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ is the s derivative of U. Hence if c is not equal to zero, we have a parabolic operator (similar to the heat operator $\partial_t - \Delta_x$). This may sound like trouble, since for the standard heat equation, we cannot pose a boundary value problem in t. However, what saves us is that the s direction of the infinite cylinder is not characteristic, since it is not orthogonal to all the directions $(k_i, 0, \ldots, 0, y_i, 0, \ldots, 0)$. The other observation is that (3.2.7) is translation-invariant in s.

There are, however, special cases of (3.2.7) where the leading term becomes elliptic. For example, if a is the identity, $b = (b_1(y'), 0, \ldots, 0)$ (called shear flow), and

$\vec{k} = (1, 0, \ldots, 0)$, then $U = U(x_1 - ct, y')$ satisfies the equation

$$\Delta_{s,y'} U + (b_1(y') + c)U_s + f(U) = 0, \tag{3.3.1}$$

which has been extensively studied by Berestycki, Larrouturou, Lions, and Nirenberg [15], [18], [19], [20], with Neumann boundary conditions on the boundary of a general bounded domain of y'.

Due to the translation-invariance in s and the strong maximum principle, solutions of (3.3.1) satisfying the boundary conditions $U(-\infty, y) = 0$, $U(+\infty, 0) = 1$ enjoy a nice monotonicity property: $U_s(s, y) > 0$ for any (s, y). Monotonicity immediately implies that limits of solutions exist as $|s| \to \infty$, and later on we will see that it also leads to front stability.

Do we have a strong maximum principle for the linear operator in (3.2.7),

$$Lu = (\nabla_y + k\partial_s)(a(y)(\nabla_y + k\partial_s)u) + b(y)^T \cdot (\nabla_y + k\partial_s)u + cu_s, \tag{3.3.2}$$

even though it is not strongly elliptic? As long as $c \neq 0$, the answer is yes, thanks to the parabolic maximum principle and the periodicity in y. Periodicity helps to overcome the degeneracy! For classical maximum principles, we refer to Protter and Weinberger [130]. Now let us take $c = -1$ for convenience and show the following proposition.

PROPOSITION 3.1. *Let u be a classical solution of the differential inequality $Lu \leq 0$ ($Lu \geq 0$) on $\mathbb{R} \times T^n$. If u achieves its minimum (maximum) at (s_0, y_0) with s_0 finite, then $u \equiv$ constant.*

Proof. We first treat the special case $n = 1, k = 1$, in which case L is given by

$$Lu = (\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)u) + b(y)(\partial_s + \partial_y)u - u_s.$$

For the time being, unfold T into $\mathbb{R}$ and regard L as an operator on $\mathbb{R}^2$. If we make the change of variables

$$s' = \frac{1}{\sqrt{2}}(s - y), \quad y' = \frac{1}{\sqrt{2}}(s + y),$$

then

$$\partial_s = \frac{1}{\sqrt{2}}(\partial_{s'} + \partial_{y'}), \quad \partial_y = \frac{1}{\sqrt{2}}(-\partial_{s'} + \partial_{y'}), \quad \partial_s + \partial_y = \sqrt{2}\partial_{y'}.$$

In terms of (s', y'), Lu becomes

$$Lu = 2(au_{y'})_{y'} - \frac{1}{\sqrt{2}}u_{s'} + \left(\sqrt{2}b - \frac{1}{\sqrt{2}}\right)u_{y'}.$$

L is a standard parabolic operator in (s', y'), elliptic in y' and parabolic in s'. By the strong maximum principle for parabolic operators, we see that if u attains its minimum at some finite point (s_0', y_0'), then

$$u \equiv \text{constant} \quad \text{if } s' \leq s_0'$$

or

$$u \equiv \text{constant} \quad \text{if } s - y \leq s_0 - y_0.$$

By the periodicity of u in y, we see that $u \equiv$ constant for all s and y. If $n \geq 2$, we can always subject y to an orthogonal transform, i.e., $y = Qy'$, and then Lu becomes

$$\begin{aligned} Lu &= (k\partial_s + Q^T\nabla_{y'})^T a(k\partial_s + Q^T\nabla_{y'})u + b^T \cdot (k\partial_s + Q^T\nabla_{y'})u - u_s \\ &= (Qk\partial_s + \nabla_{y'})^T QaQ^T(Qk\partial_s + \nabla_{y'})u + b^T \cdot Q^T(Qk\partial_s + \nabla_{y'})u - u_s. \end{aligned}$$

Choosing Q such that $Qk = e_1 = (1, 0, \ldots, 0)$, and setting $a_1 = QaQ^T$ and $b_1 = Qb$, we have

$$Lu = (e_1\partial_s + \nabla_{y'})^T a_1(e_1\partial_s + \nabla_{y'})u + b_1^T \cdot (e_1\partial_s + \nabla_{y'})u - u_s. \tag{3.3.3}$$

If we make the change of variables

$$s' = \frac{1}{\sqrt{2}}(s - y_1'), \quad z_1 = \frac{1}{\sqrt{2}}(s + y_1'), \quad z_i = \frac{1}{\sqrt{2}} y_i', \ i \geq 2,$$

then just as in the case $n = 1$, we have

$$Lu = 2\nabla_z^T(a_1 \nabla_z u) - \frac{1}{\sqrt{2}} u_{s'} + \sqrt{2} b_1^T \cdot \nabla_z u - \frac{1}{\sqrt{2}} u_{z_1}. \tag{3.3.4}$$

By the strong maximum principle for parabolic operators, if u attains its minimum at some finite point $P_0 = (s_0', z_0)$, then

$$u = \text{constant} \quad \text{if } s' \leq s_0'$$

or

$$u = \text{constant} \quad \text{if} \ \ s - y_1' \leq s_0 - y_{1,0}'.$$

In terms of (s, y), this asserts that u is a constant under some hyperplane that is not orthogonal to the s-axis. The periodicity of u in y implies that $u \equiv$ constant for all s and y. The proof is complete.

Let us outline the main ingredients of the two steps of the method. In step 1, we consider a family of elliptically regularized problems ($\epsilon > 0$, $\tau \in [0, 1]$),

$$\epsilon U_{ss} + L_\tau U + \tau f(U) = 0, \ \ (s, y) \in \Omega_a = [-a, a] \times T^n, \tag{3.3.5}$$

subject to the boundary conditions $U(-a, y) = 0$, $U(+a, y) = 1$. The operator L_τ is L with a replaced by $\langle a \rangle (1 - \tau) + \tau a$ and b replaced by τb, $\langle \cdot \rangle$ being the period average. To remove the translation-invariance of solutions, we must also impose a normalization condition: $\max_{y \in T^n} U(0, y) = \theta$. By the elliptic maximum principle, we know that U is bounded between 0 and 1 and that $U_s > 0$. Elliptic regularity also tells us that the maximum of ∇U is bounded independently of a and τ. The parameter τ links the linear problem, $\tau = 0$, with the problem of interest, $\tau = 1$. Consider the space $E = C^1(\Omega_a) \times R$. For $(v, c) \in E$, $\tau \in [0, 1]$, let $u = \varphi_\tau(v, c)$ be the unique solution of the elliptic boundary value problem

$$\epsilon u_{ss} + L_\tau u + \tau f(v) = 0$$

under the same 0 and 1 boundary conditions. Define

$$h_\tau(v, c) = \max_{y \in T^n, s = 0} \varphi_\tau(v, c).$$

Then the solution of (3.3.5) satisfies

$$u = \varphi_1(u, c), \quad h_1(u, c) = \theta. \tag{3.3.6}$$

Define $F_\tau(u, c) = (\varphi_\tau(u, c), c - h_\tau(u, c) + \theta)$, $\tau \in [0, 1]$. Now the existence of the solution is the same as that for the fixed point problem

$$F_1(u, c) = (u, c).$$

Notice that the mapping $(\tau, (u, c)) \to F_\tau(u, c)$ from $[0, 1] \times E$ to E is continuous and compact. Due to the a priori bounds on the solutions and their derivatives, the Leray–Schauder degree of the mapping $Id - F$ is well defined on a bounded closed set of the form

$$D \equiv \{(u, c) \in E, \ \|u\|_{C^1(R_a)} \leq K, \ |c| \leq K\},$$

where K is some constant larger than the bounds of the solutions. This is because the zeros of $Id - F$ cannot occur on the boundary of the set D. The degree is a measure of the number of zeros, counting multiplicity, and is invariant under a change of $\tau \in [0, 1]$; see Zeidler [166] for details. If the degree is nonzero, then we have a fixed point. This is easily checked when $\tau = 0$, since (3.3.6) is explicitly solvable, and we find that the degree is equal to 1.

In step 2, we pass first to the limit $a \to \infty$ and then to the limit $\epsilon \to 0$. To this end, the main technical work is to bound the wave speed c away from 0 and ∞ independently of both parameters. This can be achieved with the help of comparison principles of wave speeds for the $a \to \infty$ limit (see [159]) and the identity $c = -\int_{\mathbb{R}\times T^n} f(U)$ for the $\epsilon \to 0$ limit (see [160] and [84]). Thanks to the the normalization condition and $U_s \geq 0$, $U \leq \theta$ if $s \leq 0$. Hence we have a linear equation for U on $s \leq 0$. We can now look for a special decay solution of the form $\bar{U} = e^{\mu s}\psi(y)$ with $\psi(y) > 0$ and $\mu > 0$. This decay solution has a continuous limit as $\epsilon \to 0$ along a subsequence, and $\liminf_{\epsilon\to 0} \mu > 0$. It follows that the limiting solution must decay to zero as $s \to -\infty$. As $s \to +\infty$, monotonicity implies $U(s, y) \to U_+$. It is not hard to show that U_+ satisfies the elliptic equation (dropping s derivatives from (3.2.7))

$$\nabla_y \cdot (a(y)\nabla_y U) + b(y) \cdot \nabla_y U + f(U) = 0 \tag{3.3.7}$$

under periodic boundary conditions. Since $f(U) \geq 0$, the maximum principle implies that (3.3.7) has only constant nonnegative solutions. Thus U_+ equals either θ or 1. In the former case, $U \leq \theta$, and hence $f(U) \equiv 0$ for any (s, y). So $LU = 0$ for all (s, y), and thus U attains its maximum θ at a finite point $(0, y^*)$ as a result of imposing the normalization condition at $s = 0$. By the strong maximum principle property of the operator L that we just established, U must be identically equal to a constant, which is impossible since it has a zero limit at $s = -\infty$. We have constructed a desired traveling front solution with the property $U_s > 0$ (strict inequality again follows from the strong maximum principle for L).

The other bonus of the strong maximum principle for L is that by applying the sliding domain method [19], [104], we can show that traveling front solutions to (3.2.7) must be unique. The uniqueness means that there is only one value of the wave speed c for any given coefficients (a, b) and nonlinearity f of type 5. Moreover, the profile U is unique up to a constant translation in s, and it is strictly monotone in s. The basic argument for showing monotonicity is as follows. First, we compare $U(s, y)$ and its translate $U_\lambda = U(s + \lambda, y)$. For large λ, U_λ is larger than U for those points (s, y) in a bounded cylinder. The bounded cylinder is large enough so that $U(s, y)$ is close to either 0 or 1 outside of it. Then $w_\lambda \equiv U(s, y) - U(s + \lambda, y)$ satisfies the differential inequality $Lw_\lambda \leq 0$ outside of the finite cylinder. The strong maximum principle for L implies that $w_\lambda > 0$ holds at any point. Then we decrease λ to the infimum value λ_0 at which U_λ is no less than U. Now $w_{\lambda_0} \geq 0$. Again, the strong maximum principle implies that at λ_0, U and U_λ must be identical, which is possible only if $\lambda_0 = 0$, due to the front boundary conditions. We conclude that U is strictly monotone and actually has positive derivative everywhere (by invoking the minimum principle on the derivative). A similar argument can be carried out for any two profiles to show that they agree up to a constant translation; for the uniqueness of c, see [159].

What about other types of nonlinearity f? The new issues come up in step 2. If f is of type 3, there can be many nontrivial periodic states of (3.3.7), which presents a difficulty. As we know, traveling fronts may not exist for all $a(y)$ due to the existence of steady states. The convenient method for establishing traveling waves in type 3 (the bistable case) is to use the method of continuation (Xin [160], [162]) and treat a family of problems where $a(y)$ is replaced by $(1-\delta)\langle a\rangle + \delta a(y)$. We start with δ small and obtain solutions by perturbing the known one-dimensional front. The linearized operator has a simple eigenvalue at zero, and the rest of the spectrum is isolated away from zero. The monotonicity of the perturbed solutions guarantees that the same spectral property of the linearized operator remains, and so perturbation continues on δ. Since each perturbative step relies on the contraction mapping principle, there is no difficulty as $|s| \to \infty$. Of course, the same problem arises if we want to show that the continuation goes to any value $\delta \in [0,1]$, which we know is false in general. The continuation method is convenient in that it deals with the problem on the infinite domain, where estimates of solutions are usually simpler. However, it relies on good spectral properties of the linearized operators. It works for nonlinearities of both types 3 and 5, as well as other conservative-type problems [162]. To summarize, the degree method and the continuation method combined allow us to show the existence of traveling fronts as stated in Theorem 3.1.

In the special elliptic case (3.3.1), with f of type 3, whether c is zero or not makes no difference. Berestycki and Nirenberg [20] proved the following theorem.

THEOREM 3.3. *Consider* (3.3.1) *on the cylindrical domain* $\Omega = \mathbb{R} \times \omega$*, where* ω *is a bounded domain in* $\mathbb{R}^{n-1}$*. Then if* f *is bistable, there is a solution* (c, U) *of* (3.3.1) *satisfying* $U(-\infty, y') = 0$*,* $U(+\infty, y') = \psi(y')$*, and a zero Neumann boundary condition on* $\mathbb{R} \times \partial\omega$*. Here* $\psi(y')$ *is a solution of*

$$\Delta_{y'}\psi + f(\psi) = 0$$

with zero Neumann boundary condition on $\partial\omega$*. The function* ψ *satisfies either* $\psi \equiv 1$ *or* $0 < \psi < 1$*. Moreover,* $U_s > 0$*. If* ω *is a convex domain, then* $\psi = 1$ *and the solution* (c, U) *is unique.*

It is interesting that in the bistable case, the geometry of the domain plays a role.

If f is of type 1, 2, or 4, only the elliptic case has been worked out by Berestycki and Nirenberg [20] in dimensions above 1. Their result is as follows.

THEOREM 3.4. *Suppose* $f(u) > 0$ *on* $u \in (0,1)$*. Then there exists a critical speed* $c^* > 0$ *such that solutions to* (3.3.1) *exist, satisfying* $U(-\infty, y') = 0$*,* $U(+\infty, y') = 1$*, and a zero Neumann boundary condition on* $\mathbb{R} \times \partial\omega$ *if and only if* $-c \geq c^*$*. For every* $-c \geq c^*$*, there is a solution with* $U_s > 0$*. The solution is unique up to constant translation in* s *if* $f'(0) > 0$*. Furthermore, if* $f'(0) = \sup_{x\in[0,1]} u^{-1} f(u)$*, or if* f *is of type* 1*, then* c^* *is determined by the coefficient* b_1*, the domain* ω*, and* $f'(0)$*.*

The critical fronts (U^*, c^*) are constructed by truncating f into f_θ of type 5. Let χ_θ ($\theta < 0.5$) be a smooth compactly supported function such that $\chi_\theta(u) = 0$ if $u \leq \theta$, and $\chi_\theta(u) = 1$ if $u \geq 2\theta$. Then $f_\theta = \chi_\theta f$. Berestycki and Nirenberg [20] showed that the corresponding solutions (U_θ, c_θ) converge to (U^*, c^*). The faster fronts with $-c > c^*$ are constructed using U^* as upper solutions.

The type 1 case is particularly interesting. Whether $-c > c^*$, $-c = c^*$, or $-c < c^*$ is related to the existence of a positive decay solution of the form $e^{\lambda s}\psi(y')$ at $s = -\infty$ with $\lambda > 0$. The related eigenvalue problem is

$$-\Delta_{y'}\psi - f'(0)\psi \;=\; (\lambda^2 + \lambda(b_1(y') + c))\psi, \quad \psi_v|_{\partial\Omega} = 0. \tag{3.3.8}$$

Due to the condition $f'(0) > 0$, it was shown in [20] that there exists a unique value equal to c^* such that (3.3.8) has zero, one, or two principal positive eigenvalues according to $-c < c^*$, $-c = c^*$, or $-c > c^*$. We note that the occurrence of two positive principal eigenvalues is very much related to zero being the unstable node for the one-dimensional homogeneous KPP supercritical fronts discussed in section 2.

Now let us draw a connection between the above characterization of the KPP critical wave speed based on the linearization of the traveling front equation and the variational formula (3.2.9) of [88]. We consider the one-dimensional KPP equation

$$u_t = u_{xx} + f(u, x),$$

with $\mu(x) = f_u(0,x) = \sup_{x\in[0,\bar{u}(x)]} u^{-1} f(u,x)$, 1-periodic in x. Formula (3.2.9) can be rewritten as

$$\begin{aligned} c^* &= \inf_{r,\psi} \sup_{x\in[0,1]} \frac{(\psi(x)e^{rx})'' + \mu(x)\psi(x)e^{rx}}{r\psi e^{rx}} \\ &= \inf_{r,\psi} \sup_{x\in[0,1]} \frac{\psi'' + 2r\psi' + (r^2 + \mu(x))\psi}{r\psi} \end{aligned} \tag{3.3.9}$$

for any positive 1-periodic function ψ and any number $r > 0$. The supremum inside (3.3.9) is just the principal eigenvalue, call it $\Lambda = \Lambda(r)$, of the operator

$$L_r\psi = \psi'' + 2r\psi' + (\mu(x) + r^2)\psi \tag{3.3.10}$$

on the unit one-dimensional torus T. To see this, let us recall that the principal eigenvalue of a general second-order elliptic operator of the form

$$L_e = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i,x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + h(x)u \tag{3.3.11}$$

always has a simple real principal eigenvalue λ_1 on the torus T^n as a consequence of the Krein–Rutman or Perron–Frobenius theorems [146]. By Theorem 17 in chapter 2.8 of Protter and Weinberger [130], λ_1 has the upper bound (changing the sign of λ in [130] and setting their $k(x) = 1$)

$$\lambda_1 \le \inf_{w(x)\in C^2(T^n), w>0} \sup_{x\in T^n} \frac{L_e w}{w}, \tag{3.3.12}$$

which is easily seen to be an equality since λ_1 is attained by inserting the principal eigenfunction for w. Hence we have from (3.3.9)

$$c^* = \inf_{r>0} \frac{\Lambda(r)}{r}, \tag{3.3.13}$$

which will be shown in section 3.7 to agree with the formula obtained from the homogenization theory of the H-J equation or the large deviation method.

On the other hand, the equation for the traveling front ($u = U(x-ct, x)$, $s = x+ct$, $y = x$, $U(-\infty, y) = 0$, $U(+\infty, y) = 1$) is

$$(\partial_s + \partial_y)^2 U - cU_s + f(U, y) = 0,$$

which gives, upon linearizing at $U = 0$ and plugging in $e^{rs}\psi$, the eigenvalue problem at $s = -\infty$

$$\psi'' + 2r\psi' + (r^2 - rc + \mu(y))\psi = 0, \tag{3.3.14}$$

where $r > 0$, $\psi > 0$. Let us suppose for now that (3.3.14) has two positive principal eigenvalues for r if $c > c^*$, one if $c = c^*$, and none if $c < c^*$. We can regard rc as a principal eigenvalue of the operator

$$\psi'' + 2r\psi' + (r^2 + \mu(y))\psi.$$

Hence for each $c > c^*$ (for a positive principal eigenvalue r of (3.3.14)), we have

$$cr = \sup_{x\in[0,1]} \frac{\psi'' + 2r\psi' + (r^2 + \mu(x))\psi}{\psi}. \tag{3.3.15}$$

Finally, to reach c^*, we divide (3.3.15) by r and take the infimum over $r > 0$, $\psi > 0$, which recovers formula (3.3.9) or (3.2.9).

Before ending this subsection, we also mention two other methods in the literature. The first is the constraint variational method. Heinze [83] considered the traveling front problem $\Delta_{x',y} u - c u_{x'} + f(u) = 0$, $(x', y) \in \mathbb{R} \times \Omega$, $u = 0$ on $\mathbb{R} \times \partial\Omega$, where $f \in C^1[0,1]$, $f = 0$ if $u \notin [0,1]$, and $f'(0) < \mu$, μ being the lowest eigenvalue of $-\Delta_y$ on Ω with zero Dirichlet boundary condition. By the scaling $x = x'c$, $\lambda = c^{-2}$, the problem becomes

$$\begin{aligned} &u_{xx} - u_x + \lambda(\Delta_y u + f(u)) = 0, \quad \mathbb{R} \times \Omega, \\ &u = 0, \quad \mathbb{R} \times \partial\Omega, \end{aligned} \tag{3.3.16}$$

whose solution is the minimizer of the functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R} \times \Omega} (u_x)^2 e^{-x}$$

in the weighted Hilbert space $H_0^1(\mathbb{R} \times \Omega, e^{-x})$ under the constraint

$$-1 = J(u) = \int_{\mathbb{R} \times \Omega} \left(\frac{1}{2} |\nabla_y u|^2 - F(u) \right) e^{-x}.$$

The unknown λ appears as the Lagrange multiplier. Existence of a unique monotone (in x) solution in $H_0^1(\mathbb{R} \times \Omega, e^{-x})$ is proved in [83]. The front connects zero with a steady-state solution of the elliptic problem $\Delta_y v + f(v) = 0$, $u = 0$ on $\partial\Omega$. Moreover, the wave velocity has a monotone dependence on the domain size, the nonlinearity, and the boundary conditions, as a byproduct of the variational approach. It remains to find out how to apply the variational method to variable coefficient problems.

The second additional method in the literature is the discrete front approximation method of Hudson and Zinner [88], which is interesting in that it provides a min-max variational formula for the wave speed. It remains to find out how to make their min-max formula optimal for the minimum speeds in the case of nonlinearities of types 1, 2, and 4. One feature of this method is that it starts from a spatial semidiscretization of the time-dependent R-D equation and works with time-dependent solutions. Then as the discretization limit is taken, identification of the limiting time-dependent solutions follows. This is different from any of the other methods, which work with time-independent problems.

3.4. Stability and Propagation of Fronts in Periodic Media. When traveling fronts are constructed, the next question, as always, is whether they are dynamically stable with respect to initial perturbations. Let us look at a simple case, the asymptotic stability of bistable fronts in one space dimension. We consider (3.2.1) with initial data equal to a perturbed traveling front,

$$u_t = (a(x)u_x)_x + f(u), \quad u|_{t=0} = U(x,x) + u_0(x). \tag{3.4.1}$$

Let us write $u = U(x - ct, x) + v$, which in the moving frame $\xi = x - ct$ becomes $u = u(\xi, t) = U(\xi, \xi + ct) + v(\xi, t)$. We can substitute this into (3.4.1) written in the moving frame variables (ξ, t) and use (3.2.7) to find the equation for v:

$$v_t = Lv + N(v) = (a(\xi + ct)v_\xi)_\xi + cv_\xi + f'(U)v + N(v), \tag{3.4.2}$$

where the nonlinear part is $N(v) = f(U + v) - f(U) - f'(U)v$.

As described earlier, a traditional way of analyzing stability is to examine the spectral properties of the linearized operator L. Now L depends on time periodically with period $p = c^{-1}$, so it is natural to analyze the spectrum of the period map defined as e^{pL} on the space $L^2(\mathbb{R})$ or $C_{\text{unif}}(\mathbb{R})$. This is similar to analyzing stability of a periodic orbit in ODEs. The traveling front in the moving frame $U(\xi, \xi + ct)$ is a periodic orbit with period p.

The essential spectrum depends only on the large-distance behavior of L. Specifically, the essential spectrum of e^{pL} is the same as that of the operator e^{pL_∞}, where $L_\infty = (a(\xi + ct)v_\xi)_\xi + cv_\xi + (\chi_{\mathbb{R}^-} f^{'}(0) + \chi_{\mathbb{R}^+} f^{'}(1))v$. Since both $f^{'}(0)$ and $f^{'}(1)$ are negative, it is clear that e^{pL_∞} is a contraction mapping from the unit ball to its interior. Hence the essential spectrum of e^{pL} stays strictly inside the unit disc. The remaining spectrum consists of isolated eigenvalues of finite multiplicity, and there are only finitely many of them outside of the radius of the essential spectrum. A general spectral decomposition theorem of Kato [95] allows us to look at the restriction of e^{pL} (denoted by R) onto the subspace spanned by the eigenfunctions corresponding to the finitely many eigenvalues outside of the essential spectrum. Since e^{pL} is a positive operator (mapping nonnegative data into strictly positive functions), so is its restriction R. On the other hand, it is easy to check that $U_s(\xi, \xi + ct) > 0$ is an eigenfunction of e^{pL} corresponding to eigenvalue 1. By the Perron–Frobenius theorem, 1 is the simple principal eigenvalue of R, and the other eigenvalues are less than 1 in absolute value. The eigenvalue 1 is due to the invariance of (3.4.2) under the transform $\xi \to \xi - ch$, $t \to t + h$ for any h. This is analogous to the spatial translation-invariance in the case of the homogeneous media. The above spectral property of e^{pL} implies that if the initial perturbation is small enough, the solution u converges to a traveling front solution $U(x - ct + s_0, x)$ in the (x, t) variables as $t \to \infty$ for some s_0 depending on the initial perturbation. As we have seen, the strict monotonicity of the wave profile $U_s > 0$ is a crucial condition for the above stability argument.

For type 5 nonlinearity, a weighted space (as before, requiring functions to decay to zero at a certain exponential rate as $|x| \to \infty$) is necessary to isolate the essential spectrum in a disc inside the unit circle. The remaining argument is the same. In the case of the other types of nonlinearity (types 1, 2, and 4), the faster waves (faster than the minimal speeds) may be analyzed in the same manner. The dynamics of the slow waves moving at the minimal speeds requires a more delicate argument and awaits further investigation.

The stability analysis of traveling fronts is much more difficult in multidimensional periodic media. However, if we are concerned only with the large-time wave speed and not with the profile of solutions, it suffices to find good upper and lower bounds of solutions so that they exhibit the same wave speed. This idea was originally developed in Aronson and Weinberger [1] for fronts in multidimensional homogeneous media. Later on, Gärtner and Freidlin [75], [66] analyzed KPP fronts with minimal speeds for compactly supported initial data in the same spirit. Let us first look at nonlinearities of types 3 and 5, and the KPP results will come a little later.

THEOREM 3.5 (front propagation). *Consider the initial value problem for* (3.2.6) *with initial data* $0 \leq u_0(x) \leq 1$. *Let* f *be of type* 3 *with* $\mu \in (0, \frac{1}{2})$ *or of type* 5 *with* $f'(1) < 0$. *Assume in the context of type* 3 *nonlinearity that a traveling wave solution* $U(k \cdot x - c(k)t, x)$ *exists for any unit vector* $k \in \mathbb{R}^n$. *Let* $s \in \mathbb{R}$ *and let the plane orthogonal to* k *be* $S = \{y \in \mathbb{R}^n | y = x - (k \cdot x)k \ \forall x \in \mathbb{R}^n\}$.

(I) *Suppose the initial data are frontlike:* $u_0(x) \to 0$ *sufficiently fast as* $k \cdot x \to -\infty$ *and* $u_0(x) \to 1$ *sufficiently fast as* $k \cdot x \to +\infty$, *uniformly in* $S(k)$, *for some* $k \in \mathbb{R}^n$.

Then

$$\lim_{t\to\infty} u(t, skt) = \begin{cases} 1, & s > c(k), \\ 0, & s < c(k). \end{cases}$$

(II) *Suppose the initial data are pulselike: for some unit vector k, $u_0(x) \to 0$ sufficiently fast as $k \cdot x \to -\infty$; $u_0(x) > \mu + \eta$, $|k \cdot x| < L$, for some positive constants η and L (θ replacing μ for f of type 5). Then there is a positive number $L_0(\eta) > 0$ such that if $L \geq L_0$,*

$$\lim_{t\to\infty} u(t, skt) = \begin{cases} 1, & c(k) < s < -c(-k), \\ 0, & s < c(k) \text{ or } s > -c(-k). \end{cases}$$

Let us sketch the proof of statement (I) in the case of a nonlinearity f of type 5, and refer to Xin [161] for the complete proof of the theorem. The idea is to construct sub- (super-) solutions using the maximum principle and the traveling wave solutions. These sub- (super-) solutions extend those in Fife and McLeod [61], and their long-time asymptotics rely on the decay property of solutions of variable coefficient linear parabolic equations of the form

$$u_t = \nabla \cdot (a(x)\nabla u) + b(x) \cdot \nabla u, \quad \nabla \cdot b(x) = 0. \tag{3.4.3}$$

The fundamental solution of (3.4.3) has pointwise lower and upper bounds in terms of heat kernels; see Nash [120], Fabes and Stroock [54], and especially Osada [122]. To begin, let us note that due to the fast convergence of u_0 to 1 as $k \cdot x \to \infty$, we can find a number $\xi_0 > 0$ large enough and a positive spatially decaying function $q_0 = q_0(k \cdot x) < (1 - \theta)/2$ such that

$$U(k \cdot x - \xi_0, x) - q_0(k \cdot x) \leq u_0(x)$$

on $\mathbb{R}^n$. Now consider the function

$$u_l \equiv U(k \cdot x - c(k)t - \xi_1(t), x) - q_1(t, x),$$

where ξ_1 and q_1 will be chosen to satisfy

$$\xi_1'(t) > 0, \quad \xi_1(t) > 0, \quad \xi_1(t) = o(t), \quad t \to \infty.$$

We calculate

$$\begin{aligned} N[u_l] &= u_{l,t} - \nabla_x \cdot (a(x)\nabla_x u_l) - b(x) \cdot \nabla_x u_l - f(u_l) \\ &= -\xi_1'(t)U_s - q_{1,t} + \nabla_x \cdot (a(x)\nabla_x q_1) \\ &\quad + b(x) \cdot \nabla_x q_1 + f(U) - f(U - q_1). \end{aligned} \tag{3.4.4}$$

There exists $\delta \in (0, \theta)$ small enough so that if $q \in [0, (1 - \theta)/2]$ and $U \in [1 - \delta, 1]$, then

$$f(U) \leq f(U - q).$$

Since $0 \leq q \leq q_0 < (1 - \theta)/2$, we have for $U \in [1 - \delta, 1]$,

$$N[u_l] \leq -\xi_1'(t)U_s - q_{1,t} + \nabla_x \cdot (a(x)\nabla_x q_1) + b(x) \cdot \nabla_x q_1. \tag{3.4.5}$$

If $U \in [0, \delta]$, $f(U) = f(U - q_1) = 0$, so (3.4.5) holds with an equality sign. If $U \in (\delta, 1-\delta)$, then there exists $\beta > 0$ such that $U_s \geq \beta$ and $|f(U) - f(U - q_1)| \leq Kq_1$ for some $K > 0$. It follows that

$$N[u_l] \leq -\xi_1'\beta - q_{1,t} + \nabla_x \cdot (a(x)\nabla_x q_1) + b(x) \cdot \nabla_x q_1 + Kq_1. \tag{3.4.6}$$

Let us choose q_1 to satisfy the equation

$$q_{1,t} = \nabla_x \cdot (a(x)\nabla_x q_1) + b(x) \cdot \nabla_x q_1, \quad q_1|_{t=0} = q_0(k \cdot x). \tag{3.4.7}$$

To make u_l a subsolution, we just need to impose the condition

$$-\xi_1'\beta + Kq_1 \le 0, \text{ or } -\xi_1'\beta + K||q_1||_{L^\infty(\mathbb{R}^n)} = 0,$$

or

$$\xi_1' = \frac{K||q_1||_{L^\infty(\mathbb{R}^n)}}{\beta} > 0, \tag{3.4.8}$$

with $\xi_1(0) = \xi_0 > 0$. By our early comments on the fundamental solution of (3.4.7), $||q_1||_{L^\infty} = o(1)$ as $t \to \infty$. Therefore $\xi_1(t) = o(t)$. We have shown that u_l is a subsolution, and a supersolution can be constructed in a similar way. Combining these two solutions, we have (I).

For the type 3 (bistable) nonlinearity, fronts may be blocked (quenched) by the existence of steady-state solutions of (3.2.6). Let us look at the one-dimensional case. We are concerned with the solutions of the following boundary value problem (assuming $\mu \in (0, \frac{1}{2})$):

$$(a(x)u_x)_x + f(u) = 0, \quad u(-\infty) = 0, \quad u(+\infty) = 1. \tag{3.4.9}$$

If $a \equiv 1$, then multiplying (3.4.9) by u_x and integrating over $\mathbb{R}$ yields $\int_0^1 f(u)du = 0$, a contradiction. For periodically varying a, we can multiply both sides of (3.4.9) by $a(x)u_x$ and integrate to get $\int_{\mathbb{R}} a(x)u_x f(u)du = 0$. However, this need not be a contradiction. Actually, there is a competition between the closeness of the middle zero μ to $\frac{1}{2}$ and the variation of a from its mean value. The existence of a steady-state solution depends on the result of this competition. We refer to [161] for an analytical quenching example in a perturbative regime of (3.4.9), where the middle zero μ is near $\frac{1}{2}$. Front quenching examples in two space dimensions have been illustrated in Xin and Zhu [163] by numerically simulating (3.2.6). Existence of nontrivial steady states for variable R-D equations is known in cases of KPP–Fisher nonlinearity. Keller [96] showed this for the steady-state equation $Du_{xx} + s(x)u(1-u) = 0, x \in \mathbb{R}$, where $s(x)$ is the sign of x, with 0 and 1 far-field boundary conditions. He used this property to explain the polymorphism of a dialetic population resulting from the geological inhomogeneities of the habitats. The alleles are selectively favored (via the sign of S) in different places. The steady state formation for the inhomogeneous bistable case was studied in Pauwelussen [127] using a step function for the diffusion coefficient. For related works on the stability of steady states on bounded domains, see Fusco and Hale [71] and references therein.

3.5. Fast Oscillating and Slowly Varying Fronts. When the periodic media vary on a much faster scale than the width of the traveling fronts, one can upscale the faster small scale to obtain an effective front by solving a homogenization problem. Heinze [84] considered (3.2.6) with fast oscillating periodic coefficients,

$$u_t = \nabla_x \cdot (a(\epsilon^{-1}x)\nabla_x u) + \epsilon^{-1}b(\epsilon^{-1}x) \cdot \nabla_x u + f(u), \tag{3.5.1}$$

where $a(\cdot)$ and $b(\cdot)$ are the same as in (3.2.6) and $\epsilon \to 0$. Since b is divergence-free and has mean zero, there is a skew-symmetric 1-periodic matrix $B = B(x)$ such that

$$\nabla \cdot (B(\epsilon^{-1}x)\nabla u) = \epsilon^{-1}b(\epsilon^{-1}x)\nabla u.$$

If we write $A(\epsilon^{-1}x) = a(\epsilon^{-1}x) + B(\epsilon^{-1}x)$, a being symmetric and B being skew-symmetric, then (3.5.1) becomes

$$u_t = \nabla_x \cdot (A(\epsilon^{-1}x)\nabla_x u) + f(u). \tag{3.5.2}$$

Now let us look at traveling front solutions of the form $u = U(k \cdot x - c^\epsilon t, \epsilon^{-1}x)$, where c^ϵ is the wave speed, which depends on ϵ. If we let $s = \vec{k} \cdot x - c^\epsilon t$ and $y = \epsilon^{-1}x$, then $U = U^\epsilon = U(s, y, \epsilon)$ satisfies

$$\begin{aligned}(\vec{k}\partial_s + \epsilon^{-1}\nabla_y)(a(y)(\vec{k}\partial_s + \epsilon^{-1}\nabla_y)U^\epsilon) + b(y) \cdot (\vec{k}\partial_s + \epsilon^{-1}\nabla_y)U^\epsilon \\ + c^\epsilon U_s^\epsilon + f(U^\epsilon) = 0\end{aligned} \tag{3.5.3}$$

with $U^\epsilon(-\infty, y) = 0$, $U^\epsilon(+\infty, y) = 1$, and $U(s, \cdot)$ 1-periodic.

We are interested in the limit of traveling front solutions as $\epsilon \to 0$. Intuitively, we see from (3.5.2) that the oscillation wavelength is much smaller than the width of fronts determined by the magnitude of A and $f(u)$. This separation of scale means that the periodic medium can be regarded as a homogeneous medium obtained by averaging the linear part as in subsection 3.1. Let us recall the cell problem on the n-dimensional unit torus T^n,

$$\nabla \cdot (A(y)(\nabla\chi + \vec{k})) = 0, \tag{3.5.4}$$

where $\vec{k}$ is a given direction. Problem (3.5.4) has a unique smooth periodic solution χ up to an additive constant. Define the constant diffusion matrix

$$A^h\vec{k} = \langle A(\nabla\chi + \vec{k})\rangle. \tag{3.5.5}$$

It follows from (3.5.4) and (3.5.5) that

$$\begin{aligned}a^*(\vec{k}) \equiv \vec{k}^T A^h \vec{k} = \langle \vec{k}^T A(\nabla\chi + \vec{k})\rangle = \langle(\nabla\chi + \vec{k})^T A(\nabla\chi + \vec{k})\rangle \\ = \langle(\nabla\chi + \vec{k})^T a(\nabla\chi + \vec{k})\rangle \quad > 0.\end{aligned} \tag{3.5.6}$$

The main result of [84] is the following.

THEOREM 3.6. *Let f be of type* 5, *and let (U^ϵ, c^ϵ) be the unique solution of* (3.5.3) *satisfying the normalization condition* $\max_{y \in T^n} U^\epsilon(0, y) = \theta$. *Then as $\epsilon \to 0$, $c^\epsilon \to c < 0$, and u^ϵ converges to a function $u = u(s)$ weakly in $H^1(\mathbb{R} \times T^n)$ and strongly in $L^2(\mathbb{R} \times T^n)$. The pair (u, c) is the traveling wave solution of the homogenized problem $a^*(\vec{k})u'' + cu' + f(u) = 0$, $u(-\infty) = 0$, $u(+\infty) = 1$ subject to the normalization condition $u(0) = \theta$.*

A key ingredient of [84] is uniform upper and lower bounds on c^ϵ, independent of ϵ. For nonlinearities of types 1, 2, and 4, the wave speeds form a continuum and there is no upper bound. However, a similar result may hold for the wave with minimal speed. For real numbers of type 3, due to a possible quenching mechanism, there is no lower bound in general, and so the result is only possible under subtle conditions on a and b. It is also interesting to investigate the case when the mean drift $\langle b\rangle$ is not zero.

Let us turn to fronts in slowly varying media by considering the equation parametrized by a small parameter $\epsilon \to 0$,

$$u_t^\epsilon = \nabla \cdot (a(\epsilon x)\nabla u^\epsilon) + f(u^\epsilon), \quad x \in \mathbb{R}^n, \tag{3.5.7}$$

where $a(\cdot)$ is a twice continuously differentiable and positive-definite matrix and $f(u)$ is bistable (type 3). To feel the effects of slowly varying media, we have to look at a long time scale on the order $O(\epsilon^{-1})$. Let us introduce this scaling, replacing x by $\epsilon^{-1}x$ and t by $\epsilon^{-1}t$. The equation (3.5.7) then reads

$$u_t^\epsilon = \epsilon\nabla\cdot(a(x)\nabla u^\epsilon) + \epsilon^{-1}f(u^\epsilon). \tag{3.5.8}$$

Let the initial data $u^\epsilon(0,x) = g(x)$ be a compactly supported continuous function such that

$$G_0 = \{x : g(x) < \mu\}, \quad G_1 = \{x : g(x) > \mu\} \tag{3.5.9}$$

are nonempty open sets, where μ is the middle zero of $f(u)$. Since the front width is of order $O(\epsilon)$ and is much smaller than the scale of variation of $a(x)$, the effect of $a(x)$ is to slowly modulate the front speed, and the effect can be analyzed by freezing $a(x)$ as a constant matrix. Due to small diffusion of order $O(\epsilon)$ and fast reaction $O(\epsilon^{-1})$, in a short time the reaction term dominates, and u^ϵ quickly develops into transition layers or fronts connecting two stable states 0 and 1. Subsequently, the fronts move outward, that is, G_1 grows into G_0.

It is helpful to figure out the front speeds when a is a constant matrix, and the solutions behave like outward spreading fronts starting at the boundary of the support of g. If a is the identity, then for x along any direction $\vec{k}$, u^ϵ is to leading order approximated by

$$U\left(\frac{\vec{k}\cdot x - c_0|\vec{k}|t}{\epsilon}\right), \tag{3.5.10}$$

where (U, c_0) is a traveling wave solution satisfying the ODE

$$U'' + c_0U' + f(U) = 0, \;\; U(-\infty) = 1, \;\; U(+\infty) = 0, \tag{3.5.11}$$

with $U' < 0$, $c_0 > 0$. The sign of c_0 is the same as that of $\int_0^1 f(u)du$. Now for general positive-definite a, we make the change of variable $x' = a^{-1/2}x$. In terms of (x', t), the diffusion matrix becomes the identity. If x is along a vector $\vec{k}$, then x' is along $a^{-1/2}\vec{k}$, so by (3.5.10),

$$\begin{aligned} u^\epsilon &\sim U\left(\frac{\vec{k}^T a^{-1/2}x' - c_0|a^{-1/2}\vec{k}|t}{\epsilon}\right) + \cdots \\ &= U\left(\frac{\vec{k}^T a^{-1}x - c_0|a^{-1/2}\vec{k}|t}{\epsilon}\right) + \cdots . \end{aligned} \tag{3.5.12}$$

Letting $x = s\vec{k}$, we see at once that the wave speed along direction $\vec{k}$ is

$$c(\vec{k}) = \frac{c_0|a^{-1/2}\vec{k}|}{\vec{k}^T a^{-1}\vec{k}} = c_0(\vec{k}^T a^{-1}\vec{k})^{-1/2}. \tag{3.5.13}$$

Hence for slowly varying fronts, the wave speed in the direction of $\vec{k}$ at point x is given by

$$c(\vec{k}, x) = c_0(\vec{k}^T a^{-1}(x)\vec{k})^{-1/2}. \tag{3.5.14}$$

The above heuristic argument has been rigorously validated by Gärtner [76]. In fact, the reaction term in [76] can be a general Hölder continuous function $f(u, x)$ with two stable equilibria $\mu_1(x)$, $\mu_2(x)$ and one unstable equilibrium $\mu(x)$. The wave speed

$c_0 = c_0(x)$ is given by replacing $f(U)$ by $f(U, x)$ in (3.5.11), and so $c_0(x)$ may change sign. Formula (3.5.14) is used to define distances in [76]. For ease of presentation, we consider only $f = f(u)$ and $\mu_0 = 0$, $\mu_1 = 1$. We introduce the distance function for two points x, $y \in \mathbb{R}^n$,

$$\rho(x,y) \equiv \inf \int_{t_1}^{t_2} (c^*)^{-1} (\varphi_t^T \cdot a^{-1}(\varphi(t))\varphi_t)^{1/2} dt, \tag{3.5.15}$$

where the infimum is taken over all $t_1 < t_2$ and all absolutely continuous curves $\varphi(t) : [t_1, t_2] \to \mathbb{R}^n$ joining x and y. Note that the speed formula (3.5.14) is built into the definition of ρ. For any open set $G \subseteq \mathbb{R}^n$, define the sets

$$Q_0(G) = \{(x,t) \in \mathbb{R}^n \times (0, +\infty) : \rho(x, G^c) > t\}$$

and

$$Q_1(G) = \{(x,t) \in \mathbb{R}^n \times (0, +\infty) : \rho(x, G) < t\}.$$

Here is a particular but illuminating version of the main result in [76].

THEOREM 3.7. *Let u^ϵ be a classical solution of the initial value problem of* (3.5.8) *with initial data $g(x)$. Then u^ϵ converges to* 0 *uniformly on each compact subset of $Q_0(G_0)$ and to* 1 *uniformly on each compact subset of $Q_1(G_1)$.*

This theorem holds when $g(x)$ is a bounded continuous function on $\mathbb{R}^n$. Similar results hold for the KPP (type 1) nonlinearity [65], [66], [67] if c^0 above is replaced by the minimal speed and $f_u(0, x) =$ constant. If the latter condition fails, then the motion of the KPP front can have jumps or non-Markovian (history-dependent) behavior. When this happens, one cannot just freeze coefficients to find the wave speed. New sources of propagation may arise spontaneously, say, at some distance ahead of the existing front. Such a phenomenon may be attributed to the facts that the KPP traveling front with minimal speed is not as stable as the bistable front and that there exists a continuum of wave speeds. We will describe more KPP fronts with minimal speeds in the next subsection.

3.6. KPP Fronts in Slowly Varying Media. It is well known that for the KPP equation,

$$u_t = \frac{1}{2} a u_{xx} + f(u), \quad a > 0, \ \ x \in \mathbb{R},$$

if the initial condition is compactly supported or is the indicator function of the negative axis, the front solution propagates at the minimum speed, equal to $\sqrt{2af'(0)}$ for large times [100], [1], [67]. The KPP minimal speed has a linear feature in that it depends only on the derivative of f at zero, its unstable equilibrium point. This fact makes it possible to find the wave speed without knowing the wave profile under the large-time and large-space hyperbolic scaling, $t \to \epsilon^{-1} t$, $x \to \epsilon^{-1} x$. The rescaled equation reads

$$u_t = \frac{\epsilon}{2} a u_{xx} + \epsilon^{-1} f(u), \tag{3.6.1}$$

and the problem is to examine the solutions in the limit $\epsilon \to 0$. Similarly, fronts in slowly varying media can be considered as in (3.5.8).

Freidlin [65], [66] first treated the KPP minimal speed for slowly varying media using a probabilistic approach. In fact, a more general case than (3.5.8) is analyzed in [66], where f can be a function of (u, x), and a first-order advection term $b(x) \cdot \nabla u$

can be added as well. For simplicity, let us set $b = 0$ and write the equation in the form

$$u_t^\epsilon = L^\epsilon u^\epsilon + \epsilon^{-1} f(x,u), \tag{3.6.2}$$

where $L^\epsilon = \frac{\epsilon}{2}\nabla \cdot (a(x)\nabla u^\epsilon)$ and f is KPP for each x: $f(x,u) > 0$ for $u \in (0,1)$, $f(u,x) < 0$ for $u < 0$ and $u > 1$, and $f_u(x,0) = \sup_{0<u\leq 1} u^{-1} f(u,x)$. The initial data $u(x,0) = g(x)$ is nonnegative, piecewise continuous, and bounded above by 1. Let G_0 be the support of g, and $\overline{G_0}$ its closure. The operator L^ϵ generates a diffusion process denoted by (X_t^ϵ, P_x) [67], [150]. The trajectory of this process X_t^ϵ can be defined as a solution of the stochastic differential equation

$$dX_t^\epsilon = \sqrt{\epsilon}\sigma(X_t^\epsilon)dW_t + \tilde{b}(X_t^\epsilon)dt, \quad X_0^\epsilon = x, \tag{3.6.3}$$

where $\sigma(x)\sigma^T(x) = a(x)$, W_t is the Wiener process, and $\tilde{b} = \epsilon\nabla \cdot a$.

Let $c(x,u) = u^{-1} f(x,u)$. The Feynman–Kac formula then gives an implicit representation of the solution [67], [150]:

$$u^\epsilon(x,t) = E_x g(X_t^\epsilon) \exp\left(\epsilon^{-1}\int_0^t c(X_s^\epsilon, u(t-s, X_s^\epsilon))ds\right). \tag{3.6.4}$$

Since $0 < u^\epsilon \leq 1$, it follows from the KPP assumption that

$$u^\epsilon(x,t) \leq E_x g(X_t^\epsilon) \exp\left(\epsilon^{-1}\int_0^t c(X_s^\epsilon, 0)ds\right). \tag{3.6.5}$$

The expectation in (3.6.5) is well studied in the theory of large deviations (Freidlin and Wentzell [70], Varadhan [151]), and its logarithmic limit is

$$\begin{aligned} &\lim_{\epsilon\to 0} \epsilon \log E_x g(X_t^\epsilon) \exp\left(\epsilon^{-1}\int_0^t c(X_s^\epsilon, 0)ds\right) \\ &= \sup\left\{\int_0^t c(\varphi_s)ds - S_{0t}(\varphi) : \varphi_0 = x, \varphi_t \in \overline{G_0}\right\} \equiv V, \end{aligned} \tag{3.6.6}$$

where $S_{0t}(\varphi)$ is called the *action functional* for the processes (X_t^ϵ, P_x) as $\epsilon \to 0$ and is defined as

$$S_{0t}(\varphi) = \frac{1}{2}\int_0^t \dot{\varphi}_s^T a(\varphi_s)\dot{\varphi}_s ds \tag{3.6.7}$$

for absolutely continuous functions $\varphi_s : [0,t] \to \mathbb{R}^n$. It follows from (3.6.5) and (3.6.6) that

$$\lim_{\epsilon\to 0} u^\epsilon(x,t) = 0 \;\; \forall\, (x,t) \in N \equiv \{(x,t) : V(x,t) < 0\}. \tag{3.6.8}$$

The function $V(x,t)$ is continuous, and the convergence is uniform on compact subsets. To show that u^ϵ converges to 1 on any compact subset of $P = \{(x,t) : V(x,t) > 0\}$, an additional assumption (N) is needed that $V(x,t)$ is equal to

$$\sup\left\{\int_0^t c(\varphi_s)ds - S_{0t}(\varphi) : \varphi_0 = x, \varphi_t \in \overline{G_0}, (t-s, \varphi_s) \in N, s \in (0,t)\right\}$$

for any $t > 0$, $(x,t) \in \partial N$. The assumption (N) asserts that $V(x,t)$ is the supremum of the functional above over the paths where u^ϵ is nearly zero, implying that the upper bound in (3.6.5) is quite a good approximation of u^ϵ. The main result in [66] is given as the following theorem.

THEOREM 3.8. *Suppose that f is of KPP type and condition* (N) *holds. Then*

$$\lim_{\epsilon\to 0} u^\epsilon(x,t) = \begin{cases} 1, & V(x,t) > 0, \\ 0, & V(x,t) < 0, \end{cases}$$

and the convergence is uniform on compact subsets. Assumption (N) *holds in particular when $f = f(u)$.*

When $f = f(u)$, the function $V(x,t)$ is explicit:

$$V(x,t) = ct - \frac{d^2(x,G_0)}{2t},$$

where $c = f'(0)$ and $d(x,G_0)$ is the Riemannian distance in terms of the metric $ds^2 = \sum_{i,j=1}^n a_{ij}(x)dx_i dx_j$.

When f depends on x, even if (N) holds, the front speed may depend on the initial data and the front location may have jumps at certain times, violating the conventional picture (Huygens's principle) where an interface separates the reacted ($u^\epsilon \approx 1$) and unreacted ($u^\epsilon \approx 0$) regions and propagates continuously at the initial data-independent minimum speed. These interesting phenomena are related to how fast $c(x) = f_u(x,0)$ increases in x and are illustrated in [66] by one-dimensional examples (see Examples 2 and 3 in [66]), with the initial data taken as the indicator function of the left half-line.

The large deviation method of Freidlin motivates the ansatz

$$u^\epsilon \sim \exp e^{-\epsilon^{-1} I(x,t)} \tag{3.6.9}$$

and the later PDE approach developed by Evans and Souganidis [51], [52], [53]. The PDE approach is based on the logarithmic change of variable $v^\epsilon = -\epsilon \ln u^\epsilon$, due to Fleming [63], [52]. Let us look at a special case $f(x,u) = c(x)u - bu^2$, where b is a positive constant and $c(x)$ is positive, bounded, and Lipschitz continuous. For simplicity, let us further take $a = (a_{ij})$ to be a constant positive-definite matrix. Assume also that the initial condition $g(x)$ is continuous and compactly supported in $\overline{G_0}$. The function v^ϵ then satisfies the equation

$$v_t^\epsilon = \frac{\epsilon}{2} a_{ij} v^\epsilon_{x_i x_j} - \frac{1}{2} a_{ij} v^\epsilon_{x_i} v^\epsilon_{x_j} + b \exp\left\{-\frac{v^\epsilon}{\epsilon}\right\} - c,$$

$$v^\epsilon(x,0) = -\epsilon \ln g,\ x \in G_0,\ \ v^\epsilon(x,0) \to +\infty,\ \ t \to 0,\ \ x \in G_0^c. \tag{3.6.10}$$

The next step is to pass to the limit $\epsilon \to 0$ for v^ϵ. Comparison functions and maximum principles imply that the supremum norm and the Hölder norms (with exponent $\alpha \in (0,1)$) of v^ϵ are bounded in any space-time compact set. Hence v^ϵ has a uniformly convergent subsequence with limiting function v.

The function v is a viscosity solution of the variational inequality

$$\min\left[v_t + \frac{1}{2} a_{ij} v_{x_i} v_{x_j} + c, v\right] = 0, \quad x \in \mathbb{R}^n,\ t > 0. \tag{3.6.11}$$

This can be understood as follows. Fix any $T > 0$. If $v \geq 0$, then

$$v_t + \frac{1}{2} a_{ij} v_{x_i} v_{x_j} + c \geq 0,\ \ (x,t) \in \mathbb{R}^n \times (0,T] \tag{3.6.12}$$

in the viscosity sense, and on the set $\{v > 0\} \cap \mathbb{R}^n \times (0, T]$,

$$v_t + \frac{1}{2} a_{ij} v_{x_i} v_{x_j} + c = 0 \tag{3.6.13}$$

holds in the viscosity sense. The viscosity sense in (3.6.12) means that for each smooth function φ, if $u - \varphi$ has a local minimum at $(x_0, t_0) \in \mathbb{R}^n \times (0, T]$, then

$$\varphi_t(x_0, t_0) + \frac{1}{2} a_{ij} \varphi_{x_i} \varphi_{x_j}(x_0, t_0) + c \geq 0. \tag{3.6.14}$$

In (3.6.13), we have in addition that if $v - \varphi$ has a local maximum at $(x_1, t_1) \in \mathbb{R}^n \times (0, T]$ and if $v(x_1, t_1) > 0$, then

$$\varphi_t(x_1, t_1) + \frac{1}{2} a_{ij} \varphi_{x_i} \varphi_{x_j}(x_1, t_1) + c \ \leq \ 0. \tag{3.6.15}$$

We refer to Crandall, Evans, Lions, and Ishii [39], [38] for more details on the properties and applications of viscosity solutions.

To see (3.6.11), we know that $v \geq 0$ by the maximum principle. Equation (3.6.10) implies the inequality

$$v_t^\epsilon - \frac{\epsilon}{2} a_{ij} v_{x_i x_j}^\epsilon + \frac{1}{2} a_{ij} v_{x_i}^\epsilon v_{x_j}^\epsilon + c \ \geq \ 0,$$

which yields (3.6.12) as $\epsilon \to 0$ in the viscosity sense. Also, on any compact subset of $\{v > 0\}$, $b \exp\{-\epsilon^{-1} v^\epsilon\} \to 0$, implying (3.6.13) in the viscosity sense. Exploiting more properties of viscosity solutions and available bounds shows that v is Lipschitz in (x, t) and $v = 0$ on $G_0 \times \{t = 0\}$.

The variational inequality (3.6.11) with initial condition $v = 0$ on G_0, $v = +\infty$ on G_0^c has a nice interpretation in terms of the value function of a two-player, zero-sum differential game with stopping times; see Fleming and Soner [63] as well as [53]. The conclusion is that v is equal to

$$I(x, t) \ \equiv \ \sup_\theta \inf_\varphi \left\{ -\int_0^\tau c(\varphi_s) ds + S_{0\tau}(\varphi) : \varphi_0 = x, \ \ \varphi_t \in G_0 \right\}, \tag{3.6.16}$$

where $\tau = \min(t, \theta[\varphi])$ and θ is a stopping time. It follows from (3.6.16) that $I \geq 0$. Comparing (3.6.16) with (3.6.6), we see that except for the stopping time θ, I is almost $-V$. The condition (N) implies $I = \max(-V, 0)$. Hence the theorem of Freidlin above can be rephrased as $\lim_{\epsilon \to 0} u^\epsilon = 0$ uniformly on compact subsets of $I > 0$ and $\lim_{\epsilon \to 0} u^\epsilon = 1$ uniformly on compact subsets of the interior points of $I = 0$. However, the PDE approach of [51] yields this result without resorting to condition (N). In other words, the function $I(x, t)$ characterizes the limit of u^ϵ under more general circumstances. This is an advantage of the PDE approach. The large deviation approach, on the other hand, is more refined and can identify delicate front phenomena.

3.7. KPP Fronts in Periodic Media and Homogenization of H-J Equations. Let us turn to KPP fronts with minimal speed in periodic media by considering the equation

$$u_t = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) u_{x_i, x_j} + \sum_{i=1}^n b_i(x) u_{x_i} + f(x, u) \tag{3.7.1}$$

with KPP nonlinearity $f(x,u)$ and initial data $g(x)$ of compact support G_0. Gärtner and Freidlin [75], [67] solved this problem using the large deviation method and a path integral representation of solutions. Again, the nonlinearity $f(x,u)$ can be approximated by $c(x) = f_u(x,0)$ times u, so that the implicit solution formula becomes explicit, and the method of large deviations yields the long-time front speed. Here is their result.

THEOREM 3.9. *Let $z \in \mathbb{R}^n$. Define the operator*

$$L_z = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y)(\partial_{y_i} - z_i)(\partial_{y_j} - z_j) + \sum_{i=1}^{n} b_i(\partial_{y_i} - z_i) + c(y) \tag{3.7.2}$$

on 1*-periodic functions in $y \in T^n$, the n-dimensional unit torus. Let $\lambda = \lambda(z)$ be the principal eigenvalue of L_z, which can be shown to be convex and differentiable in z. Let $H(y)$ be the Legendre transform of λ,*

$$H(y) = \sup_{z\in\mathbb{R}^n}[(y,z) - \lambda(z)],$$

$y \in \mathbb{R}^n$. The function $H(y)$ is also convex and differentiable. Then for any closed $F \subseteq \{y : H(y) > 0\}$, $\lim_{t\to\infty} u(t,ty) = 0$ uniformly for $y \in F$, and for any compact $K \subseteq \{y : H(y) < 0\}$, $\lim_{t\to\infty} u(t,ty) = 1$ uniformly for $y \in K$.

It follows that the asymptotic front speed $v = v(e)$ along the unit direction e satisfies $H(ve) = 0$. If $\min_{\mathbb{R}^n} \lambda(z) > 0$, the H equation can be solved to yield

$$v = v(e) = \inf_{(e,z)>0} \frac{\lambda(z)}{(e,z)}. \tag{3.7.3}$$

In fact, $\lambda(z)$ grows quadratically in z, and so the supremum in the definition of $H(y)$ is achieved. There exists z^* such that

$$0 = H(ve) = v(e,z^*) - \lambda(z^*),$$

so $v(e,z^*) > 0$, due to $\lambda(z^*) > 0$. It follows from the inequality $\lambda(z)(v(e,z^*) - \lambda(z^*)) = 0 \geq \lambda(z^*)(v(e,z) - \lambda(z))$ that for any z satisfying $(e,z) > 0$,

$$\frac{\lambda(z)}{(e,z)} \geq \frac{\lambda(z^*)}{(e,z^*)}.$$

This implies formula (3.7.3). The assumption $\min_{\mathbb{R}^n} \lambda(z) > 0$ holds if the operator L is self-adjoint or of the form $L = \nabla\cdot(a(x)\nabla) + b(x)\cdot\nabla$, where b is an incompressible velocity field of mean zero.

Instead of going through the large deviation method, let us follow the spirit of the logarithmic transform in the PDE approach and derive the same result. First consider (3.7.1) under the scaling $x \to \epsilon^{-1}x$, $t \to \epsilon^{-1}t$. The rescaled equation reads

$$u^\epsilon_t = \frac{1}{2}\epsilon\sum_{i,j=1}^{n} a_{ij}(\epsilon^{-1}x)u^\epsilon_{x_i,x_j} + \sum_{i=1}^{n} b_i(\epsilon^{-1}x)u^\epsilon_{x_i} + \epsilon^{-1}f(\epsilon^{-1}x, u^\epsilon), \tag{3.7.4}$$

for which we make the change of variable

$$u^\epsilon = \exp(\epsilon^{-1}v^\epsilon). \tag{3.7.5}$$

Then v^ϵ satisfies the equation

$$\begin{aligned} v^\epsilon_t = \frac{\epsilon}{2}\sum_{i,j=1}^{n} a_{ij}(\epsilon^{-1}x)v^\epsilon_{x_i,x_j} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(\epsilon^{-1}x)v^\epsilon_{x_i}v^\epsilon_{x_j} \\ + \sum_{i=1}^{n} b_i(\epsilon^{-1}x)v^\epsilon_{x_i} + \frac{f(\epsilon^{-1}x, u^\epsilon)}{u^\epsilon}. \end{aligned} \tag{3.7.6}$$

The last term is bounded from above by $c(\epsilon^{-1}x) = f_u(\epsilon^{-1}x, 0)$, which also happens to be the right approximation of the nonlinearity for small values of u^ϵ. For locating the front or the region where u^ϵ is near zero, one can replace the nonlinear term by its linearization at $u^\epsilon = 0$ as we approach the front from the interior where $v^\epsilon < 0$. Then (3.7.6) looks like a homogenization problem of a viscous H-J equation, which we recall next.

The homogenization of the inviscid H-J equation was first studied by Lions, Papanicolaou, and Varadhan in [105]. They considered solutions v^ϵ of

$$v_t^\epsilon + H(\nabla v^\epsilon, \epsilon^{-1}x) = 0, \quad x \in \mathbb{R}^n \times (0, +\infty), \tag{3.7.7}$$

with initial data $v^\epsilon(x, 0) = v_0$, where H is periodic in the second variable, say, with period 1. Under the conditions that H is locally Lipschitz in all variables, $H(p, x) \to +\infty$ as $|p| \to +\infty$ uniformly in $x \in \mathbb{R}^n$, u_0 is bounded and uniformly continuous, and $\nabla v_0 \in L^\infty(\mathbb{R}^n)$, they showed that v^ϵ converges uniformly on compact sets to the viscosity solution v of the homogenized H-J equation

$$v_t + \overline{H}(\nabla v) = 0, \quad x \in \mathbb{R}^n \times (0, +\infty), \tag{3.7.8}$$

where the homogenized Hamiltonian is defined through solving the cell problem. Here is their result.

THEOREM 3.10. *For each $p \in \mathbb{R}^n$, there exists a unique real number $\overline{H}(p)$ such that the equation $H(p + \nabla w, y) = \overline{H}(p)$ has a 1-periodic solution $w = w(y)$.*

For a proof, see [105] and Evans [50]. The homogenized Hamiltonian $\overline{H}$ is convex if H is in p, but it may lose strict convexity. One example in [105] is that the $\overline{H}$ of the strictly convex function $H(p, x) = \frac{p^2}{2} + V(x)$, $x \in \mathbb{R}$, is flat near $p = 0$. In fact, let us assume $V \le 0$ and $\max V = 0$ for convenience. The cell problem reads

$$\frac{1}{2}(p + w_y)^2 + V(y) = \bar{H}, \quad y \in T,$$

which is completely solvable and gives $\bar{H} \ge 0$ such that

$$\begin{aligned} &\bar{H} = 0 \ \text{ if } |p| \le \langle \sqrt{-2V} \rangle\,; \\ &|p| = \langle \sqrt{2\bar{H} - 2V(y)} \rangle \ \text{ if } |p| > \langle \sqrt{-2V} \rangle, \end{aligned} \tag{3.7.9}$$

where $\langle \cdot \rangle$ denotes the average over one period.

The above cell problem can be derived using the ansatz

$$v^\epsilon \sim v_0(x, t) + \epsilon v_1(x, \epsilon^{-1}x, t) + \cdots. \tag{3.7.10}$$

The same ansatz is also utilized in the convergence proof of [105]. For generalizations to fully nonlinear first- and second-order equations, see [50], where a weak convergence method called the perturbed test function method is employed. Such a method incorporates the above ansatz in the structures of the test functions instead and can handle equations of first and second order in a unified way.

Now we go back to (3.7.6) with $c(\epsilon^{-1}x)$ in place of the last nonlinear term. Using the homogenization ansatz (3.7.10), it is straightforward to derive the cell problem

$$\begin{aligned} \overline{H} \; = \; & \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y) w_{y_i y_j} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y)(p_i + w_{y_i})(p_j + w_{y_j}) \\ & + \sum_{i=1}^{n} b_i(p_i + w_{y_i}) + c(y), \end{aligned} \tag{3.7.11}$$

where we solve for a periodic function w and a real constant $\overline{H}$ for given p. The homogenized equation is $v_t - \overline{H}(\nabla v) = 0$. The cell problem (3.7.11) can be transformed into an eigenvalue problem with $\overline{H}$ being the principal eigenvalue. To see this, let $\bar{w} = e^w > 0$. Then (3.7.11) in terms of $\bar{w}$ reads

$$\begin{aligned} \overline{H}\bar{w} &= \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\bar{w}_{y_i y_j} + \sum_{i,j=1}^{n} a_{ij} p_i \bar{w}_{y_j} \\ &+ \sum_{i,j=1}^{n} b_i (p_i \bar{w} + \bar{w}_{y_i}) + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} p_i p_j \bar{w} + c(y)\bar{w}. \end{aligned} \tag{3.7.12}$$

The right-hand side operator in (3.7.12) is just L_{-p}, in view of (3.7.2). Hence $\overline{H}(-z) = \lambda(z)$. To derive the front speed formula (3.7.3), consider the H-J equation

$$v_t - \overline{H}(\nabla v) = 0,$$

with initial condition $v_0(x) = 0$ if $x \in G_0$, $v_0(x) = -\infty$ otherwise. The variational Hopf formula (see Evans [48]) of the solution is

$$v(x,t) = -\inf_{y \in G_0} \overline{H}^* \left(\frac{y-x}{t} \right), \tag{3.7.13}$$

where $\overline{H}^*$ is the Legendre transform of $\overline{H}$. The function $H(y)$ in the large deviation approach is related to $\overline{H}^*$ by

$$H(y) = \sup_{-z \in \mathbb{R}^n} [(y,-z) - \lambda(-z)] = \sup_{z \in \mathbb{R}^n} [(-y,z) - \overline{H}(z)] = \overline{H}^*(-y).$$

The points (x,t) where $v < 0$ or $\lim_{\epsilon \to 0} u^\epsilon = 0$ then satisfy

$$\overline{H}^* \left(\frac{y-x}{t} \right) > 0 \quad \forall\, y \in G_0.$$

Since G_0 is compact, we can take both x and t large compared with the size of G_0. Then we drop y to get the condition

$$\overline{H}^* \left(-\frac{x}{t} \right) = H\left(\frac{x}{t} \right) > 0,$$

implying that the front speed $v(e)$ along direction e satisfies $H(v(e)e) = 0$.

Now we can also derive the earlier variational formulas (3.2.9), (3.3.9), and (3.3.13). Except for the factor $\frac{1}{2}$ (due to the $\frac{1}{2}$ in (3.7.1)) we see that (3.3.10) and (3.7.12) are the same (r becomes p). Hence the Λ of (3.3.10) is identical to the effective Hamiltonian $\bar{H}$, or $\Lambda(r) = \lambda(-r)$. Since the front described by (3.2.9) moves to the left, (3.7.3) yields (with $e = -1$)

$$c^* = \inf_{z<0} \frac{\lambda(z)}{-z} = \inf_{r>0} \frac{\lambda(-r)}{r} = \inf_{r>0} \frac{\Lambda(r)}{r},$$

which is just (3.3.13)!

Finally, putting the homogenization ansatz (3.7.10) into (3.7.5) shows that for KPP fronts in periodic media, the solution u^ϵ behaves like

$$\begin{aligned} u^\epsilon(t,x) &= e^{-\epsilon^{-1} I(t,x,\epsilon)} + \cdots, \\ I(t,x,\epsilon) &= I_0(t,x) + \epsilon I_1(t,x,\epsilon^{-1}x) + \cdots, \end{aligned} \tag{3.7.14}$$

where I can be regarded as a phase function as in a geometric optics (Wentzel–Kramers–Brillouin (WKB)) ansatz. However, for fronts of type 3 and type 5, the ansatz for u^ϵ in the same scaling ($x \to \epsilon^{-1}x$, $t \to \epsilon^{-1}t$) is

$$u^\epsilon(t,x) = U(\epsilon^{-1}\varphi(t,x,\epsilon), \epsilon^{-1}x) + \cdots,$$
$$\varphi(t,x,\epsilon) = \varphi_0(t,x) + \epsilon\varphi_1(t,x) + \cdots, \tag{3.7.15}$$

where $\varphi(t,x,\epsilon)$ is the phase variable. Plugging (3.7.15) into (3.7.1), we have

$$\frac{1}{2}(\nabla_x\varphi_0\partial_s + \nabla_y)(a(y)(\nabla_x\varphi_0\partial_s + \nabla_y)U) + b(y)\cdot(\nabla_x\varphi_0\partial_s + \nabla_y)U$$
$$- \varphi_{0,t}U_s + f(U) = 0, \tag{3.7.16}$$

where $U = U(s,y)$, $s = \frac{\varphi(t,x,\epsilon)}{\epsilon}$, $y = \frac{x}{\epsilon}$. We see that (3.7.16) is just the traveling front equation (3.2.7) with $k = \nabla_x\varphi_0$, and $c(k) = -\varphi_{0,t}$. Relating them gives the H-J equation,

$$\varphi_{0,t} + c(\nabla_x\varphi_0) = 0, \tag{3.7.17}$$

for the general front evolution. It is easy to see from (3.2.7) that $c = c(\vec{k})$ is homogeneous of degree 1 in $\vec{k}$.

3.8. KPP Fronts in Media of Separated Scales. Combining the above results on slowly and fast varying media, one can inquire about the effective fronts when the coefficients depend on both x and $\epsilon^{-1}x$. In the study of premixed flames in convecting turbulent velocity fields, one is concerned with front propagation in media with several spatial-temporal scales. When these scales are separate, the effective front can be studied using the KPP methodology discussed in the last two sections.

Majda and Souganidis [106] considered the KPP equation for the temperature field of a reacting passive scalar,

$$T^\epsilon_t + V(x,t,\epsilon^{-\alpha}x,\epsilon^{-\alpha}t)\cdot\nabla T^\epsilon \;=\; \epsilon\kappa\Delta T^\epsilon + \epsilon^{-1}f(T^\epsilon), \tag{3.8.1}$$

with compactly supported (in G_0) nonnegative initial data, a nonlinearity f of KPP type, and $\alpha \in (0,1]$. The velocity V is bounded and Lipschitz continuous and has periodic dependence on the fast oscillating scales $y \equiv \epsilon^{-\alpha}x, \tau \equiv \epsilon^{-\alpha}t$. The small parameter ϵ measures the ratio of the front thickness and large scale (dependence on (x,t)) of the velocity field, say of $O(1)$. The effective Hamiltonian $H(p,x,t)$ is defined as a solution of the following cell problem: for each $(p,x,t) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,+\infty)$ there is a unique constant $H(p,x,t)$ and a $w(y,\tau) \in C^{0,1}(\mathbb{R}^n \times (0,+\infty))$ periodic in both (y,τ) such that

$$w_\tau - a(\alpha)\kappa\Delta w - \kappa|p+\nabla w|^2 + V(x,t,y,\tau)\cdot(p+\nabla w) = -H(p,x,t), \tag{3.8.2}$$

where $a(\alpha) = 0$, if $\alpha \in (0,1)$, $a(\alpha = 1) = 1$. The case $\alpha = 1$ can be derived using an exponential change of variable and an H-J equation as in the last subsection, except that due to the time dependence, the w_τ term is added. The condition $a(\alpha) = 0$ in the case $\alpha \in (0,1)$ can be understood as follows. Ignore the slow variable (x,t) for now and change the scaling to $x = \epsilon^{-1+\alpha}x'$. Then the velocity V is $V(\epsilon^{-1}x', \epsilon^{-1}t')$ and the diffusion coefficient becomes $\epsilon^{3-2\alpha}\kappa \ll \epsilon\kappa$. Hence the diffusion term is too small to be seen at the order of the cell problem.

The function H is locally Lipschitz continuous, convex in p, and grows quadratically in $|p|$ as $|p| \to \infty$ uniformly in (x,t). The asymptotics of T^ϵ as $\epsilon \to 0$ is given by the following theorem.

THEOREM 3.11. *Let T^ϵ be a solution of* (3.8.1) *under the above assumptions. Then as $\epsilon \to 0$, $T^\epsilon \to 0$ locally uniformly in $\{(x,t) : Z < 0\}$ and $T^\epsilon \to 1$ locally uniformly in the interior of $\{(x,t) : Z = 0\}$, where $Z \in C(\mathbb{R}^n \times [0,+\infty))$ is the unique viscosity solution of the variational inequality*

$$\max(Z_t - H(\nabla Z, x, t) - f'(0), Z) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,+\infty),$$

with initial data $Z(x,0) = 0$ in G_0 and $Z(x,0) = -\infty$ otherwise. The set $\Gamma_t = \partial\{x \in \mathbb{R}^n : Z(x,t) < 0\}$ can be regarded as a front.

The authors of [106] further showed that Γ_t is a Lipschitz continuous surface and that when V is independent of (x,t) the resulting front evolves according to an H-J-type (geometric) PDE (Huygens's principle applies),

$$u_t = F(\nabla u), \tag{3.8.3}$$

with initial data $u(x,0) = g(x)$ for $x \in \mathbb{R}^n$, where $g(x)$ is uniformly continuous and positive on G_0 and negative on $\overline{G}_0^c$. The function $F = F(e)$ is the minimum speed of KPP fronts along the direction e, or the $v(e)$ in the last subsection. In general, Γ_t is shown to be bounded by surfaces defined by similar geometric PDEs.

A popular model H-J equation, the so-called G-equation, is often used in the combustion community for the study of premixed flames in turbulent media. It reads

$$G_t + V \cdot \nabla G = S_L |\nabla G|, \tag{3.8.4}$$

where S_L is the constant laminar speed, V is the incompressible fluid velocity, and the flame front is where $G = 0$. The G-equation is derived (see [97]) for combustion fronts of unit Lewis number under the assumption that the flame thickness l_F is much smaller than the integral length scale of the velocity field L_F and that thermal expansion due to heat release can be ignored. As a result, the velocity field V varies only on large integral scale. However, the G-equation has also been used for V involving several length scales much smaller than the integral scale for the purpose of averaging and for closure procedures regarding the small scales [97], [98], [128], [145], [165].

Embid, Majda, and Souganidis [46], [47] compared the results of complete averaging of the KPP equation and the averaging of the G-equation in the case of a two-scale shear velocity field of the form

$$V = \bar{\lambda}\bar{u} + \lambda(v(\epsilon^{-\alpha} x_2), 0). \tag{3.8.5}$$

Here $\bar{u} = (\cos\bar{\theta}, \sin\bar{\theta})$ and $v(x_2)$ is a 1-periodic function with unit amplitude representing a small-scale shear flow aligned with the x_1-axis. The parameter λ is the mean flow intensity, and $\bar{\theta}$ represents the angle between the mean flow and the shear. The small parameter is $\epsilon = l_F/L_F$ and $\alpha \in (0,1)$. Therefore, the velocity small scale is much smaller than the integral scale and much larger than the front thickness of order $O(\epsilon)$. The complete averaging is as we just described, and the averaging of the G-equation follows the homogenization theory of [105]. Since V is independent of the slow variables (x,t), the effective fronts can be described by the H-J equation (3.8.3). For the shear flow (3.8.5), both averagings can be calculated with exact formulas. The results of comparisons are as follows. For all values of the flow-field parameters, the enhanced flame speed predicted from the averaged G-equation always underestimates the enhanced flame speed computed from the KPP averaging. The significance of the error depends on the transverse magnitude of the mean flow $\bar{\lambda}\sin\bar{\theta}$. For $\bar{\lambda}\sin\bar{\theta} < 1$, there is excellent agreement between the two different averagings, while if $\bar{\lambda}\sin\bar{\theta} > 1$, the error can be an order of magnitude larger.

3.9. Conservative Fronts in Periodic Media. Another class of problems concerning fronts in heterogeneous media arises in transport through porous media. When solutes (ions) migrate inside porous media, some of them tend to attach onto the surface of minerals or colloids due to the existence of nonneutralized electric charges at the surface or inside these minerals [113]. This surface effect is called adsorption, which often creates a retardation on the movement of solute substance. The transport equation for the concentration of a one-species solute is based on the conservation of mass. When the adsorption reaches equilibrium, which often happens in a much shorter time than the time scale of solute migration, one arrives at the following conservative equation for the concentration C:

$$(\omega C + \rho\psi(C))_t = \nabla \cdot (\theta D \nabla C - vC), \tag{3.9.1}$$

where D is the pore-scale dispersion matrix, v is the incompressible water flow velocity, ω is the total porosity, and $\rho = (1-\omega)\rho_s$, ρ_s being the density of solid particles (minerals and colloids). The function $\psi = \psi(C)$ is called the sorption isotherm. For example, the Fruendlich isotherm is of the form $\psi(C) = \kappa C^p$, $p \in (0,1)$, where κ represents the spatial distribution of sorption sites.

Due to the heterogeneous nature of the porous medium, both v and κ are functions of the spatial variable x. The lack of detailed field information on them prompts people to make statistical assumptions, such as assuming that they are ergodic stationary random fields; see the works of van der Zee, Bosma, van Duijn, and van Riemsdijk [148], [27], [28]. In one space dimension, v is a constant, and (3.9.1) simplifies after a rescaling of constants to

$$(u + k(x)u^p)_t = (D(x)u_x)_x - u_x, \tag{3.9.2}$$

where we also make D spatially dependent. We consider the boundary conditions $u(-\infty, t) = u_l$, $u(+\infty, t) = u_r = 0$, $0 < u_l$, representing constant input of solute from the left end of a solute-free soil column. Solutions of (3.9.2) under such boundary conditions give rise to front solutions.

If k and D are constants, then by making the change of variable $v = u + ku^p$, we can write (3.9.2) as a standard conservation law:

$$v_t + (f(v) - \nu(g(v))_x)_x = 0, \quad \nu > 0,\ x \in \mathbb{R}. \tag{3.9.3}$$

An example is the well-known Burgers equation if $f = v^2/2$ and $g = v$. Front solutions $v = v(x - ct)$ are solvable in closed form, and $c = (f(u_l) - f(u_r))/(u_l - u_r)$, the so-called Rankine–Hugoniot relation. Stability of such scalar fronts is also well studied; see Il'in and Oleinik [89], Sattinger [141], Osher and Ralston [123], Gardner, Jones, and Kapitula [74], and Goodman [79], among others.

Let us now consider periodic media by supposing $k(x)$ and $D(x)$ to be 1-periodic regular functions. In periodic media, as in R-D equations, traveling fronts take the form $u = U(x - ct, x)$, which turn out to exist also for conservative equations like (3.9.2) and are asymptotically stable. The following result is obtained in Xin [162], [164].

THEOREM 3.12. *Let $k(x)$ and $D(x)$ be smooth positive functions with period* 1. *If $u_r = 0 < u_l$,* (3.9.2) *admits a Hölder continuous traveling wave solution of the form $u = U(x - st, x) \equiv U(\xi, y)$, $\xi = x - st$, $y = x$, $U(-\infty, y) = u_l$, $U(+\infty, y) = 0$, and $U(\xi, \cdot)$ has period* 1. *Such solutions are unique up to constant translations in ξ and have wave speeds*

$$s = \frac{u_l}{u_l + \langle k \rangle f(u_l)} > 0, \tag{3.9.4}$$

with $\langle k \rangle$ being the periodic mean. The wave profile U satisfies $0 \le U < u_l \ \forall \ (\xi, y)$; $U(\xi_1, y) \le U(\xi_2, y) \ \forall \ \xi_1 \ge \xi_2, \ \forall y$; $U_\xi < 0$ if $U(\xi, y) > 0$. Assume that the initial condition $u_0(x)$ satisfies $0 \le u_0(x) \le u_l, u_0 \in L^1(\mathbb{R}^+); u_0^p \in L^1(\mathbb{R}^+), u_0 - u_l \in L^1(\mathbb{R}^-)$, $u_0^p - u_l^p \in L^1(\mathbb{R}^-)$. Also let $m(u, x) = u + k(x)u^p$. Then there exists a unique number x_0 such that

$$\int_{\mathbb{R}} m(u_0(x), x) - m(U(x + x_0, x), x)dx = 0 \tag{3.9.5}$$

and such that

$$\lim_{t\to\infty} ||u(t, x) - U(x - st + x_0, x)||_1 = 0. \tag{3.9.6}$$

The construction of traveling waves uses the continuation method, and the existence result also holds in several space dimensions [162]. The Hölder continuity of solutions is a consequence of u^p being nondifferentiable at $u = 0$. The explicit effective wave speed (3.9.4) is due to the fact that (3.9.2) is conservative. Only the mean value of k contributes to the speed; the rest of the information in k influences the wave profile, however. The stability proof extends that of [123] and uses L^1 contraction of dynamics, as well as a space-time translation-invariance of the traveling fronts in the moving frame coordinate.

The dynamics of (3.9.2) with constant coefficients and spatially decaying initial data have been studied by Dawson, van Duijn, and Grundy [41], [80]. Solutions decay to zero in this regime and are known as *N waves*. See also [28] for a statistical study in heterogeneous media. The existence of traveling waves in homogeneous media for more complicated isotherms and for a solute transport system with kinetic sorption is carried out in van Duijn and Knabner [149]. The construction of traveling waves in these studies relies on dynamical systems theory for establishing connection orbits. For fronts in another conservative equation (the Richards equation of water infiltration) with more complicated dependence of wave speeds on the periodic medium, see Fennemore and Xin [58].

3.10. Summary and Further Remarks. We have used the two-scale homogenization ansatz to find the general form of traveling fronts in periodic media, discussed the construction of these fronts, shown equivalence of various variational characterizations of the front speeds, and analyzed the front stability and propagation. We have also presented results on fronts in slowly varying media, where the media slowly modulate the front speeds, in fast oscillating media, where the media homogenize much faster than the reaction scale, and in media with separated scales, where homogenization and modulation occur simultaneously. The forms of nonlinearity determine the wave speed spectrum of the fronts and the dynamics. For KPP fronts with minimum speeds, we used the reduced H-J equation and its homogenization to derive the same results as the large deviation probabilistic approach. Finally, we considered the similar form of fronts in periodic media of a conservative equation and observed that fronts are asymptotically stable and their speeds explicit.

For type 2 and type 4 nonlinearities, it remains to study the existence of traveling waves in several space dimensions and to find out whether, for compactly supported initial data, the minimum speed $c^* = c^*(k)$ defines the Hamiltonian for the asymptotic geometric front equation. This extends the geometric equation (3.8.3) of [106] on KPP fronts.

For extensions of the KPP methodology to systems of R-D equations in homogeneous and slowly varying media, see [8], [68], and [69]. These KPP systems are similar to what we described at the end of section 2.4. The solutions are uniformly bounded in time by the maximum (invariant region) principle [146]. For existence and stability of traveling fronts in related KPP systems, see also [138], [153], [154]. For a study of fronts in fast-oscillating periodic media governed by a bistable system, see [85]. For asymptotic stability of traveling fronts in infinite cylinders, see [16], [110], [136], [137].

4. Fronts in Random Media. Front propagation in random media is a very challenging and as yet largely open area. Randomness is a more physical and practical assumption in modeling heterogeneous porous structures and turbulent flows than periodicity. Since collecting field data on porosity is expensive, people can only afford to drill a limited number of wells and consequently must adopt a statistical approach to model the uncertainties [113]. Turbulent flows are well known to be intrinsically random, and it only makes sense to look at statistical quantities, such as the energy spectrum, two-point fluid velocity correlation function, and other structure functions; see Batchelor [9], among others.

Natural questions arise. How do fronts move in random media? What can we say about the front speeds, front locations, and other front characteristics? How much of what we know from periodic media generalizes to the random case? What are the new phenomena? Do fronts spread or not as a result of the competing focusing effect of nonlinearity and the spreading effect of random media?

Before answering these questions, let us first understand something about random media. We will see two large categories of random media, the tame and the wild. A tame random medium is a stochastic process $X(x,\omega)$ with finite moments (at least first and second) and short range correlations. An example is $X(x,\omega) = \xi_n(\omega)$ for $x \in [n-1,n)$, $n = 1,2,3,\ldots$, where the ξ_n are independent identically distributed (i.i.d.) random variables with finite first and second moments, $E[\xi_n] = \mu$, $E[\xi_n^2] = \sigma^2 + \mu^2$. Here the positive constant σ^2 is the variance and σ is the standard deviation. The classical central limit theorem asserts that the sum $S_n = \xi_1 + \cdots + \xi_n$ obeys

$$\frac{S_n - n\mu}{\sigma n^{1/2}} \to W_1 \tag{4.0.1}$$

as $n \to \infty$ in law, where W_1 is a unit Gaussian (standard normal distributed) random variable. The central limit theorem is a robust result and can be extended to cases with short correlations. In general, the product expectation $E[(X(s) - E[X(s)])(X(t) - E[X(t)])]$ is called the covariance function, and the correlation is the ratio of the covariance to $\sigma_s\sigma_t$, where $\sigma_s^2 = E[X(s)^2] - (E[X_s])^2$. The covariance or correlation is a measure of the degree of independence. For a stationary random process, the distribution of X is invariant under translation in x, so σ_s is a constant, and we may just use covariance for correlation if the variance is finite.

Now let us consider fronts moving in these tame random media. Intuitively, let us imagine that we are driving on a very bumpy road, with a certain distribution of endless bumps and dips. At the beginning stage, it is hard to maintain a constant speed since we always encounter something new or unexpected. However, if time is sufficiently long, we may hope that we have almost seen it all and got used to the road conditions, and are able to adjust more or less to a constant speed motion. If this is the case, then front behavior in random media is like that in periodic media, except that one has to wait much longer to reach an average speed and there will be occasional excursions from this speed due to uncertainties in the road conditions.

Motivated by this example, we can regard the location of a one-dimensional random front (defined as where the value of the solution is equal to a fixed number, say

$\frac{1}{2}$) as a time-dependent random process $S(t,\omega)$. We expect $S(t) \sim c^* t +$noise for large t, where c^* is the average constant speed. This starts to have the flavor of (4.0.1), although it is not at all clear how the short correlation and finite second moment condition are valid for S. However, the amazing thing is that the central limit theorem in fact holds for $S(t)$ in the case of one-dimensional Burgers fronts. That is,

$$\frac{S(t) - c^* t}{\sigma\sqrt{t}} \to W_1 \tag{4.0.2}$$

as $t \to \infty$ in law for some constant $\sigma > 0$. Since front problems in random media are both nonlinear and random, the robustness of the central limit theorem is remarkable. We believe that such a result holds for a much more general class of equations than the conservative PDEs of Burgers type that we are going to illustrate in section 4.1 with results obtained by the author and coworkers [155], [156], [129]. The conservative PDEs do, however, allow us to give a clean interpretation of (4.0.2) using an argument of conservation of mass. In addition, the Burgers fronts have tight widths for large times (bounded with probability arbitrarily close to 1), and so the nonlinearity overcomes the spreading effect of the random media.

Equation (4.0.2) implies that $S(t) \sim c^* t + \sqrt{t} W_1$, which is a natural generalization of the related formula $S_p(t) \sim c_p^* t + c_0$ for front locations in periodic media. Here the subscript p denotes the corresponding quantities in the periodic case, and c_0 is the constant phase shift. It is very interesting that the asymptotic expression for sums of i.i.d. random variables corresponds to the asymptotic formula for front location.

Recall that homogenization theory helped us determine the average wave speed c_p^* in the periodic case, in particular the homogenization results of the linear advection-diffusion problem and the H-J equation. We will see that their natural extensions to the random case can also help us find or interpret the speed c^*. In this sense, what we learned about fronts in periodic media carries over to the random setting. Thus a tame random medium can be thought of as being approachable from periodic media. A standard way of carrying out such an approximation is to truncate a random process and periodize it with larger and larger periods.

The homogenization process of determining the average front speed corresponds to the law of large numbers for a sum of i.i.d. random variables. However, even for tame random media, there is a new phenomenon—*front fluctuation*, manifested in the next order term $\sqrt{t} W_1$, which has no analogue in deterministic front problems. In principle, one has to understand both the mean field phenomenon (a homogenization result on the average front speed) and the statistics of the front fluctuation (Gaussian or otherwise) in order to completely describe a random front. This is certainly a more challenging task.

For R-D PDEs in tame random media, we present in section 4.2 results by Freidlin [67] and Lee and Torcaso [102] on the KPP fronts, where average front speeds were found using large deviation techniques. We derive the same results using homogenization of the related H-J equations just as in periodic media. The Donsker–Varadhan large deviation results [43], [151], [152], are used to draw the connection. We also state results by Mueller and Sowers [118] on a class of KPP equations under space-time white noise perturbations, where front speeds, finite front width, and the law of front shapes are characterized in the large-time limit. These features are consistent with the findings of the Burgers fronts.

Then there is the matter of wild random media. A wild random medium is one in which the second or first moment is infinite. In the case of $S_n = \xi_1 + \cdots + \xi_n$, a sum of i.i.d. random variables, one encounters non-Gaussian stable laws; see Breiman

[30]. If the first moment is still finite, then

$$\frac{S_n - E[\xi_1]n}{A_n} \to Y \tag{4.0.3}$$

in law, where $A_n/\sqrt{n} \to +\infty$ as $n \to \infty$. The random variable Y has a stable law of exponent $\alpha \in (1,2)$ whose characteristic function is explicit; see Theorem 9.32 of [30]. Such a random variable can be viewed as having a fractional moment α. One still has a law of large numbers, but the scaling of the fluctuation (if it exists) is a power larger than $\frac{1}{2}$, and so is anomalous.

If the probability density function has an even slower tail, so that the first moment is infinite but a fractional moment $\alpha \in (0,1)$ is finite, then

$$\frac{S_n}{A_n} \to Z, \tag{4.0.4}$$

where Z is a stable law with exponent $\alpha \in (0,1)$. In the fair coin-tossing example of [30], let S_n be the time of the nth return to equilibrium. Then (4.0.4) holds with $A_n = n^2$. If (4.0.4) with such an S_n were to represent front locations, there would be front acceleration to leading order!

Another class of wild random media is turbulent advection. The time-independent turbulent advection field is often assumed to be an incompressible stationary Gaussian random field $v(r,\omega) = (v_1, v_2, v_3)(r,\omega) \in R^3$ with mean equal to zero and covariance (see Kraichnan [101])

$$E[v_i(r)v_j(r')] = V_{ij}(r - r'),$$
$$V_{ij}(r) = D_0 \int dk\, (2\pi)^{-3}|k|^{-(3+\zeta)}\psi_0(|k|)\psi_\infty(|k|)P^{\perp}_{ij}(k)e^{ikr}, \tag{4.0.5}$$

where $\zeta \in (0,2)$, ψ_0 and ψ_∞ are infrared and ultraviolet cutoff functions so that $0 < \lambda \le |k| \le |k_\infty| < \infty$, and $P^{\perp}_{ij}(k)$ is the projection onto the subspace perpendicular to $k \in R^3$. The projection imposes incompressibility. The energy spectrum is the radial part of the integral (4.0.5) along $|k|$ after averaging out the angular contributions, and hence is const. $|k|^{-(1+\zeta)} \equiv E(k)$ for any dimension.

Since we are interested in large-scale phenomena, the small-k behavior is our concern. Taking $\lambda \to 0$, we see that the integral tending to

$$\int_{|k|\le|k_\infty|} |k|^{-1-\zeta} d|k|$$

is divergent at $k \sim 0$ for $\zeta \in (0,2)$. This means in particular that the second velocity moment $\langle |v(r)|^2 \rangle$ is infinite. It is not hard to check that the velocity difference for large separation ($|r| \gg 1$) satisfies

$$\lim_{\lambda \to 0} \langle |v(r+r') - v(r')|^2 \rangle \sim \text{const.}|r|^{\zeta}, \tag{4.0.6}$$

which shows that the velocity field viewed at large scales is Hölder continuous with exponent $\zeta/2$, called the *Hurst exponent* and denoted by H hereafter. Such a velocity field is known as fractal on the large scale.

When such a random field is considered in the advection-diffusion equation

$$u_t + v \cdot \nabla u = D\Delta u, \tag{4.0.7}$$

its effect is known to cause enhanced diffusion (eddy diffusivity) due to the increased area of the level surface of the scalar function u. In fact, the enhanced diffusion is so large that the associated space-time scaling is $x \sim D^* t^p$, with $p > 1$ the anomalous scaling. In atmospheric science, Richardson [132] discovered the anomalous diffusion law $\langle x^2 \rangle \sim O(t^3)$ in 1926. Here x can be thought of as the difference in position of two smoke particles in an advecting air flow. Similar anomalous behavior was also found in 1980 on transport through porous media; see Matheron and de Marsily [112] where $x \sim t^{3/4}$ for a milder random media. Systematic and in-depth mathematical studies have been undertaken in a series of works by Avellaneda and Majda [4], [5] and Fannjiang and Papanicolaou [56], [57], among others.

The anomalous behavior with $\zeta \in (0, 2)$ as well as the Richardson scaling law $\langle x^2 \rangle \sim O(t^3)$ can be seen from the energy spectrum and a dimensional analysis. Let us write the energy spectrum as $E(k) = Dk^{-(1+\zeta)}$. The dimension of the energy spectrum is

$$v^2/|k| = (L/T)^2 L = L^3/T^2,$$

which is equal to $DL^{1+\zeta}$ in dimension. Hence the dimension of D, denoted by $[D]$, is $L^{2-\zeta/T^2}$, implying

$$L^2 = ([D]T^2)^{2/(2-\zeta)} = [D]^{2/(2-\zeta)} T^{4/(2-\zeta)}.$$

The Richardson law follows on setting $\zeta = 2/3$, the Kolmogorov exponent.

For front propagation through a steady turbulent medium, a very useful model is the advection-diffusion-reaction equation

$$u_t + v \cdot \nabla u = D\Delta u + f(u), \tag{4.0.8}$$

where f is the reaction term, of types 1 to 5. When v is absent, we note by a simple scaling argument applied to the traveling front equation $Du'' + cu' + f(u) = 0$ that the larger the diffusion constant D, the larger the speed c. Intuitively, v causes the wrinkling of front surfaces, so much that the induced eddy diffusivity already influences the solutions on the scale $x \sim t^p$, with $p > 1$ if $\zeta \in (0, 2)$. It is then conceivable that even the leading-order front asymptotics will be altered to an anomalous expression $X \sim O(t^q)$ with $q > 1$, in analogy to the fair coin-tossing example described above.

In section 4.3, we describe the recent work of Majda and Souganidis [107] on upper bounds of KPP front speeds. The upper bounds suggest that in the turbulent (fractal) velocity regime ($H \in (0, 1)$ or $\zeta \in (0, 2)$) the front location is $X \sim O(t^q)$, with $q \in (1, 1 + H]$ to leading order. In section 4.4, we briefly describe the modeling aspect of front speeds in turbulent combustion and of front surface scaling in industrial deposition processes. In both areas, the stochastic H-J equations are used as prototype models. Summary and concluding remarks are in subsection 4.5, where we also show figures of fronts in one-dimensional random media.

4.1. Fronts in Random Burgers-Type Equations. The Burgers equation serves as an ideal candidate for understanding fronts in random media because it is solvable by the well-known Hopf–Cole formula. We will see that results on the Burgers equation also guide us in studying other scalar conservative equations with randomness.

Let us consider the celebrated Burgers equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = \nu u_{xx}, \quad \nu > 0, \tag{4.1.1}$$

with initial data

$$u(x,0) = \left(1+\exp\left(\frac{1}{2\nu}\left(x-\frac{1}{2}t\right)\right)\right)^{-1} + V_x \equiv u_s + V_x, \tag{4.1.2}$$

where u_s is the profile of a traveling front solution moving to the right with speed $\frac{1}{2}$ and connecting 1 to 0, and V_x is either white noise (formally the derivative of Brownian motion W_x) or a Gaussian process (all finite-dimensional distributions are multivariate Gaussian) with enough decay of correlations.

If V_x is a deterministic function with enough decay as $|x| \to \infty$, a classical result of Il'in and Oleinik [89] asserts that a solution $u(x,t)$ eventually converges to $u_s(x - \frac{1}{2}t + x_0)$ uniformly in x for a constant x_0. The constant x_0 depends on the integral (mass) of the initial perturbation V_x. In fact, the Burgers equation conserves the total mass $\int_{\mathbb{R}} u(x,t)\,dx$. For a bounded and decaying V, there is a unique value x_0 such that

$$\int_{\mathbb{R}} u(x,0) - u_s(x+x_0)\,dx = 0,$$

and hence by conservation of mass,

$$\int_{\mathbb{R}} \left[u(x,t) - u_s\left(x - \frac{1}{2}t + x_0\right)\right] dx = 0 \quad \forall t > 0.$$

If V is also small, then x_0 is small. Taylor expanding the above equality at $t = 0$ shows

$$\int_{\mathbb{R}} [u(x,0) - u_s(x) - u_s'(x)x_0]\,dx \sim 0,$$

implying

$$x_0 \sim \int_{\mathbb{R}} [u(x,0) - u_s(x)]\,dx = \int_{\mathbb{R}} V_x\,dx. \tag{4.1.3}$$

Thus for small perturbations, x_0 is approximately the total mass of the initial perturbation.

Now V_x is a stationary random process, and it has no decay as $|x| \to \infty$. It turns out that at time t, the truncated mass of V_x, or the integral of V_x over the interval $[-\frac{1}{2}t, \frac{1}{2}t]$, plays the role of the whole line integral (4.1.3) and causes the deviation of the front location from the mean position $\frac{1}{2}t$. It is interesting to note that the factor $\frac{1}{2}$ comes from the unperturbed front speed, and the interval $[-\frac{1}{2}t, \frac{1}{2}t]$ resembles the domain of dependence for the linear wave equation $u_{tt} - 4^{-1}u_{xx} = 0$. The picture behind this is that the perturbation gets sucked into the front from the left and the right at speed $\frac{1}{2}$. Let us calculate formally the front deviation for the white noise V_x as in (4.1.3),

$$x_0 = x_0(t,\omega) \sim \int_{-t/2}^{t/2} V_x\,dx = W_{t/2} - W_{t/2} \overset{\text{law}}{=} W_t \overset{\text{law}}{=} \sqrt{t}\,W_1, \tag{4.1.4}$$

i.e., $\sqrt{t}$ times the unit Gaussian. Thus the front location is

$$X = X(t,\omega) = \frac{t}{2} + s_0(t,\omega) \overset{\text{law}}{=} \frac{t}{2} + \sqrt{t}\,W_1. \tag{4.1.5}$$

The above heuristics are made precise in Wehr and Xin [155].

THEOREM 4.1. *Let $u(x,t)$ be the solution to the initial value problem of the Burgers equation (4.1.1)–(4.1.2), and let f be an increasing function of t. Then for the white noise perturbation V_x:*

(1) *(probing front) If $(f(t) - \frac{1}{2}t)/\sqrt{t} \to c \in \mathbb{R}$, $u(f(t),t)$ converges in distribution to a random variable equal to 0 with probability $\mathcal{N}(c)$ and equal to 1 with probability $1 - \mathcal{N}(c)$, where*

$$\mathcal{N}(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-y^2/2} dy,$$

the unit Gaussian distribution function. Given any $\epsilon \in (0,1)$, define the left and right endpoints of the interval containing the front by

$$z_-(t) = \min\{x : u(t,x) = 1 - \epsilon\}, \qquad z_+(t) = \max\{x : u(t,x) = \epsilon\},$$

so that the front width is $\{z_+(t) - z_-(t)\}$. Then

(2) *(front width) There exists a constant $t_0 > 0$ such that the random variables $\{z_+(t) - z_-(t)\}$ are tight for $t \geq t_0$; i.e., for any $\delta > 0$ there exists an M such that $\mathrm{Prob}(z_+(t) - z_-(t) > M) < \delta$ for all $t > t_0$.*

(3) *(front motion) As $s \to \infty$, there is a constant σ depending only on V_x ($\sigma = 1$ for white noise) such that*

$$\frac{z_+(t) - \frac{t}{2}}{\sigma\sqrt{t}} \overset{\text{law}}{\to} W_1,$$

and similarly for z_-.

Part (1) is a slightly weaker version of part (3), and both substantiate the formal calculation. Part (2) asserts that the noise does not spread the front width for large time, so nonlinearity dominates the randomness and preserves the coherent structure. The proof uses the Hopf–Cole formula and a Laplace method for stochastic integrals. We refer to [155] for details.

To understand a front moving in a random medium, let us now consider a Burgers equation with a spatially random flux,

$$v_t + \left(\frac{1}{2}a(x,\omega)v^2\right)_x = 0, \quad x \in \mathbb{R}, \tag{4.1.6}$$

with initial data $v(x,0) = I_{\mathbb{R}_-}(x)$, the indicator function of the negative real line, that is, a shock sitting at the origin. The random process a is positive and stationary. Equation (4.1.6) resembles the solute transport equation (3.9.2) with a random coefficient, which is to be discussed at the end this subsection. We are interested in how the initial shock moves through random media.

To find the leading front speed, an H-J homogenization argument will do. If we scale $x \to \epsilon^{-1}x$, $t \to \epsilon^{-1}t$, then (4.1.6) becomes

$$v_t + \left(\frac{1}{2}a(\epsilon^{-1}x,\omega)v^2\right)_x = 0. \tag{4.1.7}$$

Integrating (4.1.7) in x and setting $w = \int_{+\infty}^{x} v dx$, we have for w the H-J equation

$$w_t + \frac{1}{2}a(\epsilon^{-1}x,\omega)w_x^2 = 0. \tag{4.1.8}$$

Proceeding formally with the cell problem as in the periodic case, we need to find a solution to

$$\frac{1}{2}a(y,\omega)(p+w_y)^2 = \bar{H}(p) \geq 0. \tag{4.1.9}$$

We solve (4.1.9) as

$$p + w_y = \sqrt{2a^{-1}\bar{H}},$$

which gives a solution upon integrating once in y. To find a solution with the slowest (sublinear) growth in y, we must have

$$\langle -p + \sqrt{2a^{-1}\bar{H}}\ \rangle = 0, \tag{4.1.10}$$

where the bracket is the ensemble mean. It follows that

$$\bar{H} = \frac{p^2}{2}\langle a^{-1/2}\rangle^{-2}, \tag{4.1.11}$$

which means that the homogenized equation for w is $w_t + \bar{H}(w_x) = 0$. Differentiating this once, we find the homogenized equation for v,

$$v_t + \langle a^{-1/2}\rangle^{-2}\left(\frac{v^2}{2}\right)_x = 0, \tag{4.1.12}$$

which yields a shock solution for the same data with speed $\frac{1}{2}\langle a^{-1/2}\rangle^{-2}$, half of the root-harmonic mean of a.

The rigorous result on both the front speed and fluctuation is given in Wehr and Xin [156]. The assumptions on a are that the process a has a mixing property (or short-range correlation) and finite first moment of $a^{-1/2}$ and that it obeys an invariance principle (a functional central limit theorem). Under these conditions, to make a a tame random medium, we have the following.

THEOREM 4.2. *Let $2c = E[a^{-\frac{1}{2}}]^{-2}$ denote the square root-harmonic mean of the process $a(x)$. Then as $t \to \infty$,*

$$(1)\ v(\alpha t, t) \xrightarrow{\text{D}} 0 \quad for\ \alpha > c, \tag{4.1.13}$$

$$(2)\ \sqrt{a(\alpha t)}v(\alpha t, t) \xrightarrow{\text{D}} \sqrt{2c} \quad for\ \alpha < c, \tag{4.1.14}$$

$$(3)\ \sqrt{a(ct + z\sqrt{t})}v(ct + z\sqrt{t}, t) \xrightarrow{\text{D}} X, \tag{4.1.15}$$

where X is a random variable equal to $\sqrt{2c}$ with probability $\mathcal{N}(\mu^2\sigma^{-1}z)$ and equal to 0 with probability $1 - \mathcal{N}(\mu^2\sigma^{-1}z)$, where $\mathcal{N}(s) = (2\pi)^{-1/2}\int_{-\infty}^{s} e^{-s'^2/2}\,ds'$ is the error function.

The theorem also validates the homogenization of the random H-J equation (4.1.8). A general theory of homogenization of random H-J equations is, however, not yet known.

The idea of the proof is to regularize (4.1.6) with a special viscous term in the parabolic equation

$$v_t + \left(\frac{1}{2}a(x)v^2\right)_x = \nu(\sqrt{a(x)}(\sqrt{a(x)}v)_x)_x, \tag{4.1.16}$$

where $\nu > 0$. Making the change of variables $u = \sqrt{a(x)}v$, we get an equation for u, $u_t/\sqrt{a(x)} + (\frac{1}{2}u^2)_x = \nu(\sqrt{a(x)}u_x)_x$. If we make the further change of variables

$$\xi = \int_0^x \frac{1}{\sqrt{a(x')}}\, dx', \tag{4.1.17}$$

we find that the equation for u in the variables (ξ, t) becomes the standard viscous Burgers equation

$$u_t + \left(\frac{1}{2}u^2\right)_\xi = \nu u_{\xi\xi} \tag{4.1.18}$$

with the new initial condition

$$u(\xi, 0) = \sqrt{a(x(\xi))} I_{\mathbb{R}_-}(\xi). \tag{4.1.19}$$

We then work our way back from the Hopf–Cole formula of (4.1.19) to solutions of (4.1.16) and pass to the limit $\nu \to 0$.

Since the speed of a shock front for the Burgers equation is equal to its height divided by 2 as a consequence of the conservation law, intuitively the asymptotic speed of the front, arising from our random initial condition in the ξ variable, equals half of its average height, i.e.,

$$\frac{1}{2}\lim_{L\to\infty}\frac{1}{L}\int_{-L}^{0}\sqrt{a(x(\xi))}\, d\xi,$$

which, after changing the variable of integration to x, gives

$$\frac{1}{2}\lim_{L\to\infty}\frac{-x(-L)}{L} = \frac{1}{2}E[a^{-\frac{1}{2}}]^{-1}.$$

To get from this the front speed in the x variable, we divide this value by $E[a^{-\frac{1}{2}}]$ in view of (4.1.17) and arrive at the speed $c = \frac{1}{2}E[a^{-\frac{1}{2}}]^{-2}$, half of the square root-harmonic mean of the process $a(x)$.

These results for the random Burgers equation help us formulate a front fluctuation theory for the solute transport equation below, which is not integrable. Postel and Xin [129] numerically simulated the random fronts of the solute transport equation with Langmuir isotherm:

$$\left(u + k(x)\frac{u}{1+u}\right)_t = (D(x)u_x)_x - u_x, \quad x \in \mathbb{R}, \tag{4.1.20}$$

where $k(x)$ and $D(x)$ are random processes with finite correlation lengths. The boundary conditions are $u(t, -\infty) = u_l > 0$ and $u(t, +\infty) = 0$. They found that the front location $X(t)$ obeys a law

$$\frac{X(t) - st}{\sqrt{t}} \to \text{Gaussian} \tag{4.1.21}$$

as $t \to \infty$, where s is the constant mean front speed given by the same formula as for periodic media (see (3.9.4)), simply replacing the cell average there by the ensemble mean. This suggests that the Gaussian front asymptotics are valid for nonintegrable one-dimensional scalar conservation laws. Moreover, numerical solutions support the following theory of front fluctuation based on the random Burgers fronts. First the conserved quantity is $\int_{\mathbb{R}} u + k(x)f(u)$. With $u_r = 0$, the mass density behind the front is $u_l + k(x)f(u_l)$, which can be written as

$$u_l + \langle k \rangle f(u_l) + (k(x) - \langle k \rangle)f(u_l),$$

i.e., mean plus fluctuation. Let s be the mean front velocity. By analogy with the Burgers equation, the front deviation is approximately equal to

$$\int_{-st}^{st} (k(x) - \langle k \rangle) f(u_l) dx. \tag{4.1.22}$$

For a stationary process with enough decay of correlations, the invariance principle holds (see [25]) and gives

$$\frac{\int_{-st}^{st} (k(x) - \langle k \rangle) dx}{\sigma_a \sqrt{2st}} \to W_1 \tag{4.1.23}$$

in law as $t \to +\infty$, where W_1 is the unit Gaussian and σ_a is the velocity autocorrelation, defined as

$$\sigma_a^2 = \int_{-\infty}^{\infty} E[(k(0) - \langle k \rangle)(k(x) - \langle k \rangle)]\, dx. \tag{4.1.24}$$

For the standard deviation we have the formula

$$\sigma_x = f(u_l)\sigma_a \sqrt{2st} \tag{4.1.25}$$

for large times. Thus the $\sqrt{t}$-normalized front standard deviation σ is

$$\sigma = \lim_{t \to \infty} \frac{\sigma_x}{\sqrt{t}} = f(u_l)\sigma_a \sqrt{2s}, \tag{4.1.26}$$

which will be seen to agree very well with the numerically obtained empirical formula.

4.2. KPP Fronts in Random Media. Let us turn to a case of KPP fronts in one-dimensional random media solved by Freidlin [67]. Consider the KPP equation

$$u_t = \frac{1}{2} u_{xx} + \xi(x, \omega) u(1 - u), \tag{4.2.1}$$

with deterministic initial data $u(x, 0) = g(x) \in [0, 1]$ compactly supported on the positive x-axis. We are interested in fronts moving to the right. Here the random function $\xi(x, \omega) \geq 0$ is stationary and ergodic (short-range correlation) in x, so that the moment condition

$$Ee^{t\xi(0)} < +\infty$$

holds for all t.

The stochastic representation of the solution is

$$u(x, t) = E_x \exp\left(\int_0^t c(W_s, u(t - s, W_s)) ds \right) g(W_t), \tag{4.2.2}$$

where W_x is the one-dimensional Brownian motion (Wiener process) starting at x. From here, Freidlin derived upper and lower bounds to show convergence of u to 0 or 1 almost surely (a.s.; used in the usual probabilistic sense [30]) as $t \to \infty$. To introduce the result, let us define the function $\mu(z)$ by

$$\mu(z) = \lim_{t\to\infty} \frac{1}{t} \ln E_t \exp\left(\int_0^{\tau_0} [\xi(W_s) + z] ds \right) \quad \text{a.s.,} \tag{4.2.3}$$

where τ_0 is the first hitting time of the Wiener process to zero and $W_0 = t$. This function is convex, lower semicontinuous, and monotone nondecreasing. Since $\xi(x)$ is ergodic, $\mu(z)$ is deterministic. In view of (4.2.3), $\mu = \infty$ if $z > 0$. It can be shown that there is a number $\bar{g} \le 0$ such that $\mu(z)$ is differentiable and $-\sqrt{2|z|} \le \mu(z) \le 0$ for $z \le \bar{g}$, while $\mu(z) = +\infty$ if $z > \bar{g}$.

The asymptotic front speed on the positive x-axis is then given by

$$v^* = \inf_{z \le \bar{g}} \frac{z}{\mu(z)}, \tag{4.2.4}$$

which appears in the following front propagation result.

THEOREM 4.3. *For any $v > v^*$, $\lim_{t\to+\infty} \sup_{x\ge vt} u(x,t) = 0$ a.s. If for any $v \in \mathbb{R}$,*

$$\limsup_{t\to\infty} \frac{1}{t} \ln E_{vt} \exp\left(-\int_0^{t} \xi(W_s) ds \right) < 0 \quad a.s., \tag{4.2.5}$$

then for any $v \in (0, v^*)$, $\lim_{t\to+\infty} \inf_{0\le x\le vt} u(x,t) = 1$ a.s.

The strict inequality (4.2.5) is a nondegeneracy condition on ξ, since the left-hand side is always ≤ 0 due to $\xi \ge 0$.

By the Feynman–Kac formula, the expectation

$$E_x \exp\left(\int_0^{\tau_0} [\xi(W_s) + z] ds \right), \quad x \ge 0, \tag{4.2.6}$$

when it is finite (that is, when $z < \bar{g}$), satisfies the equation

$$\frac{1}{2} \frac{d^2 u}{dx^2} + [\xi(x) + z] u = 0. \tag{4.2.7}$$

It is now clear that the function $\mu(z)$ is simply the decay rate of a positive solution of (4.2.7) on the positive real line. In the special case in which $\xi(x)$ is a periodic function, such a decaying positive solution has the form $u(x) \sim \psi(x) e^{\mu x}$. Substitution of this form into (4.2.7) gives

$$\frac{1}{2}(d/dx + \mu)^2 \psi + \xi(x) u = (-z) u, \quad x \in T, \tag{4.2.8}$$

where $-z$ becomes the unique principal eigenvalue of the left-hand-side elliptic operator over the unit circle. Comparing with the function $\lambda(z)$, the principal eigenvalue of the operator L_z of (3.7.2) with $a = Id$, $b = 0$, $c = \xi$, we see that

$$\lambda(-\mu) = -z, \quad \mu = -\lambda^{-1}(-z). \tag{4.2.9}$$

It is easy to see that the seemingly different variational formula (4.2.4) is just

$$v^* = \inf_{z \le \bar{g}} \frac{-z}{\lambda^{-1}(-z)} = \inf_{r \equiv \lambda^{-1}(-z) = -\mu > 0} \frac{\lambda(r)}{r}, \tag{4.2.10}$$

exactly the KPP front speed formula (3.7.3) of section 3!

Since the KPP results in periodic media can be derived from the periodic homogenization of the H-J equation, it is natural to ask whether formula (4.2.4) can be derived from the homogenization of a random H-J equation. We have seen that in (4.1.7)–(4.1.12), the periodic cell problem can be extended to yield a meaningful solution if we replace the periodic boundary condition by a slow growth condition. However, in the case of the principal eigenvalue problem (4.2.8), there is no straightforward extension because in the random case there is, generally speaking, no simple eigenvalue at the edge of the spectrum. In spite of the complexity of the spectrum, the leading edge can still be characterized for random media with short-range correlation. This is provided by the Donsker–Varadhan large deviation theory and the resulting generalized variational formula.

Let us scale space and time by ϵ and follow the same methodology as in (3.7.4)–(3.7.6). The H-J approximation of the KPP equation in this scaling limit is

$$v_t - \frac{1}{2}v_x^2 = \frac{\epsilon}{2}v_{xx} + \xi(\epsilon^{-1}x)v, \tag{4.2.11}$$

with initial data tending to zero on the support of $u(x,0) = g(x)$ and $-\infty$ otherwise. Let us carry out our calculation for a more general random H-J homogenization problem,

$$\alpha_t + \frac{|\nabla\alpha|^2}{2} + V(\epsilon^{-1}x) = \frac{\epsilon}{2}\Delta\alpha, \tag{4.2.12}$$

which becomes a higher dimensional extension of (4.2.11) on setting $\alpha = -v$, $V = \xi$. Since our goal is to find the homogenized Hamiltonian, it is convenient to consider initial data of the plane wave form

$$\alpha(x,0) = p \cdot x. \tag{4.2.13}$$

We notice that (4.2.12) can be linearized via a logarithmic transform just as in the Hopf–Cole formula of the Burgers equation. Making the change of variable

$$\alpha = -\epsilon \log w,$$

we find that w satisfies

$$w_t = \frac{\epsilon}{2}\Delta w + \epsilon^{-1}V(\epsilon^{-1}x)w, \quad w(0,x) = \exp(-\epsilon^{-1}p \cdot x). \tag{4.2.14}$$

By the Feynman–Kac formula,

$$w(T,x) = E_x\left\{\exp\left(\frac{-p\cdot\xi(\epsilon,T)}{\epsilon} + \epsilon^{-1}\int_0^T V\left(\frac{\xi(\epsilon,\tau)}{\epsilon}\right)d\tau\right)\right\}, \tag{4.2.15}$$

where $\xi(\epsilon,t) = x + \sqrt{\epsilon}W$, $W(0) = 0$, the n-dimensional Brownian motion. Putting this expression in (4.2.15) and changing the variable τ to $\epsilon\tau$ in the integral, we get

$$w(T,x) = E\left\{\exp\left(\frac{-p\cdot x}{\epsilon} - \frac{p\cdot W(T)}{\epsilon^{1/2}} + \int_0^{\epsilon^{-1}T} V(W(\tau) + \epsilon^{-1}x)d\tau\right)\right\}, \tag{4.2.16}$$

or by Brownian scaling,

$$w(T,x) = E\left\{\exp\left(\frac{-p\cdot x}{\epsilon} - p\cdot W(\epsilon^{-1}T) + \int_0^{\epsilon^{-1}T} V(W(\tau) + \epsilon^{-1}x)d\tau\right)\right\}. \tag{4.2.17}$$

Using decay of correlation of the V process, it can be shown for small ϵ and $x \neq 0$ ($x = 0$ is obvious) that

$$\alpha^{\epsilon} = p \cdot x - \epsilon \log E \left\{ \exp \left(-p \cdot W(\epsilon^{-1}T) + \int_0^{\epsilon^{-1}T} V(W(\tau)) d\tau \right) \right\} + o(1). \tag{4.2.18}$$

We show that $\alpha = \alpha^{\epsilon}$ converges to

$$p \cdot x - \bar{H}T,$$

which solves the homogenized H-J equation

$$\alpha_t + \bar{H}(\nabla \alpha) = 0 \tag{4.2.19}$$

with initial data $p \cdot x$. To this end, we show that the limit of the second term in (4.2.18) divided by $-T$ has a well-defined limit $\bar{H}$ as $\epsilon \to 0$. This gives, in view of (4.2.18), with $t = \epsilon^{-1}T$,

$$\bar{H} = \lim_{t \to \infty} t^{-1} \log E \left\{ \exp \left(-p \cdot W(t) + \int_0^t V(W(\tau)) d\tau \right) \right\}, \tag{4.2.20}$$

our first formula for $\bar{H}$. If we define $\lambda(z) = \bar{H}(-z)$, then as in section 3, we have the analogue of formula (3.7.3),

$$v^*(e) = \inf_{(e,z)>0} \frac{\lambda(z)}{(e,z)} = \inf_{(e,z)>0} \frac{\bar{H}(-z)}{(e,z)}. \tag{4.2.21}$$

The connection between $\bar{H}$ and μ is $\mu^{-1}(z) = -\bar{H}(z)$ or $\mu(-z) = \bar{H}^{-1}(z)$. In fact, formally inverting (4.2.3), we get the negative of the expression in the exponential of (4.2.20) with $V = \xi$, except that $W(t)$ replaces t and t replaces τ_0. This is easy to understand because the Wiener process of (4.2.3) starts at t and gets conditioned at zero, while that of (4.2.20) starts at zero and we look at its position $W(t)$ at time t. Essentially this is the reason why $\mu(z) \sim O(\sqrt{|z|})$ and $\bar{H}(z) \sim O(z^2)$ for large z.

One way to justify the inverse relation between μ and $\bar{H}$ is to work out a formula for H so that it reduces to the right answer in the periodic case and so that its value in the random case can be approximated by periodizing the potential V. Let us write

$$\bar{H} = \lim_{t \to \infty} t^{-1} \log E \left\{ \exp \left(-\int_0^t p dW + \int_0^t V(W(\tau)) d\tau \right) \right\}$$

and use the Girsanov formula to change the measure from Wiener to $P(\mu)$ corresponding to the process

$$d\mu = -p dt + dW. \tag{4.2.22}$$

Then

$$\bar{H} = \lim_{t \to \infty} t^{-1} \log \tilde{E} \left\{ \exp \left(\frac{1}{2} \int_0^t |p|^2 + \int_0^t V(\mu(\tau)) d\tau \right) \right\}, \tag{4.2.23}$$

where $\tilde{E}$ is the expectation with respect to $P(\mu)$.

Thus we have the formula

$$\bar{H} = \frac{|p|^2}{2} + \lim_{t \to \infty} t^{-1} \log \tilde{E} \left\{ \exp \left(\int_0^t V(\mu(\tau)) d\tau \right) \right\}, \tag{4.2.24}$$

where $\mu = -pt + DW(t)$. This is the type of integral studied extensively by Donsker and Varadhan [43], [151], [152]. In the periodic case, (4.2.24) asserts that $\bar{H}$ is $|p|^2/2$ plus the principal eigenvalue of the operator $\frac{1}{2}\Delta - p \cdot \nabla + V(y)$ on the torus T^n. This is exactly the result of the periodic homogenization! In the random setting, the limit (4.2.24) has been shown to exist a.s. by Donsker and Varadhan for short-range potential V using large-deviation techniques. Moreover, a variational formula is available for $\bar{H}$ [151], which shows that the periodization approximation converges in the infinite period limit.

Rigorous justification of the above derivation of the effective Hamiltonian as well as convergence for general initial data is in progress. In particular, we can show that the formula (3.7.9) remains true if V is a two-state Markov process, by using (4.2.24) and taking the inviscid limit. It is interesting to see if the flat piece $\bar{H}$ persists for other random potential V.

The extension of Freidlin's one-dimensional KPP result has been carried out by Lee and Torcaso [102] to a d-dimensional ($d \geq 2$) lattice KPP equation of the form

$$u_t = \tilde{\Delta} u + \xi(x) u(1-u), \quad t > 0, \ \ x \in Z^d, \tag{4.2.25}$$

with initial condition $u(0, x) = 1$ if $x = 0$, $u(0, x) = 0$ otherwise. Here

$$\tilde{\Delta} f(x) = \frac{1}{2d} \sum_{e \in Z^d : |e|=1} f(x+e) - f(x),$$

the discrete Laplacian. The random variables $\xi(x)$ are i.i.d., bounded, and nonnegative. If A is the essential supremum of $\xi(0)$, then they showed that for each vector $e \in Z^d$,

$$\mu(z; e) = \lim_{t \to +\infty} \frac{1}{t} \log E_{te} e^{\int_0^{\tau_0} (\xi(\eta_s) + z) ds}$$

is a nonrandom convex function for $z < -A$. Here η_s is the strong Markov process on Z^d corresponding to the generator $\tilde{\Delta}$ starting at te when $s = 0$, and τ_0 is the first time at which η_s hits zero. The large-time front speed in direction e is

$$v_e = \inf_{z < -A} \frac{z}{\mu(z; e)}.$$

Another class of random KPP equations arising from the limit of certain interacting particle systems is of the type

$$u_t = u_{xx} + a(u) + \epsilon b(u) \dot{W}, \tag{4.2.26}$$

where $a(u)$ is a KPP nonlinearity, typically equal to $u(1-u)$, $b(u)$ is a Lipschitz continuous function of u, and $\dot{W} = \dot{W}(x, t)$ is a space-time white noise. Recently, Mueller and Sowers [118] investigated random traveling fronts in a special form of (4.2.26),

$$u_t = u_{xx} + u - u^2 + \epsilon \sqrt{u(1-u)} \dot{W}, \tag{4.2.27}$$

with the initial data $u_0 = I_{(-\infty, a)}$.

Equation (4.2.27) is closely related to the so-called *historical process*; see Dawson and Perkins [42] for details. Simply put, the historical process is a measure on the sets of paths over a time period $[0, t_0]$, which represents the past history up to time t_0 of a cloud of infinitesimally small particles whose density at time t is $u(x, t)$. The particles move according to independent Brownian motions, and they give birth and die. There is an excess of births over deaths, which has a size of $1-u$. An interesting property of this process is that a small collection of particles dies out quickly for large values of x for the given initial density u_0, which implies that solution u has a compact support on the positive x-axis. By symmetry of the solutions, u is also equal to 1 outside a compact interval. Dawson and Perkins then define the location of the wave front to be

$$b(t) = \sup\{x \in \mathbb{R} : u(x,t) > 0\}. \tag{4.2.28}$$

The main results proved in [118] can be summarized as follows.

THEOREM 4.4. *Consider a solution u of (4.2.27) with initial data $I_{(-\infty,a)}$, $a > 0$. With probability 1, $0 \le u \le 1$ for all (x,t). For ϵ small enough, the solution u behaves like a moving front with the following properties.*

(1) *(front speed and shape) With probability 1, $\lim_{t\to\infty} b(t)/t$ exists and lies in $(0,+\infty)$, and this limit depends on ϵ. The law of the front profile $v(x,t) = u(b(t)+x,t)$ tends towards a stationary limit as $t \to \infty$.*

(2) *(front width) Let $I(t) = [a(t), b(t)]$ be the smallest closed interval such that $u = 1$ for $x < a(t)$ and $u = 0$ for $x > b(t)$. Then with probability 1, $I(t)$ is a compact interval for all $t \ge 0$.*

This theorem seems to be the first on KPP random fronts that provides information on front shape and width in addition to front speed. The almost-sure finite front width property is reminiscent of the viscous Burgers front under white noise perturbation. It would be interesting to find out if $b(t)$ obeys a central limit theorem and whether ϵ-dependent upper and lower bounds on the asymptotic wave speed can be obtained.

4.3. KPP Fronts in Turbulent Shear Flows. Recently, Majda and Souganidis [107] studied upper bounds on front speeds in the KPP equation with fractal and smooth shear flow advection field,

$$T_t - \frac{\kappa}{2}\Delta T + v_\lambda(x) T_y + w T_x + K T(T-1) = 0, \quad \mathbb{R}^2 \times (0,+\infty), \tag{4.3.1}$$

with initial data $T(x,y,0) = T_0 = 1$ if $y < 0$, $T(x,y,0) = T_0 = 0$ if $y > 0$. Here T is the temperature of a combustion front, $\kappa > 0$ is the diffusion constant, $K > 0$ is the reaction rate constant, and $w > 0$ is the constant sweeping velocity. Shear flow is a special incompressible field depending spatially only on one variable x. The velocity field $v_\lambda(x)$ is a stationary zero-mean Gaussian random field with the spectral representation

$$v_\lambda(x) = (2\pi)^{-1/2} V_0 \int_{|k|\ge\lambda} e^{ixk} \psi^{1/2}(|k|)|k|^{-1/2-H} W(dk), \tag{4.3.2}$$

where the parameter H, the Hurst exponent, satisfies $H < 1$, with $H \in (0,1)$ being the usual turbulence regime. In (4.3.2), $W(dk)$ is a complex Gaussian white noise with

$$\langle W(dk) W(dk') \rangle = \delta(k+k')\, dk\, dk'.$$

Here $\langle\cdot\rangle$ means averaging with respect to velocity statistics. The function ψ is a nonnegative, continuous, rapidly decreasing function with $\psi(0) = 1$. It is used to prevent divergence for large $|k|$, hence it is called the ultraviolet cutoff. The infrared cutoff for small $|k|$ is the parameter λ, $\lambda \ll 1$. Majda and Souganidis were interested in the limit as $\lambda \to 0$ of solutions of (4.3.1) to understand how front propagation is influenced by a fully developed turbulent shear flow.

The behavior of the velocity field v_λ as $\lambda \to 0$ is completely different depending on whether H satisfies $H < 0$ or $0 < H < 1$. For $H < 0$, the second velocity moment or mean energy $\langle v_\lambda^2(0)\rangle$ has a finite limit as $\lambda \to 0$, while if $H \in (0,1)$, there is infrared divergence of the mean energy, and the velocity difference satisfies for $|x| \gg 1$

$$\lim_{\lambda\to 0}\langle (v_\lambda(x+x') - v_\lambda(x'))^2\rangle = C_H^2 V_0^2 |x|^{2H}, \tag{4.3.3}$$

where C_H is a universal constant. Due to (4.3.2) and the scaling properties of W, $v_\lambda(\lambda^{-1}x) = \lambda^{-H} V_\lambda(x)$ in law, where

$$V_\lambda(x) = (2\pi)^{-1/2} V_0 \int_{|k|\geq 1} e^{ixk}\psi^{1/2}(\lambda|k|)|k|^{-1/2-H} W(dk), \tag{4.3.4}$$

whose correlation function is

$$\langle V_\lambda(x) V_\lambda(0)\rangle = \int_{|k|\geq 1} e^{ixk}\psi(\lambda|k|)|k|^{-1-2H} dk, \tag{4.3.5}$$

where $|k|^{-1-2H}$ is the energy spectrum ($H = \frac{1}{3}$ is the Kolmogorov exponent). It follows from (4.3.5) that if $H \in (0,1)$, V_λ converges to V locally uniformly with probability 1, and V is Hölder continuous with exponent arbitrarily close to H. As a consequence, one expects that there is anomalous enhancement of the front speeds due to front wrinkling at all scales by the shear flow field. On the other hand, if $H < 0$, the velocity field has finite mean energy and its sample paths are smooth (due to the cutoff ψ, the correlation function of v_λ is smooth). The authors illustrated this transitional phenomenon through the behavior of upper bounds of front speeds in the y direction. The following scaling of solutions is introduced:

$$T^\lambda(x,y,t) = T(\lambda^{-1}x, \alpha(\lambda)^{-1}y, \beta(\lambda)^{-1}t). \tag{4.3.6}$$

The results of [107] can be summarized as follows.

THEOREM 4.5. (1) *(fractal case, $H \in (0,1)$) Choose $\alpha(\lambda) = \lambda^{1+H}$ and $\beta(\lambda) = \lambda$ in* (4.3.6)*. There is a universal constant $\tilde{C}_H$ depending on $|x| + t$ and C_H in* (4.3.3) *such that as $\lambda \to 0$, $T^\lambda \to 0$ a.s. for large t, provided*

$$y > V_0 \tilde{C}_H (2\kappa K)^{H/2} t^{1+H}. \tag{4.3.7}$$

For the ensemble-averaged solutions, choose $\beta(\lambda) = \alpha(\lambda)^{2/3}\lambda^{-2H/3}$, so that $\lambda\beta^{-1}(\lambda) \to 0$ as $\lambda \to 0$. Then as $\lambda \to 0$, $\langle T^\lambda\rangle \to 0$ for

$$y > (2\kappa K C_0)^{1/2} t^{3/2}, \tag{4.3.8}$$

with $C_0 = V_0^2 \int_{|k|\geq 1} |k|^{-1-2H} dk$.

(2) *(smooth case, $H < 0$) Choose $\alpha(\lambda) = \lambda|\ln\lambda|^{1/2}$ and $\beta(\lambda) = \lambda$. Then for almost every realization of the velocity field, there is a constant C depending on that realization and an absolute constant c_0 such that as $\lambda \to 0$, $T^\lambda(x,y,t) \to 0$ for large t, provided*

$$y > C V_0 c_0 (\ln(|x| + t))^{1/2} t. \tag{4.3.9}$$

For ensemble-averaged solutions, choose $\alpha(\lambda) = \lambda$ and $\beta(\lambda) = \lambda$. Then as $\lambda \to 0$, $\langle T^\lambda\rangle \to 0$ for

$$y > (2\kappa K)^{1/2} t. \tag{4.3.10}$$

The upper bound (4.3.7) suggests that in the fractal regime, turbulent fronts accelerate with the anomalous propagation law $y = O(t^p)$, $p \in (1, 1 + H]$. We see from (4.3.7) and (4.3.8) that in the fractal regime, the front speed scaling in the averaged front is very different from that of individual realization, which is called non-self-averaging and is attributed to strong intermittency of the flow field.

To find the above upper bounds, the authors worked with the upper solution $\bar{T}^\lambda$ of T^λ. In the fractal regime and the related scaling, T^λ satisfies

$$T^\lambda - \frac{\kappa}{2}(\lambda T^\lambda_{xx} + \lambda^{1+2H} T^\lambda_{yy}) + V_\lambda(x) T^\lambda_y + w T^\lambda_x + \frac{K}{\lambda} T^\lambda (T^\lambda - 1) = 0. \tag{4.3.11}$$

The upper solution $\bar{T}^\lambda$ solves the linear advection-diffusion equation

$$\bar{T}^\lambda - \frac{\kappa}{2}(\lambda \bar{T}^\lambda_{xx} + \lambda^{1+2H} \bar{T}^\lambda_{yy}) + V_\lambda(x) \bar{T}^\lambda_y + w \bar{T}^\lambda_x - \frac{K}{\lambda} \bar{T}^\lambda = 0 \tag{4.3.12}$$

with the same initial data. Equation (4.3.12) can be analyzed via the Feynman–Kac path integral representation of solutions and the almost sure regularity and growth estimates of velocity field. The other upper bounds are obtained in the same manner.

Certainly, finding the same type of lower bounds will be delicate and quite necessary for confirming the front acceleration in the fractal regime. Note also that the above upper bounds hold for all the other non-KPP nonlinearities as well.

We mention in passing that Malham and Xin [109] recently studied the front solutions to the coupled Navier–Stokes and R-D system (Boussinesq system)

$$\begin{aligned} \psi_t + u \cdot \nabla \psi &= \Delta \psi - \psi \theta^m, \quad \nabla \cdot u = 0, \\ \theta_t + u \cdot \nabla \theta &= d \Delta \theta + \psi \theta^m, \\ u_t + u \cdot \nabla u &= \nu \Delta u - \nabla p + \sigma \theta e_z \end{aligned} \tag{4.3.13}$$

on the infinite tube $\Omega = \{(x, y) \in \Omega' \times \mathbb{R}\}$, with Ω' a bounded domain in $\mathbb{R}^{n-1}$, $n = 2, 3$, with front initial data. For all $m \geq 1$ if $n = 2$, and for $m = 1, 2, 3$ if $n = 3$, the front speeds are bounded from above by $O(e^{ct})$, $c = c(d, \nu, \sigma) > 0$. Here $d > 0$ is the Lewis number, $\nu > 0$ is the fluid viscosity, and $\sigma > 0$ is the Rayleigh number. In the laminar regime, when ν is suitably bounded away from zero, under the additional assumptions of small coupling σ and $d = 1$, they showed that front speeds are $O(1)$. These are complementary deterministic results.

4.4. Modeling Fronts in Random Media. There is a rich literature of modeling activities in the combustion community on premixed turbulent flame fronts. We refer the readers to the excellent recent review article by Ronney [134]. Some of the fundamental questions are the existence of a turbulent burning velocity (S_T), the role of the velocity spectrum on S_T, and quenching of flames by turbulence. These questions can be asked also for the passive reaction-diffusion-advection equation

$$u_t + v \cdot \nabla u = \nu \Delta u + f(u), \tag{4.4.1}$$

where the incompressible random velocity v is assumed to have the turbulent spectrum and ensemble mean zero; f is the reaction term of interest, and $\nu > 0$ is the viscosity coefficient. For applications, the number of space dimensions is two or three. Can one find conditions on v so that an effective front speed S_T is well defined for large-time front propagation? If so, how does it depend on the spectrum of v?

A "folklore" result for S_T in the regime of small flame thickness is

$$S_T / S_L = A_T / A_L, \tag{4.4.2}$$

where A_T is the surface area of the wrinkled front due to turbulent velocity v, A_L is the cross-sectional area with respect to the direction of front propagation, and S_L is the laminar velocity (say, the natural chemical front speed when $v = 0$). When the flame thickness is small, a flame front can be approximated by a surface that is wrinkled by the turbulent velocity. The problem is reduced to determining the surface area increase due to wrinkling. Yet it is highly nontrivial how A_T depends on v. Let U be the ratio of the rms's of v and S_L. The Clavin–Williams [36] relation for small U is

$$S_T/S_L \sim 1 + U^2, \tag{4.4.3}$$

which has been confirmed in the case of a periodic flow field by Papanicolaou and Xin [126]. Recently, Kerstein and Ashurst [98] suggested that the Clavin–Williams relation may be applicable only to periodic flows (see also [3]). They proposed instead

$$S_T/S_L \sim 1 + U^{4/3} \tag{4.4.4}$$

for random flows. Their simulation also showed that the front has to propagate a long distance, on the order of $O(U^{-2/3}L_I)$, before reaching the steady flame speed in (4.4.4). Here L_I is the integral scale of the turbulent velocity v. For small U, this time scale is very large for computation as well as experiments. It will be extremely interesting to establish (4.4.4) in a rigorous way. For large values of U, the enhancement of propagation speed is shown experimentally to level off, as seen in Figure 3 of [134].

Another feature of the combustion modeling is that instead of simulating a reaction-diffusion-advection equation like (4.4.1), a popular H-J (so-called) *G-equation*

$$G_t + v \cdot \nabla G = S_L |\nabla G| \tag{4.4.5}$$

is used. The G-equation was proposed by Kerstein, Ashurst, and Williams [97] so that the level set of G represents a flame surface under the wrinkling of turbulent velocity v. Direct numerical simulations have been performed for small values of U where S_L is not too small compared with v. In the context of the G-equation, S_T is defined as $S_L\langle|\nabla G|\rangle$. Yakhot [165] used a renormalization expansion procedure to find the more general relation for S_T as

$$U_T = e^{(U/U_T)^p} \tag{4.4.6}$$

with $p = 2$, where $U_T = S_T/S_L$. For large U, (4.4.5) implies that U_T is approximately $U/\sqrt{\ln U}$. In [98], (4.4.6) is modified to a power $p = 4/3$ inside the exponential. Peters [128] studied the mean and variance of G within closure approximations and also modeled additional effects such as curvature and local flow straining.

What determines the different power laws in (4.4.3), (4.4.4), and (4.4.6) is recognized to be the energy spectrum of v, i.e., the way the fluid energy is distributed in wavenumber space [134]. Typically, high Reynolds number flows have a broad range of scales, which tends to increase the wrinkled flame surface areas and so S_T. On the other hand, if too much energy is concentrated in a narrow range of scales, islands of reactants can form and decrease the effective flame surface area, making S_T smaller. An analysis of island formation is given in Joulin and Sivashinsky [92]. For an interesting experimental comparison of flame speeds in the presence of single- and multiple-scale Taylor–Couette flows (flows in the annulus between two concentric cylinders), see Shy et al. [143], Ronney, Haslam, and Rhys [135], and Figure 4 of [134].

H-J equations with random coefficients similar to the G-equation (4.4.5) also arise in understanding the formation and roughening of nonequilibrium interfaces; see Kardar, Parisi, and Zhang (KPZ) [94], Barabási and Stanley [6], Family and Vicsek [55], and references therein. These random H-J equations, known as *KPZ equations*, are of the form

$$h_t = \nu_x h_{xx} + \nu_y h_{yy} + \frac{\lambda_x}{2} h_x^2 + \frac{\lambda_y}{2} h_y^2 + \eta, \tag{4.4.7}$$

where the constants ν_x, λ_x, etc., are measurable parameters of the underlying physical processes and $\eta = \eta(x, y, t)$ is a Gaussian noise. A common property of most rough interfaces observed experimentally or in discrete models is that their roughening obeys simple power laws. The morphology and dynamics of a rough interface can be characterized by the surface width $w(t, L)$, defined as

$$w^2(t, L) = \langle [h(x, t) - \bar{h}(t)]^2 \rangle = L^{2\alpha} f(t/L^z), \tag{4.4.8}$$

where α is the roughness exponent of the interface described by its height $h(x, t)$, z is the dynamic exponent characterizing the scaling of the relaxation times with the system size L, $\bar{h}(t)$ is the mean height of the interface at time t, and $\langle \cdot \rangle$ denotes both ensemble and space average. The scaling function $f = f(u)$ behaves like $f \sim u^{2\alpha/z}$ for small u and $f \sim$ const. for large u. Such a surface roughening phenomenon occurs in technologically important processes such as sputter etching, where surface morphology evolves due to erosion. Usually one relies on numerical simulation and physical reasoning or a formal renormalization method to substantiate (4.4.8).

A scaling and roughening study of flame fronts is discussed in Provatas et al. [131] using a model of two coupled R-D equations for temperature and reactant concentration,

$$\begin{aligned} T_t &= D\Delta_x T - \Gamma(T - T_0) + R(T, C), \quad x \in \mathbb{R}^2, \\ \lambda_1 C_t &= -R(T, C), \end{aligned} \tag{4.4.9}$$

where $R(T, C) = \lambda_2 T^{3/2} e^{-A/T} C$, the Arrhenius reaction, λ_2 and A are positive constants, T_0 is the background temperature, and random concentration initial data $C_0(x)$ are given. The function C_0 is equal to 1 with probability c and 0 with probability $1 - c$. The average initial concentration is c. Interestingly, (4.4.9) can model forest fires with appropriate values of its parameters. The authors showed that there is a threshold c^* below which no temperature fronts propagate and above which fronts propagate. A mean field theory of percolation is developed to explain the transition. For $c > c^*$, the moving T interface develops large fluctuations and appears rough. The roughness scales like the solutions of the KPZ equation. In the long-wavelength and almost-uniform background concentration limit, the KPZ equation is derived for the interface.

4.5. Summary, Figures, and Concluding Remarks. We have shown that there are two types of random media, the tame and the wild. Tame random media satisfy finite moment conditions and exhibit short correlations. There is a nice analogy between fronts in tame random media and the classical central limit theorem in probability. We showed that the front locations in equations of Burgers type obey Gaussian statistics and the central limit theorem. The Burgers equation with random flux is also a place to apply the homogenization of the random H-J equation, which is employed shortly after to derive the KPP front speeds in random media obtained by the large deviation approach. We also presented a model KPP equation with space-time

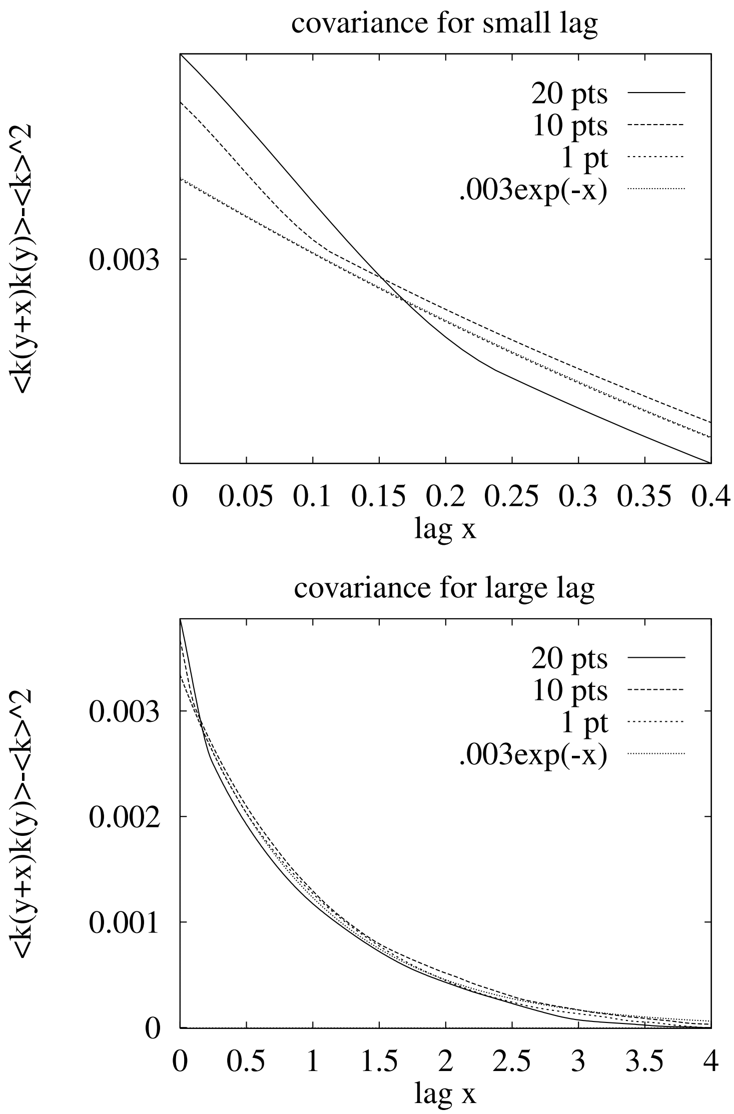

Fig. 4 *Covariance function for the random coefficients.* Adapted from M. Postel and J. Xin, *Comp. Geosci.,* 1 (1997), pp. 251–270, by permission of Baltzer Science Publishers.

white noise, where the front width is a.s. finite and the front speed and shape exist in a proper sense.

For wild (turbulent) random media, we motivated the anomalous phenomena with stable laws in probability and the linear advection-diffusion problem. Later the upper bounds of front speeds in a turbulent shear flow suggested that front speed anomalies exist and front acceleration occurs. We finally illustrated the related modeling issues in turbulent premixed combustion and the usefulness of the KPZ equation (a noisy H-J equation) in understanding the evolution of stochastic interfaces in industrial applications.

Let us show some figures on fronts to contrast different media and models and also to help bring out some future research problems. Figures 4–8 come from simulations

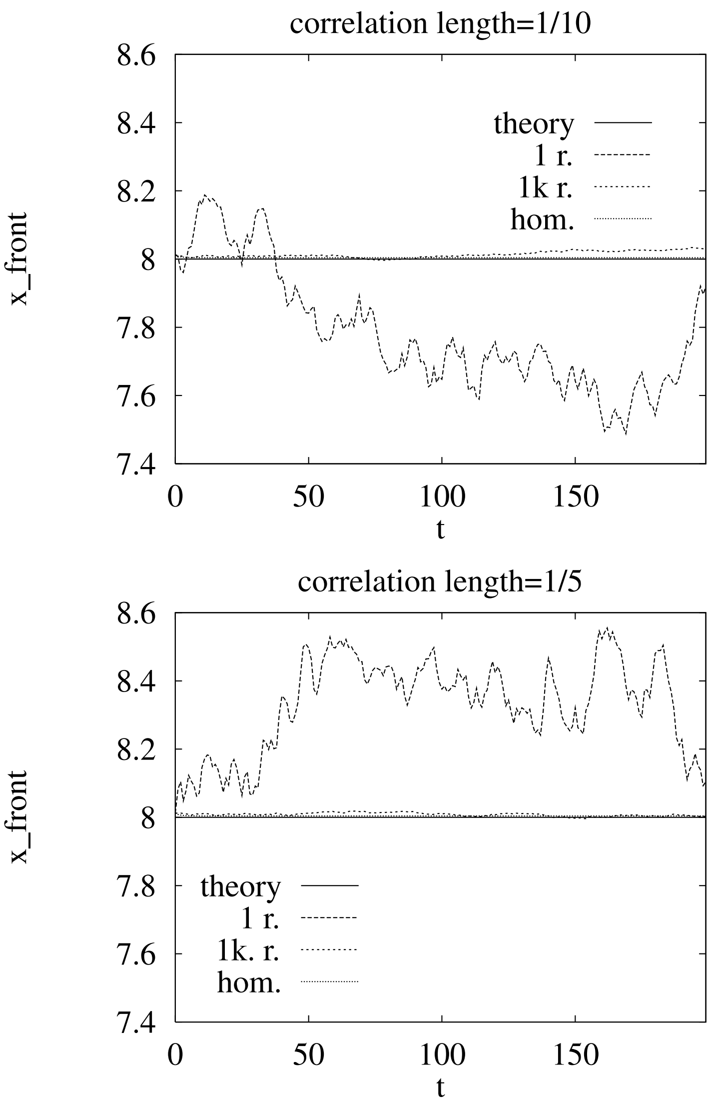

Fig. 5 *Front position for correlation lengths* 1/5 *and* 1/10. Adapted from M. Postel and J. Xin, *Comp. Geosci.,* 1 (1997), pp. 251–270, by permission of Baltzer Science Publishers.

of the random conservative equation

$$\left(u + k(x,\omega)\frac{u}{1+u}\right)_t \; = \; 0.02u_{xx} - u_x$$

in the moving frame $y = x - st$, with boundary conditions $u(t, -\infty) = 1$ and $u(t, +\infty) = 0$. The random stationary process $k(x, \omega)$ has mean $\langle k \rangle = 1$ and standard deviation $\sigma_k = 0.29$. It is constructed as follows. First, we generate the values of k at discrete points with spacing δ_r by $k_i = \beta k_{i-1} + \eta_i$, where $\beta \in (0, 1)$ and the η_i's are uniformly distributed random variables. The variables k_i are correlated with covariance $\langle k_i - \langle k \rangle\rangle \cdot \langle k_{i+n} - \langle k \rangle\rangle = \sigma_k^2 \beta^n$ and correlation length $l_c \equiv -\delta_r / \log \beta$.

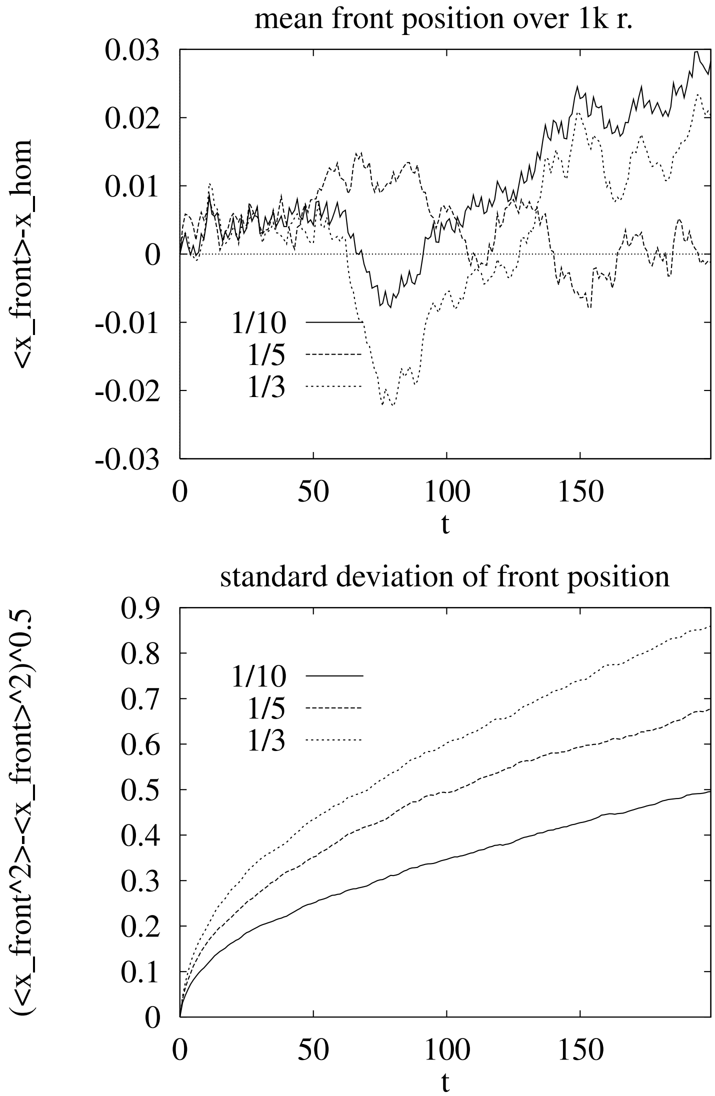

Fig. 6 *Mean and variance of the front position for correlation lengths* $1/5$ *and* $1/10$. Adapted from M. Postel and J. Xin, *Comp. Geosci.,* 1 (1997), pp. 251–270, by permission of Baltzer Science Publishers.

Then we put numerical grid points in the spacing δ_r, with grid size $\delta_y = M^{-1}\delta_r$, $M \geq 1$. The values of k at the grid points are obtained from linear interpolation. Figure 4 shows the covariance function of the process k with $M = 1$, 10, and 20 for small and large lag (separation). The curves decay to zero exponentially for large lag in the same way and differ only slightly for small lag.

For the generated k, the front speed is calculated theoretically according to formula (3.9.4). The numerical domain is the interval $y \in [0, 17.5]$, with $u = 1$ at $y = 0$, $u = 0$ at $y = 17.5$, and $\delta_y = 0.025$, $\delta_t = 0.1$. An upwind finite difference approximation is used. The initial front starts at $y = 8$. We see in Figure 5 that the front

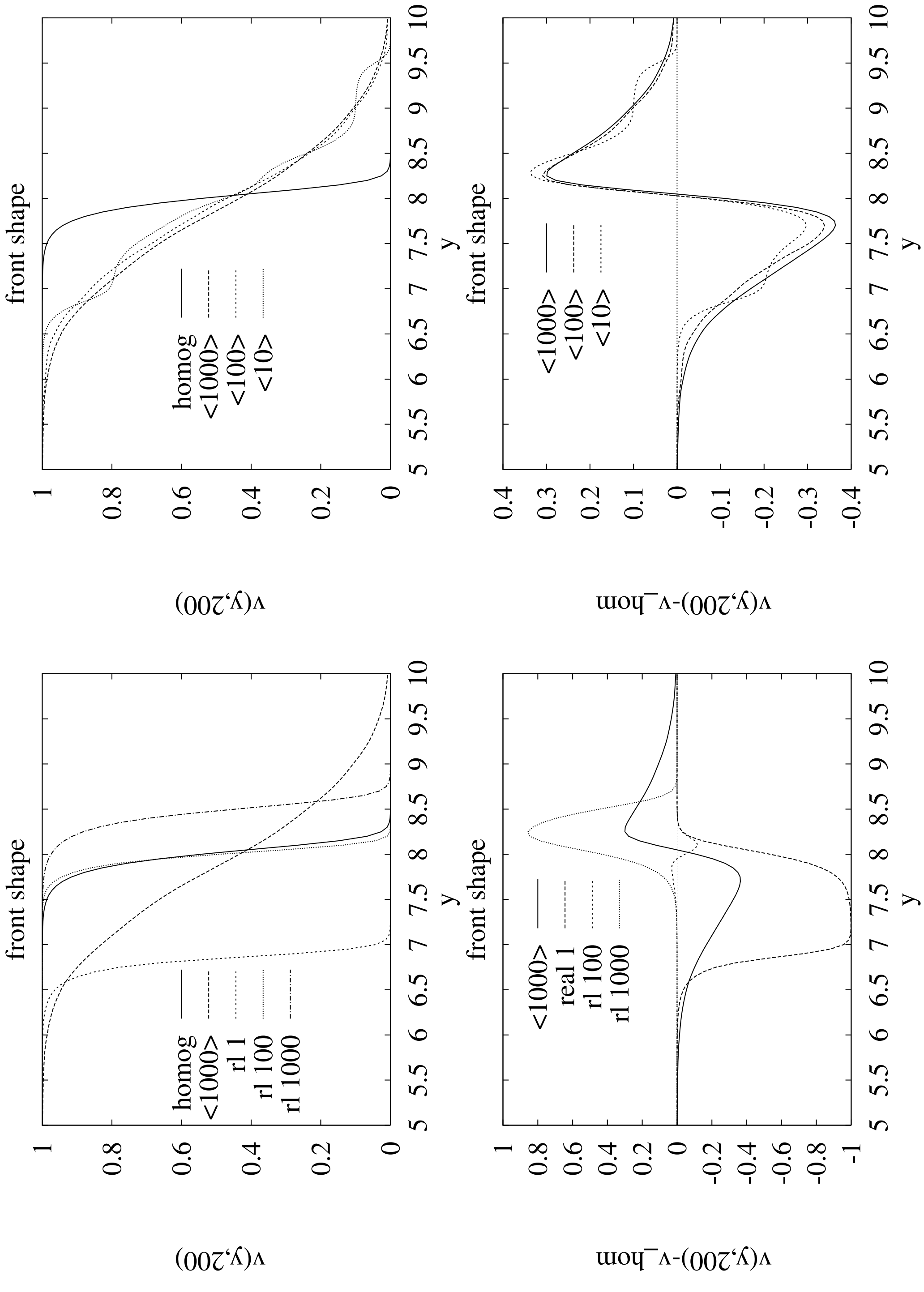

Fig. 7 *Influence of randomness on front shape.* Adapted from M. Postel and J. Xin, *Comp. Geosci.*, 1 (1997), pp. 251–270, by permission of Baltzer Science Publishers.

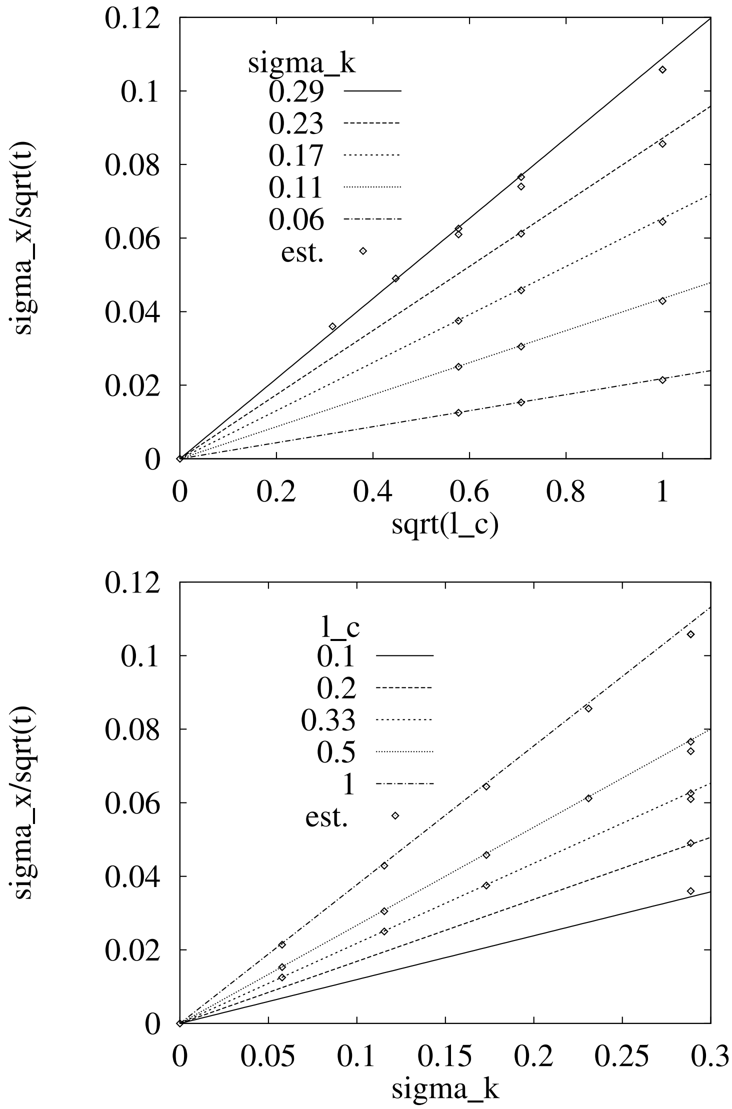

Fig. 8 *Influence of the standard deviation and correlation length of the Langmuir coefficient on the standard deviation of the front position.* Adapted from M. Postel and J. Xin, *Comp. Geosci.*, 1 (1997), pp. 251–270, by permission of Baltzer Science Publishers.

location of a single realization fluctuates about the mean position, which is the one predicted by (3.9.4), the homogeneous front location, and is very close to the averaged (over 1000 realizations) front location. For different realizations and different correlation lengths, we can have either the up or the down fluctuation. In Figure 6, we show the mean front location and the standard deviations averaged over 1000 realizations for three correlation lengths. The parabolic shapes of the curves of standard deviation vs. time show the $\sqrt{t}$ scaling. In Figure 7, we show the front shapes. In the top-left frame, three single realizations jump about the mean position. The fronts look the same since the front width is comparable to the spatial scale of k. In the top-right frame, we average the fronts over 10, 100, and 1000 realizations. In the 10-realization averaging, we see the appearance of spatial structures inside the fronts as they are being widened. The bottom two frames are the corresponding differences between the random and homogeneous fronts. In Figure 8, we plot the $\sqrt{t}$-normalized front deviation σ_x vs. $\sqrt{l_c}$ and σ_k. The linear dependence in either case is evident, and suggests the empirical formula $\sigma_x = \text{const.}\sigma_k\sqrt{l_c}$, which is derivable from (4.1.26). In fact, summing up $\sigma_k^2\beta^n/n$ provides a discrete approximation to σ_a^2. So, for small β,

$$\sigma_a^2 = \sigma_k^2(1-\beta)^{-1} \sim \frac{-\sigma_k^2}{\log\beta} = \sigma_k^2 l_c/\delta_r,$$

and upon substitution into (4.1.26) we get the $\sigma_k\sqrt{l_c}$ factor. In fact, with the simulation parameters, the prefactor constant obtained from the above approximation and (4.1.26) is 2.357, while direct simulation gives 2.65, a very good agreement [129].

These computational results suggest the following problems.

• Study the statistics of front locations for convex scalar conservation laws with randomness and establish a central limit theorem. Also study the statistics of R-D fronts. A first step is to carry out numerical simulations of KPP fronts to collect evidence of a Gaussian law.

• Study further the qualitative properties of R-D front speeds, such as enhancement phenomena, by employing variational methods. In the KPP case, it would be interesting to obtain bounds on the μ function for a given random medium.

• Investigate the proper extension of the periodic homogenization results of [105] to the random H-J equation and analyze properties of the effective Hamiltonian. Equally interesting is the problem of establishing the KPP front speeds via the H-J formalism that we adopted here to derive the known KPP results. Since random H-J equations arise in many other applications as well, they play the role of universal equations for front propagation problems.

• Study the anomalous propagation law for fronts in turbulent media.

It is clear that studies of fronts in heterogeneous and, in particular, in random media are at the beginning stage. We hope that the ideas, techniques, and results discussed in this review will benefit future researchers in this rich and exciting area.

Acknowledgments. I would like to thank Prof. George Papanicolaou for suggesting this survey project and for his helpful comments. I would also like to thank G. Eyink, T.-Y. Lee, J. Wehr, and B. Zinner for many interesting discussions. It is a pleasure to thank all the anonymous referees and Prof. Nick Trefethen for their constructive comments on the early manuscripts that led to many improvements in the paper.

REFERENCES

[1] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. Math., 30 (1978), pp. 33–76.

[2] D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve propagation.* Lecture Notes in Math. 446, Springer-Verlag, Berlin, 1975, pp. 5–49.

[3] W. T. Ashurst and G. I. Sivashinsky, *On the flame propagation through periodic flow fields*, Combustion Sci. Technol., 80 (1991), pp. 159–164.

[4] M. Avellaneda and A. J. Majda, *Mathematical models with exact renormalization for turbulent transport*, Comm. Math. Phys., 131 (1990), pp. 381–429.

[5] M. Avellaneda and A. J. Majda, *Mathematical models with exact renormalization for turbulent transport* II*: Fractal interfaces, non-Gaussian statistics and the sweeping effect*, Comm. Math. Phys., 146 (1992), pp. 139–204.
[6] A.-L. Barabási and H. E. Stanley, *Fractal Concepts in Surface Growth*, Cambridge University Press, Cambridge, UK, 1995.
[7] G. I. Barenblatt, Y. B. Zeldovich, and A. G. Istranov, *On the diffusional-thermal stability of a laminar flame*, Zh. Prikl. Mekh. Tekh. Fiz., 4 (1962), pp. 21–26.
[8] G. Barles, L. C. Evans, and P. E. Souganidis, *Wavefront propagation for reaction-diffusion systems of PDE*, Duke Math. J., 61 (1990), pp. 835–859.
[9] G. K. Batchelor, *The Theory of Homogeneous Turbulence,* Cambridge Science Classics, Cambridge University Press, Cambridge, UK, 1982.
[10] R. D. Benguria and M. C. Depassier, *Validity of the linear speed selection mechanism for fronts of the nonlinear diffusion equation*, Phys. Rev. Lett., 73 (1994), pp. 2272–2274.
[11] R. D. Benguria and M. C. Depassier, *Variational characterization of the speed of propagation of fronts for the nonlinear diffusion equation*, Comm. Math. Phys., 175 (1996), pp. 221–227.
[12] R. D. Benguria and M. C. Depassier, *Speed of fronts of the reaction-diffusion equation*, Phys. Rev. Lett., 77 (1996), pp. 1171–1173.
[13] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, Stud. Math. Appl. 5, North-Holland, Amsterdam, 1978.
[14] H. Berestycki and B. Larrouturou, *Quelque aspects mathématiques de la propagation des flames prémélangée*, Pitman Res. Notes Math. Ser. 220, H. Brezis and J.-L. Lions, eds., 1991, p. 65.
[15] H. Berestycki, B. Larrouturou, and P.-L. Lions, *Multi-dimensional travelling-wave solutions of a flame propagation model*, Arch. Rational Mech. Anal., 111 (1990), pp. 33–49.
[16] H. Berestycki, B. Larrouturou, and J.-M. Roquejoffre, *Stability of traveling fronts in a model for flame propagation, part* 1*: Linear stability*, Arch. Rational Mech. Anal., 117 (1992), pp. 97–117.
[17] H. Berestycki, B. Nicolaenko, and B. Scheurer, *Traveling wave solutions to combustion models and their singular limits*, SIAM J. Math. Anal., 16 (1985), pp. 1207–1242.
[18] H. Berestycki and L. Nirenberg, *Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains*, in Analysis et cetera, P. Rabinowitz and E. Zehnder, eds., Academic Press, New York, 1990, pp. 115–164.
[19] H. Berestycki and L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat., 22 (1991), pp. 1–37.
[20] H. Berestycki and L. Nirenberg, *Traveling fronts in cylinders*, Ann. Inst. H. Poincaré Anal. Non Linéare, 9 (1992), pp. 497–572.
[21] J. Billingham and D. J. Needham, *A note on the properties of a family of traveling wave solutions arising in cubic autocatalysis*, Dynam. Stability Systems, 6 (1991), pp. 33–49.
[22] J. Billingham and D. J. Needham, *The development of traveling waves in quadratic and cubic autocatalysis with unequal diffusion rates,* I. *Permanent form traveling waves*, Philos. Trans. Roy. Soc. London Ser. A, 334 (1991), pp. 1–24.
[23] J. Billingham and D. J. Needham, *The development of traveling waves in quadratic and cubic autocatalysis with unequal diffusion rates,* II. *The initial value problem with an immobilized or nearly immobilized autocatalyst.*, Philos. Trans. Roy. Soc. London Ser. A, 336 (1991), pp. 497–539.
[24] J. Billingham and D. J. Needham, *The development of traveling waves in quadratic and cubic autocatalysis with unequal diffusion rates,* III. *Large time development in quadratic autocatalysis*, Quart. Appl. Math., 1 (1992), pp. 343–372.
[25] P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York, 1968.
[26] A. Bonnet, B. Larrouturou, and L. Sainsaulieu, *On the stability of multiple steady planar flames when the Lewis number is less than one*, Phys. D, 69 (1993), pp. 345–352.
[27] W. Bosma and S. van der Zee, *Transport of reacting solute in a one-dimensional chemically heterogeneous porous media*, Water Resources Research, 29 (1993), pp. 117–131.
[28] W. Bosma, S. van der Zee, and C. J. van Duijn, *Plume development of a nonlinearly adsorbing solute in heterogeneous porous formations*, Water Resources Research, 32 (1996), pp. 1569–1584.
[29] M. Bramson, *Convergence of Solutions of the Kolmogorov Equations to Traveling Waves*, Mem. Amer. Math. Soc. 44, Providence, RI, 1983.
[30] L. Breiman, *Probability*, Classics Appl. Math., SIAM, Philadelphia, 1992.
[31] J. Bricmont and A. Kupiainen, *Stability of moving fronts in the Ginzburg–Landau equations*, Comm. Math. Phys., 159 (1994), pp. 287–318.

[32] J. Bricmont and A. Kupiainen, *Renormalizing partial differential equations*, in Constructive Physics, Results in Field Theory, Statistical Mechanics and Condensed Matter Physics. Proceedings of the Conference, Springer, Berlin, 1995, pp. 83–115.

[33] J. Bricmont, A. Kupiainen, and J. Xin, *Global large time self-similarity of a thermal-diffusive combustion system with critical nonlinearity*, J. Differential Equations, 130 (1996), pp. 9–35.

[34] L.-Y. Chen, N. Goldenfeld, and Y. Oono, *Renormalization group theory and variational calculations for propagating fronts*, Phys. Rev. E, 49B (1994), pp. 4502–4511.

[35] L.-Y. Chen, N. Goldenfeld, Y. Oono, and G. Paquette, *Selection, stability and renormalization*, Phys. A, 204 (1994), pp. 111–133.

[36] P. Clavin and F. A. Williams, *Theory of premixed-flame propagation in large-scale turbulence,* J. Fluid Mech., 90 (1979), pp. 598–604.

[37] P. Collet and J. Xin, *Global existence and large time asymptotic bounds of L^∞ solutions of thermal diffusive combustion systems on $\mathbb{R}^n$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 23 (1996), pp. 625–642.

[38] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc., 27 (1992), pp. 1–67.

[39] M. G. Crandall, L. C. Evans, and P.-L. Lions, *Some properties of viscosity solutions of Hamilton–Jacobi equations,* Trans. Amer. Math. Soc., 282 (1984), pp. 487–502.

[40] M. C. Cross and P. C. Hohenberg, *Pattern formation out of equilibrium*, Rev. Modern Phys., 65 (1993), pp. 851–1112.

[41] C. N. Dawson, C. J. van Duijn, and R. E. Grundy, *Large time asymptotics in contaminant transport in porous media*, SIAM J. Appl. Math., 56 (1996), pp. 965–993.

[42] D. A. Dawson and E. Perkins, *Historical Processes*, Mem. Amer. Math. Soc. 0065–9266, Providence, RI, 1991.

[43] M. D. Donsker and S. R. S. Varadhan, *Asymptotics for the Wiener sausage*, Comm. Pure Appl. Math., 28 (1975), pp. 525–565.

[44] R. Durrett, *Probability: Theory and Examples*, 2nd ed., Wadsworth and Brooks, Belmont, CA, 1995.

[45] J.-P. Eckmann and C. E. Wayne, *The nonlinear stability of front solutions for parabolic pde's*, Comm. Math. Phys., 161 (1994), pp. 323–334.

[46] P. F. Embid, A. J. Majda, and P. E. Souganidis, *Comparison of turbulent flame speeds from complete averaging and the G-equation*, Phys. Fluids, 7 (1995), pp. 2052–2060.

[47] P. F. Embid, A. J. Majda, and P. E. Souganidis, *Effective geometric front dynamics for premixed turbulent combustion with separated velocity scales*, Combustion Sci. Techn., 103 (1994), pp. 85–100.

[48] L. C. Evans, *Partial Differential Equations*, AMS, Providence, RI, 1998.

[49] L. C. Evans, *Weak Convergence Methods for Nonlinear PDEs*, NSF-CBMS Regional Conference Ser. Math. 74, Providence, RI, 1990.

[50] L. C. Evans, *Periodic homogenization of certain nonlinear PDE's*, Proc. Roy. Soc. Edinburgh Sect. A, 120 (1992), pp. 245–265.

[51] L. C. Evans and P. E. Souganidis, *A PDE approach to geometric optics for certain semilinear parabolic equations*, Indiana Math. J., 38 (1989), pp. 141–172.

[52] L. C. Evans and P. E. Souganidis, *A PDE approach to certain large deviation problems for systems of parabolic equations*, Ann. Inst. H. Poincaré Anal. Non Linéare, 6 (1989), Suppl., pp. 229–258.

[53] L. C. Evans and P. E. Souganidis, *Differential games and representation formulas for solutions of Hamilton–Jacobi–Isaacs equations*, Indiana Math. J., 33 (1984), pp. 773–797.

[54] E. B. Fabes and D. W. Stroock, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Rational Mech. Anal., 96 (1986), pp. 327–338.

[55] F. Family and T. Vicsek, eds., *Dynamics of Fractal Surfaces*, World Scientific, Singapore, 1991.

[56] A. Fannjiang and G. Papanicolaou, *Diffusion in Turbulence*, Probab. Theory Related Fields, 105 (1996), pp. 279–334.

[57] A. Fannjiang and G. Papanicolaou, *Convection enhanced diffusion for random flows*, J. Statist. Phys., 88 (1997), pp. 1033–1076.

[58] G. Fennemore and J. X. Xin, *Wetting fronts in one dimensional periodically layered soils,* SIAM J. Appl. Math., 58 (1998), pp. 387–427.

[59] P. C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, Lecture Notes in Biomath. 28, Springer-Verlag, New York, 1979.

[60] P. C. Fife, *Dynamics of Internal Layers and Diffusive Interfaces*, CBMS-NSF Regional Conf. Ser. Appl. Math. 53, SIAM, Philadelphia, 1988.

[61] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to traveling front solutions*, Arch. Rational Mech. Anal., 65 (1977), pp. 335–362.

[62] R. A. Fisher, *The wave of advance of advantageous genes,* Ann. Eugenics, 7 (1937), pp. 355–369.

[63] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Appl. Math. 25, Springer-Verlag, New York, 1993.

[64] S. Focant and Th. Galley, *Existence and stability of propagating fronts for an autocatalytic reaction-diffusion system*, Phys. D, 120 (1998), pp. 346–368.

[65] M. I. Freidlin, *Propagation of a concentration wave in the presence of random motion associated with the growth of a substance*, Soviet Math. Dokl., 20 (1979), pp. 503–507.

[66] M. I. Freidlin, *Limit theorems for large deviations and reaction-diffusion equations*, Ann. Probab., 13 (1985), pp. 639–675.

[67] M. I. Freidlin, *Functional Integration and Partial Differential Equations*, Ann. Math. Stud. 109, Princeton University Press, Princeton, NJ, 1985.

[68] M. I. Freidlin and T.-Y. Lee, *Large deviation principle for the diffusion-transmutation processes and Dirichlet problem for PDE systems with small parameter*, Probab. Theory Related Fields, 105 (1996), pp. 227–254.

[69] M. I. Freidlin and T.-Y. Lee, *Wave front propagation and large deviations for diffusion-transmutation process*, Probab. Theory Related Fields, 106 (1996), pp. 39–70.

[70] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Springer-Verlag, New York, 1984.

[71] G. Fusco and J. K. Hale, *Stable equilibrium in a scalar parabolic equation with variable coefficients*, SIAM J. Math. Anal., 16 (1995), pp. 1152–1164.

[72] Th. Gallay, *Local stability of critical fronts in nonlinear parabolic pde's*, Nonlinearity, 7 (1994), pp. 741–764.

[73] R. Gardner, *Existence of multidimensional traveling wave solutions of an initial boundary value problem,* J. Differential Equations, 61 (1986), pp. 335–379.

[74] R. Gardner, C. K. R. T. Jones, and T. Kapitula, *Stability of traveling waves for non-convex scalar viscous conservation laws*, Comm. Pure. Appl. Math., 46 (1993), pp. 505–526.

[75] J. Gärtner and M. I. Freidlin, *On the propagation of concentration waves in periodic and random media*, Soviet Math. Dokl., 20 (1979), pp. 1282–1286.

[76] J. Gärtner, *Bistable reaction-diffusion equations and excitable media*, Math. Nachr., 112 (1983), pp. 125–152.

[77] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, New York, 1983.

[78] R. E. Goldstein, P. Nelson, T. Powers, and U. Seifert, *Front propagation in the pearling instability of tubular vesicles*, J. Physique II, 6 (1996), pp. 767–796.

[79] J. Goodman, *Stability of viscous shock profiles in several space dimensions*, Trans. Amer. Math. Soc., 311 (1989), pp. 683–695.

[80] R. E. Grundy, C. J. van Duijn, and C. N. Dawson, *Asymptotic profiles with finite mass in one-dimensional contaminant transport through porous media: Fast reaction case*, Quart. J. Mech. Appl. Math., 47 (1994), pp. 69–106.

[81] K. P. Hadeler and F. Rothe, *Traveling fronts in nonlinear diffusion equations*, J. Math. Bio., 2 (1975), pp. 251–263.

[82] B. G. Haslam and P. D. Ronney, *Fractal properties of propagating fronts in a strongly stirred fluid*, Phys. Fluids, 7 (1995), pp. 1931–1937.

[83] S. Heinze, *A Variational Approach to Traveling Waves*, preprint, 1996.

[84] S. Heinze, *Homogenization of Flame Fronts*, preprint, 1996.

[85] S. Heinze, *Wave Solutions to Reaction-Diffusion Systems in Perforated Domains*, preprint, 1996.

[86] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer-Verlag, New York, 1981.

[87] D. Horvath, V. Petrov, S. Scott, and K. Schowalter, *Instabilities in propagating reaction-diffusion fronts*, J. Chem. Phys., 98 (1993), pp. 6332–6343.

[88] W. Hudson and B. Zinner, *Existence of traveling waves for reaction diffusion equations of Fisher type in periodic media*, in Boundary Problems for Functional Differential Equations, World Scientific, Singapore, 1995, pp. 187–199.

[89] A. M. Il'in and O. A. Oleinik, *Behavior of the solutions of the Cauchy problem for certain quasilinear equations for unbounded increase of time*, AMS Trans. Ser. 2, 42 (1964), pp. 19–23.

[90] C. Jones, *Stability of traveling wave solutions of the FitzHugh–Nagumo system*, Trans. Amer. Math. Soc., 286 (1984), pp. 431–469.
[91] G. Joulin and P. Clavin, *Linear stability analysis of nonadiabatic flames: Diffusion-thermal model*, Combust. and Flame, 35 (1979), pp. 139–153.
[92] G. Joulin and G. I. Sivashinsky, *Pockets in premixed flames and combustion-rate*, Combustion Sci. Tech., 77 (1991), pp. 329–335.
[93] Ja. Kanel, *Stabilization of solutions of the Cauchy problem for equations encountered in combustion theory*, Mat. Sb., 59 (1962), pp. 245–288.
[94] M. Kardar, G. Parisi, and Y.-C. Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett., 56 (1986), pp. 889–892.
[95] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.
[96] J. B. Keller, *Genetic variability due to geographical inhomogeneity*, J. Math. Biol., 20 (1984), pp. 223–230.
[97] A. R. Kerstein, W. T. Ashurst, and F. A. Williams, *Field equations for interface propagation in an unsteady homogeneous flow field*, Phys. Rev. A, 37 (1988), pp. 2728–2731.
[98] A. R. Kerstein and W. T. Ashurst, *Propagation rate of growing interfaces in stirred fluids*, Phys. Rev. Lett., 68 (1992), p. 934.
[99] K. Kirchgässner, *On the nonlinear dynamics of traveling fronts*, J. Differential Equations, 96 (1992), pp. 256–278.
[100] A. N. Kolmogorov, I. G. Petrovsky, and N. S. Piskunov, *Etude de l'eq́uation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Moskow Univ. Math. Bull., 1 (1937), pp. 1–25.
[101] R. H. Kraichnan, *Small-scale structure of a scalar field convected by turbulence*, Phys. Fluids, 11 (1968), pp. 945–963.
[102] T.-Y. Lee and F. Torcaso, *Wave propagation in a lattice KPP equation in random media*, Ann. Probab., 26 (1998), pp. 1179–1197.
[103] C. D. Levermore and J. Xin, *Multidimensional stability of traveling waves in a bistable reaction-diffusion equation*, Part II, Comm. Partial Differential Equations, 17 (1992), pp. 1901–1924.
[104] C. Li, *Monotonicity and symmetry on bounded and unbounded domains*, I & II, Comm. Partial Differential Equations, 16 (1991), pp. 491–526, 585–615.
[105] P. L. Lions, G. C. Papanicolaou, and S. Varadhan, *Homogenization of Hamilton–Jacobi Equations*, unpublished.
[106] A. J. Majda and P. E. Souganidis, *Large scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales*, Nonlinearity, 7 (1994), pp. 1–30.
[107] A. J. Majda and P. E. Souganidis, *Bounds on enhanced turbulent flame speeds for combustion with fractal velocity fields*, J. Statist. Phys., 83 (1996), pp. 933–954.
[108] A. Malevanets, A. Careta, and R. Kapral, *Biscale chaos in propagating front,* Phys. Rev. E, 52 (1995), pp. 4724–4735.
[109] S. Malham and J. Xin, *Global solutions to a reactive Boussinesq system with front data on an infinite domain*, Comm. Math. Phys., 193 (1998), pp. 287–316.
[110] J.-F. Mallordy and J.-M. Roquejoffre, *A parabolic equation of the KPP type in higher dimensions*, SIAM J. Math. Anal., 26 (1996), pp. 1–20.
[111] M. Marion, *Qualitative properties of a nonlinear system for laminar flames without ignition temperature*, Nonlinear Anal., 11 (1983), pp. 1269–1292.
[112] G. Matheron and G. de Marsily, *Is transport in porous media always diffusive? A counterexample*, Water Resources Research, 16 (1980), pp. 901–917.
[113] G. de Marsily, *Quantitative Hydrogeology—Groundwater Hydrology for Engineers*, Academic Press, New York, 1986.
[114] R. Martin and M. Pierre, *Nonlinear reaction-diffusion systems*, in Nonlinear Equations in the Applied Sciences, W. F. Ames and C. Rogers, eds., Academic Press, Boston, 1992.
[115] H. McKean, *Application of Brownian motion to the equation of KPP*, Comm. Pure Appl. Math., 28 (1975), pp. 323–331.
[116] M. Metcalf, J. Merkin, and S. Scott, *Oscillating wave fronts in isothermal chemical systems with arbitrary powers of autocatalysis*, Proc. Roy. Soc. London Ser. A, 447 (1994), pp. 155–174.
[117] C. Mueller and E. Perkins, *The compact support property of solutions to the heat equations with noise*, Probab. Theory Related Fields, 93 (1992), pp. 1–29.
[118] C. Mueller and R. Sowers, *Random travelling waves for the KPP equations with noise*, J. Funct. Anal., 128 (1995), pp. 439–498.
[119] J. D. Murray, *Mathematical Biology*, 2nd ed., Lecture Notes in Biomath. 19, Springer-Verlag, New York, 1993.

[120] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., 80 (1958), pp. 931–954.
[121] S. Nii, *Stability of the traveling multiple-front (multiple-back) wave solutions of the FitzHugh–Nagumo equations*, SIAM J. Math. Anal., 28 (1997), pp. 1094–1112.
[122] H. Osada, *Diffusion processes with generators of generalized divergence form*, J. Math. Kyoto Univ., 27 (1987), pp. 597–619.
[123] S. Osher and J. Ralston, *Stability of traveling waves in convective porous medium equation*, Comm. Pure Appl. Math., 35 (1982), pp. 737–749.
[124] G. C. Paquette and Y. Oono, *Structural stability and selection of propagating fronts in semilinear parabolic partial differential equations*, Phys. Rev. E, 49 (1994), pp. 2368–2388.
[125] G. Papanicolaou and S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, Colloq. Math. Soc. János Bolyai 27, Random Fields, Esztergom, Hungary, 1979, North Holland, Amsterdam, 1982, pp. 835–873.
[126] G. Papanicolaou and X. Xin, *Reaction-diffusion fronts in periodically layered media*, J. Stat. Phys., 63 (1991), pp. 915–931.
[127] J. Pauwelussen, *Nerve impulse propagation in a branching nerve system: A simple model*, Phys. D., 4 (1981), pp. 67–88.
[128] N. Peters, *A spectral closure for premixed turbulent combustion in the flamelet regime*, J. Fluid Mech., 242 (1992), pp. 611–629.
[129] M. Postel and J. Xin, *A numerical study of fronts in random media using a reactive solute transport model*, Comp. Geosci., 1 (1997), pp. 251–270.
[130] M. Protter and H. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1967.
[131] N. Provatas, T. Ala-Nissila, M. Grant, K. Elder, and L. Piché, *Scaling, propagation and kinetic roughening of flame fronts in random media*, J. Statist. Phys., 81 (1995), pp. 737–759.
[132] L. F. Richardson, *Atmospheric diffusion shown on a distance-neighbour graph*, Proc. Roy. Soc. London Ser. A, 110 (1926), pp. 709–737.
[133] L. C. Rogers and D. Williams, *Diffusion, Markov Processes and Martingales*, 1, Foundations, John Wiley, New York, 1994.
[134] P. Ronney, *Some Open Issues in Premixed Turbulent Combustion*, Lecture Notes in Phys. 449, Springer-Verlag, Berlin, 1995, pp. 1–22.
[135] P. D. Ronney, B. G. Haslam, and N. O. Rhys, *Front propogation rates in randomly stirred media*, Phys. Rev. Lett., 74 (1995), pp. 3804–3807.
[136] J.-M. Roquejoffre, *Stability of traveling fronts in a model for flame propagation, part 2: Nonlinear stability*, Arch. Rational Mech. Anal., 117 (1992), pp. 119–153.
[137] J.-M. Roquejoffre, *Convergence to traveling waves for solutions of a class of semilinear parabolic equations*, J. Differential Equations, 108 (1996), pp. 262–295.
[138] J.-M. Roquejoffre, *Global stability of traveling waves and convergence towards stacked families of waves in monotone parabolic systems*, SIAM J. Math. Anal., 27 (1996), pp. 1261–1269.
[139] J. Rubinstein, P. Sternberg, and J. Keller, *Fast Reaction, Slow Diffusion, and Curve Shortening*, SIAM J. Appl. Math., 49 (1989), pp. 116–133.
[140] B. Sandstede, *Stability of N-fronts bifurcating from a twisted heteroclinic loop and an application to the FitzHugh–Nagumo equation*, SIAM. J. Math. Anal., 29 (1998), pp. 183–207.
[141] D. H. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Adv. Math., 22 (1976), pp. 312–355.
[142] N. Shigesada, K. Kawasaki, and E. Teramoto, *Traveling periodic waves in heterogeneous environments*, Theor. Population Biol., 30 (1986), pp. 143–160.
[143] S. S. Shy, P. D. Ronney, S. G. Buckley, and V. Yakhot, 24th *International Symposium on Combustion*, Combustion Institute, Pittsburgh, PA, 1992, pp. 543–551.
[144] G. I. Sivashinsky, *Instabilities, pattern formation, and turbulence in flames*, Ann. Rev. Fluid Mech, 15 (1983), pp. 179–199.
[145] G. I. Sivashinsky, *Cascade renormalization theory of turbulent flame speed*, Combustion Sci. Tech, 62 (1988), pp. 77–96.
[146] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.
[147] K. Uchiyama, *The behavior of some nonlinear diffusion equations for large time*, J. Math. Kyoto Univ., 18 (1978), pp. 453–508.
[148] S. E. A. T. M. van der Zee and W. H. van Riemsdijk, *Transport of reactive solute in spatially variable soil systems*, Water Resources Research, 23 (1987), pp. 2059–2069.
[149] C. J. van Duijn and P. Knabner, *Solute transport in porous media with equilibrium and*

nonequilibrium multiple sites adsorption, travelling waves, J. Reine Angew. Math., 415 (1991), pp. 1–49.

[150] S. R. S. Varadhan, *Stochastic Processes*, Courant Institute Lecture Notes, New York University, 1968.

[151] S. R. S. Varadhan, *Large Deviations and Applications*, CBMS-NSF Regional Conf. Ser. Appl. Math. 46, SIAM, Philadelphia, 1984.

[152] S. R. S. Varadhan, *Stochastic differential equations—Large deviations*, in Critical Phenomenon, Random Systems, Gauge Theories, K. Osterwalder and R. Stora, eds., Les Houches, 1984.

[153] A. Volpert, V. Volpert, and V. Volpert, *Traveling Wave Solutions of Parabolic Systems*, Transl. Math. Monographs 140, AMS, Providence, RI, 1994.

[154] V. Volpert and A. Volpert, *Existence and stability of stationary solutions for a class of semilinear parabolic systems*, Comm. Partial Differential Equations, 18 (1993), pp. 2051–2069.

[155] J. Wehr and J. Xin, *White noise perturbation of the viscous shock fronts of the Burgers equation*, Comm. Math. Phys., 181 (1996), pp. 183–203.

[156] J. Wehr and J. Xin, *Front speed in the Burgers equation with a random flux*, J. Statist. Phys., 88 (1997), pp. 843–871.

[157] J. Xin, *Existence and stability of travelling waves in periodic media governed by a bistable nonlinearity*, J. Dynam. Differential Equations, 3 (1991), pp. 541–573.

[158] J. X. Xin, *Multidimensional stability of traveling waves in a bistable reaction-diffusion equation, Part* I, Comm. Partial Differential Equations, 17 (1992), pp. 1889–1899.

[159] J. X. Xin, *Existence and uniqueness of travelling wave solutions in a reaction-diffusion equation with combustion nonlinearity*, Indiana Math. J., 40 (1991), pp. 985–1008.

[160] J. X. Xin, *Existence of planar flame fronts in convective-diffusive periodic media*, Arch. Rational Mech. Anal., 121 (1992), pp. 205–233.

[161] J. X. Xin, *Existence and nonexistence of traveling waves and reaction-diffusion front propagation in periodic media*, J. Statist. Phys., 73 (1993), pp. 893–926.

[162] J. X. Xin, *Existence of multidimensional traveling waves in transport of reactive solutes through periodic porous media*, Arch. Rational Mech. Anal., 128 (1994), pp. 75–103.

[163] J. X. Xin and J. Zhu, *Quenching and propagation of bistable reaction-diffusion fronts in multidimensional periodic media*, Phys. D, 81 (1995), pp. 94–110.

[164] J. X. Xin, *Stability of traveling waves in a solute transport equation*, J. Differential Equations, 135 (1997), pp. 269–298.

[165] V. Yakhot, *Propagation velocity of premixed turbulent flames*, Combustion Sci. Tech., 60 (1988), pp. 191–214.

[166] E. Zeidler, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1984.