Journal of Statistical Physics, Vol. 73, Nos. 5/6, 1993

Existence and Nonexistence of Traveling Waves and Reaction-Diffusion Front Propagation in Periodic Media

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Received March 2, 1993; final July 27, 1993

We review the existence results of traveling wave solutions to the reaction-diffusion equations with periodic diffusion (convection) coefficients and combustion (bistable) nonlinearities. We prove that whenever traveling waves exist, the solutions of the initial value problem with either frontlike or pulselike data propagate with the constant effective speeds of traveling waves in all suitable directions. In the case of bistable nonlinearity and one space dimension, we give an example of nonexistence of traveling waves which causes "quenching" ("localization") of wavefront propagation. Quenching (localization) only occurs when the variations of the media from their constant mean values are large enough. Our related numerical results also provide evidence for this phenomenon in the parameter regimes not covered by the analytical example. Finally, we comment on the role of the effective wave speeds in determining the effective wavefront equation (Hamilton-Jacobi equation) of the reaction-diffusion equations under the small-diffusion, fast-reaction limit with a formal geometric optics expansion.

KEY WORDS: Reaction-diffusion eqvations; homogenization; traveling waves; maximum principle; Hamilton-Jacobi equations.

1. INTRODUCTION

In this paper, we consider the long time asymptotic wavefront propagation of the solutions of the following reaction-diffusion (RD) equations:

$$\begin{aligned} u_t &= \nabla_x \cdot (a(x)\,\nabla_x u) + b(x) \cdot \nabla_x u + f(u) \\ u|_{t=0} &= u_0(x) \end{aligned} \tag{1.1}$$

under the following assumptions:

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0022-4715/93/1200-0893$07.00/0

A1. $a(x) = (a_{ij}(x))$, $x = (x_1, x_2, ..., x_n) \in R^n$ is a smooth positive-definite matrix on R^n, 1-periodic in each direction x_i.

A2. $b(x) = (b_j(x))$ is a smooth divergence-free vector field, 1-periodic in each direction x_i, and has mean equal to zero.

A3. $f(u)$ is either a combustion nonlinearity with ignition temperature cutoff, i.e., $f(u) = 0, u \in [0, \theta]$, for some $\theta \in (0, 1)$, $f(u) > 0$, $u \in (0, 1)$, $f(1) = 0$, $f(u) \in C^1[0, 1]$; or a bistable nonlinearity, i.e., $f(u) = u(1-u)(u-\mu)$, for some $\mu \in (0, 1/2)$.

We will remark on the case of the Lipschitz-continuous combustion-type nonlinearity later. The initial condition $u_0(x)$ is continuous and ranges between 0 and 1.

Equations of the form (1.1) appear in the study of premixed flame propagation through turbulence,[10] where u is the temperature of the combustible gas, $b(x)$ is the prescribed turbulent incompressible fluid velocity field with zero ensemble mean, $f(u)$ is the Arrhenius reaction term, and $a(x)$ is taken as a constant matrix. Based on their formal asymptotic analysis in the large-activation-temperature limit, Clavin and Williams[10] found that the temperature "profile" u propagates with effective turbulent flame speed. To understand the basic propagation features of solutions of (1.1) in inhomogeneous media from a rigorous analysis viewpoint, we consider both $a(x)$ and $b(x)$ to be periodic here. This is a special case of the general assumption that they are stationary ergodic random fields. We hope that our results in the periodic case will provide some insight into the more challenging random case.

The random diffusion matrix $a(x)$ arises in solute transport problems of hydrology,[11] where u is the concentration of the solute substance, $b(x)$ is the steady incompressible fluid velocity with homogeneous statistics, and $a(x)$ is the so-called pore-scale dispersion tensor. Equation (1.1) is thus general enough to relate to both of these applications above.

The combustion nonlinearity we consider here is an approximation of the Arrhenius reaction with an ignition temperature cutoff, which is commonly used in the literature to avoid the so-called "cold boundary difficulty" (see, e.g., refs. 4 and 8). The bistable nonlinearity comes about as a result of chemical reactions of overall order 3. The other important nonlinearity (Kolmogorov–Petrovsky–Piskunov nonlinearity), e.g., $f(u) = u(1-u)$ is due to second-order chemical reaction.[25,17]

Equation (1.1) with KPP nonlinearity was studied extensively by Freidlin,[13] using large-deviation techniques. He treated the periodic case in any space dimension and the random case in one space dimension. As will be clear later, the long-time asymptotic behavior of solutions of Eq. (1.1) with combustion or bistable nonlinearity is governed by a different form of geometric optics than the exponential-type geometric optics in the KPP case, which is amenable to the large-deviation methods. Our approach begins with constructing special traveling wave solutions to (1.1) and deriving their basic properties, in particular, the monotonicity property.[26–28] This is done by making use of maximum principle and applying the beautiful sliding domain techniques of Berestycki, Nirenberg, and Li.[5–7,19] Then we use the traveling wave solutions to estimate the solutions of the initial value problem (1.1) by the maximum principle. This is carried out by extending the classical results of Fife and McLeod[12] for the homogeneous bistable reaction-diffusion equation in one space dimension. Due to the presence of periodic coefficients in the problem, proving asymptotic stability of traveling waves in several space dimensions is very hard if not impossible. Motivated by the work of Aronson and Weinberger[1] and Freidlin,[13] we instead focus on the wave speed in the long-time asymptotic wavefront propagation and ignore the more delicate problem of studying the wave profile. This is achieved by taking long-time limits of solutions along rays. In practice, the wave speed is also a more interesting and observable quantity than the wave profile.[23,29] We find that as long as traveling waves exist, the solutions of (1.1) propagate with the speeds of traveling waves even though they may not converge to the profiles of traveling waves, which is what one tries to study in the stability analysis.

Traveling waves are shown to exist for Eq. (1.1) for general $a(x)$ and $b(x)$ under our earlier assumptions if $f(u)$ is a combustion nonlinearity.[28] However, if $f(u)$ is a bistable nonlinearity, then traveling waves in general exist only up to a certain level of variations of coefficients from their mean states. If traveling waves cease to exist, then there is no wave propagation, and the solutions are localized in space. In the combustion language, this is called "quenching." We illustrate the quenching phenomenon by both analytical and numerical examples. A related study on quenching in the one-dimensional bistable reaction-diffusion equation can be found in Pauwelussen,[20] where the diffusion coefficient is a piecewise constant function with one jump at $x=0$. If the jump is large enough, then traveling waves cannot pass the origin, and so quenching is referred to as wave blocking in ref. 20. We believe that quenching is a rather generic phenomenon, which depends mainly on the degree of spatial inhomogeneity, and not on the specific form of coefficients, be they a piecewise constant function, a periodic function, or otherwise. Yet it is interesting to see this happen for smooth coefficients.

The rest of the paper is organized as follows. In Section 2, we review the known existence results of traveling waves and their main properties. In Section 3, we construct subsolutions and supersolutions for solutions of

Eq. (1.1), using the traveling wave profiles, and establish the long-time asymptotic wavefront propagation (Theorem 3.1). In Section 4, we give an analytical example on quenching, using perturbation analysis. In Section 5, we present some numerical results illustrating quenching and its properties for different parameter regimes from that of Section 4. In Section 6, we show that long-time wavefront propagation is related to the large-space, large-time limit or homogenization limit of (1.1). Upon space-time rescaling, (1.1) is in the small-diffusion, fast-reaction form with rapidly oscillatory coefficients. We give a formal geometric optics expansion and derive the effective wavefront equation (the eikonal equation), which is a Hamilton–Jacobi equation with its Hamiltonian being the effective wave speed as a function of wave numbers (i.e., the dispersion relation).

2. EXISTENCE, UNIQUENESS, AND MONOTONICITY OF TRAVELING WAVES

We consider the traveling wave solutions of Eq. (1.1) of the form $u(x, t) = U(k \cdot x - ct, x)$, where k is a constant vector in R^n, and c is a unknown constant, the wave speed; U, as a function of $s = k \cdot x - ct$ and $y = x$, satisfies the boundary conditions $U(-\infty, y) = 0$, $U(+\infty, y) = 1$, and $U(s, \cdot)$ has period 1. Upon substitution into Eq. (1.1), we obtain the following equation for $U = U(s, y)$ and c:

$$(k\partial_s + \nabla_y)[a(y)(k\partial_s + \nabla_y)\, U] + b(y) \cdot (k\partial_s + \nabla_y)\, U + cU_s + f(U) = 0$$
$$U(-\infty, y) = 0, \qquad U(+\infty, y) = 1, \qquad U(s, \cdot) \text{ has period } 1 \qquad (2.1)$$

Equation (2.1) is a degenerate elliptic equation on the infinite cylinder $R^1 \times T^n$, T^n being the n-dimensional unit torus. We have the following results.

Theorem 2.1 (Existence, Combustion Case). Under assumptions A1–A3 and that $f'(1) < 0$, there exist classical solutions $(U(s, y), c)$ of Eq. (2.1) satisfying all the boundary conditions and such that

$$0 < U < 1, \qquad \forall (s, y) \in R^1 \times T^n \qquad (2.2)$$

$$U_s > 0, \qquad \forall (s, y) \in R^1 \times T^n \qquad (2.3)$$

$$c < 0 \qquad (2.4)$$

For a proof, see Xin.[28]

Theorem 2.2 (Existence, Bistable Case). Let $\bar{a} = \int_{T^n} a(x)\,dx$, and assume that A1–A3 are valid. There is a positive number δ_{cr} such that if

$$\|a(x) - \bar{a}\|_{H^s(T^n)} < \delta_{cr} \qquad \text{and} \qquad \|b(x)\|_{H^s(T^n)} < \delta_{cr}$$

where $s = s(n) > n + 1$, then there exist classical solutions $(U(s, y), c)$ of Eq. (2.1) satisfying all the boundary conditions (2.2)–(2.4).

Proof. By Xin,[26] there is $\delta > 0$ such that if

$$\|a(x) - \bar{a}\|_{H^s} < \delta, \qquad \|b(x)\|_{H^s} < \delta$$

then Eq. (2.1) admits classical solutions. Following Xin,[28] the set $\delta > 0$ for which solutions of (2.1) exist is open. So there exists a critical value δ_{cr} such that if $\delta < \delta_{cr}$, classical solutions exist. The rest of the theorem can be proved as in Xin.[27]

Theorem 2.3 (Monotonicity and Uniqueness). Under assumptions A1–A3 and that $f'(1) < 0$, any classical solutions (U, c) satisfy (2.2)–(2.4). Moreover, if (U', c') is another classical solution, then $c' = c$ and $U'(s, y) = U(s + s_0, y)$ for some $s_0 \in R^1$.

We refer to Xin[27] for a proof.

Remark 2.1. By Theorems 2.1–2.3, the wave speed c is a well-defined function of wave vector k, i.e., $c = c(k)$. A simple scaling argument shows that $c(\alpha k) = \alpha c(k)$ for any $\alpha > 0$. Thus $c(k)$ is homogeneous of degree 1. It is also not hard to show that under A1–A3, $c = c(k) \in C(R^n)$. This relation between c and k is often called the dispersion relation.

Following Proposition 1.1 in Xin,[28] we have:

Proposition 2.1. Assume that A1–A3 hold and that $f'(1) < 0$. Then any classical solution (U, c) of Eq. (2.1) satisfies:

$$0 < U \leqslant Ce^{\lambda s}, \qquad \forall s \leqslant s_1 \tag{2.5}$$

$$0 < 1 - U \leqslant Ce^{-\lambda s}, \qquad \forall s \geqslant s_2 \tag{2.6}$$

$$0 < U_s \leqslant Ce^{-\lambda |s|}, \qquad \forall\ |s| \geqslant s_3 \tag{2.7}$$

for some positive constants C, $-s_1$, s_2, s_3, and λ.

3. LARGE-TIME WAVEFRONT PROPAGATION

In this section, we construct subsolutions and supersolutions for Eq. (1.1), using traveling wave solutions as described in Section 2. The basic idea goes back to Fife and McLeod[12] and their Lemmas 4.1 and 6.1. The wavefront propagation will then follow directly from the subsolutions and supersolutions. We first treat the bistable case. Due to the monotonicity of traveling waves, the construction in ref. 12 carries through with minor modifications. We give all the details for the sake of completeness. Then we discuss the combustion case, where several new arguments have to be presented, and some more restrictions on the initial data are required.

Proposition 3.1 (Frontlike Data). Consider Eq. (1.1) with $f(u) = u(1-u)(u-\mu)$, $\mu \in (0, 1/2)$, and initial data $0 \leqslant u_0 \leqslant 1$. Assume that the traveling wave solutions exist. Let k be a unit vector in R^n, and define

$$S = \{ y \in R^n \mid y = x - (k \cdot x)\, k, \forall x \in R^n \}$$

Suppose that

$$\limsup_{k \cdot x \to -\infty} u_0(x) < \mu \tag{3.1}$$

$$\liminf_{k \cdot x \to +\infty} u_0(x) > \mu \tag{3.2}$$

uniformly in S. Then there exist smooth functions $\xi_i = \xi_i(t)$, $q_i = q_i(t)$, $i = 1, 2$, such that

$$U(k \cdot x - c(k)\, t - \xi_1, x) - q_1 \leqslant u(t, x) \leqslant U(k \cdot x - c(k)\, t + \xi_2, x) + q_2 \tag{3.3}$$

where for $i = 1, 2$,

$$\xi_i'(t) > 0, \qquad \xi_i(t) > 0, \qquad \sup_{t>0} |\xi_i(t)| < +\infty \tag{3.4}$$

$$q_i(t) > 0, \qquad q_i(t) \text{ is nonincreasing in } t \text{ and } = O(e^{-\gamma t}) \tag{3.5}$$

as $t \to +\infty$, for some $\gamma > 0$. Here, prime means time derivative.

Proof. Let us construct subsolution first. By (3.2), $\exists X_0 > 0$ such that if $k \cdot x \geqslant X_0$, then $u_0(x) > \mu$. Choose $q_1(0, x) \equiv q_1^{in}$, so that $\mu < 1 - q_1^{in} < u_0(x)$, if $k \cdot x \geqslant X_0$. Thus for any $\xi_1^{in} \equiv \xi_1(0)$, we have

$$U(k \cdot x - \xi_1^{in}, x) - q_1^{in} \leqslant 1 - q_1^{in} < u_0(x)$$

if $k \cdot x \geqslant X_0$. If $k \cdot x \leqslant X_0$, $\exists X_1 > 0$, $X_1 = X_1(X_0)$, such that if $\xi_1^{in} \geqslant X_1$,

$$U(k \cdot x - \xi_1^{in}, x) \leqslant U(X_0 - X_1, x) \leqslant q_1^{in}$$

Combining the above, we have

$$U(k \cdot x - \xi_1^{in}, x) - q_1^{in} \leqslant u_0(x) \tag{3.6}$$

Let $u_l \equiv U(k \cdot x - ct - \xi(t), x) - q(t)$, where $\xi'(t) > 0$, $\xi(t) > 0$, and $0 < q(t) \leqslant q_1^{in}$. We will derive the equations for $\xi(t)$ and $q(t)$ so that (3.4) and (3.5) hold and u_l is a subsolution. Define

$$N[u] = u_t - \nabla_x \cdot (a(x) \nabla_x u) - b(x) \cdot \nabla_x u - f(u) \tag{3.7}$$

and let $s = k \cdot x - ct - \xi(t)$ and $y = x$ when differentiating U. Using Eq. (2.1) for the traveling wave profile, we have

$$\begin{aligned} N[u_l] &= [-c - \xi'(t)]\, U_s - q_t - \nabla_x \cdot [a(x)(k\partial_s + \nabla_y)\, U] \\ &\quad + \nabla_x \cdot [a(x) \nabla_x q] - b(x) \cdot (k\partial_s + \nabla_y)\, U + b(x) \cdot \nabla_x q - f(U - q) \\ &= [-c - \xi'(t)]\, U_s - q_t - (k\partial_s + \nabla_y) \cdot [a(y)(k\partial_s + \nabla_y)\, U] \,|_{y=x} \\ &\quad + \nabla_x \cdot [a(x) \nabla_x q] - b(y) \cdot (k\partial_s + \nabla_y)\, U \,|_{y=x} \\ &\quad + b(x) \cdot \nabla_x q - f(U - q) \\ &= -\xi'(t)\, U_s - q_t + f(U) - f(U - q) \end{aligned} \tag{3.8}$$

Since $q \in [0, q_1^{in}]$ and $f'(1) < 0$, there exists $\delta > 0$ such that if $U \in [1, 1 - \delta]$, $f(U) - f(U - q) \leqslant \alpha q$ for some $\alpha < 0$, depending only on f. Similarly, since $f'(0) < 0$, there exists $\delta' > 0$ such that for $U \in [0, \delta']$, $f(U) - f(U - q) \leqslant \alpha' q$, $\forall q \in [0, q_1^{in}]$, for some $\alpha' < 0$, depending only on f.

Now for $U \in [1, 1 - \delta]$,

$$N[u_l] \leqslant -\xi'(t)\, U_s - q_t + \alpha q \tag{3.9}$$

and for $U \in [0, \delta']$ we have the same inequality with α replaced by α'. For $U \in [\delta', 1 - \delta]$, due to monotonicity of U and $f \in C^1$, we get

$$U_s \geqslant \beta > 0, \qquad f(U) - f(U - q) \leqslant Kq$$

for some $\beta > 0$, $K > 0$. Note that the inequality $U_s \geqslant \beta > 0$ is not true for all (x, t). Since $U_s \to 0$ if and only if $U \to 0$ or $U \to 1$, this inequality only holds for those (x, t) such that $U = U(x, t)$ stay away from 0 and 1. Here we restrict $U \in [\delta', 1 - \delta]$.

Let us choose $q(t, x)$ to satisfy

$$\begin{aligned} q_t &= -\gamma q \\ q\,|_{t=0} &= q_1^{in} \end{aligned} \tag{3.10}$$

where $-\gamma = \max(\alpha, \alpha') < 0$. Thus

$$0 < q(t) \leqslant q_1^{in} e^{-\gamma t}$$

It follows from (3.8) that

$$N[u_l] \leqslant -\xi'(t)\,\beta + \gamma q + Kq \tag{3.11}$$

Let us choose

$$\xi'(t) = \frac{(K+\gamma)\,q(t)}{\beta}, \qquad \xi(0) = \xi_1^{in} \tag{3.12}$$

Then $\xi(t)$ and $q(t, x)$ are as desired in (3.4) and (3.5). We see from (3.9)–(3.12) that with above choice of ξ and q, $N[u_l] \leqslant 0$, and u_l is a subsolution. The construction of the supersolution is similar. The proof is complete.

Proposition 3.2 (Pulselike data). Consider Eq. (1.1) with $f(u) = u(1-u)(u-\mu)$, $\mu \in (0, 1)$, and the initial data $u_0(x)$, $0 \leqslant u_0(x) \leqslant 1$; and assume that traveling wave solutions $U(k \cdot x - c(k)\,t, x)$ exist. Suppose that $u_0(x)$ satisfies

$$\limsup_{|k \cdot x| \to +\infty} u_0(x) < \mu \qquad \text{uniformly in } S \tag{3.13}$$

$$u_0(x) > \mu + \eta \qquad \text{for} \quad |k \cdot x| < L \tag{3.14}$$

where η and L are positive constants. Then there exists $L_0 = L_0(\eta)$ such that if $L \geqslant L_0$, there are smooth functions $\xi_i(t)$, $q_i(t)$, $i = 1, 2$, so that

$$\begin{aligned} U_+(k \cdot x - c(k)\,t - \xi_1(t), x) + U_-(-k \cdot x - c(-k)\,t - \xi_1(t), x) - 1 - q_1(t) \\ \leqslant u(t, x) \leqslant U_+(k \cdot x - c(k)\,t + \xi_2(t), x) \\ + U_-(-k \cdot x - c(-k)\,t + \xi_2(t), x) - 1 + q_2(t) \end{aligned} \tag{3.15}$$

for all $t \geqslant 0$. The subscripts $+$ and $-$ denote waves going along forward (or k) and backward (or $-k$) directions, respectively. Moreover, we have for $i = 1, 2$

$$(-1)^i\,\xi_i(t) > 0, \qquad \xi_i'(t) > 0, \qquad \sup_{t > 0} |\xi_i(t)| < +\infty \tag{3.16}$$

$$q_i(t) > 0, \qquad q_i(t) \text{ is nonincreasing in } t \text{ and of } O(\varepsilon^{-\gamma t}) \tag{3.17}$$

as $t \to +\infty$, for some $\gamma > 0$.

Proof. Let us first construct the inequality (3.15), then fit it to the initial data. By (3.13) and Proposition 3.1, we get

$$u(x, t) \leqslant U_+(k \cdot x - c(k)\,t + \alpha_1(t), x) + q_1^+(t) \tag{3.18}$$

and applying the same argument to $u(-x, t)$, we get

$$u(x, t) \leqslant U_-(-k \cdot x - c(-k)\, t + \alpha_2(t), x) + q_1^-(t) \tag{3.19}$$

where α_i, $i = 1, 2$, q_1^+, and q_1^- are like the corresponding functions in (3.4) and (3.5). If $k \cdot x > 0$, then Proposition 2.1 implies that

$$1 - U_+(k \cdot x - c(k)\, t + \alpha_1(t), x) \leqslant 1 - U_+(-c(k)\, t + \alpha_1(t), x) \leqslant Ke^{\lambda c(k) t} \tag{3.20}$$

for some $K > 0$, $\lambda > 0$. So by (3.19) and that $c(k) < 0$ for all unit vectors $k \in R^n$, we have

$$\begin{aligned} u(x, t) &\leqslant U_-(-k \cdot x - c(-k)\, t + \alpha_2(t), x) + U_+(k \cdot x - c(k)\, t + \alpha_1(t), x) \\ &\quad - 1 + Ke^{\lambda c(k) t} + q_1^-(t) \\ &\leqslant U_-(-k \cdot x - c(-k)\, t + \xi_2(t), x) + U_+(k \cdot x - c(k)\, t + \xi_2(t), x) \\ &\quad - 1 + q_2(t) \end{aligned} \tag{3.21}$$

where $q_2(t) = Ke^{\lambda c(k) t} + q_1^-(t)$ and $\xi_2(t) = \max\{\alpha_1(t), \alpha_2(t)\}$ are as required in (3.16) and (3.17). If $k \cdot x \leqslant 0$, with U_- replacing U_+ in (3.20) and (3.18) replacing (3.19), we arrive at a similar inequality as (3.21). In any case, we end up with the right-hand inequality in (3.15).

To get the left-hand inequality, consider

$$u_l \equiv U_+(k \cdot x - c(k)\, t - p(t), x) + U_-(-k \cdot x - c(-k)\, t - p(t), x) - 1 - q(t) \tag{3.22}$$

for some functions p and q as described in (3.16) and (3.17) with $i = 1$. Then

$$\begin{aligned} N[u_l] &= u_{l,t} - \nabla_x \cdot (a(x) \nabla_x u_l) - b(x) \cdot \nabla_x u_l - f(u_l) \\ &= -p'(t)(U_{+,s} + U_{-,s}) - q_t \\ &\quad + f(U_+) + f(U_-) - f(U_+ + U_- - 1 - q) \end{aligned} \tag{3.23}$$

If $k \cdot x > 0$, then, by (3.20),

$$\begin{aligned} 1 - U_+(k \cdot x - c(k)\, t - p(t), x) &\leqslant 1 - U_+(-c(k)\, t - p(t), x) \\ &\leqslant Ke^{\lambda(c(k) t + p(t))}, \qquad \lambda > 0 \end{aligned} \tag{3.24}$$

We will make $q(t) \leqslant q(0) \leqslant 1 - \mu - \eta$. So if $-p(t)$ is large enough, $0 \leqslant 1 - U_+ + q \leqslant q_1$ for all $t \geqslant 0$, where $q_1 = 1 - \mu - \eta/2$. There exists a small $\delta > 0$, depending only on f, so that if $U_- \in [0, \delta]$, or $U_- \in [1-\delta, 1]$, then

$$f(U_-) - f(U_- - (1 - U_+ + q)) \leqslant -\mu_1(1 - U_+ + q) \tag{3.25}$$

for some $\mu_1 > 0$. It follows that

$$N[u_l] \leqslant -p'(t)(U_{+,s} + U_{-,s}) - q_t + f(U_+) - \mu_1(1 + q - U_+) \tag{3.26}$$

while

$$f(U_+) \leqslant \mu_2(1 - U_+) \tag{3.27}$$

for some $\mu_2 > 0$. Thus, we get by (3.24),

$$\begin{aligned} N[u_l] &\leqslant -p'(t)(U_{+,s} + U_{-,s}) - q_t \\ &\quad + (\mu_2 - \mu_1)(1 - U_+) - \mu_1 q \\ &\leqslant |\mu_2 - \mu_1| \, Ke^{\lambda(c(k)t + p(t))} - q_t - \mu_1 q \end{aligned} \tag{3.28}$$

Now let q satisfy the following equation:

$$q_t = -\mu_0 q \tag{3.29}$$

where $0 < \mu_0 < \min\{\mu_1, |\lambda c(k)|\}$. The function q then satisfies (3.17) and that

$$0 \leqslant q(t) \leqslant (1 - \mu - \eta)\, e^{-\mu_0 t}$$

It follows from (3.28) and (3.29) that for some constant $C_1 > 0$

$$\begin{aligned} N[u_l] &\leqslant e^{\lambda p(t)} K \,|\mu_2 - \mu_1|\, e^{\lambda c(k)t} - C_1 \varepsilon^{-\mu_0 t} \\ &\leqslant (e^{\lambda p(t)} K \,|\mu_2 - \mu_1| - C_1)\, e^{-\mu_0 t} \\ &\leqslant 0 \end{aligned} \tag{3.30}$$

if $-p(t)$ is large enough. Suppose now $U_- \in [\delta, 1-\delta]$; then there exist positive constants β, C_2 such that

$$U_{-,s} \geqslant \beta$$

$$f(U_-) - f(U_- - (1 + q - U_+)) \leqslant C_2(1 + q - U_+)$$

From (3.24) and (3.27) we also have that

$$f(U_+) \leqslant C_3 e^{-\lambda |c(k)t + p(t)|}$$

for some constant $C_3 > 0$. It follows from (3.23) that

$$\begin{aligned} N[u_l] &\leqslant -p'(t)\,\beta + \mu_0 q + C_3 e^{-\lambda|c(k)t+p(t)|} + C_2(1+q-U_+) \\ &\leqslant -p'(t)\,\beta + C_4 q + C_4 e^{-\lambda|c(k)t+p(t)|} \\ &\leqslant -p'(t)\,\beta + C_4 C_1 e^{-\mu_0 t} + C_4 e^{\lambda c(k)t} \\ &\leqslant -p'(t)\beta + C_4(C_1+1)\,e^{-\mu_0 t} \end{aligned} \tag{3.31}$$

where C_4 is some positive constant. Now choose $p(t)$ to satisfy

$$p'(t) = \frac{C_4(C_1+1)}{\beta}\,e^{-\mu_0 t} \tag{3.32}$$

and $p(0)$ is sufficiently negative. Then $p'(t) > 0$ and $p(t)$ is also sufficiently negative for all $t \geqslant 0$. Combining the above arguments, we have shown that for $k \cdot x > 0$, u_l is a subsolution with p and q given in (3.29) and (3.32). The case $k \cdot x \leqslant 0$ is similar, and we just need to switch everywhere from U_+ to U_-. If we call the larger of the p's and q's obtained in the two cases ξ_1 and q_1, then we have shown the left-hand inequality in (3.15).

Finally, we fit (3.15) to the initial data. The right-hand side, being deduced from Proposition 3.1, is no problem. For the left-hand side, at $t=0$, if $|k \cdot x| \leqslant L$, we have by (3.14) that

$$\begin{aligned} &U_+(k \cdot x - \xi_1(0), x) + U_-(-k \cdot x - \xi_1(0), x) - 1 - q_1(0) \\ &\quad \leqslant 1 - q_1(0) \leqslant \mu + \eta < u_0(x) \end{aligned}$$

where we chose $q_1(0) \equiv q_1^{in} = 1 - \mu - \eta$ for all x. If $|k \cdot x| > M = M(\xi_1(0), \eta)$, we have

$$U_+ + U_- - 1 - q_1(0) \leqslant \tfrac{1}{2} q_1^{in} - q_1^{in} < 0 \leqslant u_0(x) \tag{3.33}$$

Thus if $L \geqslant L_0 \equiv M(\xi_1(0), \eta)$, (3.15) holds for all $t \geqslant 0$. We finish the proof.

Before presenting the results for the combustion case, we first show the following lemmas.

Lemma 3.1. Consider the Cauchy problem of the following linear parabolic equation:

$$\begin{aligned} &u_t = \nabla_x \cdot (a(x)\,\nabla_x u) + b(x) \cdot \nabla_x u, \qquad x \in R^n \\ &u|_{t=0} = u_0(x) \end{aligned} \tag{3.34}$$

where $u_0(x) \in L^2 \cap L^\infty(R^n)$, and is continuous; $a(x)$, $b(x)$ are smooth, uniformly bounded, and $a(x)$ is positive definite (uniformly bounded away from zero). Then

$$\lim_{t \to \infty} \|u(t, x)\|_{L^\infty(R^n)} = 0 \tag{3.35}$$

Proof. Existence of bounded smooth solutions of Eq. (3.34) is standard, and they can be represented by the fundamental solution of (3.34), denoted by $\Gamma(t, x, y)$, as

$$u(t, x) = \int_{R^n} \Gamma(t, x, y)\, u_0(y)\, dy \tag{3.36}$$

where $\Gamma(t, x, y) \geqslant 0$ is continuous in t, x, y, and satisfies the estimate

$$\Gamma(t, x, y) \leqslant M t^{-n/2} \exp\left(-\frac{\alpha}{2t} \|x-y\|^2\right) \tag{3.37}$$

for some positive constants M and α depending only on the coefficients $a(x)$ and $b(x)$. In (3.37), $\|\cdot\|$ is the usual Euclidean distance in R^n. For details on the existence of $\Gamma(t, x, y)$, its construction, and the estimate (3.37), we refer to Friedman[16] and Varadhan.[24] It follows from (3.36) and (3.37) that

$$|u(t, x)| \leqslant M \int_{R^n} t^{-n/2} \exp\left(-\frac{\alpha}{2t} \|x-y\|^2\right) |u_0(y)|\, dy \tag{3.38}$$

We notice that the right-hand side of (3.38) is, up to a multiplicative constant, just a solution of a usual heat equation with constant diffusion coefficients and initial data $|u_0(y)|$. By a straightforward Fourier analysis of the heat equation with $L^2(R^n)$ initial data, one sees that the solution decays to zero in L^∞ via L^2 decay and Sobolev imbedding thanks to the parabolic regularity. We finish the proof by applying inequality (3.38).

Lemma 3.2. Consider the solution $u(t, x)$ of Eq. (3.34) and the assumptions on $a(x)$ and $b(x)$ there; however, $u_0 = u_0(x_1) \in L^2(R^1) \cap L^\infty(R^1)$ and is continuous in x_1. Then

$$\lim_{t \to \infty} \|u(t, x)\|_{L^\infty(R^n)} = 0 \tag{3.39}$$

Proof. Due to $u_0 = u_0(x_1)$, inequality (3.37) can be simplified to

$$|u(t, x)| \leqslant M_1 \int_{R^1} t^{-1/2} \exp\left[-\frac{\alpha}{2t} (x_1 - y_1)^2\right] |u_0(y_1)|\, dy_1 \tag{3.40}$$

for some constant $M_1 > 0$. The right-hand side of inequality (3.40) decays to zero in $L^\infty(R^1)$, which implies (3.39). The proof is complete.

Lemma 3.3. Consider the Cauchy problem of the equation

$$u_t = \nabla_x \cdot (a(x)\, \nabla_x u) + b(x) \cdot \nabla_x u + \varepsilon e^{-\alpha_0 t} f_0(k \cdot x)$$
$$u|_{t=0} = u_0(x) \tag{3.41}$$

where $a(x)$ and $b(x)$ are as in Lemmas 3.1 and 3.2; α_0 is a positive constant; $\varepsilon \in (0, 1)$, a positive constant; $f_0(s) = e^{-\gamma_0 |s|}$ for some constant $\gamma_0 > 0$; k is any unit vector in R^n; $k \cdot x$ is the inner product of k and x; and $u_0(x)$ is as in Lemma 3.1 or 3.2. Then

$$\lim_{t \to \infty} \|u(t, x)\|_{L^\infty(R^n)} = 0 \tag{3.42}$$

and

$$\|u(t, x)\|_{L^\infty(R^n)} \leqslant \|u_0(x)\|_{L^\infty(R^n)} + \varepsilon M c(\alpha_0), \qquad \forall t \geqslant 0 \tag{3.43}$$

where M is the constant in (3.38), depending only on the fundamental solution, and $c(\alpha_0) = \alpha_0^{-1}(2\pi/\alpha_0)^{n/2}$.

Proof. Solutions of Eq. (3.41) can be written as

$$u(t, x) = \int_{R^n} \Gamma(t, x, y)\, u_0(y)\, dy$$
$$+ \varepsilon \int_0^t \int_{R^n} \Gamma(t - s, x, y)\, \varepsilon^{-\alpha_0 s} f_0(k \cdot y)\, dy\, ds \tag{3.44}$$

$$= \int_{R^n} \Gamma(t, x, y)\, u_0(y)\, dy$$
$$+ \varepsilon \int_0^t ds\, e^{-\alpha_0 (t-s)} \int_{R^n} \Gamma(s, x, y)\, f_0(k \cdot y)\, dy \tag{3.45}$$

where the first integral decays to zero uniformly in x as $t \to \infty$, by Lemma 3.1 or 3.2. The inner layer integral of the second term can be estimated using (3.37), to yield

$$\left| \int_{R^n} \Gamma(s, x, y)\, f_0(k \cdot y)\, dy \right|$$
$$\leqslant M s^{-n/2} \int_{R^n} \exp\left(- \frac{\alpha_0}{2s} \|x - y\|^2 \right) f_0(k \cdot y)\, dy \tag{3.46}$$

Making the change of variables inside the right-hand side integral in (3.46) by letting $\xi = Qy$, $\xi_1 = k \cdot y$, where Q is an orthogonal matrix with k as a first row vector, we get

$$\begin{aligned} & s^{-n/2} \int_{R^n} \exp\left(-\frac{\alpha_0}{2s} \|x-y\|^2 \right) f_0(k \cdot y)\, dy \\ & = s^{-n/2} \int_{R^n} \exp\left(-\frac{\alpha_0}{2s} \|Qx-\xi\|^2 \right) f_0(\xi_1)\, d\xi \\ & = \left(\frac{2\pi}{\alpha_0}\right)^{(n-1)/2} \int_{R^1} \exp\left[-\frac{\alpha_0}{2s} (k\cdot x - \xi_1)^2 \right] f_0(\xi_1)\, d\xi_1 \\ & \equiv \beta(s) \end{aligned} \tag{3.47}$$

where $\beta(s)$ is a positive function, $\beta(s) \leqslant (2\pi/\alpha_0)^{n/2} \|f_0\|_{L^\infty}$, $\beta(s) \to 0$, as $s \to +\infty$. Thus the second integral in (3.45) is bounded up to a factor of M by

$$\begin{aligned} & \int_0^t ds\, e^{-\alpha_0(t-s)} \beta(s)\, ds \\ & = \int_0^{t/2} ds\, e^{-\alpha_0(t-s)} \beta(s) + \int_{t/2}^t e^{-\alpha_0(t-s)} \beta(s)\, ds \\ & \leqslant \left(\frac{2\pi}{\alpha_0}\right)^{n/2} \|f_0\|_{L^\infty} \int_{t/2}^t e^{-\alpha_0 s}\, ds \\ & \quad + \sup_{s \in [t/2, t]} \beta(s) \int_0^{t/2} e^{-\alpha_0 s}\, ds \to 0 \end{aligned} \tag{3.48}$$

as $t \to \infty$, from which (3.42) follows. Combining (3.45)–(3.47), we obtain (3.43). The proof is complete.

Below, we construct subsolutions and supersolutions for the combustion case.

First, we have:

Proposition 3.3. (Frontlike Data). Consider Eq. (1.1) with combustion nonlinearity (see A3), $f'(1) < 0$, and initial data $u_0(x)$, $0 \leqslant u_0 \leqslant 1$. Let k be a unit vector in R^n, and define

$$S = \{ y \in R^n \mid y = x - (k \cdot x)\, k, \forall x \in R^n \} \tag{3.49}$$

Suppose that

$$\lim_{k \cdot x \to -\infty} u_0(x) = 0 \qquad \text{uniformly in } S$$
$$u_0(x)\, \chi(-\infty, 0)(k \cdot x) \in L^2(R^1_{k \cdot x}) \qquad \text{uniformly in } S \tag{3.50}$$

$$\lim_{k \cdot x \to +\infty} u_0(x) = 1 \qquad \text{uniformly in } S$$
$$[1 - u_0(x)]\, \chi_{(0, +\infty)}(k \cdot x) \in L^2(R^1_{k \cdot x}) \qquad \text{uniformly in } S \tag{3.51}$$

Then there exist smooth functions $\xi_i = \xi_i(t)$, $q_i = q_i(t, x)$, $i = 1, 2$, such that

$$U(k \cdot x - c(k)\, t - \xi_1, x) - q_1 \leqslant u(t, x) \leqslant U(k \cdot x - c(k)\, t + \xi_2, x) + q_2 \tag{3.52}$$

where for $i = 1, 2$,

$$\xi_i'(t) > 0, \qquad \xi_i(t) > 0, \qquad \xi_i(t) = o(t) \tag{3.53}$$

$$q_i(t, x) > 0, \qquad \|q_i\|_{L^\infty(R^n)} \text{ is nonincreasing in } t \text{ and } = o(1) \tag{3.54}$$

as $t \to \infty$.

Proof. Let us consider a subsolution. By (3.51), $\exists X_0 > 0$, if $k \cdot x \geqslant X_0$, then there is a function

$$q_0 = q_0(k \cdot x)$$
$$q_0(k \cdot x)\, \chi_{(X_0, +\infty)}(k \cdot x) \in L^2(R^1_{k \cdot x})$$
$$0 < q_0 < 1 - \theta - \varepsilon, \qquad \varepsilon \in (0, 1 - \theta/2)$$

such that $u_0 \geqslant 1 - q_0(k \cdot x)$. It follows that $u_0(x) \geqslant U(k \cdot x - \xi_0, x) - q_0(k \cdot x)$ for any ξ_0. For $k \cdot x \in (-\infty, X_0)$, choose $\xi_0 \geqslant X_1 = X_1(X_0) > 0$ such that

$$U(k \cdot x - \xi_0, x) \leqslant C \exp(-\lambda\, |k \cdot x - \xi_0|)$$

$\forall k \cdot x \in (-\infty, X_0)$, by Proposition 2.1. It is easy to define $q_0(k \cdot x)$ for $k \cdot x \in (-\infty, X_0)$ such that $q_0 \in L^2 \cap L^\infty(R^1_{k \cdot x})$, $0 < q_0 < 1 - \theta - \varepsilon$, and $q_0(k \cdot x) \geqslant U(k \cdot x - \xi_0, x)$. Thus,

$$U(k \cdot x - \xi_0, x) - q_0(k \cdot x) \leqslant 0 \leqslant u_0(x)$$

on $k \cdot x \in (-\infty, X_0)$. For the above ξ_0 and q_0, we have

$$U(k \cdot x - \xi_0, x) - q_0(k \cdot x) \leqslant u_0(x)$$

on R^n.

Now consider the function

$$u_l \equiv U(k \cdot x - c(k)\, t - \xi_1(t), x) - q_1(t, x)$$

where ξ_1 and q_1 will be chosen as in (3.53) and (3.54). We then have

$$\begin{aligned} N[u_l] &= u_{l,t} - \nabla_x \cdot (a(x) \nabla_x u_l) - b(x) \cdot \nabla_x u_l - f(u_l) \\ &= -\xi_1'(t)\, U_s - q_{1,t} + \nabla_x \cdot (a(x) \nabla_x q_1) \\ &\quad b(x) \cdot \nabla_x q_1 + f(U) - f(U - q_1) \end{aligned} \tag{3.55}$$

There exists $\delta \in (0, \theta)$ small so that if $q \in [0, 1-\theta-\varepsilon]$ and $U \in [1-\delta, 1]$, then

$$f(U) \leqslant f(U - q)$$

Since $0 \leqslant q \leqslant q_0 < 1 - \theta - \varepsilon$, we have for $U \in [1-\delta, 1]$

$$N[u_l] \leqslant -\xi_1'(t)\, U_s - q_{1,t} + \nabla_x \cdot (a(x) \nabla_x q_1) + b(x) \cdot \nabla_x q_1 \tag{3.56}$$

If $U \in [0, \delta]$, then $f(U) = f(U - q_1) = 0$, so (3.56) holds with the equality sign. If $U \in (\delta, 1-\delta)$, $\exists \beta > 0$ such that $U_s \geqslant \beta$ and $|f(U) - f(U - q_1)| \leqslant K q_1$ for some $K > 0$. It follows that

$$N[u_l] \leqslant -\xi_1' \beta - q_{1,t} + \nabla_x \cdot (a(x) \nabla_x q_1) + b(x) \cdot \nabla_x q_1 + K q_1 \tag{3.57}$$

Let us choose q_1 to satisfy the equation

$$\begin{aligned} q_{1,t} &= \nabla_x \cdot (a(x) \nabla_x q_1) + b(x) \cdot \nabla_x q_1 \\ q_1|_{t=0} &= q_0(k \cdot x) \end{aligned} \tag{3.58}$$

To make u_l a subsolution, we just need to impose that

$$-\xi_1' \beta + K q_1 \leqslant 0 \qquad \text{or} \qquad -\xi_1' \beta + K \|q_1\|_{L^\infty(R^n)} = 0$$

or

$$\xi_1' = \frac{K \|q_1\|_{L^\infty(R^n)}}{\beta} > 0 \tag{3.59}$$

with $\xi_1(0) = \xi_0 > 0$. Lemma 3.2 implies that $\|q_1\|_{L^\infty} = o(1)$ as $t \to \infty$. So $\xi_1(t) = o(t)$. We showed that u_l is a subsolution, and a supersolution can be constructed in a similar way. We complete the proof.

Proposition 3.4 (Pulselike Data). Consider Eq. (1.1) with combustion nonlinearity (see A2), $f'(1) < 0$, and the initial data u_0, $0 \leqslant u_0(x) \leqslant 1$. Suppose that for some unit vector $k \in R^n$, u_0 satisfies

$$\lim_{|k \cdot x| \to \infty} u_0(x) = 0 \qquad u_0(x) \in L^2(R^1_{k \cdot x}), \qquad \text{uniformly in } S \tag{3.60}$$

$$u_0(x) > \theta + \eta \qquad \text{for} \quad |k \cdot x| \leqslant L \tag{3.61}$$

where $\eta \in (0, 1-\theta)$, $L > 0$, and S is that of (3.49). Then $\exists L_0 = L_0(\eta, f)$ so that if $L \geqslant L_0$, there are smooth functions $\xi_i = \xi_i(t)$, $q_i = q_i(t, x)$, $i = 1, 2$, such that

$$\begin{aligned} &U_+(k \cdot x - c(k)\, t - \xi_1, x) + U_-(-k \cdot x - c(-k)\, t - \xi_1, x) - 1 - q_1 \\ &\quad \leqslant u(t, x) \leqslant U_+(k \cdot x - c(k)\, t + \xi_2, x) \\ &\quad\quad + U_-(-k \cdot x - c(-k)\, t + \xi_2, x) - 1 + q_2 \end{aligned} \tag{3.62}$$

for all $t \geqslant 0$. The subscripts + and − denote the waves going along forward (k) and backward $(-k)$ directions, respectively. Moreover, we have for $i = 1, 2$

$$(-1)^i \, \xi_i(0) > 0, \qquad \xi_i'(t) > 0, \qquad |\xi_i(t)| = o(t) \tag{3.63}$$

$$q_i(t, x) > 0, \qquad \|q_i(t, x)\|_{L^\infty(R^n)} = o(1) \tag{3.64}$$

as $t \to \infty$.

Proof. Along direction k, by (3.60) and Proposition 3.3, we have that

$$u(x, t) \leqslant U_+(k \cdot x - c(k)\, t + \xi_2^+(t), x) + q_2^+(t, x) \tag{3.65}$$

where ξ_2^+ and q_2^+ are as in (3.53) and (3.54) with $i = 2$. Applying the same argument to $u(-x, t)$ gives

$$u(x, t) \leqslant U_-(-k \cdot x - c(-k)\, t + \xi_2^-(t), x) + q_2^-(t, x) \tag{3.66}$$

where ξ_2^- and q_2^- are similar to ξ_2^+ and q_2^+. If $k \cdot x > 0$, then

$$1 - U_+(k \cdot x - c(k)\, t + \xi_2^+(t), x) \leqslant K \exp[-\lambda\, |c(k)\, t|] \tag{3.67}$$

for some $K > 0$, $\lambda > 0$. Combining (3.66) and (3.67), we have

$$\begin{aligned} u(x, t) \leqslant\; & U_-(-k \cdot x - c(-k)\, t + \xi_2^-(t), x) + U_+(k \cdot x - c(k)\, t + \xi_2^+(t), x) \\ & - 1 + K \exp[-\lambda\, |c(k)\, t|] + q_2^-(t, x) \end{aligned} \tag{3.68}$$

which implies the right-hand inequality of (3.62) if we choose

$$\xi_2(t) \geqslant \max(\xi_2^+(t), \xi_2^-(t)$$

and

$$q_2(t, x) = K \exp[-\lambda\, |c(k)\, t|] + q_2^-(t, x)$$

A similar inequality holds if $k \cdot x \leqslant 0$. Combining the two cases, and adjusting ξ_2 and q_2 if necessary, we end up with the right-hand inequality of (3.62). For the left-hand inequality, consider the function

$$\begin{aligned} u_l \equiv\ & U_+(k \cdot x - c(k)\, t - p(t), x) + U_-(-k \cdot x - c(-k) \\ & \times t - p(t), x) - 1 - q(t, x) \end{aligned} \tag{3.69}$$

where p and q are as in (3.63) and (3.64) with $i = 1$, $-p(0) \gg 1$, and $0 < q(t, x) \leqslant 1 - \theta - \eta/2$, for all t and x. Direct calculation shows that

$$\begin{aligned} N[u_l] = & -p'(t)(U_{+,s} + U_{-,s}) - q_t + \nabla_x \cdot (a(x)\, \nabla_x q) + b(x) \cdot \nabla_x q \\ & + f(U_+) + f(U_-) - f(U_+ + U_- - 1 - q) \end{aligned} \tag{3.70}$$

where we ignore the arguments of the functions. If $k \cdot x > 0$, let q satisfy the equation

$$\begin{aligned} & q_t = \nabla_x \cdot (a(x)\, \nabla_x q) + b(x) \cdot \nabla_x q + \varepsilon \exp(-\alpha_0 t - \gamma\, |k \cdot x|) \\ & q\,|_{t=0} = q(0, x) \end{aligned} \tag{3.71}$$

where $\alpha_0 = \lambda\, |c(k)|/2$, $\lambda > 0$ being the decay rate of U_+ near positive infinity, $\gamma \in (\lambda/2, \lambda)$. Choose $\|q(0, x)\|_{L^\infty} \leqslant 1 - \theta - \eta$. By Lemma 3.3, if ε is small enough, then $\|q(t, x)\|_{L^\infty(R^n)} \leqslant 1 - \theta - \eta/2$ for all $t \geqslant 0$. The function $p(t)$ is to be determined so that

$$-p(t) - c(k)\, t \geqslant -c(k)\, t/2 + m, \qquad \forall t \geqslant 0 \tag{3.72}$$

where m is a positive number as large as we please. Suppose this is done; then

$$k \cdot c - c(k)\, t - p(t) \geqslant -c(k)\, t - p(t) \gg 1$$

There exists a positive constant C, independent of $p(t)$, such that

$$1 - U_+(k \cdot x - c(k)\, t - p(t), x) \leqslant C \exp\{-\lambda[k \cdot x - c(k)\, t - p(t)]\} \tag{3.73}$$

for all t and x. Below, we will use C as a generic constant independent of $p(t)$. If m is large enough, it follows that

$$0 \leqslant 1 - U_+ + q \leqslant 1 - \theta - \frac{\eta}{4} \tag{3.74}$$

Suppose $U_- \in [0, \theta)$; then

$$\begin{aligned} f(U_-) &= f(U_+ + U_- - 1 - q) = f(U_- - (1 - U_+ + q)) = 0 \\ N[u_l] &\leqslant -p'(t)(U_{+,s} + U_{-,s}) - q_t \\ &\quad + \nabla_x \cdot (a(x) \nabla_x q) + b(x) \cdot \nabla_x q + f(U_+) \end{aligned} \tag{3.75}$$

while

$$f(U_+) \leqslant C(1 - U_+) \leqslant C \exp\{-\lambda[k \cdot x - c(k)\, t - p(t)]\} \tag{3.76}$$

Thus,

$$\begin{aligned} N[u_l] &\leqslant -p'(t)(U_{+,s} + U_{-,s}) - q_t + \nabla_x \cdot (a(x) \nabla_x q) \\ &\quad + b(x) \cdot \nabla_x q + C \exp\{-\lambda[k \cdot x - c(k)\, t - p(t)]\} \\ &\leqslant -q_t + \nabla_x \cdot (a(x) \nabla_x q) + b(x) \cdot \nabla_x q \\ &\quad + C \exp\{-\lambda[k \cdot x - c(k)\, t - p(t)]\} \end{aligned} \tag{3.77}$$

There exists δ small enough so that if $U_- \in [1 - \delta, 1]$, then

$$f(U_-) - f(U_- - q) \leqslant 0 \qquad \text{for} \quad q \leqslant 1 - \theta - \frac{\eta}{4}$$

Suppose that $U_- \in [1 - \delta, 1]$, then by (3.74)

$$f(U_-) - f(U_+ + U_- - 1 - q) = f(U_-) - f(U_- - (1 + q - U_+)) \leqslant 0$$

which implies that

$$N[u_l] \leqslant -p'(t)(U_{+,s} + U_{-,s}) - q_t + \nabla_x \cdot (a(x) \nabla_x q) + b(x) \cdot \nabla_x q + f(U_+) \tag{3.78}$$

which is just like (3.75), and so (3.77) holds also.

Now suppose $U_- \in [\theta, 1 - \delta)$; then $\exists \beta > 0$, such that $U_{-,s} \geqslant \beta > 0$, and (3.70) gives

$$\begin{aligned} N[u_l] &\leqslant -p'(t)\,\beta - q_t + \nabla_x\cdot(a(x)\,\nabla_x q) + b(x)\cdot\nabla_x q \\ &\quad + f(U_+) + f(U_-) - f(U_- - (1 - U_+ + q)) \\ &\leqslant -p'(t)\,\beta - q_t + \nabla_x\cdot(a(x)\,\nabla_x q) + b(x)\cdot\nabla_x q \\ &\quad + C\exp\{-\lambda[k\cdot x - c(k)\,t - p(t)]\} + C(1 - U_+ + q) \\ &\leqslant -p'(t)\,\beta - q_t + \nabla_x\cdot(a(x)\,\nabla_x q) + b(x)\cdot\nabla_x q \\ &\quad + C\exp\{-\lambda[k\cdot x - c(k)\,t - p(t)]\} + Cq \end{aligned} \tag{3.79}$$

Now choose $p(t)$ to satisfy

$$-p'(t)\,\beta + C\,\|q\|_\infty = 0 \tag{3.80}$$

By Lemma 3.3 (with the required initial data given shortly), we get

$$\begin{aligned} &p'(t) = K\,\|q\|_{L^\infty} \\ &p(t) = o(t) \qquad \text{as} \quad t\to\infty \end{aligned} \tag{3.81}$$

where $K \equiv C/\beta$ is independent of p.

Let us prove (3.72). Consider the function

$$\begin{aligned} G(t) \equiv K\int_0^t \|q\|_{L^\infty}(s)\,ds + \frac{c(k)}{2}\,t \\ - \sup_{t\in[0,t_0]}\left[K\int_0^t \|q\|_{L^\infty}(s)\,ds + \frac{c(k)}{2}\,t\right] \end{aligned} \tag{3.82}$$

where t_0 is a number such that if $t\geqslant t_0$, then

$$G'(t) = K\,\|q\|_{L^\infty}(t) + \frac{c(k)}{2} \leqslant 0$$

Since $\|q\|_{L^\infty}(t)\to 0$ as $t\to\infty$, such t_0 exists. Therefore, $G(t)\leqslant 0$ for all $t\geqslant 0$. Let

$$m_0 \equiv \sup_{t\in[0,t_0]}\left[K\int_0^t \|q\|_{L^\infty}(s)\,ds + \frac{c(k)}{2}\,t\right]$$

Then

$$-K\int_0^t \|q\|_{L^\infty}(s)\,ds \geqslant \frac{c(k)\,t}{2} - m_0 \qquad \forall t\geqslant 0 \tag{3.83}$$

It follows that

$$-p(t)-c(k)\,t=-\int_0^t K\,\|q\|_\infty\,(s)\,ds-p(0)-c(k)\,t$$
$$\geqslant\frac{c(k)\,t}{2}-m_0-p(0)-c(k)\,t$$
$$=-\frac{c(k)\,t}{2}-m_0-p(0) \qquad (3.84)$$

Since m_0 is a constant independent of $p(0)$, we arrive at (3.72) by setting $m=-m_0-p(0)$. With the above choice of $p(t)$, both (3.79) and (3.77) imply

$$N[u_l]\leqslant-\varepsilon\exp(-\alpha_0 t-\gamma\,|k\cdot x|)$$
$$+C\exp\{-\lambda[k\cdot x-c(k)\,t-p(t)]\}$$
$$=-\varepsilon\exp\left[\frac{c(k)\,\lambda t}{2}-\gamma\,|k\cdot x|\right]$$
$$+C\exp\{-\lambda[k\cdot x-c(k)\,t-p(t)]\},\qquad \forall k\cdot x\geqslant 0 \quad (3.85)$$

By our choice of γ, we see that

$$\exp(-\gamma\,|k\cdot x|)\geqslant\exp(-\lambda k\cdot x)$$

In view of (3.84)

$$N[u_l]\leqslant\exp[-\lambda(k\cdot x)]$$
$$\times\left(-\varepsilon\exp\left[\frac{c(k)\,\lambda t}{2}\right]+C\exp\left\{\lambda\left[\frac{c(k)\,t}{2}+m_0+p(0)\right]\right\}\right)$$
$$\leqslant\exp[-\lambda(k\cdot x)]\exp\left[\frac{c(k)\,\lambda t}{2}\right](-\varepsilon+C\exp\{\lambda[m_0+p(0)]\})$$
$$\leqslant 0$$

if $-p(0)$ is large enough. The case $k\cdot x\leqslant 0$ can be shown in a similar way. Thus, letting $\xi_1(t)$ and q_1 be no less than the larger of the p's and q's of the two cases, we have shown that u_l is a subsolution; the left-side inequality of (3.62) holds if u_l can be fit to the initial data.

If $|k\cdot x|\leqslant L$, choose $q_1(0,x)=q_1^{in}=1-\theta-\eta$; then by (3.61)

$$U_+(k\cdot x-\xi_1(0),x)+U_-(-k\cdot x-\xi_1(0),x)-1-q_1(0,x)\leqslant 1-q_1(0,x)$$
$$\leqslant\theta+\eta\leqslant u_0(x) \qquad (3.86)$$

If $|k \cdot x| \geqslant M \equiv M(\xi_1(0), \eta)$, then

$$\begin{aligned} U_+ + U_- - 1 - q_1(0, x) &\leqslant C \exp[-\lambda_1 |k \cdot x - \xi_1(0)|] \\ &\quad + C \exp[-\lambda_2 |k \cdot x + \xi_1(0)|] - q_1(0, x) \\ &\leqslant 0 \leqslant u_0(x) \end{aligned} \tag{3.87}$$

where $\lambda_i > 0$, $i = 1, 2$, are constants, and $q_1(0, x) > 0$ is easily defined to decay slower than exponential but faster than an $L^2(R^1_{k \cdot x})$ function, i.e., $q_1(0, x) \in L^2 \cap L^\infty(R^1_{k \cdot x})$. So $q_1(0, x)$ satisfies the conditions in Lemmas 3.2–3.3 ([whose proofs are easily modified for data in $L^2 \cap L^\infty(R^1_{k \cdot x})$]. Now, if $L > M$, u_l is a subsolution. The proof is complete.

Remark 3.1. In Propositions 3.3 and 3.4, the condition $f'(1) < 0$ is only used to ensure the existence of traveling waves. The proofs only need the fact that traveling waves that satisfy monotonicity and Proposition 2.1 exist. For example, such traveling waves are shown to exist in Xin[27] for a slightly different reaction-diffusion equation; however, uniqueness is not known without assuming $f'(1) < 0$. Besides implying a propagation theorem, Proposition 3.3 also provides an indirect proof of the uniqueness of the traveling wave speed.

Remark 3.2. During the preparation of our work, we received the preprint of Roquejoffre,[22] where different subsolutions and supersolutions are constructed for the combustion model of Berestycki *et al.*[8,9] In the combustion case (or case B in ref. 22), the construction of ref. 22 requires the initial data to decay by a particular exponential rate and the assumption $f'(1) < 0$. It also depends more on the structure of traveling wave profiles near infinities. The result of ref. 22 is stronger than ours here in that the functions $q(t, x)$ decay exponentially in time and that the $\xi(t)$ do not grow in time.

As a consequence of Propositions 3.2–3.4, we have:

Theorem 3.1 (Large-Time Wavefront Propagation). Consider the reaction-diffusion equation (1.1) with initial data u_0 either frontlike or pulselike, and assume that the conditions in Propositions 3.1–3.4 hold. Let $s \in R^1$. Then:

I. For frontlike data along direction k

$$\lim_{t\to\infty} u(t, skt) = \begin{cases} 1 & \text{if } s > c(k) \\ 0 & \text{if } s < c(k) \end{cases}$$

II. For pulselike data along direction k

$$\lim_{t\to\infty} u(t, skt) = \begin{cases} 1 & \text{if } c(k) < s < -c(-k) \\ 0 & \text{if } s < c(k) \text{ or } s > -c(-k) \end{cases}$$

Remark 3.3. Theorem 3.1 holds also for Lipschitz-continuous combustion nonlinearity. We can always approximate it from above and below with C^1 combustion nonlinearities. The solutions of the Cauchy problem with the same initial data and these approximate nonlinearities also bound the solution under the Lipschitz-continuous nonlinearity from above and below. Since the traveling waves in the approximate problems have speeds approaching that of the traveling wave in the Lipschitz-continuous combustion nonlinearity, we conclude.

4. AN EXAMPLE OF QUENCHING

In this section, we show that quenching occurs in the one-dimensional bistable reaction-diffusion equations with periodic coefficients when the variation of the coefficients from their mean values are large enough for a given nonlinearity. In other words, traveling wave solutions cease to exist, and stationary solutions exist instead. We only consider a case where perturbation analysis is available.

Let us consider the problem

$$\begin{gathered} (a(x)\, u_x)_x + \mu^2 f(u) = 0 \\ u((-\infty) = 0, \qquad u(+\infty) = 0, \qquad u(0) = 1/2 \end{gathered} \tag{4.1}$$

where $a(x) = 1 + \delta\lambda a_1(x)$, $|\delta| \ll 1$, $\lambda \in R^1$, and $a_1(x)$ is a smooth 1-periodic function; $f(u) = u(1-u)(u - \frac{1}{2} + \delta)$. We will look for a solution to (4.1) when $|\delta| \ll 1$ for suitable μ, λ, and $a_1(x)$.

Write $u = \varphi^\mu(x) + \delta v(x)$ and $f(u) = f_0(u) + \delta u(1-u)$, where $f_0(u) = u(1-u)(u - \frac{1}{2})$, and $\varphi^\mu(x)$ is the known solution of

$$\begin{gathered} \varphi^\mu_{xx} + \mu^2 f_0(u) = 0 \\ \varphi^\mu(-\infty) = 0, \qquad \varphi^\mu(+\infty) = 1, \qquad \varphi^\mu(0) = \tfrac{1}{2} \end{gathered} \tag{4.2}$$

Moreover, $\varphi_x^\mu > 0$, $\varphi^\mu(x) = \varphi_1(\mu x)$, $\varphi_1(x)$ being the solution to (4.2) with $\mu = 1$, and $\varphi_x^\mu(-x) = \varphi_x^\mu(x)$. Substituting v for u in (4.1) gives

$$((1+\delta\lambda a_1))_x (\varphi_x^\mu + \delta v_x) + (1+\delta\lambda a_1)(\varphi_{xx}^\mu + \delta v_{xx}) + \mu^2 f_0(\varphi^\mu + \delta v) + \mu^2\delta(1-\varphi^\mu - \delta v)(\varphi^\mu + \delta v) = 0 \tag{4.3}$$

Using (4.2), we simplify (4.3) to

$$\delta\lambda a_{1,x}(\varphi_x^\mu + \delta v_x) + \delta v_{xx} + \delta\lambda a_1(\varphi_{xx}^\mu + \delta v_{xx}) + \mu^2\,\delta v\, f_0'(\varphi^\mu) + \frac{\mu^2}{2} f_0''(\varphi^\mu)(\delta v)^2 - \mu^2(\delta v)^3 + \mu^2\delta(1-\varphi^\mu - \delta v)(\varphi^\mu + \delta v) = 0$$

or

$$v_{xx} + \lambda a_1(\varphi_{xx}^\mu + \delta v_{xx}) + \lambda a_{1,x}(\varphi_x^\mu + \delta v_x) + \mu^2 f_0'(\varphi^\mu)\, v + \frac{\delta\mu^2}{2} v^2 f_0''(\varphi^\mu) - \mu^2\delta^2 v^3 + \mu^2(1-\varphi^\mu - \delta v)(\varphi^\mu + \delta v) = 0$$

or

$$v_{xx} + \mu^2 f_0'(\varphi^\mu)\, v \equiv R = -\lambda(a_1\varphi_x^\mu)_x - \lambda\delta(a_1 v_x)_x - \frac{\mu^2\delta}{2} v^2 f_0''(\varphi^\mu) + \mu^2\delta^2 v^3 - \mu^2(1-\varphi^\mu - \delta v)(\varphi^\mu + \delta v) \tag{4.4}$$

The operator $d_{xx}\cdot + \mu^2 f_0'(\varphi^\mu)\cdot$ has one-dimensional null-function $\varphi_x^\mu > 0$. So to invert it, the right-hand side of (4.4) must satisfy the solvability condition

$$0 = \int_{R^1} R\varphi_x^\mu \, dx \tag{4.5}$$

We choose λ so that (4.5) holds. Since

$$\int_{R^1} \mu^2(1-\varphi^\mu)\,\varphi^\mu\varphi_x^\mu\, dx = \int_0^1 \mu^2(1-\varphi^\mu)\,\varphi^\mu\, d\varphi^\mu = \frac{\mu^2}{6}$$

(4.5) can be written as

$$-\lambda\int_{R^1} (a_1\varphi_x^\mu)_x\,\varphi_x^\mu\, dx - \delta\lambda\int_{R^1} (a_1 v_x)_x\,\varphi_x^\mu\, dx = \frac{\mu^2\delta}{2}\int_{R^1} v^2 f_0''(\varphi^\mu)\,\varphi_x^\mu\, dx + \frac{\mu^2}{6} + \delta\int_{R^1} [\mu^2\,\delta v^3\,\varphi_x^\mu + \mu^2 v(1-\varphi^\mu)\,\varphi_x^\mu - \mu^2 v(\varphi^\mu + \delta v)\,\varphi_x^\mu]\, dx \tag{4.6}$$

For given v, (4.6) can be solved for λ if

$$\int_{R^1} \varphi_x^\mu (a_1 \varphi_x^\mu)_x \, dx \neq 0 \tag{4.7}$$

When (4.7) is satisfied, it is easy to see that Eq. (4.4) admits solution in $H^1(R^1)$ by the contraction mapping principle for some $\lambda = \lambda(\delta, \mu)$ and $\delta = \delta(\mu)$, $|\delta| \ll 1$. Now (4.7) is just

$$\begin{aligned}\int_{R^1} (a_1 \varphi_x^\mu)_x \, \varphi_x^\mu \, dx &= -\int_{R^1} \varphi_{xx}^\mu (a_1 \varphi_x^\mu) \, dx \\ &= -\int_{R^1} a_1 \left(\frac{(\varphi_x^\mu)^2}{2} \right)_x dx = \frac{1}{2} \int_{R^1} (\varphi_x^\mu)^2 \, a_{1,x} \, dx \end{aligned} \tag{4.8}$$

We show that there exist a mean-zero 1-periodic function $b(x)$ $[b(x) = a_{1,x}(x)]$ so that

$$\int_{R^1} (\varphi_x^\mu)^2 \, b(x) \, dx \neq 0 \tag{4.9}$$

if μ is chosen large enough. More specifically, we show that

$$\int_{R^1} (\varphi_x^\mu)^2 \cos x \, dx \neq 0 \tag{4.10}$$

Since φ_x^μ is an even function, (4.10) is the same as

$$\int_{R^1} (\varphi_x^\mu)^2 \, e^{ix} \, dx \neq 0 \tag{4.11}$$

whose left-hand side is equal to

$$\int_{R^1} \mu^2 (\varphi_{1,x}(\mu x))^2 \, e^{ix} \, dx = \mu \int_{R^1} (\varphi_{1,x}(x))^2 \, e^{ix/\mu} \, dx \tag{4.12}$$

Now $(\varphi_{1,x}(x))^2 > 0$ and decays exponentially to zero at infinities, and its Fourier transform is smooth and positive at zero. So there exists $\tilde{x} = \tilde{x}_0 \in (0, 1)$ such that

$$\int_{R^1} (\varphi_{1,x}(x))^2 \, e^{i\tilde{x}_0 x} \, dx \neq 0 \tag{4.13}$$

Finally, we choose $\mu = 1/\tilde{x}_0$ and see from (4.11)–(4.13) that (4.7) holds with this value of μ and $a_1(x) = \sin x$. To summarize, we have:

Proposition 4.1. There are positive numbers $\mu > 0$ and $\delta_0 = \delta_0(\mu) \neq 0$ such that if $\delta \in (0, \delta_0)$, there exists a nonzero real number $\lambda_1 = \lambda_1(\delta, \mu)$ so that Eq. (4.1) with $a(x) = 1 + \delta\lambda_1 \sin x$ admits a solution satisfying all the boundary conditions.

Proof. The existence of solutions follows from our discussion above, and since the homogeneous problem [i.e., $a(x) = 1$, $\delta \neq 0$] has no stationary solution taking zero and one at infinities, $\lambda_1 \neq 0$.

Remark 4.1. The proposition above implies that Eq. (4.1) also has a solution satisfying $u_1(-\infty) = 1$, $u_1(0) = 1/2$, and $u_1(\infty) = 0$.

Remark 4.2. The periodicity condition on $a(x)$ is not essential. One can find a smooth nonperiodic function $\tilde{a}(x)$ so that

$$\int_{R^1} (\varphi_x^\mu)^2 \, \tilde{a}_x \, dx \neq 0$$

This means that quenching is only related to the degree of inhomogeneities. In Proposition 4.1, λ_1 may be rather small in absolute value, but already is large enough compared to the degree of inhomogeneity causing quenching for the given nonlinearity, which is very close to the derivative of a potential with two equal wells.

Due to the fact that solutions of (4.1) satisfy the maximum principle, we have:

Corollary 4.1. Under the conditions of Proposition 4.1, Eq. (4.1) does not have the wavefront propagation phenomenon; in particular, it does not admit traveling wave solutions.

Remark 4.3. Note that Proposition 4.1 holds when $a(x) = 1 + \delta\lambda_1 \sin x$ for $\lambda_1 \neq 0$ and $\lambda_1 = \lambda_1(\delta, \mu)$. If we further decrease $|\lambda|$ for fixed δ, then by Theorem 2.2, $\exists \lambda_0 > 0$ such that if $|\lambda| \in (0, \lambda_0)$, traveling wave solutions exist. What happens for $|\lambda| \in (\lambda_0, |\lambda_1|)$, and $|\lambda| \in (|\lambda_1|, \infty)$? We expect that for $\lambda > 0$ there is a single transition point $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$, then traveling waves exist, and if $\lambda \geqslant \lambda^*$, stationary solutions exist and waves are quenched. That the nonexistence domain is closed follows from Xin.[28] We observe this type of transition in our numerical computations to be presented in the next section. The case $\lambda < 0$ is similar.

5. NUMERICAL RESULTS ON QUENCHING

In this section, we present some numerical results on quenching in the parameter regime away from that of the example in Section 4.

We use an implicit second-order finite-difference method to calculate the solutions of Eq. (4.1) with frontlike initial data. The spatial domain is $[-N, N]$, and boundary conditions are $u(-N)=0$, $u(N)=1$. To minimize boundary effects, we put a small parameter $\varepsilon=0.25$ in front of the diffusion term, $1/\varepsilon=4$ before the reaction term, and choose a steep front as initial data. The example we show here is

$$\begin{gathered} u_t = 0.25[(1+\delta \sin 20x)\, u_x]_x + 4u(1-u)(u-\mu) \\ u(-5)=0, \qquad u(+5)=1 \end{gathered} \tag{5.1}$$

We choose sin $20x$ instead of sin x in the coefficient to see some oscillation inside the transition layers of fronts. The adjustable parameters are $\delta \in (0, 1)$, which measures the degree of inhomogeneity, and $\mu \in (0, 0.5)$, which measures how close f is to the derivative of an equal well potential.

Let us fix $\mu=0.365$. When we set $\delta=0.96$, Fig. 1 shows that wavefronts form and propagate to the left from the initial profile centered at $x=3$. We plot the history of propagation every 8 unit time intervals up to $t=80$. We observe the steady propagation of wiggling wavefronts (as opposed to the monotone wave profile in the spatially homogeneous case). Increasing the spatial domain or the computation time does not change the picture. Then we tune up δ to 0.97; Fig. 2 shows that the steady front

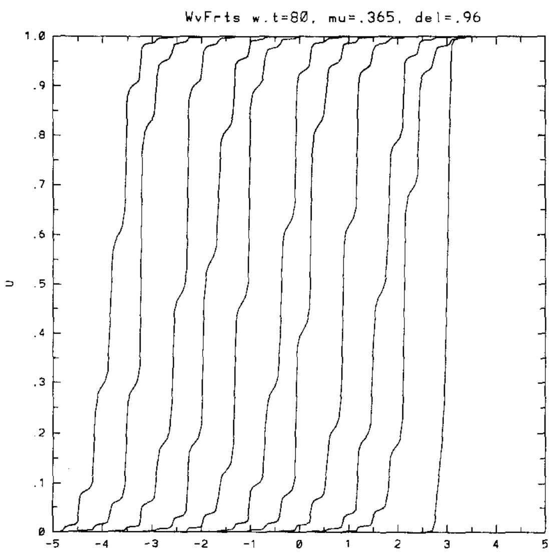

Fig. 1. Plot of solutions of Eq. (5.1) with $\mu=0.365$, $\delta=0.96$ for $t=8i$, $i=0$, 1,..., 10. Time step = 0.025, spatial step = 0.05.

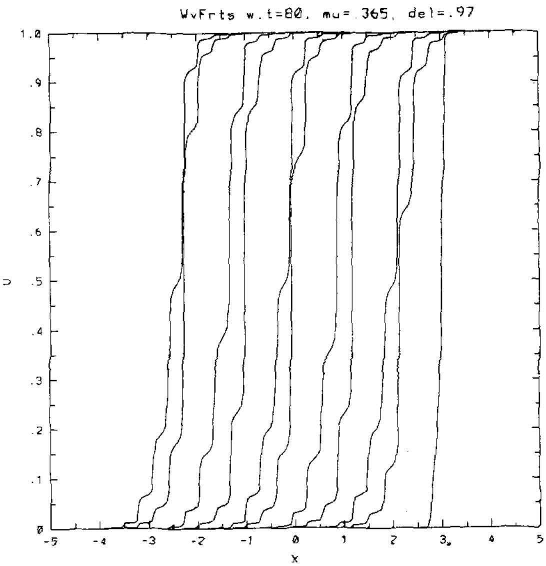

Fig. 2. Plot of solutions of Eq. (5.1) with $\mu = 0.365$, $\delta = 0.97$ for $t = 8i$, $i = 0, 1, \ldots, 10$. Time step = 0.025, spatial step = 0.05.

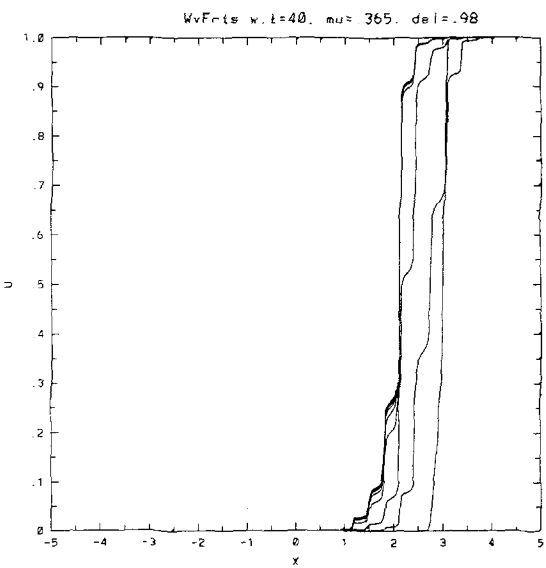

Fig. 3. Plot of solutions of Eq. (5.1) with $\mu = 0.365$, $\delta = 0.98$ for $t = 4i$, $i = 0, 1, \ldots, 10$. Time step = 0.025, spatial step = 0.05.

propagation still persists, but with a slower speed. Next, we tune up δ further to 0.98; Fig. 3 shows that the front forms and starts to move to the left, but finally stops near $x=2$, where quenching occurs. The total time in Fig. 3 is $t=40$, and the solutions are plotted every 4 unit time intervals. To make sure that we reach the steady state, we run up to $t=80$, and find the same quenching location, as shown in Fig. 4. When we increase δ above 0.98, quenching remains, which is seen in Fig. 5 with $\delta=0.99$. The same thing happens for other values of $\delta \in (0.98, 1)$. This confirms what we expected on the nonexistence interval of traveling waves at the end of the last section. Figure 6 illustrates a more remarkable appearance of quenching, where waves are localized amazingly close to the initial data, which is centered at $x=0$. The parameters there are $\mu=0.43$, $\delta=0.96$, the total running time is $t=40$, and the solutions are drawn every 4 unit time intervals.

The quenching phenomenon here bears a lot of resemblance to the localization in random Schrödinger operators, although the former occurs for the bistable nonlinearity already in periodic media, while the latter happens only in random media. What is more interesting is whether one has quenching (localization) for the combustion nonlinearity in random media. Quenching (localization) in this case does not happen in periodic

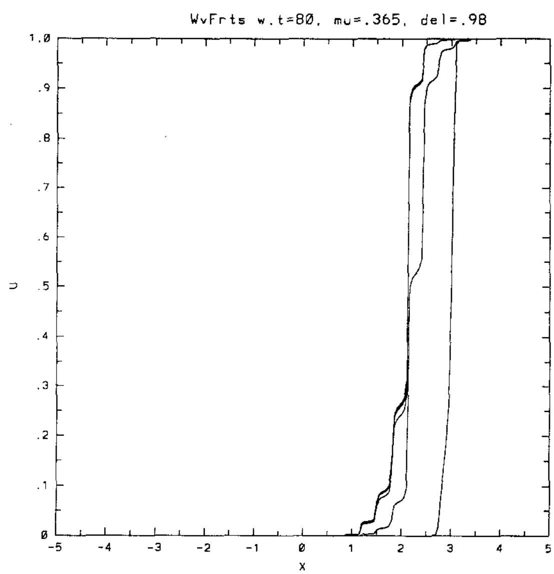

Fig. 4. Plot of solutions of Eq. (5.1) with $\mu=0.365$, $\delta=0.98$, for ten samples of time evolution from $t=0$ to $t=80$. Time step $=0.025$, spatial step $=0.05$.

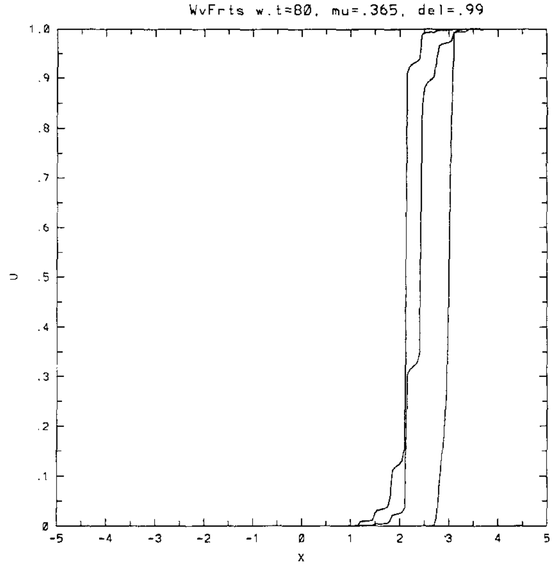

Fig. 5. Plot of solutions of Eq. (5.1) with $\mu = 0.365$, $\delta = 0.99$ for ten samples of time evolution from $t = 0$ to $t = 80$. Time step $= 0.025$, spatial step $= 0.05$.

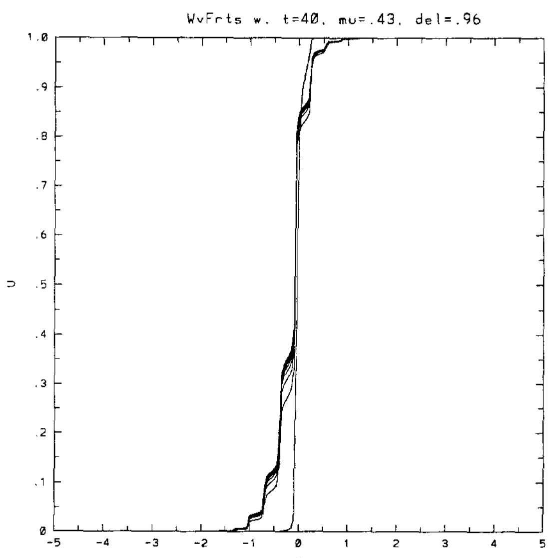

Fig. 6. Plot of solutions of Eq. (5.1) with $\mu = 0.43$, $\delta = 0.96$ for $t = 4i$, $i = 0, 1, \ldots, 10$. Time step $= 0.025$, spatial step $= 0.05$.

media, as shown in Xin,[28] yet the proof relies strongly on the periodicity assumption. It is thus conceivable that quenching may occur for flame fronts in random media. Also, what is quenching like in several space dimensions? We leave these issues to a future publication.

6. HOMOGENIZATION AND GEOMETRIC OPTICS

In this section, we scale Eq. (1.1) to a homogenization problem, give a formal geometric optics expansion, and derive the effective wavefront equation.

Let us consider the large-space, large-time or the homogenization limit of $u(t, x)$ of Eq. (1.1). Replacing x by x/ε and t by t/ε and denoting the scaled u by $u^\varepsilon(t, x)$, we see that $u^\varepsilon(t, x)$ satisfies

$$\begin{aligned} u^\varepsilon &= \varepsilon\nabla_x \cdot \left[a\left(\frac{x}{\varepsilon}\right) \nabla_x u^\varepsilon \right] + b\left(\frac{x}{\varepsilon}\right) \cdot \nabla_x u^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon) \\ u|_{t=0} &= u_0\left(\frac{x}{\varepsilon}\right) \end{aligned} \tag{6.1}$$

which becomes the familiar-looking homogenization problem with rapidly oscillating coefficients as $\varepsilon \to 0$. Due to the small parameter ε before the diffusion term and the large parameter $1/\varepsilon$ before the reaction term, this limit is also called the small-diffusion, fast-reaction limit. Notice that under the small-diffusion, fast-reaction limit, the transition layer of the wave profile is of $O(\varepsilon)$, which is the same order as that of the oscillation wavelengths in the coefficients. We look for a geometric optics expansion of the form

$$\begin{aligned} u^\varepsilon(t, x) &= U\left(\frac{\varphi(t, x, \varepsilon)}{\varepsilon}, \frac{x}{\varepsilon}, t, x\right) + \text{h.o.t.} \\ \varphi(t, x, \varepsilon) &= \varphi_0(t, x) + \varepsilon\varphi_1(t, x) + \text{h.o.t.} \end{aligned} \tag{6.2}$$

where $\varphi(t, x, \varepsilon)$ is the phase variable and U is the amplitude. This is a hybrid expansion based on the usual geometric optics and the homogenization expansions. Substituting (6.2) into (6.1) gives to leading order

$$\begin{aligned} &(\nabla_x \varphi_0 \partial_s + \nabla_y)[a(y)(\nabla_x \varphi_0 \partial_s + \nabla_y)\, U] + b(y) \cdot (\nabla_x \varphi_0 \partial_s + \nabla_y)\, U \\ &\quad - \varphi_{0,t} U_s + f(U) = 0 \end{aligned} \tag{6.3}$$

where $U = U(s, y, x, t)$, $s = \varphi(t, x, \varepsilon)/\varepsilon$, $y = x/\varepsilon$, and x, t are slow variables. We see that (6.3) is just Eq. (2.1) for the traveling waves with $k = \nabla_x \varphi_0$ and $c(k) = -\varphi_{0,t}$, which yields the "eikonal equation":

$$\varphi_{0,t} + c(\nabla_x \varphi_0) = 0 \tag{6.4}$$

This is a Hamilton–Jacobi equation with $c = c(k)$ as an effective Hamiltonian. The set $\{(t, x) \mid \varphi_0(t, x) = 0\}$ characterizes the effective wavefront separating the space-time regions where u^ε is near zero or one. In fact, Eq. (6.3) also defines the amplitude U, which is different from the usual geometric optics, where one gets an equation for the amplitude from the next-order terms of the expansion. It is straightforward to adapt Theorem 3.1 to one for the limit $\lim_{\varepsilon \to 0} u^\varepsilon(t, x)$, which we leave to the reader.

The geometric optics expansion in the KPP case is of the form

$$
\begin{aligned}
u_\varepsilon(t, x) &= \exp\left[-\frac{I(t, x, \varepsilon)}{\varepsilon} \right] + \text{h.o.t} \\
I(t, x, \varepsilon) &= I_0(t, x) + \varepsilon I_1\left(t, x, \frac{x}{\varepsilon}\right) + \text{h.o.t}
\end{aligned} \tag{6.5}
$$

which is very different from (6.2). The form (6.5) falls under the large-deviation framework, and $I(t, x, \varepsilon)$ is just the large-deviation rate function. It turns out that $I_0(t, x)$ also satisfies a Hamilton–Jacobi equation and the effective wave speeds can be determined independently of the wave shapes. We refer to Freidlin[13,14] and Gartner and Freidlin[15] for the probabilistic aspects and Barles *et al.*[2] (and references therein) for the PDE aspects of the theory. In contrast in the bistable or the combustion case, the wave speeds and wave shapes are coupled and have to be determined together.

Finally, we remark that the limit $\lim_{\varepsilon \to 0} u^\varepsilon(t, x)$ is a space-time almost everywhere strong limit due to the sharpening effects of wavefronts as ε goes to zero, regardless of the presence of the oscillating coefficients.

ACKNOWLEDGMENTS

I wish to thank Profs. G. Papanicolaou and T. Spencer for their interest in this work and several helpful conversations on related problems. I thank Dr. J.-M. Roquejoffre for sending me preprints of his work and Prof. M. Brio for his suggestions on numerics. Part of the work was done during the author's visit to the Institute for Advanced Study, Princeton, New Jersey, supported by NSF grant DMS-9100383. Last but not the least, I would like to express my deep appreciation to Lily Xie for her support, and thank the referees for their constructive comments.

REFERENCES

1. D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. Math.* **30**:33–76 (1978).
2. G. Barles, L. C. Evans, and P. E. Souganidis, Wavefront propagation for reaction-diffusion systems of PDE, *Duke Math. J.* **61**(3) (1990).
3. A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures* (North-Holland, Amsterdam, 1978).
4. H. Berestycki, B. Nicolaenko, and B. Scheurer, Travelling wave solutions to combustion models and their singular limits, *SIAM J. Math. Anal.* **16**(6):1207–1242 (1985).
5. H. Berestycki and L. Nirenberg, Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains, in *Analysis, et Cetera*, P. Rabinowitz *et al.* eds. (Academic Press, 1990), pp. 115–164.
6. H. Berestycki and L. Nirenberg, On the method of moving planes and the sliding method, *Bol. Soc. Bras. Mat.* **22**:1–37 (1991).
7. H. Berestycki and L. Nirenberg, Traveling fronts in cylinders (1992), preprint.
8. H. Berestycki and B. Larrouturou, A semi-linear elliptic equation in a strip arising in a two-dimensional flame propagation model, *J. Reine Angew. Math.* **396**:14–40 (1989).
9. H. Beresticki, B. Larrouturou, and P. L. Lions, Multi-dimensional travelling-wave solutions of a flame propagation model, *Arch. Rat. Mech. Anal.* **111**:33–49 (1990).
10. P. Clavin, and F. A. Williams, Theory of premixed-flame propagation in large-scale turbulence, *J. Fluid Mech.* **90**:598–604 (1979).
11. G. Dagan, and S. P. Neuman, Nonasymptotic behavior of a common Eulerian approximation for transport in random velocity fields, *Water Resources Res.* **27**(12):3249–3256 (1991).
12. P. C. Fife and J. B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, *Arch. Rat. Mech. Anal.* **65**:355–361 (1977).
13. Mark Freidlin, *Functional Integration and Partial Differential Equations* (Princeton University Press, Princeton, New Jersey, 1985).
14. M. I. Freidlin, Geometric optics approach to reaction-diffusion equations, *SIAM J. Appl. Math.* **46**(2) (1986).
15. J. Gartner and M. I. Freidlin, On the propagation of concentration waves in periodic and random media, *Dokl. Akad. Nauk SSSR* **249**:521–525 (1979).
16. A. Friedman, *Partial Differential Equations of Parabolic Type* (Pentice-Hall, Englewood Cliffs, New Jersey, 1964).
17. J. Hanna, A. Saul, and K. Showalter, Detailed studies of propagating fronts in the iodate oxidation of arsenous acid reaction, *J. Phys. Chem.* **90**:225 (1986).
18. A. Kolmogorov, I. Petrovskii, and N. Piskunov, A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem, *Bjul. Mosk. Gos. Univ.* **1**(7):1–26 (1937).
19. C. Li, Ph.D. Thesis, Courant Institute, NYU (1989).
20. J. Pauwelussen, Nerve impulse propagation in a branching nerve system: A simple model, *Physica D* **1981**(4):67–88.
21. G. Papanicolaou and X. Xin Reaction-diffusion fronts in periodically layered media, *J. Stat. Phys.* **63**(5/6):915–931 (1991).
22. J.-M. Roquejoffre, Eventual monotonicity and convergence to traveling fronts for the solutions of parabolic equations in cylinders (1992, preprint.
23. G. I. Sivashinsky, Cascade-renormalization theory of turbulent flame speed, *Combust. Sci. Tech.* **62**:77–96 (1988).
24. S. R. S. Varadhan, On the behavior of the fundamental solutions of the heat equation with variable coefficients, *Commun-Pure Appl. Math.* **20**(2):431–455 (1967).

25. S. Yates and C. Ensfield, Transport of dissolved substance with second order reaction, *Water Resources Res.* **25**(7):1757–1762 (1990).
26. X. Xin, Existence and stability of travelling waves in periodic media governed by a bistable nonlinearity, *J. Dynam. Differential Equations* **3**(4):541–573 (1991).
27. X. Xin, Existence and uniqueness of traveling waves in a reaction-diffusion equation with combustion nonlinearity, *Indiana Univ. Math. J.* **40**(3):985–1008 (1991).
28. J. X. Xin, Existence of planar flame fronts in convective diffusive periodic media, *Arch. Rat. Mech. Anal.* **121**:205–233 (1992).
29. V. Yakhot, Propagation velocity of premixed turbulent flames, *Combustion Sci. Technol.* **60**:191–214 (1988).