Self-Similar Decay in the Kraichnan Model of a Passive Scalar

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Abstract

We study the two-point correlation function of a freely decaying scalar in Kraichnan's model of advection by a Gaussian random velocity field, stationary and white-noise in time but fractional Brownian in space with roughness exponent $0 < \zeta < 2$, appropriate to the inertial-convective range of the scalar. We find all self-similar solutions, by transforming the scaling equation to Kummer's equation. It is shown that only those scaling solutions with scalar energy decay exponent $a \leq (d/\gamma)+1$ are statistically realizable, where d is space dimension and $\gamma = 2-\zeta$. An infinite sequence of invariants $J_p,\ p = 0, 1, 2, ...$ is pointed out, where J_0 is Corrsin's integral invariant but the higher invariants appear to be new. We show that at least one of the invariants J_0 or J_1 must be nonzero (possibly infinite) for realizable initial data. Initial data with a finite, nonzero invariant—the first being J_p—converge at long times to a scaling solution Φ_p with $a = (d/\gamma) + p,\ p = 0, 1$. The latter belong to an exceptional series of self-similar solutions with stretched-exponential decay in space. However, the domain of attraction includes many initial data with power-law decay. When the initial data has all invariants zero or infinite and also it exhibits power-law decay, then the solution converges at long times to a non-exceptional scaling solution with the same power-law decay. These results support a picture of a "two-scale" decay with breakdown of self-similarity for a range of exponents $(d+\gamma)/\gamma < a < (d+2)/\gamma$, analogous to what has recently been found in decay of Burgers turbulence.

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1 Introduction

A key hypothesis in the theory of turbulence decay is that the energy spectrum has asymptotically a *self-similar* form

$$E(k,t) = v^2(t)\ell(t)F(k\ell(t)) \tag{1.1}$$

where $\ell(t)$ is a suitable length-scale and $v(t)$ a velocity scale. Equivalently, the hypothesis may be made for the longitudinal velocity two-point correlation $B_{LL}(r,t) := \hat{r}_i\hat{r}_j\langle v_i(\mathbf{x})v_j(\mathbf{x}+\mathbf{r})\rangle$, that it obey:

$$B_{LL}(r,t) = v^2(t)f(r/\ell(t)). \tag{1.2}$$

Historically, hypotheses (1.1) and (1.2) were first proposed for freely decaying turbulence in 1938 by von Kármán and Howarth [1]. They referred to the *Ansätze* (1.1) and (1.2) as "self-preservation hypotheses," since the shape of the spectrum and correlation function are thereby preserved in the decay process. Detailed discussion of such self-preservation hypotheses is to be found in [2], Chapter 16. Since the early work in turbulence, corresponding hypotheses have been proposed for many other nonequilibrium processes, e.g. surface growth [3] and phase-ordering dynamics [4]. In those fields the assumption is usually called *dynamic self-similarity* or *dynamic scaling*. Essentially, the hypothesis amounts to the statement that there is only one relevant length-scale in the decay process. For example, in turbulence decay this is plausibly the integral length-scale $L(t) := \frac{1}{B_{LL}(0,t)}\int_0^\infty dr\ B_{LL}(r,t)$. Althoughly widely employed in nonequilibrium physics, the validity of such self-similarity hypotheses is still debated and their foundations poorly understood. This is particularly true when the random initial data of the system exhibit long-range power-law correlations.

Recently, the validity of the self-similarity has been examined in a soluble model, the decaying Burgers turbulence [5]. Those authors solved exactly for the two-point correlations and energy spectra of the one-dimensional Burgers equation with initial energy spectra exhibiting a low-wavenumber power-law form, $E(k,t_0) \sim Ak^n$ for $kL_0 \ll 1$, and thus a power-law decay in the spatial velocity correlation function, $B(r,t_0) \sim A'r^{-(n+1)}$ for $r \gg L_0$ (when n is not a positive, even integer). What was discovered by the authors of [5] was that the hypothesis of dynamic self-similarity was violated when $1 < n < 2$. Instead, a new length-scale $L_*(t)$ developed dynamically which was much larger asymptotically than the integral length-scale $L(t)$. The new length-scale was characterized by the property that the initial low-wavenumber power-law spectrum was preserved only for $kL_*(t) \ll 1$. This preservation was traditionally believed to hold for all $kL(t) \ll 1$, which has been called the principle of *permanence of large-eddies* [6]. The development of two distinct length-scales had important consequences for the decay process. For example, the rate of decay was found to be different than that predicted by traditional phenomenological theory.

Another exactly soluble turbulence model is available, a model of a turbulently advected scalar proposed by R. H. Kraichnan in 1968 [7]. This model corresponds to a stochastic advection-diffusion equation

$$\partial_t \theta(\mathbf{r},t) + (\mathbf{v}(\mathbf{r},t)\boldsymbol{\cdot}\boldsymbol{\nabla})\theta(\mathbf{r},t) = \kappa \triangle \theta(\mathbf{r},t), \tag{1.3}$$

in which the advecting field $\mathbf{v}(\mathbf{r},t)$ is a "synthetic turbulence", specifically, a Gaussian random velocity field with zero mean and covariance

$$\langle v_i(\mathbf{r},t) v_j(\mathbf{r}',t')\rangle = D_{ij}(\mathbf{r}-\mathbf{r}')\delta(t-t') \tag{1.4}$$

which is white-noise in time. The remarkable feature of the model which Kraichnan discovered is that there is no closure problem. In particular, Kraichnan showed that the 2-point correlation function $\Theta(\mathbf{r},t) := \langle\theta(\mathbf{r},t)\theta(\mathbf{0},t)\rangle$ obeys the following closed equation in homogeneous scalar decay:

$$\partial_t \Theta(\mathbf{r},t) = [D_{ij}(\mathbf{0}) - D_{ij}(\mathbf{r})]\boldsymbol{\nabla}_{\mathbf{r}_i}\boldsymbol{\nabla}_{\mathbf{r}_j}\Theta(\mathbf{r},t) + 2\kappa \triangle \Theta(\mathbf{r},t). \tag{1.5}$$

Recently the study of the Kraichnan model has undergone a renaissance, impelled by the observation [8] that the Nth-order statistical correlations for $N > 2$ should exhibit "anomalous scaling", not predicted by naive dimensional analysis as in Kolmogorov's 1941 theory. In particular, attention has been focused on the *inertial-convective range* of the model, in which the molecular diffusivity $\kappa \to 0$ and the velocity covariance in space has a power-law form

$$D_{ij}(\mathbf{r}) \sim D_{ij}(\mathbf{0}) - D_1 \cdot r^\zeta \left[\delta_{ij} + \frac{\zeta}{d-1}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right] + O\left(\frac{r}{L}\right)^2, \tag{1.6}$$

for $0 < \zeta < 2$ and $r \ll L$, where the latter length-scale is the velocity integral scale. In this relation statistical isotropy as well as homogeneity has been assumed. The formula (1.6) mimicks the situation in a real turbulent scalar decay when the scalar is spectrally supported on length-scales L_θ much smaller than the velocity integral scale L but yet much larger than the dissipation length-scale η_d set by the molecular diffusivity κ. In this situation, the calculation of the anomalous exponents has proved to be possible analytically by perturbation expansion in three regimes: small space Hölder exponent $H = \frac{\zeta}{2}$ corresponding to a "rough" velocity field [9, 10], small exponent $1 - H$ coresponding to expansion about a "smooth" velocity field or so-called Batchelor limit [11, 12], and expansion in $\frac{1}{d}$ with d the space dimension [13, 14].

Our interest here is instead to study the Kraichnan model as a soluble test case of turbulent decay. Passive scalars undergo a turbulent decay which is similar in many respects to that of the velocity itself. This is very well described in [15]. The decrease in the scalar energy or intensity $E_\theta(t) := \frac{1}{2}\langle\theta^2(\mathbf{0},t)\rangle$ is by a process of progressive degradation of the scalar at the higher wavenumbers. In the course of this decay process, the scalar integral length-scale $L_\theta(t) = \frac{1}{\Theta(0,t)}\int_0^\infty dr\ \Theta(r,t)$ grows as the spectral support of the scalar is shifted progressively to lower wavenumbers. On dimensional grounds, this growth of the length-scale is governed by

$$\frac{1}{L_\theta(t)}\frac{dL_\theta(t)}{dt} \propto D_1 L_\theta^{-\gamma}(t), \tag{1.7}$$

with $\gamma := 2 - \zeta$, which leads to the relation $L_\theta(t) \propto (D_1(t-t_0))^{1/\gamma}$. As has been emphasized in [15], this can be thought of as a "Richardson diffusion" of the scalar integral length-scale up through the velocity inertial range (which is here taken to be statistically stationary). This growth law for the scalar length-scale can be converted into an energy decay law under two additional assumptions. First, if one considers initial scalar spectra of a power-law form $E_\theta(k,t_0) \propto Ak^n$ for low-wavenumbers $kL_\theta(t_0) \ll 1$, then the hypothesis of permanence of large eddies would imply that this low-wavenumber spectrum persists with a time-independent constant A for $kL^*_\theta(t) \ll 1$, if $n < d+1$. Second, the hypothesis of dynamic self-similarity would imply that there is only one relevant length-scale, the scalar integral length $L_\theta(t)$, so that $L^*_\theta(t) = L_\theta(t)$ up to a constant factor. Under these two assumptions, the scalar energy may be estimated to order of magnitude by integrating over the low-wavenumber range up to $L_\theta^{-1}(t)$, with the result

$$E_\theta(t) \propto AL_\theta^{-p}(t) \propto A(D_1(t-t_0))^{-p/\gamma}, \tag{1.8}$$

for $p = n+1$. Thus, the decay rate is non-universal, and depends upon the low-wavenumber spectral exponent.

The main aim of this work is to examine the decay problem in the Kraichnan model, to investigate the universality of the "two-scale" phenomenon discovered in [5]. Because the standard phenomenology is common to both velocity and passive scalar decay, we may use the Kraichnan model as a source of insight. There seems to have been less work on the decay problem in the Kraichnan model than on the statistical steady-state and most of this in the Batchelor limit $\zeta = 2$, the so-called viscous-convective range. In addition to the early work of Kraichnan [16], the decay of the passive scalar in the Batchelor limit has been recently examined by Son [17]. Our work here will be devoted instead to the inertial-convective range of the scalar in which the velocity correlator is given by (1.6) with $0 < \zeta < 2$. A few preliminary investigations on this problem have been reported in a very recent review article of Majda and Kramer [18], Section 4.2, but no exhaustive study seems yet to have been made. Our interest is thus in the low-order $N = 2$ correlator, rather than in the higher-order statistics. In the same inertial-convective range considered above, with the assumptions of homogeneity and isotropy, equation (1.5) for the 2-point correlator simplifies to

$$\partial_t\Theta(r,t) = \frac{D_1}{r^{d-1}}\frac{\partial}{\partial r}\left[r^{d+\zeta-1}\frac{\partial\Theta}{\partial r}(r,t)\right]. \tag{1.9}$$

We shall be particularly interested in the issue of self-similarity of the decay process. In fact, one of our main results will be an analytical construction and complete classification of *all* self-similar decay solutions of equation (1.9), along with an analysis of their domains of attraction.

The reader should note that in equation (1.9) the limit has been taken of vanishing molecular diffusivity. We have done so in order to focus on the turbulent dissipation of the scalar, which leads to a decaying scalar energy $E_\theta(t) \to 0$ as $t \to \infty$ even in the limit $\kappa \to 0$. This corresponds to a famous conjecture on the three-dimensional energy cascade put forth by Onsager [19], who proposed that the limiting turbulent ensemble in the limit of vanishing viscosity $\nu \to 0$ should consist of realizations of the inviscid Euler equations which dissipate energy. Of course, these must be weak or distributional solutions, not classical solutions. Onsager coined the term "ideal turbulence" for this limiting dissipative ensemble governed by the ideal fluid equations. More recently, this ideal mechanism of dissipation has been called the "dissipative anomaly," since Polyakov pointed out a close similarity to conservation-law anomalies in quantum field theory [20]. This specifically turbulent mechanism of dissipation is well-illustrated by the Kraichnan model. Since the operator on the RHS of (1.9) is homogeneous degree $-\gamma$, it follows using $E_\theta(t) = \frac{1}{2}\Theta(0,t)$ that

$$\frac{dE_\theta}{dt}(t) = \frac{D_1}{2r^{d-1}}\frac{\partial}{\partial r}\left[r^{d+\zeta-1}\frac{\partial\Theta}{\partial r}(r,t)\right]\Big|_{r=0} = 0 \tag{1.10}$$

when the 2nd-order structure function

$$\begin{aligned} S_2(r,t) &:= \langle[\theta(\mathbf{r},t)-\theta(\mathbf{0},t)]^2\rangle \\ &= 2[\Theta(0,t)-\Theta(r,t)] \sim Cr^{\xi}, \end{aligned} \tag{1.11}$$

with $\xi > \gamma$. Thus, some critical degree of singularity is required for turbulent dissipation. Our self-similar decay solutions—which are exact solutions of the zero-diffusion Kraichnan equations—explicitly illustrate this ideal dissipation mechanism. The implications for the theory of weak solutions of the hyperbolic stochastic PDE, the $\kappa \to 0$ limit of (1.3), will be discussed in a forthcoming work [21].

2 Background Material

(2.1) Review of the Kraichnan Model

The model we consider is the stochastic partial differential equation

$$d\theta(\mathbf{r},t) = \kappa \triangle \theta(\mathbf{r},t)dt - (\mathbf{W}(\mathbf{r},dt)\boldsymbol{\cdot}\boldsymbol{\nabla})\theta(\mathbf{r},t), \tag{2.1}$$

where $\mathbf{W}(\cdot,t)$ is a Wiener process in the function space $C(\mathbb{R}^d,\mathbb{R}^d)$, with covariance function

$$\langle W_i(\mathbf{r},t)W_j(\mathbf{r}',t')\rangle = D_{ij}(\mathbf{r}-\mathbf{r}')t\wedge t'. \tag{2.2}$$

The stochastic PDE is interpreted in the Stratonovich sense. See [22], Chapter 6, and [23] for a more detailed discussion of the mathematical foundations. The spatial covariance matrix $\mathbf{D}$ we consider is defined by the Fourier integral

$$D_{ij}(\mathbf{r}) = D\int d^d\mathbf{k}\ \frac{P_{ij}(\mathbf{k})}{\left(k^2+k_0^2\right)^{(d+\zeta)/2}}e^{i\mathbf{k}\boldsymbol{\cdot}\mathbf{r}}. \tag{2.3}$$

where $0<\zeta<2$. The constant k_0 is an infrared cutoff for the velocity field, proportional to the inverse velocity integral length $k_0 \propto L^{-1}$. $P_{ij}(\mathbf{k})$ is the projection in $\mathbb{R}^d$ onto the subspace perpendicular to $\mathbf{k}$. Thus (2.3) automatically defines a suitable positive-definite, symmetric matrix-valued function, divergence-free in each index. We have made the choice (2.3) just for specificity. In fact, any velocity covariance with the properties discussed next would suffice.

The matrix $D_{ij}(\mathbf{r})$ can be written as $D_{ij}(\mathbf{r}) = P_{ij}(\boldsymbol{\nabla}_{\mathbf{r}})K(\mathbf{r})$, or as

$$D_{ij}(\mathbf{r}) = K(r)\delta_{ij} + \partial_i\partial_j H(r), \tag{2.4}$$

where the function $K(r)$ is defined by the integral

$$K(r) = D\int d^d\mathbf{k}\ \frac{e^{i\mathbf{k}\boldsymbol{\cdot}\mathbf{r}}}{\left(k^2+k_0^2\right)^{(d+\zeta)/2}} \tag{2.5}$$

and $H(r)$ is given by the (for $d=2$, principal part) integral

$$H(r) = D\int d^d\mathbf{k}\ \frac{1}{k^2}\cdot\frac{e^{i\mathbf{k}\boldsymbol{\cdot}\mathbf{r}}}{\left(k^2+k_0^2\right)^{(d+\zeta)/2}}, \tag{2.6}$$

so that $-\triangle H = K$. The scalar function $K(r)$ is essentially just the standard Bessel potential kernel [24], and may thus be expressed in terms of a modified Bessel function:

$$K(r) = D\frac{2^{1-(\zeta/2)}k_0^{-\zeta}\pi^{d/2}}{\Gamma\left(\frac{d+\zeta}{2}\right)} \cdot (k_0 r)^{\zeta/2} K_{\zeta/2}(k_0 r). \tag{2.7}$$

The Hessian matrix $\partial_i\partial_j H(r)$ of the function H of magnitude $r = |\mathbf{r}|$ alone is

$$\partial_i\partial_j H(r) = \delta_{ij} J(r) + \widehat{r}_i\widehat{r}_j \cdot r\frac{dJ}{dr}(r), \tag{2.8}$$

with $J(r) = H'(r)/r$ and $\widehat{\mathbf{r}} = \mathbf{r}/r$. However, because $\mathrm{Tr}\,(\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} H) = -K$, a Cauchy-Euler equation follows for $J(r)$:

$$r\frac{dJ}{dr}(r) + d \cdot J(r) = -K(r). \tag{2.9}$$

Due to the rapid decay of its Fourier transform, the function $J(r)$ is continuous. Thus, the relevant solution is found to be

$$J(r) = -r^{-d}\int_0^r \rho^{d-1} K(\rho) d\rho. \tag{2.10}$$

in terms of $K(r)$. Using this expression for $J(r)$, along with Eq.(2.8), we thus find

$$D_{ij}(\mathbf{r}) = (K(r) + J(r))\delta_{ij} - (K(r) + d \cdot J(r))\widehat{r}_i\widehat{r}_j, \tag{2.11}$$

which gives D_{ij} as an explicit linear functional of K. Cf. [2], equations (14.1),(14.3).

We are interested to consider the model in the range of length-scales $r \ll L$. We require some asymptotic expressions in that range:

$$K(\mathbf{r}) = K_0 - K_1 r^\zeta + O\left(k_0^2 r^2\right), \tag{2.12}$$

with

$$K_0 = D\frac{\Gamma\left(\frac{\zeta}{2}\right)\pi^{d/2}}{\Gamma\left(\frac{d+\zeta}{2}\right)} \cdot k_0^{-\zeta} \tag{2.13}$$

and

$$K_1 = D\frac{\Gamma\left(\frac{\gamma}{2}\right)\pi^{d/2}}{2^\zeta \cdot \zeta \cdot \Gamma\left(\frac{d+\zeta}{2}\right)}. \tag{2.14}$$

Also,

$$D_{ij}(\mathbf{r}) = D_0\delta_{ij} - D_1 \cdot r^{\zeta} \cdot \left[\delta_{ij} + \frac{\zeta}{d-1}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right] + O\left(k_0^2 r^2\right) \tag{2.15}$$

with

$$D_0 = \frac{d-1}{d} K_0 \tag{2.16}$$

and

$$D_1 = \frac{d-1}{d+\zeta} K_1 \tag{2.17}$$

The first equation is derived by means of the known Frobenius series expansion for the modified Bessel functions (e.g. [25], section 7.2.2, equations (12),(13)). These give

$$z^{\nu} K_{\nu}(z) = \frac{\Gamma(\nu)}{2^{1-\nu}} - \frac{\Gamma(1-\nu)}{\nu \cdot 2^{1+\nu}} z^{2\nu} + O\left(z^2\right). \tag{2.18}$$

From this expansion for $K_\nu(z)$ and from the representation (2.7) for $K(r)$ we obtain the asymptotic expression (2.12). Next, using (2.11), we observe that, if $K(r)$ has a power-law form, $K(r) = Kr^{\xi}$, then it is easy to calculate that

$$D_{ij}(\mathbf{r}) = Kr^{\xi} \frac{d-1}{d+\xi}\left[\delta_{ij} + \frac{\xi}{d-1}\left(\delta_{ij} - \widehat{r}_i \widehat{r}_j\right)\right]. \tag{2.19}$$

Since $D_{ij}(r)$ is linearly related to $K(r)$, we may apply this formula to the first two terms in the expansion (2.12), taking first $\xi = 0$ and then $\xi = \zeta$. This yields the second expansion (2.15).

As already shown by Kraichnan [7], the equation for the 2-point correlation functions in the model (2.1) is closed. This was subsequently generalized to the N-point correlations [26, 27]. By now, these equations have been derived by several arguments and in many places, e.g. [28]. Therefore, we shall give no derivation here. However, we make a few comments on the physical interpretation in the case $N = 2$. Since $D_{ij}(\mathbf{0}) = D_0\delta_{ij}$, the two terms from $D_{ij}(\mathbf{0}) - D_{ij}(\mathbf{r})$ may be treated separately, with the result that (1.5) may be written as

$$\partial_t \Theta(\mathbf{r}, t) = (D_0 + 2\kappa) \triangle \Theta(\mathbf{r}, t) - D_{ij}(\mathbf{r}) \boldsymbol{\nabla}_{\mathbf{r}_i} \boldsymbol{\nabla}_{\mathbf{r}_j} \Theta(\mathbf{r}, t). \tag{2.20}$$

Hence, we see that the first term gives essentially an augmentation of the molecular diffusivity, i.e. it produces an "eddy diffusivity" $\kappa_{eddy} = \frac{1}{2} D_0$. In fact, it is the same eddy diffusivity which appears in the $N = 1$ equation [28]. The second term, as we discuss in more detail below, represents additional triadic interactions between one velocity mode and two scalar modes. From (2.13)-(2.17) we see that both of these terms are separately infrared divergent, in the limit $k_0 \to 0$, but that this infrared divergence cancels in the equation (1.5) for $\Theta(\mathbf{r}, t)$. Since we are only interested in the inertial-convective range behavior of the scalar, when $L_\theta(t) \ll L$, it is convenient for us to take the limit $k_0 \to 0$. This has been done in deriving (1.9), which follows easily from (1.5) and (2.15).

We shall also need below the equation for the spectral scalar energy transfer. We introduce the Fourier transform

$$\widehat{\Theta}(\mathbf{k}, t) := \frac{1}{(2\pi)^d} \int \Theta(\mathbf{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{r}. \tag{2.21}$$

In terms of this, the scalar energy spectrum is

$$E_\theta(k, t) := \frac{1}{2}\omega_{d-1} k^{d-1} \widehat{\Theta}(k, t), \tag{2.22}$$

where $\omega_{d-1} = 2\pi^{d/2}/\Gamma\left(\frac{d}{2}\right)$ is the $(d-1)$-dimensional measure of the unit sphere in d-dimensions and, if the scalar statistics are not isotropic, $\widehat{\Theta}(k, t)$ is a spherical average. It is straightforward to Fourier transform (1.5), with the result

$$\begin{aligned} \partial_t \widehat{\Theta}(\mathbf{k}, t) &= -k_i k_j \int d^d\mathbf{q} \, \widehat{D}_{ij}(\mathbf{q}) \left[\widehat{\Theta}(\mathbf{k}, t) - \widehat{\Theta}(\mathbf{k} - \mathbf{q}, t) \right] - 2\kappa k^2 \widehat{\Theta}(\mathbf{k}, t) \\ &= -(D_0 + 2\kappa) k^2 \widehat{\Theta}(\mathbf{k}, t) + \int d^d\mathbf{q} \, (\mathbf{k}^\top \widehat{\mathbf{D}}(\mathbf{q}) \mathbf{k}) \cdot \widehat{\Theta}(\mathbf{k} - \mathbf{q}, t), \end{aligned} \tag{2.23}$$

where $\widehat{D}_{ij}(\mathbf{q}) = \widehat{K}(q) P_{ij}(\hat{\mathbf{q}})$. Notice that this last $d \times d$ matrix is positive semidefinite. Also, for each $\mathbf{k}$, $\widehat{\Theta}(\mathbf{k}, t) \geq 0$, as a statistical realizability requirement. Thus, we can now see that in the spectral representation for each wavevector $\mathbf{k}$, the first D-term is always negative and represents a "loss" term, while the second D-term is always positive and gives a "gain" term. The first is an "eddy diffusivity" effect, as we have already discussed. The second can be seen to result from triadic interactions of a velocity mode with wavevector $\mathbf{q}$ and two scalar modes with wavevectors $\mathbf{k} - \mathbf{q}$ and $\mathbf{k}$. It is easy to derive from this an expression for the transfer term

$T_\theta(k,t)$ in the spectral energy balance equation

$$\partial_t E_\theta(k,t) = T_\theta(k,t) - 2\kappa k^2 E_\theta(k,t). \tag{2.24}$$

There are "loss" and "gain" terms which are both infrared divergent in the limits $k_0 \to 0$, but the infrared divergence cancels exactly in the equation for $E_\theta(k,t)$.

(2.2) Phenomenology of Turbulent Scalar Decay

As discussed briefly in the Introduction, the standard phenomenology of turbulent decay is built upon two fundamental hypotheses: the *permanence of large eddies* (PLE) and *dynamic self-similarity* (DSS). We shall review each of these topics here in turn, commenting upon the original motivations for these hypotheses and their dynamical justification (or not) within the Kraichnan model.

First, we consider the permanence of large eddies. The motivation for this hypothesis lies in the phenomenon of *spectral backtransfer of energy.* As was first shown by Proudman and Reid in a calculation with the quasinormal closure for decaying three-dimensional, homogeneous turbulence [29], the rate of change of the energy spectrum asymptotically at very low wavenumbers is dominated by a small but significant source of energy, which arises from nonlinear interactions of energy-range modes. Because the source-term is positive, and hence opposite in sign to the forward-cascading transfer through the inertial subrange, this phenomenon is called "backtransfer". According to calculations within spectral closures— such as quasinormal closure or its more sophisticated descendants, such as eddy-damped quasinormal markovian (EDQNM) closure—the transfer rate $T(k,t)$ is a power-law form $\dot{B}(t)k^{d+1}$ in d space-dimensions. See [29] and also [2], Sections 15.5-15.6. The same phenomenon for scalar transfer that $T_\theta(k,t) \sim \dot{B}_\theta(t)k^{d+1}$ at low k was subsequently found by Reid using again the quasinormal closure [30]. See also [15] and [2], Section 19.4. These closure calculations lead one to expect that, if the initial scalar spectrum has a power-law form, $E_\theta(k,t_0) \sim Ak^n$ for $kL_\theta(t_0) \ll 1$, then this state of affairs will be preserved in time for $n < d+1$. Indeed, with the latter assumption, the initial spectrum will dominate the time-integral $\int_{t_0}^t ds\ T_\theta(k,s) \sim B_\theta(t)k^{d+1}$ asymptotically for small enough k. Hence, one may expect that $E_\theta(k,t) \sim A_\theta k^n$ for $kL_\theta(t) \ll 1$ with a constant independent of time. This is the usual statement of the hypothesis of permanence of large eddies, in a spectral formulation.

One may also formulate a permanence hypothesis in physical space. Thus, in the scalar case, one may suppose that, if $\Theta(r,t) \sim A'_\theta r^{-p}$ for $r \gg L(t)$ at the initial time $t = t_0$, then this relation will persist with the same constant A'_θ at later times $t > t_0$, at least when $p < d+2$. In the case of velocity correlations for Navier-Stokes turbulence, Proudman and Reid showed that pressure forces induce a long-range power-law $r^{-(d+2)}$ at any positive time, even if such correlations are not present initially. This is the exact physical-space analogue of the k^{d+1} spectral backtransfer. Thus, when $p < d+2$ the correlations present initially shall dominate for r sufficiently large. Similar arguments apply to the scalar correlations. Although the long-range pressure forces drop out of the expression for the first time-derivative of the scalar correlation, they appear in the second- and higher-order derivatives. Thus, a similar power-law decay is expected there.

Although there is a formal correspondence between the spectral space and physical space formulations of the hypothesis of permanent large eddies, the two versions are *not* equivalent, as has been emphasized in [5]. Formally (e.g. see [31], Ch.IX.6, Theorem 4),

$$E_\theta(k,t) \sim A_\theta k^n \quad \text{for} \quad kL_\theta(t) \ll 1 \tag{2.25}$$

corresponds to

$$\Theta(r,t) \sim A'_\theta r^{-p} \quad \text{for} \quad r \gg L_\theta(t), \tag{2.26}$$

with $p = n+1$ and

$$A'_\theta = 2^p \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-p}{2}\right)} A_\theta. \tag{2.27}$$

Even formally, one can see that there is a problem when $p - d = 2m$, an even, nonnegative integer, since then $A'_\theta = 0$ given by the above formula, when A_θ is finite. In fact, this case corresponds to $\widehat{\Theta}(k,t) \sim (k^2)^m$ for $kL_\theta(t) \ll 1$, which is thus analytic at small k. Hence, one may expect the decay of $\Theta(r,t)$ in physical space to be faster than any power, consistent with the vanishing of A'_θ in (2.27). It is therefore quite possible that $E_\theta(k,t)$ exhibits a power-law (with or without permanent coefficient) and that $\Theta(r,t)$ has no power-law behavior whatsoever.

A case in point is when the function $\Theta(r,t)$ is integrable, with a nonzero integral,

$$0 < K(t) := \frac{1}{(2\pi)^d}\int d\mathbf{r}\ \Theta(\mathbf{r},t) < \infty. \tag{2.28}$$

This is consistent with rapid decay in physical space faster than any power, e.g. exponential. However, the energy spectrum in this case exhibits a power-law at low wavenumber. In fact,

$$E_\theta(k,t) \sim A(t)k^{d-1} \tag{2.29}$$

with

$$A(t) := \frac{1}{2}\omega_{d-1}\cdot K(t), \tag{2.30}$$

just using the definition of the energy spectrum. This is sometimes called an "equipartition spectrum", because it represents an average modal energy which is the same for each Fourier mode $\mathbf{k}$. Not only is there a power-law spectrum, but, in fact, the coefficient $A(t)$ is independent of time t, i.e. spectral PLE holds. This is true because $K(t)$ is known to be a constant of the motion, called the *Corrsin invariant.* It was first derived for the true passive scalar by S. Corrsin in 1951 [32, 33], who employed the equation

$$\partial_t\Theta(r,t) = \frac{2}{r^{d-1}}\frac{\partial}{\partial r}\left[r^{d-1}\left(B(r,t) + \kappa\frac{\partial\Theta}{\partial r}(r,t)\right)\right], \tag{2.31}$$

with

$$B(r,t) := \widehat{\mathbf{r}}\boldsymbol{\cdot}\langle\mathbf{v}(\mathbf{r},t)\theta(\mathbf{r},t)\theta(\mathbf{0},t)\rangle. \tag{2.32}$$

This equation plays the role of the von Kármán-Howarth equation for the temperature field and it is analogous to the equation (1.9) in the Kraichnan model. It is not hard to see that this equation implies that $dK(t)/dt = 0$ when $B(r,t) = o\left(r^{-(d-1)}\right)$. See [2], Section 15.2. Furthermore, the quasinormal closure calculations suggest that the indirect effect of pressure forces in $\partial_t B(r,t)$ (i.e. in the higher-order time-derivatives) lead to a decay at least as $B(r,t) =$

$O\left(r^{-(d+1)}\right)$, provided that the decay was no slower initially. See [2], Section 15.6. The induced power-law decay in the scalar correlation itself is $\Theta(r,t) = O\left(r^{-(d+2)}\right)$, which is integrable and thus the Corrsin integral should still be finite.

All of these arguments, which are derived under some closure assumptions for true passive scalars, are derived more directly and convincingly in the Kraichnan model. For example, if initially there is a power-law decay of correlations in space as in (2.26) in the Kraichnan model, then

$$\partial_t \Theta(r,t_0) \sim -D_1 p(d+\gamma-p)A'_\theta r^{-(p+\gamma)} \quad \text{for} \quad r \gg L_\theta(t_0), \tag{2.33}$$

since the operator on the RHS of (1.9) is homogeneous degree $-\gamma$. Since $\gamma > 0$, this decay is faster, and it does not seem possible for the initial power-law to be upset at large enough r. Notice that in the Kraichnan model—unlike for true turbulent scalars— there is no action of pressure forces on the scalar correlators whatsoever. Hence, in the Kraichnan model, spatial PLE should hold for *all* $p > 0$ and not just for $0 < p < d+2$. In the spectral formulation, however, there is such a restriction. It follows from (2.23) that in the limit as $k \to 0$,

$$\partial_t E_\theta(k,t) \sim -(D_0 + 2\kappa)k^2 E_\theta(k,t) + \dot{B}_\theta(t)k^{d+1} \tag{2.34}$$

with

$$\dot{B}_\theta(t) := \int d^d\mathbf{q}\ (\widehat{\mathbf{k}}^\top \widehat{\mathbf{D}}(\mathbf{q})\widehat{\mathbf{k}}) \cdot \widehat{\Theta}(\mathbf{q},t) \geq 0 \tag{2.35}$$

and $\widehat{\mathbf{k}}$ any unit vector. Thus, the "gain" term provides exactly the k^{d+1} power-law backtransfer term expected at low wavenumber k. Hence, spectral PLE should hold when there is a power-law spectrum of the form (2.25) initially, with $n < d+1$. Finally, the theorem on the Corrsin invariant is easily seen to follow by multiplying (1.9) by r^{d-1} and integrating over r. The result follows as long as $\Theta(r,t)$ is $o\left(r^{-(d-\gamma)}\right)$ near $r=0$ and $r=\infty$. The moral is that PLE, whether in physical or spectral space, should hold when the necessary power-laws are present initially.

The second major hypothesis invoked in the traditional theory of turbulence decay is dynamic self-similarity (DSS). Mathematically, for the scalar decay problem, it amounts to the assumption that $\Theta(r,t)$ can be reduced to a function of a single variable $\Phi(\rho)$ through a suitable choice of the length and scalar concentration scales $\ell = \ell(t)$ and $\vartheta = \vartheta(t)$:

$$\Theta(r,t) = \vartheta^2(t)\Phi\left(\frac{r}{\ell(t)}\right). \tag{2.36}$$

The function $\Phi(\rho)$ is called the *scaling function.* There is an equivalent spectral space version in which one assumes that

$$E_\theta(k,t) = \vartheta^2(t)\ell(t)F(k\ell(t)) \tag{2.37}$$

for a spectral scaling function $F(\kappa)$. The scale $\vartheta(t)$ can be taken to be the rms scalar intensity $\vartheta(t) := [\langle\theta^2(t)\rangle]^{1/2} = [\Theta(\mathbf{0},t)]^{1/2}$. The length-scale can be taken to be, for example, the scalar integral length $L_\theta(t)$. Within the validity of the hypothesis, all other relevant lengths are either $0,\infty$, or differ merely by a constant factor. It is natural to take $\ell(t) \propto L_\theta(t)$, because we want to consider a limit in which all dissipative length-scales go to zero and we also want to capture the energetics of the decay.

Clearly, the justification of the DSS hypothesis is more difficult than that of PLE. Mathematically, it provides a natural simplifying assumption, but truth and simplicity need not coincide. Our main purpose here is to investigate the validity of DSS in the specific context of the Kraichnan model. Although the methods we use are very specific to the model, they allow us to draw some conclusions that reasonably apply to other problems. The present work should also provide a testbed for general frameworks of understanding DSS. For example, it is possible that dynamic renormalization group methods can provide a more universal foundation [34].

Note: Hereafter energy spectra, integral lengths, etc. will refer to the scalar field only and not to the velocity field. Hence we drop the subscript θ without any possibility of confusion.

3 Self-Similar Decay and Its Breakdown

(3.1) Derivation & Solution of the Scaling Equation

In this section we shall completely identify and classify all of the self-similar decay solutions in the Kraichnan model. Following the discussion in section 2.2, we look for solutions to (1.9) in the form

$$\Theta(r,t) = \vartheta^2(t)\Phi\left(\frac{r}{L(t)}\right). \tag{3.1}$$

As there, we choose

$$\vartheta^2(t) = \Theta(0,t) \tag{3.2}$$

and thus

$$\Phi(0) = 1 \tag{3.3}$$

by definition. Subsituting the *Ansatz* (3.1) into (1.9) gives

$$\left(\frac{2\dot{\vartheta}(t)}{D_1\vartheta(t)L^{\zeta-2}(t)}\right)\Phi(\rho) + \left(\frac{-\dot{L}(t)}{D_1L^{\zeta-1}(t)}\right)\rho\frac{\partial\Phi}{\partial\rho}(\rho) = \frac{1}{\rho^{d-1}}\frac{\partial}{\partial\rho}\left(\rho^{d+\zeta-1}\frac{\partial\Phi}{\partial\rho}\right). \tag{3.4}$$

The only way that this can hold with $\Phi(\rho)$ a function independent of time t is if

$$\frac{2\dot{\vartheta}(t)}{D_1\vartheta(t)L^{\zeta-2}(t)} = -\alpha \tag{3.5}$$

and

$$\frac{\dot{L}(t)}{D_1L^{\zeta-1}(t)} = \beta \tag{3.6}$$

for some constants α, β. Indeed, the Wronskian of the two functions $\Phi(\rho)$ and $\rho\Phi'(\rho)$ in terms of the logarithmic variable $\xi = \ln\rho$ is $\Phi^2(\xi)\frac{d^2}{d\xi^2}\ln\Phi(\xi)$, so that they are linearly independent on any interval where $\Phi(\rho) \neq 0$ and is not a pure power-law. In order that $\frac{dE}{dt}(t) < 0$, we must have $\alpha > 0$ (which explains our choice of sign in (3.5)). We see that the second equation (3.6) is identical with (1.7) postulated for the scalar integral length, up to the factor of β. The solution is $L(t) = [L_0^\gamma + \beta\gamma D_1(t-t_0)]^{1/\gamma}$. We may always choose β to be unity by a suitable choice of lengthscale in the *Ansatz*. In other words, there are many lengths growing according to the equation (3.6) with some β (e.g. the integral length), but we choose the one obeying the equation with $\beta = 1$. This amounts to a rescaling $\beta \to \beta' = 1, L \to L' = \beta^{-1/\gamma} L$. This transformation takes also $\alpha \to \alpha' = \alpha/\beta$. We shall always assume hereafter that this rescaling has been done and simply write $\beta' = 1, L' = L, \alpha' = \alpha$. Then we arrive at the equation

$$-\alpha\Phi - \rho\frac{\partial\Phi}{\partial\rho} = \frac{1}{\rho^{d-1}}\frac{\partial}{\partial\rho}\left(\rho^{d+\zeta-1}\frac{\partial\Phi}{\partial\rho}\right). \tag{3.7}$$

The first coefficient equation (3.5) may then be written using the second (3.6) as

$$\frac{2\dot{\vartheta}(t)}{\vartheta(t)} = -\alpha\frac{\dot{L}(t)}{L(t)}, \tag{3.8}$$

whose solution is

$$\vartheta^2(t) = A \cdot L^{-\alpha}(t) \tag{3.9}$$

for some constant A with dimensions $[\Theta \cdot L^{\alpha}]$. It is clear that any value of the constant A is possible, because the equation (1.9) is linear and homogeneous. However, for any solution we are always free to chose units of the scalar field (e.g. temperature scale) so that $A \equiv 1$. Then (3.9) is identical with (1.8) obtained from the PLE hypothesis, if we take $\alpha = p$. In fact, we can verify this directly from equation (3.7), if we substitute the asymptotic formula $\Phi(\rho) \sim C\rho^{-p}$ for $\rho \gg 1$. We obtain

$$(\alpha - p)\rho^{-p} \sim p(d + \gamma - p)\rho^{-(p+\gamma)} \tag{3.10}$$

for large ρ. Since $\gamma > 0$, the righthand side is asymptotically negligible compared with the left, and we see that $\alpha = p$. Of course, we do not mean to imply that PLE is necessarily true for self-similar solutions, either spatially or spectrally. What is shown above is only that, *if* spatial PLE holds, then we have the identification $\alpha = p$. We may now write out the equation for the scaling function in final form, as

$$\rho^{\zeta}\Phi''(\rho) + [(d + \zeta - 1)\rho^{\zeta-1} + \rho]\Phi'(\rho) + \alpha\Phi(\rho) = 0. \tag{3.11}$$

This equation can be solved in terms of confluent hypergeometric functions by means of the substitution $x = -\frac{\rho^{\gamma}}{\gamma}$. In fact, making this substitution into (3.11) gives:

$$x\frac{\partial^2\Phi}{\partial x^2} + \left[\frac{d}{\gamma} - x\right]\frac{\partial\Phi}{\partial x} - \frac{\alpha}{\gamma}\Phi = 0. \tag{3.12}$$

This is a second-order differential equation with coefficients linear in the variable x. Any such equation can be solved in terms of confluent hypergeometric functions. In fact, (3.12) is *Kummer's equation* with $a = \alpha/\gamma$, $c = d/\gamma$ ([25], Ch.6). This equation has two solutions which are traditionally denoted $\Phi(a, c; x)$ and $\Psi(a, c; x)$. The former is an entire function of x defined by the power series

$$\Phi(a, c; x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots \tag{3.13}$$

The second can be defined as

$$\Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\Phi(a-c+1, 2-c; x) \tag{3.14}$$

when c is not an integer, and otherwise by a limit of this expression ([25],section 6.5). We see that for $c > 1$ and c non-integer, the second solution has a power-law divergence at $x = 0$, while for integer c the divergence is logarithmic ([25],section 6.7.1). The Wronskian of the two solutions is $-\frac{\Gamma(c)}{\Gamma(a)}$, ([25], section 6.7) which are thus independent for $a, c > 0$. These conditions are satisfied in our problem since $c = \frac{d}{\gamma} > 1$ and $a = \frac{\alpha}{\gamma} > 0$. Thus, we may summarize the main conclusion of this section as follows: *The unique solution of the scaling equation (3.11) satisfying the boundary condition* $\Phi(0) = 1$ *is*

$$\Phi(\rho) = \Phi\left(\frac{\alpha}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right). \tag{3.15}$$

Then, $\Theta(r, t) := \vartheta^2(t)\Phi(r/L(t))$ *with* $\vartheta(t), L(t)$ *solutions of (3.5) and (3.6) for* $\beta = 1$ [1] *is an exact solution of (1.9).* Hence, we have identified all possible scaling solutions in the Kraichnan model. As an interesting historical note, we observe that von Kármán and Howarth already reduced their scaling equation, with neglect of triple correlations, to Whittaker's form of the confluent hypergeometric equation [1]. Kraichnan found the *energy spectrum* of the white-noise model to obey Kummer's equation in the Batchelor limit $\zeta \to 2, \gamma \to 0$ [16]. Also, in the steady-state with random forcing and molecular diffusion, the solution of the 2-point correlation was given as a Kummer function by Chertkov et al. in [13].

[1] If we had used the original length-scale $L(t)$ satisfying (3.6) in defining the scaling solution, rather that $L'(t)$ with β set equal to 1, then the result would have been instead $\Phi\left(\frac{\alpha}{\beta\gamma}, \frac{d}{\gamma}; -\frac{\beta}{\gamma}\rho^\gamma\right)$.

(3.2) Asymptotic Behaviors & Permanence of Large Eddies

The behavior of $\Phi(\rho)$ for small and large ρ can be obtained from the known asymptotics of the Kummer function. For small ρ it follows from (3.13) that

$$\Phi(\rho) = 1 - \frac{\alpha}{d\gamma}\rho^{\gamma} + O(\rho^{2\gamma}). \tag{3.16}$$

This is the scaling that we expect for a dissipative solution. Because the Kummer function is entire, the only singularity of the scaling function is at $\rho = 0$, due to the fractional power γ. Hence, we may apply Theorem 4 of Ch.IX.6 of [31] to obtain, for $0 < \gamma < 2$, that

$$\widehat{\Phi}(\kappa) \sim \frac{\alpha}{d\gamma} \cdot \frac{2^{\gamma}}{\pi^{\frac{1}{2}(d+2)}}\Gamma\left(\frac{\gamma+2}{2}\right)\Gamma\left(\frac{d+\gamma}{2}\right)\sin\left(\frac{\pi\gamma}{2}\right)k^{-(d+\gamma)} \tag{3.17}$$

for $\kappa \to \infty$. Hence, from (3.16) the spectral scaling function goes as

$$F(\kappa) \sim \alpha \cdot 2^{\gamma}\frac{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{d+\gamma}{2}\right)}{4\pi\Gamma\left(\frac{d+2}{2}\right)}\sin\left(\frac{\pi\gamma}{2}\right)\kappa^{-(1+\gamma)} \tag{3.18}$$

for $\kappa \gg 1$. Of course, this is the spectral law one would expect in the inertial-convective range of the white-noise model. In fact, going back to dimensionful quantities, we get

$$E(k,t) \sim \frac{1}{2}\alpha C(\gamma,d)\vartheta^2(t)L^{-\gamma}(t)\kappa^{-(1+\gamma)} \tag{3.19}$$

with $C(\gamma,d) := 2^{\gamma}\frac{\Gamma(\frac{\gamma}{2})\Gamma(\frac{d+\gamma}{2})}{2\pi\Gamma(\frac{d+2}{2})}\sin\left(\frac{\pi\gamma}{2}\right)$ for $kL(t) \gg 1$. However, referring to the equation (3.5) and using the definition of the scalar dissipation $\chi(t) = -\frac{d}{dt}\left(\frac{1}{2}\vartheta^2(t)\right)$, one finds that

$$E(k,t) \sim C(\gamma,d)\frac{\chi(t)}{D_1}k^{-(1+\gamma)} \tag{3.20}$$

for $kL(t) \gg 1$. This is identical to the result that holds in the forced steady state, with the same (universal) value of the constant $C(\gamma,d)$. In fact, the corresponding spatial result is

$$\Theta(r,t) \sim \vartheta^2(t) - \frac{\chi(t)}{2\gamma d \cdot D_1}r^{\gamma} \tag{3.21}$$

for $r \ll L(t)$, which coincides exactly with the result in equation (1.19b) of [13].

The large ρ behavior of $\Phi(\rho)$ is obtained from the asymptotics of the Kummer function for large negative arguments ([25], section 6.13.1):

$$\Phi(a,c;x) = \frac{\Gamma(c)}{\Gamma(c-a)}(-x)^{-a}\left[1+O(|x|^{-1})\right] \tag{3.22}$$

as $\mathrm{Re}\, x \to -\infty$. Thus, we see that

$$\Phi(\rho) \sim \gamma^{\alpha/\gamma}\frac{\Gamma\left(\frac{d}{\gamma}\right)}{\Gamma\left(-\frac{\nu}{\gamma}\right)}\rho^{-\alpha} \tag{3.23}$$

for $\rho \gg 1$. We have set

$$\nu := \alpha - d \tag{3.24}$$

so that $\alpha = d+\nu$. We see that $\Phi(\rho)$ has the power-law form presumed in the PLE hypothesis, at least when $\nu \neq \gamma\ell,\ \ell = 0,1,2,....$ Otherwise, the coefficient of the above asymptotic expression vanishes and it no longer gives the leading behavior at large ρ. We shall examine that case in detail below. However, for all but such exceptional α values the spatial PLE in fact holds. This can be seen by returning to the dimensionful variables and using (3.9), which gives

$$\Theta(r,t) \sim \gamma^{\alpha/\gamma}\frac{\Gamma\left(\frac{d}{\gamma}\right)}{\Gamma\left(-\frac{\nu}{\gamma}\right)}r^{-\alpha} \tag{3.25}$$

for $r \gg L(t)$. This verifies the spatial PLE, for the non-exceptional values $\alpha \neq d+\gamma\ell,\ \ell = 0,1,2,...$, since the coefficient of the asymptotic power is explicitly independent of the time.

Let us now consider the exceptional cases, $\alpha = d+\gamma\ell,\ \ell = 0,1,2,....$ It turns out that these are given in terms of elementary functions. In fact, for $\ell = 0,1,2,...$

$$\begin{aligned}\Phi(c+\ell,c;-x) &= \frac{1}{(c)_\ell x^{c-1}}\frac{d^\ell}{dx^\ell}\left[x^{c+\ell-1}e^{-x}\right] \\ &= \frac{\ell!}{(c)_\ell}L_\ell^{c-1}(x)e^{-x}\end{aligned} \tag{3.26}$$

where $(c)_\ell = c(c+1)\cdots(c+\ell-1)$ and $L_\ell^{c-1}(x)$ is the *generalized Laguerre polynomial* of degree ℓ. The first line follows from [25], 6.9.2(36) and the Kummer transformation, [25], 6.3(7). The second line follows either from the Rodriguez formula for the Laguerre polynomial or from [25], 6.4(11) with $a = c$. Using (3.26) we see that

$$\Phi_\ell(\rho) := \Phi\left(\frac{d}{\gamma} + \ell, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right) = \frac{\ell!}{\left(\frac{d}{\gamma}\right)_\ell} L_\ell^{\frac{d-\gamma}{\gamma}}\left(\frac{\rho^\gamma}{\gamma}\right) e^{-\frac{\rho^\gamma}{\gamma}}. \tag{3.27}$$

Explicitly, the first few functions are, for $\ell = \nu = 0$,

$$\Phi_0(\rho) = e^{-\frac{\rho^\gamma}{\gamma}}, \tag{3.28}$$

for $\ell = 1, \nu = \gamma$

$$\Phi_1(\rho) = \left[1 - \frac{\rho^\gamma}{d}\right] e^{-\frac{\rho^\gamma}{\gamma}}, \tag{3.29}$$

and for $\ell = 2, \nu = 2\gamma$,

$$\Phi_2(\rho) = \left[1 - \frac{2\rho^\gamma}{d} + \frac{\rho^{2\gamma}}{d(d+\gamma)}\right] e^{-\frac{\rho^\gamma}{\gamma}}. \tag{3.30}$$

We see that all of the scaling functions in the exceptional cases $\alpha = d + \gamma\ell,\ \ell = 0, 1, 2, ...$ do not behave as power-laws at large ρ, but, instead, have a stretched-exponential decay. Thus, spatial PLE cannot hold.

The case $\ell = \nu = 0$ when $\alpha = d$ corresponds to a finite, nonvanishing Corrsin invariant K. In fact, the dimensionless version

$$\widetilde{K} = \frac{\omega_{d-1}}{(2\pi)^d}\int_0^\infty e^{-\frac{\rho^\gamma}{\gamma}} \rho^{d-1} d\rho \tag{3.31}$$

can be calculated explicitly by substituting $t = \rho^\gamma/\gamma$ to yield a Gamma integral:

$$\widetilde{K} = \frac{\gamma^{\frac{d}{\gamma}-1}}{2^{d-1}\pi^{d/2}} \frac{\Gamma\left(\frac{d}{\gamma}\right)}{\Gamma\left(\frac{d}{2}\right)}. \tag{3.32}$$

Thus, the spectral scaling function has the expected equipartition power-law at low-wavenumbers:

$$F(\kappa) \sim \frac{1}{2}\omega_{d-1}\widetilde{K}\kappa^{d-1} = \frac{\gamma^{\frac{d}{\gamma}-1}}{2^{d-1}} \frac{\Gamma\left(\frac{d}{\gamma}\right)}{\left[\Gamma\left(\frac{d}{2}\right)\right]^2} \cdot \kappa^{d-1}. \tag{3.33}$$

Although spatial PLE does not hold, the spectral PLE is valid. Indeed, the dimensionful Corrsin integral is

$$K(t) = \widetilde{K}\vartheta^2(t)L^d(t) = \widetilde{K} \tag{3.34}$$

using (3.9) for $\alpha = d$ (with our convention that $A \equiv 1$). The integral is explicitly independent of time, in agreement with the general theorem on invariance. As an aside, we remark that the stretched exponential solution was noted by Majda and Kramer [18]. In fact, it was found for $\gamma = 2/3$ already by Batchelor in 1952 in a slightly different context [35]. He considered (1.9) as an equation for the particle pair-separation distribution, in which case it is just the scale-dependent diffusion equation proposed by Richardson [36]. However, along with the new physical interpretation, there are also different mathematical requirements on the solutions than those we impose. To represent a probability distribution function only positive solutions with unit integral are admissable.

Let us investigate the validity of the spectral PLE more generally. It is clear from (3.23) that the spectral scaling function $F(\kappa)$ is not C^∞ at $\kappa = 0$. In fact, the Fourier transform of the radially symmetric function $\Phi(\rho)$ is given by

$$\widehat{\Phi}(\kappa) = \frac{1}{(2\pi)^{d/2}\kappa^{(d-2)/2}} \int_0^\infty \rho^{d/2}\Theta(\rho)J_{(d-2)/2}(\kappa\rho)\, d\rho \tag{3.35}$$

where $J_\nu(z)$ is the Bessel function [37]. Then, by using the Frobenius series expansion of the Bessel function ([25],7.2.1(2)), one obtains the formal Taylor series of $\widehat{\Phi}(\kappa)$ in powers of κ^2:

$$\widehat{\Phi}(\kappa) \sim \sum_{j=0}^{\infty} B_j \kappa^{2j}, \tag{3.36}$$

with

$$B_j := \frac{(-1)^j}{\pi^{d/2} 2^{2j+d-1} j! \Gamma\left(\frac{d+2j}{2}\right)} \int_0^\infty \rho^{2j}\Phi(\rho)\rho^{d-1}\, d\rho. \tag{3.37}$$

Because of the power-law decay with exponent $-\alpha$ in (3.23), one expects that B_j diverges for $2j \geq \nu \neq \gamma\ell$, $\ell = 0, 1, 2, ...$ This will be verified below. Hence, derivatives $\widehat{\Phi}^{(r)}(0)$ of order $r \geq \nu/2$ do not exist. However, it seems reasonable to assume that $\kappa = 0$ is the only singularity of $\widehat{\Phi}(\kappa)$, which is C^∞ elsewhere. In that case, Theorem 4 of Ch.IX.6 of [31] can be employed to infer that the following asymptotic expansion holds for $\nu \neq 2j, \gamma\ell$, $j, \ell = 0, 1, 2, ...$ and for $\kappa \ll 1$

$$\widehat{\Phi}(\kappa) \sim \sum_{j=0}^{[\![\frac{\nu}{2}]\!]} B_j \kappa^{2j} + B(\nu)\kappa^\nu \tag{3.38}$$

with

$$B(\nu) = \frac{\gamma^{\alpha/\gamma}}{\pi^{d/2}2^{\alpha}} \frac{\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{d}{\gamma}\right)}{\Gamma\left(-\frac{\nu}{\gamma}\right)\Gamma\left(\frac{\alpha}{2}\right)}. \tag{3.39}$$

In the case $\nu = 2j, \nu \neq \gamma\ell,\ j, \ell = 0, 1, 2, ...,$ the second term after the sum is changed to $B_*(\nu)\kappa^\nu \log \kappa$, modified by a logarithm. The summation itself represents the Taylor polynomial of degree $\left[\!\left[\frac{\nu}{2}\right]\!\right]$ of the contribution to $\widehat{\Phi}(\kappa)$ analytic at $\kappa = 0$. Its coefficients B_j may be obtained from the formula (3.37) using with $\beta = 2j$ the integral

$$\int_0^\infty \rho^\beta \Phi\left(\frac{\alpha}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right) \rho^{d-1}\, d\rho = \begin{cases} \frac{\Gamma\left(\frac{d+\beta}{\gamma}\right)\Gamma\left(\frac{d}{\gamma}\right)\Gamma\left(\frac{\nu-\beta}{\gamma}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)\Gamma\left(\frac{-\beta}{\gamma}\right)}\gamma^{\frac{d+\beta}{\gamma}-1} & \text{for } \beta < \nu \\ \pm\infty & \text{for } \beta \geq \nu, \end{cases} \tag{3.40}$$

valid for the nonexceptional values $\nu \neq \gamma\ell,\ \ell = 0, 1, 2, ...$ The sign $\pm$ in the second case of (3.40) is given by $\operatorname{sgn}\Gamma\left(-\frac{\nu}{\gamma}\right)$, the same as the power-law tail in (3.23). For the exceptional values $\nu = \gamma\ell,\ \ell = 0, 1, 2, ...$ the first line in (3.40) is valid in both cases $\beta < \nu$ and $\beta \geq \nu$, with the convention that $\Gamma\left(-\frac{\beta}{\gamma} + \ell\right)/\Gamma\left(-\frac{\beta}{\gamma}\right) = \left(-\frac{\beta}{\gamma} + \ell - 1\right)\ldots\left(-\frac{\beta}{\gamma} + 1\right)\left(-\frac{\beta}{\gamma}\right) = \left(-\frac{\beta}{\gamma}\right)_\ell$. The proof of this integral formula proceeds by making the change of variables $t = \rho^\gamma/\gamma$ and using the Laplace transform of the Kummer function

$$\begin{aligned} \int_0^\infty e^{-st} t^{b-1} \Phi(a, c; -t)\, dt &= \frac{\Gamma(b)\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(1+s)^{c-a-b} F\left(c-a, 1-a; b-a+1; -s\right) \\ &\quad + \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)} s^{a-b} F\left(a, 1+a-c; a-b+1; -s\right) \end{aligned} \tag{3.41}$$

which is given in terms of the hypergeometric function F and valid for $\mathrm{Re}s > 0$. This formula follows from [25],6.10(5) and 2.10(4). Notice that for the exceptional cases $a = c + \ell, \ell = 0, 1, 2, ...$ the second term in (3.41) vanishes. The limit $s \downarrow 0$ may be obtained recalling that $F(0) = 1$, with the result given in (3.40). We see that in general for $\nu > 0$, there are nonzero B_j terms from the analytic contribution which dominate the Fourier transform at small κ. However, it is interesting to note that the lowest coefficient B_0, which coincides with the dimensionless Corrsin integral $\widetilde{K}$, always vanishes for $\nu > 0$. Thus, the Fourier transform $\widehat{\Phi}(\kappa)$ always vanishes at $\kappa = 0$ for $\nu > 0$ and the second singular term proportional to κ^ν is formally the leading-order contribution for all $\nu < 2$.

We may now return to the question of spectral PLE. The asymptotic low-wavenumber expansion for the spectral scaling function, corresponding to (3.38), is

$$F(\kappa) \sim \sum_{j=0}^{[\![\frac{\nu}{2}]\!]} A_j \kappa^{d+2j-1} + A(\nu)\kappa^{\alpha-1} \tag{3.42}$$

valid for $\kappa \ll 1$, with $A_j := \frac{1}{2}\omega_{d-1}B_j$ and

$$A(\nu) := \frac{\gamma^{\alpha/\gamma}}{2^\alpha}\frac{\Gamma\left(-\frac{\nu}{2}\right)}{\Gamma\left(-\frac{\nu}{\gamma}\right)} \cdot \frac{\Gamma\left(\frac{d}{\gamma}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}. \tag{3.43}$$

If we return to dimensionful variables and use again (3.9), we obtain

$$E(k,t) \sim \sum_{j=0}^{[\![\frac{\nu}{2}]\!]} A_j L^{-(\nu-2j)}(t)\kappa^{d+2j-1} + A(\nu)k^{\alpha-1} \tag{3.44}$$

for $kL(t) \ll 1$. Spectral PLE appears to hold, in the sense that the coefficient $A(\nu)$ of the singular term is explicitly time-independent, for all but possibly the exceptional values $\nu = 2j, \gamma\ell,\ j,\ell = 0,1,2,....$ Of course, the singular term is the leading one for $\nu < 2$ and then spectral PLE appears to hold in the standard sense.

However, this is wrong for an important reason. While $A(\nu) > 0$ for $-d < \nu < \gamma$, it becomes negative at rather higher values of ν: $A(\nu,d) < 0$ for $\gamma < \nu < 2\min\{1,\gamma\}$! Thus, realizability of the scaling solution is violated in this case. This is a crucial issue which we have neglected up until now. In fact, only solutions with positive spectra over the whole wavenumber range are physically admissable. Thus, we have reached one of the important conclusions of this work: *No self-similar decay is possible in the Kraichnan model with* $\gamma < \nu < 2\min\{1,\gamma\}$, *since in that range the scaling solution has a negative spectrum at low wavenumbers.* We have proved this subject to a single assumption, that the origin is the only singular point for the Fourier transform of the scaling function. The result is verified by explicit computations for the special case $\gamma = 1$ in an Appendix. In fact, we shall prove below that self-similar decay occurs for no exponent $\nu > \gamma$. The physics of this phenomenon will be discussed in section 3.4 below.

(3.3) Realizability of the Scaling Solutions

We must now examine more closely the issue of realizability. As already shown by Kraichnan [7], the 2-point correlation of the statistical problem (1.3) must satisfy the closed partial differential equation (1.9). However, there can be solutions of the PDE which do *not* correspond to any solution of the statistical problem. The necessary and sufficient condition for a solution of (1.9) to be realized as a solution of the statistical problem (1.3) is that it be positive-definite. Necessity is obvious. To see sufficiency, take for any positive-definite initial data $\Theta(r,0)$ the Gaussian measure μ_0 over scalar fields $\theta_0(\mathbf{r})$ with zero mean and with the given positive-definite function as its 2-point covariance. Then, the solution $\Theta(r,t)$ of the PDE (1.9) with the specified initial datum $\Theta(r,0)$ will be the same, by Kraichnan's result, as the 2-point correlation of the statistical problem posed by (1.3) with random initial data $\theta_0(\mathbf{r})$ distributed according to μ_0. It is a corollary of this remark that the PDE (1.9) is *positive-definiteness preserving*, that is, positive-definite solutions result from positive-definite initial data. The upshot is that not every scaling solution of (1.9) that we found in Section 3.1 can necessarily be realized as a solution of the Kraichnan model (1.3). This will only be true if it is positive-definite, and this question must now be addressed.

First, we give a general proof of realizability when $0 < \alpha \leq d$ or $-d < \nu \leq 0$. We must show that $\Phi(\rho)$ is positive-definite as a function on d-dimensional Euclidean space. This is proved in two steps. We observe first that the $\nu = 0$ function $\Phi\left(\frac{d}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right) = e^{-\frac{\rho^\gamma}{\gamma}}$ is positive-definite for $0 < \gamma \leq 2$. In fact, for $0 < \gamma \leq 2$ the functions $e^{-\frac{\rho^\gamma}{\gamma}}$ are the characteristic functions (Fourier transforms) of positive probability densities, the spherically symmetric stable distributions of parameter γ. General multivariate stable laws were first investigated by Lévy [38] and Feldheim [39]; for an introduction to their basic theory, see, for example, the monograph of Zolotarev [40], Section I.6. The positive-definiteness of the characteristic functions $e^{-\frac{\rho^\gamma}{\gamma}}$ for the spherically symmetric stable distributions can be obtained by an easy modification of the proof of Bochner for the 1-dimensional case [41]. However, with this result, we may then use a standard integral representation for the Kummer functions

$$\Phi(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{ux} u^{a-1}(1-u)^{c-a-1} du \tag{3.45}$$

valid for $\mathrm{Re}\, c > \mathrm{Re}\, a > 0$. See [25], 6.5(1). From this we see that

$$\Phi\left(\frac{\alpha}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right) = \frac{\Gamma\left(\frac{d}{\gamma}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)\Gamma\left(-\frac{\nu}{\gamma}\right)} \int_0^1 e^{-u\frac{\rho^\gamma}{\gamma}} u^{\frac{\alpha}{\gamma}-1}(1-u)^{-\frac{\nu}{\gamma}-1} du, \tag{3.46}$$

when $-d < \nu < 0$. Hence, $\Phi\left(\frac{\alpha}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right)$ is a convex combination of positive-definite functions, in fact, averaged over a Beta distribution. Thus, this scaling function is always positive-definite for $-d < \nu \leq 0$.

We now show that the scaling functions are positive-definite—and thus realizability holds—for the larger range $-d < \nu \leq \gamma$ or $0 < \alpha \leq d + \gamma$. As we have seen above, the low-wavenumber spectrum remains positive in the range $-d < \nu < \gamma$. Furthermore, we prove now that, if the scaling function for $\nu = \gamma$ is positive definite, then so are all the functions for $-d < \nu < \gamma$. This follows from the identity [2]

$$\Phi(c+\ell, c; x) = \frac{\Gamma(c+1)}{\Gamma(c+\ell)\Gamma(1-\ell)} \int_0^1 \Phi(c+1, c; ux) u^{c+\ell-1}(1-u)^{-\ell} du \tag{3.47}$$

for $1 > \mathrm{Re}\, \ell > -\mathrm{Re}\, c$. This is most easily proved by expanding both sides in a power-series in x and comparing the coefficients. One may also give a proof based upon [25], 6.4(12) and 6.5(1). Thus,

$$\Phi\left(\frac{d+\nu}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right) = \frac{\Gamma\left(\frac{d+\gamma}{\gamma}\right)}{\Gamma\left(\frac{d+\nu}{\gamma}\right)\Gamma\left(\frac{\gamma-\nu}{\gamma}\right)} \int_0^1 \Phi\left(\frac{d+\gamma}{\gamma}, \frac{d}{\gamma}; -u\frac{\rho^\gamma}{\gamma}\right) u^{\frac{d+\nu}{\gamma}-1}(1-u)^{-\frac{\nu}{\gamma}} du, \tag{3.48}$$

when $-d < \nu < \gamma$. Hence, if the scaling function for $\nu = \gamma$, $\Phi\left(\frac{d+\gamma}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right)$, is positive-definite, then, as weighted integrals of it with respect to a Beta distribution, so are the scaling functions for $-d < \nu < \gamma$.

[2] Although it is not hard to prove, we did not find this formula in standard treatises on Kummer functions. We would be grateful for any reference.

It therefore becomes important to answer whether $\Phi_1(\rho) = \Phi\left(\frac{d}{\gamma}+1, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right)$ is positive-definite or not. We shall show that this question is related to properties of the equipartition solution $\Phi_0(\rho) = \Phi\left(\frac{d}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right)$. In fact, it is not hard to show by direct calculation that

$$d \cdot \Phi_1(\rho) = \left[d + \rho\frac{d}{d\rho}\right]\Phi_0(\rho). \tag{3.49}$$

This is actually a particular example of a standard relation between Kummer functions, [25], 6.4(11), transformed by the substitution $x = -\rho^\gamma/\gamma$. What is useful here is that the righthand side of (3.49) above is also just minus the lefthand side of (3.7), in the equipartition case $\alpha = d$. However, the expression in (3.7) arose by differentiating the scaling *Ansatz* with respect to time and dividing by $D_1 L^{-\gamma}(t)$. Thus, we infer that

$$\Phi_1(\rho) = \frac{-\frac{d}{dt}\Theta_0(r,t)}{d \cdot D_1 L^{-\gamma}(t)}, \tag{3.50}$$

where $\Theta_0(r,t) = \vartheta^2(t)\Phi_0(r/L(t))$. This relation may be Fourier-transformed easily, giving

$$\widehat{\Phi}_1(\kappa) = \frac{-\frac{d}{dt}\widehat{\Theta}_0(k,t)}{d \cdot D_1 L^{-\gamma}(t)}, \tag{3.51}$$

If we employ the relation $\widehat{\Theta}_0(k,t) = \widehat{\Phi}_0(kL(t))$, which follows using (3.9) for $\alpha = d$, and (3.6) for the time-derivative of $L(t)$, then (3.51) yields the final result

$$\widehat{\Phi}_1(\kappa) = -\frac{1}{d}\kappa\frac{d\widehat{\Phi}_0}{d\kappa}(\kappa). \tag{3.52}$$

From this we can see that realizability holds for the threshold case $\ell = 1, \nu = \gamma$ when the Fourier transform $\widehat{\Phi}_0(\kappa)$ is a monotone nonincreasing function of spectral radial coordinate κ.

To complete the proof, we must verify this property. In fact, the monotone nonincreasing of the density with respect to the radial cooordinate is equivalent—for spherically symmetric functions—to the property of *unimodality* of a multivariate density, as it has been defined by Olshen and Savage [42]. For a general introduction to the subject of unimodality and to its proof for stable probability laws in particular, see [40], Section 2.7, and [43]. The subject has a rather colorful history, involving a series of published false proofs and claims by eminent mathematicians (including Kolmogorov), which is summarized in those works. The first proof of unimodality of the symmetric, one-dimensional stable distributions was given by Wintner in 1936 [44]. It was then widely conjectured that *all* one-dimensional stable distributions are unimodal, but it took over forty years until a correct proof was found in 1978 by Yamazato [45]. The proof of unimodality of spherically symmetric stable distribution functions in multi-dimensions was given about the same time, by S. J. Wolfe [46]. This is exactly the property we need to guarantee realizability for the threshold case $\nu = \gamma$, and thence, by equation (3.48), for all ν in the range $-d < \nu \leq \gamma$.

On the other hand, it is reasonable to conjecture that no scaling solutions with $\nu > \gamma$ are realizable. We have already established that realizability fails when $\gamma < \nu < 2\min\{1, \gamma\}$, by showing that the low-wavenumber spectrum becomes negative. The same argument does not work for all $\nu > \gamma$, because (i) the coefficient of the singular contribution to the spectrum in (3.42) oscillates in sign as ν is increased and (ii) the singular term is not the leading-order term at low-wavenumbers for $\nu > 2$. However, we shall now prove that the scaling solutions for $\nu > \gamma$ are indeed non-realizable. The proof is based upon the fact that, for $\nu > \gamma$, two integrals vanish, namely:

$$\int_{\mathbb{R}^d} \Phi(\rho)\, d^d\boldsymbol{\rho} = \int_{\mathbb{R}^d} \rho^\gamma \Phi(\rho)\, d^d\boldsymbol{\rho} = 0. \tag{3.53}$$

This is a direct consequence of the general formula (3.40) for the case $\beta = \gamma$. We shall now show (following a suggestion of D. Thomson [47, 48]) that these two conditions are equivalent to

$$\int_{\mathbb{R}^d} \kappa^{-(d+\gamma)} \widehat{\Phi}(\kappa)\, d^d\boldsymbol{\kappa} = 0, \tag{3.54}$$

when realizability (or nonnegativity of $\widehat{\Phi}$ pointwise) is assumed. To prove (3.54) we make use of the following generating functional for the moment-integrals in (3.53):

$$G(t) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t\frac{\rho^\gamma}{\gamma}} \Phi(\rho)\, d^d\boldsymbol{\rho}. \tag{3.55}$$

Then it is easy to see that (3.53) is equivalent to $G(0) = \dot{G}(0) = 0$ or, as well, $\lim_{t\to 0} \frac{G(t)}{t} = 0$.

Next we make use of Parseval's theorem to rewrite (3.55) as

$$G(t) := \int_{\mathbb{R}^d} \frac{1}{t^{d/\gamma}} \widehat{\Phi}_0\left(\frac{\kappa}{t^{1/\gamma}}\right) \widehat{\Phi}(\kappa)\, d^d\boldsymbol{\kappa}, \tag{3.56}$$

where $\widehat{\Phi}_0(\kappa)$ is the Fourier transform of the scaling function for $\nu = 0$, i.e. the density of the multidimensional Lévy stable distribution with parameter γ. This step is justified because, clearly, $\Phi_0 \in L^2$, and because, for general $\nu > 0$, boundedness and the large-ρ decay in (3.23) imply that as well $\Phi \in L^2$. Next we observe from (3.17) that $\widehat{\Phi}_0(\kappa) \sim c \cdot \kappa^{-(d+\gamma)}$ as $\kappa \to \infty$, for some positive constant $c > 0$. Thus,

$$\lim_{t\to 0} \frac{1}{t^{(d+\gamma)/\gamma}} \widehat{\Phi}_0\left(\frac{\kappa}{t^{1/\gamma}}\right) = c \cdot \kappa^{-(d+\gamma)}. \tag{3.57}$$

We see finally, by Fatou's lemma, that

$$\begin{aligned} 0 = \liminf_{t\to 0} \frac{G(t)}{t} &= \liminf_{t\to 0} \int_{\mathbb{R}^d} \frac{1}{t^{(d+\gamma)/\gamma}} \widehat{\Phi}_0\left(\frac{\kappa}{t^{1/\gamma}}\right) \widehat{\Phi}(\kappa)\, d^d\boldsymbol{\kappa} \\ &\geq \int_{\mathbb{R}^d} \liminf_{t\to 0} \frac{1}{t^{(d+\gamma)/\gamma}} \widehat{\Phi}_0\left(\frac{\kappa}{t^{1/\gamma}}\right) \widehat{\Phi}(\kappa)\, d^d\boldsymbol{\kappa} \\ &= c \int_{\mathbb{R}^d} \kappa^{-(d+\gamma)} \widehat{\Phi}(\kappa)\, d^d\boldsymbol{\kappa} \geq 0, \end{aligned} \tag{3.58}$$

where the last inequality holds by the assumed nonnegativity of $\widehat{\Phi}(\kappa)$. Thus, the identity (3.54) is established. However, it follows immediately then that $\widehat{\Phi}(\kappa) = 0$ for *a.e.* κ, which is a clear contradiction. Thus, the assumption that $\widehat{\Phi}(\kappa) \geq 0$ for all κ cannot be correct.

We may summarize the conclusions of this section as follows: *The scaling functions* $\Phi(\rho) = \Phi\left(\frac{\alpha}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right)$ *are positive-definite, radially-symmetric functions on* $\mathbb{R}^d$ *for all* α *in the interval* $(0, d+\gamma]$ *and for no* α *in the range* $(d+\gamma, \infty)$*. In particular, the scaling solutions* $\Theta(r,t) = \vartheta^2(t)\Phi(r/L(t))$ *for* $\alpha \leq d+\gamma$ *are all realized as solutions of the statistical problem posed by the equation (1.3) with the random initial data* μ_0. It is interesting to note that, for cases $\nu > \gamma$ when the low-wavenumber spectrum is positive, there must be at least one negative spectral interval away from the origin. This is explicitly verified in the Appendix for $\gamma = 1$, in which case the spectral scaling functions $F(\kappa)$ can be calculated in a closed form (related to the multidimensional Cauchy distribution) for all of the exceptional values $\nu = \ell,\ \ell = 0, 1, 2, ...$

(3.4) A Physical Explanation of the Results

We have now shown there are realizable self-similar solutions for $-d < \nu \leq \gamma$ but not for $\nu > \gamma$. Although $\nu = \gamma$ corresponds to a realizable DSS solution, something a bit strange must occur in that case. For example, PLE cannot hold either spatially or spectrally. If there is a power-law in $E_1(k,t)$ at low-wavenumber, it must be distinct from the power $k^{d+\gamma-1}$ which is naively expected. Indeed, if that naive power-law occurred, then it would imply a corresponding spatial decay $r^{-(d+\gamma)}$, since γ lies in the range $0 < \gamma < 2$ where this deduction is correct. However, we know that $\Theta_1(r,t)$ decays faster than *any* inverse power of r. Hence, if any power-law at all occurs in the spectrum, it must be different from the naive one and, in fact, it must correspond to ν an even, positive integer. We show now that there is a spectral power-law k^{d+1} (naively corresponding to $\nu = 2$) and representing scalar backtransfer.

More precisely, we show that

$$\widehat{\Phi}_1(\kappa) \sim \frac{\gamma^{\frac{d+\zeta}{\gamma}}}{d(4\pi)^{d/2}} \frac{\Gamma\left(\frac{d+2}{\gamma}\right)}{\Gamma\left(\frac{d+2}{2}\right)} \kappa^2 \tag{3.59}$$

for $\kappa \ll 1$. To prove this, we use (3.51) again, but calculate now the time-derivative using the transfer equation discussed in Section 2.1:

$$\partial_t \widehat{\Theta}_0(k,t) = -k_i k_j \int d^d\mathbf{q}\ \widehat{D}_{ij}(\mathbf{q}) \left[\widehat{\Theta}_0(k,t) - \widehat{\Theta}_0(|\mathbf{k}-\mathbf{q}|,t)\right], \tag{3.60}$$

where $\widehat{D}_{ij}(\mathbf{q}) = \widehat{K}(q) P_{ij}(\hat{\mathbf{q}})$. We can take the limit as $k \to 0$ in this expression, with the result that

$$\partial_t \widehat{\Theta}_0(k,t) \sim -\frac{d-1}{d} k^2 \int d^d\mathbf{q}\ \widehat{K}(q) \left[\widehat{\Theta}_0(0,t) - \widehat{\Theta}_0(q,t)\right], \tag{3.61}$$

where the factor $(d-1)/d$ comes from the angular average of $P_{ij}(\hat{\mathbf{q}})$. Fourier inversion gives

$$\int d^d\mathbf{q}\ \widehat{K}(q) \left[\widehat{\Theta}_0(0,t) - \widehat{\Theta}_0(q,t)\right] = \frac{1}{(2\pi)^d} \int d^d\mathbf{r}\ [K(0) - K(r)]\Theta_0(r,t). \tag{3.62}$$

In this form, the limit as $k_0 \to 0$ is easy to evaluate, with the result

$$\lim_{k_0 \to 0} \int d^d\mathbf{r}\ [K(0) - K(r)]\Theta_0(r,t) = K_1 \int d^d\mathbf{r}\ r^{\zeta}\Theta_0(r,t), \tag{3.63}$$

for $K_1 = \left(\frac{d+\zeta}{d-1}\right) D_1$. By substituting the scaling *Ansatz* and using again (3.9), one finds

$$\int d^d\mathbf{r}\ r^\zeta \Theta_0(r,t) = L^\zeta(t)\cdot \omega_{d-1}\int_0^\infty \ \rho^{d+\zeta-1}\Phi_0(\rho)\,d\rho. \tag{3.64}$$

The right side of (3.64) can be reduced to a Gamma integral by the subsitution $t = \rho^\gamma/\gamma$, giving

$$\int_0^\infty \ \rho^{d+\zeta-1}\Phi_0(\rho)\,d\rho = \gamma^{\frac{d+\zeta}{\gamma}-1}\Gamma\left(\frac{d+\zeta}{\gamma}\right). \tag{3.65}$$

Putting together (3.61)-(3.65) gives finally

$$\begin{aligned}\partial_t\widehat{\Theta}_0(k,t) &\sim -\frac{D_1}{(2\pi)^d}\left(\frac{d-1}{d}\right)\left(\frac{d+\zeta}{d-1}\right)\omega_{d-1}\gamma^{\frac{d+\zeta}{\gamma}-1}\Gamma\left(\frac{d+\zeta}{\gamma}\right)L^\zeta(t)(\kappa/L(t))^2 \\ &= -D_1 L^{-\gamma}(t)\cdot\gamma^{\frac{d+\zeta}{\gamma}}\Gamma\left(\frac{d+2}{\gamma}\right)\kappa^2\Big/(4\pi)^{d/2}\Gamma\left(\frac{d+2}{2}\right).\end{aligned} \tag{3.66}$$

Substituting this into (3.51) yields (3.59), as claimed.

The result may be easily understood in terms of our general result (3.38) for the low-wavenumber asymptotics. The coefficient $B(\nu)$ of the singular contribution vanishes at $\nu=\gamma$, because of the Gamma function $\Gamma\left(-\frac{\nu}{\gamma}\right)$ in the denominator of (3.39). Thus, the leading term ought to be $B_1\kappa^2$. If we substitute $\beta=2, \nu=\gamma$ in (3.40), the integral is found to be $-\frac{2}{d}\Gamma\left(\frac{d+2}{\gamma}\right)\gamma^{\frac{d+2}{\gamma}-1}$. This yields $B_1 = \frac{\gamma^{\frac{d+\zeta}{\gamma}}}{d(4\pi)^{d/2}}\frac{\Gamma\left(\frac{d+2}{\gamma}\right)}{\Gamma\left(\frac{d+2}{2}\right)}$ in (3.37), in exact agreement with (3.59).

In terms of the spectral scaling function, the result for our DSS solution is

$$F_1(\kappa)\sim \frac{\gamma^{\frac{d+2}{\gamma}-1}}{2^{d+1}}\frac{\Gamma\left(\frac{d+2}{\gamma}\right)}{\left[\Gamma\left(\frac{d+2}{2}\right)\right]^2}\cdot\kappa^{d+1} \tag{3.67}$$

for $\kappa \ll 1$. Although self-similar, spectral PLE must be violated in this solution. In fact, when the naive power is replaced by any other, the DSS solution will automatically develop a time-dependent low-wavenumber coefficient, in contrast to (3.44). To see this, we may return to dimensionful variables in (3.67), using now (3.9) with $\alpha = d+\gamma$. The result is

$$E_1(k,t)\sim A(\gamma,d)L^\zeta(t)k^{d+1} \tag{3.68}$$

for $kL(t)\ll 1$, with $A(\gamma,d) := \frac{\gamma^{\frac{d+2}{\gamma}-1}}{2^{d+1}}\frac{\Gamma\left(\frac{d+2}{\gamma}\right)}{\left[\Gamma\left(\frac{d+2}{2}\right)\right]^2}$. Because of the "leftover" factor of $L^\zeta(t)$, the coefficient of the asymptotic k^{d+1} power-law is now explicitly time-dependent. In fact, it exhibits power-law growth $L^{\zeta}(t) \sim [\gamma D_1(t-t_0)]^{\zeta/\gamma}$ for long times $t - t_0 \gg L^{\gamma}(t_0)/D_1$. This increase is consistent with the interpretation of the low-wavenumber spectrum as arising from scalar backtransfer.

Since there are no realizable self-similar solutions for $\nu > \gamma$, one is led to the following question: how shall an initial scalar spectrum $E(k,t_0) \sim Ak^{\alpha_0 - 1}$ at small k with $\alpha_0 = d + \nu_0$ decay asymptotically at long times? According to the traditional view, when $\nu_0 \geq 2$, then the decay shall be asymptotically self-similar at long times described by the spectrum $E_1(k,t)$ for the threshold case $\nu = \gamma$ above. We see no reason to doubt the validity of this view. On the other hand, when $-d < \nu_0 < 2$, the traditional view states that the decay will be described at long times by the self-similar solution with $\nu = \nu_0$. This is perfectly consistent when $-d < \nu_0 < \gamma$ and, again, we see no reason to doubt the traditional picture of the decay. However, when $\gamma < \nu_0 < 2$, then there is no realizable self-similar solution with $\nu = \nu_0$! Hence, the traditional view must be wrong for $\gamma < \nu_0 < 2$.

We cannot from our present analysis say what happens in the case $\gamma < \nu_0 < 2$, because it lies outside consideration of the self-similar solutions themselves. We shall treat this dynamical problem in Section 4. However, we shall now show that the scenario proposed in [5] for the Burgers decay gives a completely consistent account of the known facts also in the Kraichnan model. The picture that is proposed is of a *two-scale decay.* In addition to the integral length $L(t)$, there is another length-scale $L_*(t) \gg L(t)$, which separates an inner solution $E_{in}(k,t)$ for $kL_*(t) \gg 1$ and an outer solution $E_{out}(k,t)$ for $kL_*(t) \ll 1$. The inner solution is just the self-similar decay solution $E_1(k,t)$ for $\nu = \gamma$. The outer solution is the same as the initial spectrum $E_{out}(k,t) \sim Ak^{\alpha_0 - 1}$ for $kL_*(t) \ll 1$. Hence, spectral PLE holds, but only in the outer region. Now by matching the inner and outer solutions, one can find the crossover length-scale $L_*(t)$, or, equivalently, its associated wavenumber $k_*(t)$, as

$$A[k_*(t)]^{d+\nu_0-1} \sim D(t-t_0)^{\zeta/\gamma}[k_*(t)]^{d+1}, \tag{3.69}$$

or

$$L_*(t) \sim (t - t_0)^{\frac{\zeta}{(2-\nu_0)\gamma}}. \tag{3.70}$$

Since $L(t) \sim (t - t_0)^{1/\gamma}$, the inequality $L_*(t) \gg L(t)$ necessary for validity of this picture only holds if $\zeta/(2 - \nu_0) > 1$ or if $2 > \nu_0 > \gamma$. It is a highly nontrivial test of consistency that the critical value $\nu_0 = \gamma$, above which the ratio $R(t) := L_*(t)/L(t)$ grows, coincides exactly with the value $\nu = \gamma$, above which there is no realizable self-similar decay solution. This gives us some confidence in the correctness of the picture proposed. If this picture is correct, then at very long times DSS is restored and the decay is described by the inner solution spectrum $E_1(k,t)$, since $L_*(t) \to +\infty$ in units of $L(t)$. As $\nu_0 \to 2-$, the growth rate of $L_*(t)$ becomes infinitely fast and the outer solution region disappears, in agreement with the traditional view. Thus, PLE and DSS both hold for $\gamma < \nu_0 < 2$, but the decay is not what one would naively expect for DSS+ PLE, because the outer range where PLE holds is not part of the inner-range DSS solution.

A similar picture may be developed in physical space, but, as noted in [5], the separation of the length-scales is not as sharp. Now one would expect $\Theta_{in}(r,t) = \Theta_1(r,t)$ for $r \ll L_*(t)$ and $\Theta_{out}(r,t) \sim A' r^{-(d+\nu_0)}$ for $r \gg L_*(t)$. Note that for $r \approx L(t)$,

$$\Theta_{in}(r,t) \sim [L(t)]^{-(d+\gamma)} \gg \Theta_{out}(r,t) \sim [L(t)]^{-(d+\nu_0)}. \tag{3.71}$$

The crossover occurs at a larger length-scale $L_*(t)$, which is found by the matching condition

$$[L(t)]^{-(d+\gamma)} \exp[-(1/\gamma)(L_*(t)/L(t))^{\gamma}] \sim [L_*(t)]^{-(d+\nu_0)}. \tag{3.72}$$

It is easy to see that this implies a solution for the ratio $R_*(t) = L_*(t)/L(t)$ of the form

$$R(t) \sim [\log(t - t_0)]^{1/\gamma}. \tag{3.73}$$

Hence, $L_*(t)$ is only larger than $L(t)$ by a logarithmic term in physical space.

4 Convergence to Self-similar Solutions

(4.1) The Long-time Scaling Limit

In this section, we identify the time-dependent solutions of the Θ equation (1.9) which show eventually a self-similar form of decay. That is, we find domains of attraction of the self-similar solutions constructed in Section 3. We are interested in observing the solutions on a range of length-scales comparable to $L(t)$ and at a level of scalar amplitude comparable to the rms fluctuation $\vartheta(t)$. Hence, we consider the rescaled solutions

$$\Theta(r,t) = \vartheta^2(t)\Gamma(r/L(t), \tau(t)), \tag{4.1}$$

where $\frac{\dot{\vartheta}}{\theta} = -\frac{\alpha}{2}\frac{\dot{L}}{L}$, $\frac{\dot{L}}{L} = D_1 L^{\zeta-2}$ are as in (3.5),(3.6) and

$$\tau(t) := \log L^\gamma \sim \log(t - t_0). \tag{4.2}$$

The function Γ solves:

$$\gamma\frac{\partial\Gamma}{\partial\tau}(\rho,\tau) = \frac{1}{\rho^{d-1}}\frac{\partial}{\partial\rho}\left[\rho^{d+\zeta-1}\frac{\partial\Gamma}{\partial\rho}(\rho,\tau)\right] + \rho\frac{\partial\Gamma}{\partial\rho}(\rho,\tau) + \alpha\Gamma(\rho,\tau), \tag{4.3}$$

where $\rho = r/L(t)$. This change of variables may be made for any $\alpha > 0$, and, if we wish to make this explicit we shall refer to the above function as $\Gamma_{(\alpha)}(\rho,\tau)$. Of course, it is only possible that a nontrivial scaling limit is obtained as $\tau \to \infty$ for one exponent α. If a nontrivial limit is obtained for α, then, for any other exponent, say, α', it follows that

$$\lim_{\tau\to\infty} \Gamma_{(\alpha')}(\rho,\tau) = \begin{cases} \infty & \text{if } \ \alpha' > \alpha \\ 0 & \text{if } \ \alpha' < \alpha \end{cases} \tag{4.4}$$

Of course, this means that if the scaling limit $\Gamma_{(\alpha')}(\rho,\tau) \to 0$ (resp. ∞) for α', then an exponent α larger (resp. smaller) than α' is required for a nontrivial limit (if this is possible at all).

To analyze the equation (4.3), we make the change of variables $x = \gamma^{-1}\rho^\gamma$, which is the same as in Section 3 up to a sign. We obtain

$$\frac{\partial\Gamma}{\partial\tau}(x,\tau) = x\frac{\partial^2\Gamma}{\partial x^2}(x,\tau) + (c+x)\frac{\partial\Gamma}{\partial x}(x,\tau) + a\Gamma(x,\tau), \tag{4.5}$$

with $a = \alpha/\gamma, c = d/\gamma$. Let $\Phi_{(\alpha)}(x) := \Phi(a, c; -x)$ be the self-similar solution discussed in Section 3, for the same choice of α as employed in $\Gamma_{(\alpha)}$. Then Φ is a steady state solution of (4.5). We define $A(x,\tau)$ via $\Gamma(x,\tau) = \Phi(x)A(x,\tau)$. Clearly, to establish convergence $\Gamma(x,\tau) \to \Phi(x)$, it is enough to show that $A(x,\tau) \to 1$. We find for A that:

$$\frac{\partial A}{\partial \tau}(x,\tau) = x\frac{\partial^2 A}{\partial x^2}(x,\tau) + \left(c + x + 2x\frac{\Phi'(x)}{\Phi(x)}\right)\frac{\partial A}{\partial x}(x,\tau) \tag{4.6}$$

The transformation to $A(x,t)$ has removed the explicit α dependence, which is now represented only through $\Phi_{(\alpha)}$.

We shall now obtain characterizations of the domains of convergence of the self-similar solutions for each α. In this analysis, it is important to distinguish two general classes of initial data: those of "rapid decay" for which

$$\int_{\mathbb{R}^d} |\Theta(r,0)|^2 e^{\frac{r^\gamma}{\gamma}} d\mathbf{r} < \infty \tag{4.7}$$

and those of "slow decay" for which the above integral is infinite. Thus, the initial data showing rapid decay belong to L^2 with a stretched-exponential weight and must decay at least as fast as the weight. This turns out to be a useful formal criterion for "rapid-decay". Initial data with power-law decay at large r—which are of particular interest in view of the question of validity of spatial PLE—are classified as "slow decay" functions. Convergence results will be established for both classes of initial data below.

(4.2) Initial Data with Rapid Decay

We analyze first the case of "rapid decay". If we make the scaling with $\alpha = d$ and define correspondingly $\Gamma(x,\tau) = A(x,\tau)\Phi_0(x)$, then (4.6) becomes

$$\frac{\partial A}{\partial \tau}(x,\tau) = x\frac{\partial^2 A}{\partial x^2}(x,\tau) + (c - x)\frac{\partial A}{\partial x}(x,\tau) := \mathcal{L}_0 A(x,t) \tag{4.8}$$

We used $\Phi_0(x) = e^{-x}$. The operator $\mathcal{L}_0$ has as its eigenfunctions the generalized Laguerre polynomials $L_\ell^{c-1}(x)$ with eigenvalues $-\ell$. Indeed, the Laguerre polynomial is characterized as the unique solution $y = L_\ell^{c-1}(x)$ of the second-order equation

$$xy'' + (c - x)y' + \ell y = 0 \tag{4.9}$$

which is regular at the origin. See [25], 6.9.2(36) or [49], 5.1, 5.3. In terms of $A(x,0)$, the rapid decay criterion (4.7) becomes

$$\int_0^\infty |A(x,0)|^2 e^{-x} x^{c-1} dx < \infty \tag{4.10}$$

It is known that the generalized Laguerre polynomials $\left\{L_\ell^{c-1}(x) : \ell = 0, 1, 2, ...\right\}$ are a complete, orthogonal set on the interval $(0, \infty)$ with the weight $w_c(x) = x^{c-1}e^{-x}$. See [49], Theorem 5.7.1. Thus, it follows from the above remarks that the solution $A(x,\tau)$ of (4.8) with initial datum $A(x,0)$ also satisfies the condition (4.10) and, furthermore, has the expansion

$$A(x,\tau) = \sum_{\ell=0}^{\infty} a_\ell e^{-\ell\tau} L_\ell^{c-1}(x) \tag{4.11}$$

which converges in the L^2-sense with weight $w_c(x)$. The expansion coefficients are given in terms of the initial datum by

$$a_\ell = \frac{\ell!}{\Gamma(c+\ell)} \int_0^\infty A(x,0) L_\ell^{c-1}(x)\ e^{-x} x^{c-1} dx. \tag{4.12}$$

We assume the standard normalization of the Laguerre polynomials, [49], 5.1.1.

An important conclusion follows immediately from the fact that the expansion coefficients may also be obtained from the solution $A(x,\tau)$:

$$a_\ell e^{-\ell\tau} = \frac{\ell!}{\Gamma(c+\ell)} \int_0^\infty A(x,\tau) L_\ell^{c-1}(x)\ e^{-x} x^{c-1} dx. \tag{4.13}$$

If we recall that $\Gamma(x,\tau) = A(x,\tau)e^{-x}$ and that $e^\tau = L^\gamma(t)$, then (4.12) and (4.13) together imply that

$$[L(t)]^{\ell\gamma} \int_0^\infty \Gamma(x,\tau) L_\ell^{c-1}(x)\ x^{c-1} dx = \int_0^\infty \Gamma(x,0) L_\ell^{c-1}(x)\ x^{c-1} dx. \tag{4.14}$$

In other words, the lefthand side of this equation is for each $\ell = 0, 1, 2, ...$ an invariant of motion of the equation (4.5) (with $a = c$). If we return to the unscaled solution, these invariants, with an appropriate choice of normalization, take the form

$$J_\ell(t) := \int_{\mathbb{R}^d} L^{\ell\gamma}(t) \frac{\ell!}{(c)_\ell} L_\ell^{c-1}\left(\frac{r^\gamma}{\gamma L^\gamma(t)}\right) \Theta(r,t)\ d^d\mathbf{r} \tag{4.15}$$

for $\ell = 0, 1, 2, ...$. *Thus, there is an infinite sequence of integral invariants of the equation (1.9) for* $\Theta(r, t)$. The first such invariant for $\ell = 0$ is nothing but the Corrsin invariant $J_0 = K$. The next two are, for $\ell = 1$,

$$J_1(t) := \int_{\mathbb{R}^d} \left[L^\gamma(t) - \frac{r^\gamma}{d} \right] \Theta(r, t)\ d^d\mathbf{r} \tag{4.16}$$

and for $\ell = 2$,

$$J_2(t) := \int_{\mathbb{R}^d} \left[L^{2\gamma}(t) - \frac{2L^\gamma(t) r^\gamma}{d} + \frac{r^{2\gamma}}{d(d+\gamma)} \right] \Theta(r, t)\ d^d\mathbf{r}. \tag{4.17}$$

These are really "generalized invariants", because they depend not only upon the solution $\Theta(r, t)$ but also explicitly upon the time t. However, we see that, in the subspace defined by the vanishing of the first p invariants, $J_0 = J_1 = \cdots = J_{p-1} = 0$, the pth integral $J_p \propto \int r^{p\gamma}\Theta(r, t)\ d^d\mathbf{r}$ and is an ordinary invariant. This may also be inferred directly from equation (1.9), by using the fact that its righthand side defines an operator homogeneous of degree $-\gamma$ and using integration by parts.

These new invariants play a key role in the problem of the convergence in the scaling limit. We see using the relation $\Gamma(x, \tau) = A(x, \tau)e^{-x}$, the definition of $\Phi_\ell(x)$, and the expansion formulae (4.11),(4.12) that

$$\Gamma(x, \tau) = \sum_{\ell=0}^{\infty} c_\ell [L(t)]^{-\gamma\ell} \Phi_\ell(x) \tag{4.18}$$

with convergence of the summation in the L^2-sense on $(0, \infty)$ for the weight $\tilde{w}_c(x) = e^x x^{c-1}$. The expansion coefficient

$$c_\ell = \frac{1}{\Gamma(c)} \int_0^\infty \Gamma(x, 0) L_\ell^{c-1}(x) x^{c-1} dx \tag{4.19}$$

is proportional to the invariant J_ℓ, $\ell = 0, 1, 2, ...$. Now suppose that the first p invariants vanish: $J_0 = J_1 = \cdots = J_{p-1} = 0$. Since the solution $\Phi_p(x)$ corresponds to $\alpha = d + p\gamma$, we see that if we scale according to that α, we obtain

$$\Gamma(x, \tau) = \sum_{\ell=0}^{\infty} c_{p+\ell} [L(t)]^{-\gamma\ell} \Phi_{p+\ell}(x). \tag{4.20}$$

Hence, it follows that $\lim_{\tau\to\infty} \Gamma(x, \tau) = c_p \Phi_p(x)$. Gathering together the above results, we may state the following proposition: *Suppose that the initial datum* $\Theta(r, 0)$ *lies in the* L^2 *space with* *stretched exponential weight* $e^{\frac{r^\gamma}{\gamma}}$ *and that the first* p *invariants vanish* $J_0 = J_1 = \cdots = J_{p-1} = 0$ *but* $J_p \neq 0$*. If one scales the solution* $\Theta(r,t)$ *with* $\alpha = d + p\gamma$*, then*

$$\lim_{\tau\to\infty} \Gamma_{(\alpha)}(x,\tau) = c_p \Phi_p(x) \tag{4.21}$$

with convergence in the L^2 *sense on* $(0,\infty)$ *with weight* $\tilde{w}_c(x) = e^x x^{c-1}$*. The constant* c_p *is given by the ratio of the pth invariants* J_p *of the initial datum* $\Gamma(x,0)$ *and* $\tilde{J}_p$ *of the equilibrium solution* $\Phi_p(x)$*.* It only remains to justify the last claim. In fact, with the normalization of the invariants adopted here

$$\begin{aligned} \tilde{J}_p &= \frac{1}{\Gamma(c)} \int_0^\infty \Phi_p(x) L_p^{c-1}(x) x^{c-1} dx \\ &= \frac{p!}{\Gamma(p+c)} \int_0^\infty \left[L_p^{c-1}(x)\right]^2 e^{-x} x^{c-1} dx \\ &= 1. \end{aligned} \tag{4.22}$$

This completes the proof.

It is interesting to note that $J_0 = J_1 = 0$ but $0 < J_p < \infty$ for some $p \geq 2$ is not consistent with realizable initial data, due to the nonrealizability of scaling solutions $\Phi_p(x)$ for $p > 1$. Indeed, if p is the least integer p for which $J_p \neq 0$, then our preceding result implies *a fortiori* that $\Gamma(x,\tau) \to c_p\Phi_p(x)$ in L^2 with respect to the finite measure $e^{-x}x^{c-1}dx$ and hence, along a subsequence of times τ_k, $k = 1, 2, ...$, convergence for a.e. x. If the initial data were realizable (positive-definite), then, since positive-definiteness is preserved by the dynamics and by pointwise limits, the limit $c_p\Phi_p(x)$ would be positive-definite as well. However, this contradicts our earlier result. Thus, no positive-definite $\Theta(r,0)$ can have $J_0 = J_1 = 0$ but $0 < J_p < \infty$ for some $p \geq 2$. Our argument here is rather indirect, using the equation (1.9), but the conclusion involves no dynamics. In fact, it follows directly by the same argument used in section 3.3 to prove nonrealizability of the scaling solutions for $\nu > \gamma$ that any initial datum $\Theta(0) \in L^2$ (unweighted) with $J_0 = J_1 = 0$ cannot be realizable (positive-definite). The condition $J_0 = J_1 = 0$ is precisely equivalent to the condition of vanishing moments, (3.53), employed there.

There is a direct connection (pointed out to us by K. Gawędzki) of our generalized invariants with the "slow modes" in 2-particle Lagrangian statistics that were discovered in [50]. It is a consequence of that work that there are homogeneous moment functions $\phi_{0,p}(\mathbf{r})$ of degree $\sigma_{0,p},\ p = 0,1,2,...,$ whose integrals over any initial 2-point function evolve in time as pure degree p polynomials:

$$\int \phi_{0,p}(\mathbf{r})\ \Theta(\mathbf{r},t)\ d^d\mathbf{r} = \sum_{q=0}^{p} c_{p,q}\, t^{p-q} \int \phi_{0,q}(\mathbf{r})\,\Theta(\mathbf{r},0)\ d^d\mathbf{r}, \tag{4.23}$$

for some computable constants $c_{p,q} = \int \overline{\psi_{0,q}(\mathbf{r},1)}\,\phi_{0,p}(\mathbf{r})d^d\mathbf{r}$, in the notations of [50]. As these constants are manifestly independent of initial data, it is not hard to infer from (4.23) the existence of an associated sequence of "generalized invariants". These "slow modes" were constructed in [50] for every angular momentum sector $\ell = 0,1,2,...,$ but for the rotationally-invariant sector $\ell = 0$ they are particularly simple, given just by the powers $\phi_{0,p}(r) = r^{p\gamma}$. See Appendix A of [50]. It can then be easily shown that the associated sequence of generalized invariants in the $\ell = 0$ sector coincides with the sequence $J_p,\ p = 0,1,2,...$ we found above.

(4.3) Initial Data with Slow Decay & A Finite Invariant

The previous results do not allow us to address the question whether permanence of large eddies (PLE) holds in the space domain. For this, we must consider initial data with only power-law decay at large distances. Such data with slow decay fall themselves into two broad classes: those with one of the invariants $J_\ell,\ \ell = 0,1,2,...$ finite and those with no finite invariants. Here by "finite" we mean both non-zero and non-infinite. It will be shown below that the class of initial data with a finite invariant behaves very similarly—as far as the leading-order behavior is concerned—to the initial data with rapid decay.

To study such initial data, a new technique is required. An important observation is that (4.6) is the backward equation corresponding to the *Fokker-Planck equation* on the half-line $x > 0$

$$\frac{\partial P}{\partial \tau}(x,\tau) = -\frac{\partial}{\partial x}\left(a(x)P(x,\tau)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(b(x)P(x,\tau)\right) \tag{4.24}$$

with drift

$$a(x) = c + x + 2x\frac{\Phi'(x)}{\Phi(x)} \tag{4.25}$$

and diffusion

$$b(x) = \sigma^2(x) = x. \tag{4.26}$$

Thus, it follows immediately that

$$A(x,\tau) = E[A(X_{x,\tau},0)] = \int_0^\infty dy\ A(y,0)P(y,\tau|x,0) \tag{4.27}$$

where $X_{x,\tau}$ is the diffusion starting at x at $\tau = 0$ and obeying the Ito equation

$$dX_\tau = a(X_\tau)d\tau + \sigma(X_\tau)dW_\tau, \tag{4.28}$$

and $P(y,\tau|x,0)$ is the transition probability kernel for the process. The formal invariant density $P(x)$ of the Fokker-Planck equation is

$$P(x) \propto x^{c-1}e^x\Phi^2(x). \tag{4.29}$$

However, this is only a normalizable probability density for the exceptional cases when $\nu = \gamma\ell,\ \ell = 0,1,2,...$, in which case we refer to the density as $P_\ell(x),\ \ell = 0,1,2,....$ This can be easily understood from the character of the drift term. When $\Phi(x) \propto x^{-a}$ at large x, then $a(x) \sim c - 2a + x$ for $x \gg 1$, which is unstable. However, when $\Phi(x) = p(x)e^{-x}$ for some polynomial $p(x)$—as for the exceptional series—then $a(x) \sim a - x$ for $x \gg 1$, which is stable. [3]

[3]These observations give another method to infer the existence of the sequence of generalized invariants $J_p,\ p = 0,1,2,...$ In fact, using the invariant density in (4.29), equation (4.6) may be rewritten as

$$\frac{\partial A}{\partial\tau}(x,\tau) = \frac{x^{1-c}e^{-x}}{\Phi^2(x)}\frac{\partial}{\partial x}\left[x^ce^x\Phi^2(x)\frac{\partial A}{\partial x}(x,\tau)\right].$$

It then follows by a formal integration by parts argument that

$$J(\tau) := \int_0^\infty dx\ x^{c-1}e^xA(x,\tau)\Phi^2(x)$$

is an invariant for *all* α. However, only for the exceptional values $\alpha = d + \gamma p,\ p = 0,1,2,...$ do the integrands decay rapidly enough to justify this argument. These give the familiar sequence of invariants $J_p,\ p = 0,1,2,...$

In particular, it is true for the equipartition solution $\Phi_0(x) = e^{-x}$ that

$$a(x) = c - x \tag{4.30}$$

and the invariant distribution is given by

$$P_0(x) = \frac{1}{\Gamma(c)} x^{c-1} e^{-x}. \tag{4.31}$$

For this $\ell = 0$ process we can also calculate the transition probability kernel by means of expansion in the eigenfunctions of the backward operator $\mathcal{L}_0$ in section 4.1:

$$P(y,\tau|x,0) = \frac{y^{c-1}e^{-y}}{\Gamma(c)} \sum_{\ell=0}^{\infty} \frac{L_\ell^{c-1}(y) L_\ell^{c-1}(x)}{\binom{\ell+c-1}{\ell}} e^{-\ell\tau}. \tag{4.32}$$

This sum can be evaluated in closed form, using [25], 10.12(20), with $\alpha = c-1, z = e^{-\tau}$:

$$P(y,\tau|x,0) = \frac{y^{c-1}e^{-y}}{1-e^{-\tau}} \exp\left\{-\frac{x+y}{e^{\tau}-1}\right\} \left(xye^{-\tau}\right)^{-(c-1)/2} I_{c-1}\left\{\frac{2\left(xye^{-\tau}\right)^{1/2}}{1-e^{-\tau}}\right\}, \tag{4.33}$$

in terms of the modified Bessel function $I_{c-1}(z)$. Cf. [13], equation (2.15b).

Using these results, we can establish our first main convergence result of this section: *If $\Theta(0) \in L^\infty \bigcap L^1$ and the Corrsin invariant of the initial datum is non-zero, then the limit exists,*

$$\lim_{\tau\to\infty} \Gamma_{(d)}(x,\tau) = c_0 \Phi_0(x)$$

uniformly on compacts in x. The constant c_0 is the ratio of the Corrsin invariants K of the initial data $\Gamma(x,0)$ and $\tilde{K}$ of the equipartition scaling solution $\Phi_0(x)$. The condition $\Theta(0) \in L^\infty$ is natural, since $|\Theta(r,0)| \leq \Theta(0,0) < \infty$ for any positive-definite initial data with finite energy. This theorem includes the case of power-law decay $\Theta(r,0) \sim Ar^{-\alpha}$ for $r \gg L_0$ with $\alpha > d$, which guarantees integrability. Because $\Theta(0) \in L^1$, the Corrsin invariant must be finite. The theorem remains true even if the Corrsin invariant is zero initially, but in that case it yields a trivial (null) scaling limit.

We prove the result first for bounded $A(0)$. This will follow from standard convergence results for time-dependent distributions of one-dimensional diffusion processes. For example,

[51], Section 23, Theorem 3 implies that for any $A(0) \in L^\infty$ and uniformly on compacts in x,

$$\lim_{\tau\to\infty} E\left[A\left(X_{x,\tau},0\right)\right] = \int A(y,0)\, P(y)\, dy \tag{4.34}$$

where

$$P(x) \propto \frac{1}{\sigma^2(x)} \exp\left\{\int_{x_0}^{x} \frac{2a(t)}{\sigma^2(t)} dt\right\}. \tag{4.35}$$

is the invariant measure of the process, if the diffusion is defined with instantaneous reflection upon hitting the boundaries. Such boundary conditions must therefore be checked to be satisfied. We show, in fact, that the process $X_{x,\tau}$ has zero probability of reaching either of the boundaries of the semi-infinite interval, 0 or ∞, at any finite time. Thus, it is a special case of reflected b.c., with no reflection ever required, and the Theorem 3 of [51] applies.

We will treat the boundary conditions in a generality that will permit to discuss later cases as well. According to [51], Section 21, Theorem 1, for any $b \in (0,\infty)$, if

$$L_1^- = \int_0^b \exp\left\{-\int_b^x \frac{2a(t)}{\sigma^2(t)} dt\right\} dx = +\infty, \tag{4.36}$$

then the process $X_{x,\tau}$ attains the point b before 0 a.s., for any $x \in (0,b)$. In that case, the process never hits 0 a.s., because continuity in time requires that it pass through the interval $(0,b)$ to reach 0 and the Markov property requires that each time it re-enters the interval it must exit through b. The same statement holds for the right endpoint: if

$$L_1^+ = \int_b^\infty \exp\left\{-\int_b^x \frac{2a(t)}{\sigma^2(t)} dt\right\} dx = +\infty, \tag{4.37}$$

then the process $X_{x,\tau}$ attains the point b before reaching ∞ a.s., for any $x \in (b,\infty)$, and thus never reaches ∞ in any finite time a.s. We note in general that

$$\psi(x;b) := \exp\left\{-\int_b^x \frac{2a(t)}{\sigma^2(t)} dt\right\} = C\left[\sigma^2(x)P(x)\right]^{-1}, \tag{4.38}$$

using (4.35), where the constant $C = \sigma^2(b)P(b)$. Thus, for any of the processes (4.24), (4.25), it follows that

$$\psi(x;b) = C' \frac{e^{-x}}{x^c \Phi^2(x)} \tag{4.39}$$

using $\sigma^2(x) = 2x$ and the formula (4.29) for the invariant measure. Clearly, $L_1^- = \int_0^b \psi(x;b)\ dx$ and $L_1^+ = \int_b^\infty \psi(x;b)\ dx$.

In the case at hand, for the equipartition solution, we see that $\psi_0(x;b) = Cx^{-c}e^x$. Since $c \geq 1$, both $L_1^- = +\infty$ and $L_1^+ = +\infty$. Thus, we conclude that the process never reaches the boundary in finite time a.s. This completes the convergence proof for the case of bounded $A(0)$. By itself, this is only a strengthening of the result in section 4.1, since $A(0) \in L^\infty$ implies that $A(0)$ is L^2 with respect to the weight $w_c(x) = x^{c-1}e^{-x}$. However, the sense of convergence is stronger, being now pointwise in x uniformly on compacts.

Having proved the result for bounded $A(0)$, we now extend to the case where $\Gamma(0) \in L^\infty$ and $\Gamma(0) \in L^1$ with respect to weight x^{c-1}. We have, using $\Gamma(x,0) = A(x,0)e^{-x}$, that

$$|A(x,0)| \leq \|\Gamma(0)\|_{L^\infty} e^x \tag{4.40}$$

and

$$\int_0^\infty |A(x,0)|P_0(x)\ dx = \frac{1}{\Gamma(c)}\int_0^\infty |\Gamma(x,0)|\ x^{c-1}dx < \infty. \tag{4.41}$$

Because of the latter result, we may choose $M > 1$ so large that for any small $\epsilon > 0$,

$$\int_M^\infty |A(x,0)|P_0(x)\ dx < \epsilon. \tag{4.42}$$

Let us then take

$$A^{(M)}(x,0) := \begin{cases} A(x,0) & x \leq M \\ 0 & x > M \end{cases} \tag{4.43}$$

Thus, by (4.40), $A^{(M)}(0)$ is bounded: $\|A^{(M)}(0)\|_{L^\infty} \leq \|\Gamma(0)\|_{L^\infty} e^M$. By triangle inequality

$$\begin{aligned} &\left|\int_0^\infty dy\ A(y,0)P(y,\tau|x,0) - \int_0^\infty dy\ A(y,0)P_0(y)\right| \\ &\quad\leq \left|\int_0^\infty dy\ A^{(M)}(y,0)P(y,\tau|x,0) - \int_0^\infty dy\ A^{(M)}(y,0)P_0(y)\right| \\ &\qquad + \int_M^\infty dy\ |A(y,0)|P(y,\tau|x,0) + \int_M^\infty dy\ |A(y,0)|P_0(y). \end{aligned} \tag{4.44}$$

To control the second term we employ the estimate

$$\sup_{y>0} \frac{P(y,\tau|x,0)}{P_0(y)} \leq \frac{e^x}{(1-e^{-\tau})^c}. \tag{4.45}$$

This is proved using the inequality ([25], 7.3.2(4) and 7.2.2(12)):

$$|z^{-\nu} I_\nu(z)| \le \frac{e^{|\mathrm{Re}\, z|}}{2^\nu \Gamma(\nu+1)} \tag{4.46}$$

with $z = \frac{2\left(xye^{-\tau}\right)^{1/2}}{1-e^{-\tau}}$ and $\nu = c-1$ in (4.33). Thus,

$$P(y,\tau|x,0) \le \frac{P_0(y)}{(1-e^{-\tau})^c} \exp\left\{ \frac{-(x+y) + 2\left(xye^{\tau}\right)^{1/2}}{e^\tau - 1} \right\}. \tag{4.47}$$

The maximum of the exponent is found to occur at $y = xe^\tau$, yielding the estimate (4.45). That inequality can then be used to bound

$$\int_M^\infty dy\ |A(y,0)| P(y,\tau|x,0) \le \frac{e^x}{(1-e^{-\tau})^c}\epsilon \tag{4.48}$$

because of the condition (4.42) on M. Since $A^{(M)}(0)$ is bounded for a fixed M, our preliminary convergence result applies and we obtain

$$\limsup_{\tau\to\infty} \left| \int_0^\infty dy\ A(y,0)P(y,\tau|x,0) - \int_0^\infty dy\ A(y,0)P_0(y) \right| \le (e^x+1)\epsilon. \tag{4.49}$$

Since ϵ was arbitrary, we get

$$\lim_{\tau\to\infty} \int_0^\infty dy\ A(y,0)P(y,\tau|x,0) = \int_0^\infty dy\ A(y,0)P_0(y) := c_0. \tag{4.50}$$

and therefore

$$\lim_{\tau\to\infty} \Gamma(x,\tau) = c_0 \Phi_0(x) \tag{4.51}$$

uniformly on compact subsets of $(0,\infty)$.

To identify the constant c_0, we note that the Corrsin invariant of the initial data is

$$K(0) = \gamma^{c-1}\omega_{d-1} \int_0^\infty A(x,0)\Phi_0(x)\ x^{c-1}\ dx. \tag{4.52}$$

Furthermore, $P_0(x) = \frac{1}{\widetilde{K}}\gamma^{c-1}\omega_{d-1}x^{c-1}\Phi_0(x)$, where $\widetilde{K}$ is the Corrsin invariant of the equipartition solution Φ_0. Thus,

$$K(0) = \widetilde{K} \cdot \int_0^\infty A(x,0)P_0(x)\ dx = \widetilde{K} \cdot c_0. \tag{4.53}$$

Of course, this is consistent with the time-invariance of the Corrsin integral, which, by (4.51), gives $K(\tau) = c_0\widetilde{K}$ for all $\tau \geq 0$.

If the Corrsin invariant is finite but vanishes, then the results in the preceding Section 4.2 suggest that the asymptotic behavior will be described by Φ_p, if J_p is the first non-vanishing invariant. We next prove such a result, which is relevant to initial conditions with power-law decay for $\nu > \gamma$. The theorem we establish is the following: *Suppose that* $\Theta(0) \in L^\infty$ *and* $\int_{\mathbb{R}^d} d\mathbf{r}\ r^{p\gamma}|\Theta(r,0)| < \infty$. *If* $J_\ell = 0,\ \ell = 0, 1, ..., p-1$ *but* $J_p \neq 0$, *then the solution rescaled appropriate to parameter* $\alpha = d + p\gamma$ *satisfies*

$$\lim_{\tau\to\infty} \Gamma_{(\alpha)}(x,\tau) = c_p\Phi_p(x) \tag{4.54}$$

uniformly on compact subsets of x. *Furthermore, the constant* c_p *is the ratio* $J_p/\widetilde{J}_p$, *where* $\widetilde{J}_p$ *is the value of the invariant for the scaling solution* $\Theta_p(r,t) = \vartheta^2(t)\Phi_p(r/L(t))$.

We give the proof first for the bounded $A(0)$. We wish to make a proof very similar to the previous one. In fact,

$$A(x,\tau) = E[A(X_{x,\tau}, 0)] \tag{4.55}$$

where $X_{x,\tau}$ is the diffusion process appropriate to $\alpha = d + p\gamma$. It is enough to show that

$$\lim_{\tau\to\infty} A(x,\tau) = \int_0^\infty A(x,0)P_p(x)\ dx := c_p \tag{4.56}$$

uniformly on compact sets of x with

$$P_p(x) = \frac{p!}{\Gamma(c+p)}\left[L_p^{c-1}(x)\right]^2 e^{-x}x^{c-1} \tag{4.57}$$

the invariant measure of the process. The complication is that the diffusion in all the cases $p \geq 1$ has singular points and decomposes into $p+1$ simple pieces, each with its own invariant measure. In fact, using (4.24) for the drift $a(x)$ and (3.26) for $\Phi_p(x)$,

$$a(x) = c - x + \sum_{k=1}^{p} \frac{2x}{x - x_k} \tag{4.58}$$

where $x_k,\ k = 1, ..., p$ are the p roots of the generalized Laguerre polynomial $L_p^{c-1}(x)$. We recall that these are all real and simple, and located in the interior of the interval $(0,\infty)$. Cf. [49], Theorem 3.3.1. We may label them in increasing order $x_1 < x_2 < \cdots < x_p$. Because of the pole terms, the zeros are repulsive singularities of the drift field $a(x)$ and the regular set of points of the process decomposes into a disjoint union of $p+1$ open intervals, $I_k = (x_k, x_{k+1}),\ k = 0, 1, .., p$, with $x_0 = 0$ and $x_{p+1} = \infty$. It is easy to calculate that for each $x, b \in I_k$,

$$\psi(x; b) = \frac{Cx^{-c}e^x}{\prod_{k=1}^p (x - x_k)^2}, \tag{4.59}$$

with C some constant. Thus, $\int_{x_k}^b \psi(x; b)\ dx = +\infty$ and $\int_b^{x_{k+1}} \psi(x; b)\ dx = +\infty$, so that, again by [51], Section 21, Theorem 1, the boundary points of each interval are inaccessible in finite time a.s. Thus, the process is not ergodic but instead there exist $p+1$ distinct, ergodic invariant distributions $P_p^{(k)}$ supported on the intervals $I_k,\ k = 0, 1, ..., p$. Up to a normalization factor $w_k = (\int_{I_k} P_p(x)dx)^{-1}$, these coincide with P_p restricted to the interval I_k, i.e.

$$P_p^{(k)}(x) = w_k P_p(x)|_{I_k} \tag{4.60}$$

On each interval separately, the Theorem 3, Section 23 of [51] applies. Thus, for $x \in I_k$

$$\lim_{\tau\to\infty} E\left[A(X_{x,\tau}, 0)\right] = w_k \int_{I_k} A(x, 0)P_p(x) := c_{p,k}. \tag{4.61}$$

We need to show that the constants $c_{p,k}$ are, in fact, independent of k.

By assumption, the initial data has the first p invariants vanishing, $J_0 = \cdots = J_{p-1} = 0$ but $J_p \neq 0$:

$$\frac{p!}{\Gamma(p+c)} \int_0^\infty A(x, 0)L_\ell^{c-1}(x)L_p^{c-1}(x)e^{-x}x^{c-1}\, dx = 0 \tag{4.62}$$

for $\ell = 0, ..., p-1$, and

$$\frac{p!}{\Gamma(p+c)} \int_0^\infty A(x, 0) \left[L_p^{c-1}(x)\right]^2 e^{-x}x^{c-1}\, dx = J_p. \tag{4.63}$$

The uniform bound $|A(x, \tau)| \leq \|A(0)\|_{L^\infty}$ follows from (4.55). Thus, the $J_\ell,\ \ell = 0, 1, ...p$ are rigorously dynamical invariants and the equations (4.62),(4.63) hold with $A(x, 0)$ replaced by $A(x, \tau)$. Then we may apply (4.61) and Lebesgue's theorem to conclude

$$\frac{p!}{\Gamma(p+c)} \sum_{k=0}^p c_{p,k} \int_{x_k}^{x_{k+1}} L_\ell^{c-1}(x)L_p^{c-1}(x)e^{-x}x^{c-1}\, dx = 0 \tag{4.64}$$

for $\ell = 0, ..., p-1$, and

$$\frac{p!}{\Gamma(p+c)} \sum_{k=0}^{p} c_{p,k} \int_{x_k}^{x_{k+1}} \left[L_p^{c-1}(x)\right]^2 e^{-x} x^{c-1}\, dx = J_p. \tag{4.65}$$

These may be summarized as a matrix equation $\mathbf{M}\mathbf{c}_p = \mathbf{J}$ with

$$M_{\ell,k} := \int_{I_k} L_\ell^{c-1}(x) L_p^{c-1}(x) e^{-x} x^{c-1}\, dx. \tag{4.66}$$

By orthogonality of Laguerre polynomials one solution is $c_{p,k} = J_p/\tilde{J}_p$ for all k, where recall

$$\begin{aligned} \tilde{J}_p &:= \frac{1}{\Gamma(c)} \int_0^\infty \Phi_p(x) L_p^{c-1}(x) x^{c-1}\, dx \\ &= \frac{p!}{\Gamma(p+c)} \int_0^\infty \left[L_p^{c-1}(x)\right]^2 e^{-x} x^{c-1}\, dx = 1 \end{aligned} \tag{4.67}$$

is the pth invariant of the scaling solution Φ_p itself. This is the unique solution if the matrix $\mathbf{M}$ is nonsingular. That is equivalent to the statement that an arbitrary polynomial $p(x)$ of degree p can satisfy

$$\int_{I_k} p(x) L_p^{c-1}(x) x^{c-1}\, dx = 0, \quad k = 0, 1, ..., p \tag{4.68}$$

only if $p(x) \equiv 0$. In fact, this is true, because the polynomial has at most p real roots but there are $p+1$ intervals. Hence, there is at least one interval $I_k,\ k = 0, 1, ..., p$ on which it does not change sign. Then (4.68) implies that $p(x) \equiv 0$ on that interval, and, hence, everywhere. Thus, we conclude. As a by-product of this argument, we have shown that

$$c_p = \int_0^\infty A(x,0) P_p(x)\ dx = \frac{1}{\Gamma(c)} \int_0^\infty \Gamma(x,0) L_p^{c-1}(x)\ x^{c-1} dx \tag{4.69}$$

is given by $c_p = J_p/\widetilde{J}_p$, as claimed.

Having proved the result for bounded $A(0)$, we now extend to the case where $\Gamma(0) \in L^\infty$ and $\Gamma(0) \in L^1$ with respect to weight x^{p+c-1}. Here we make use of the observation that the scalings by $\alpha_p = d + p\gamma$ and $\alpha_0 = d$ are simply related by $\Gamma_{(\alpha_p)}(x,\tau) = e^{p\tau}\Gamma_{(d)}(x,\tau)$, where we have made the α-dependence explicit. After this we shall employ the equipartition scaling and the corresponding definition of $\Gamma_{(d)}(x,\tau) = A_{(d)}(x,t)e^{-x}$, so, when no α-dependence is given, $\alpha = d$ is implied. Then, we must show that

$$\lim_{\tau\to\infty} e^{p\tau} A(x,\tau) = c_p \frac{p!}{(c)_p} L_p^{c-1}(x) \tag{4.70}$$

uniformly on compacts, where

$$c_p = \frac{1}{\Gamma(c)}\int_0^\infty \Gamma(x,0) L_p^{c-1}(x)\ x^{c-1} dx. \tag{4.71}$$

We define the following auxilliary function:

$$H(y,0) := (-1)^p \int_y^\infty dy_1 \int_{y_1}^\infty dy_2 \cdots \int_{y_{p-1}}^\infty dy_p\ y_p^{c-1}\Gamma(y_p,0). \tag{4.72}$$

It is not hard to see that reversing orders of integrations by the Tonelli theorem gives

$$\begin{aligned}\int_0^\infty |H(y,0)|\ dy &\leq \int_0^\infty dy_p \int_0^{y_p} dy_{p-1} \cdots \int_0^{y_1} dy\ y_p^{c-1}|\Gamma(y_p,0)| \\ &= \frac{1}{p!}\int_0^\infty |\Gamma(y,0)|\ y^{p+c-1} dy < \infty. \end{aligned} \tag{4.73}$$

Of course, it follows directly from the definition that

$$\Gamma(y,0) = \frac{1}{y^{c-1}} \frac{d^p}{dy^p} H(y,0). \tag{4.74}$$

Because of the vanishing of the first p invariants, one may readily check that

$$\left.\frac{d^\ell H}{dy^\ell}(y,0)\right|_{y=0} = 0, \quad \ell = 0,1,...,p-1. \tag{4.75}$$

Thus, making p integrations by parts with $\frac{d^p}{dy^p} L_p^{c-1}(y) = (-1)^p$

$$\begin{aligned}\frac{1}{\Gamma(c)}\int_0^\infty H(y,0)\ dy &= \frac{1}{\Gamma(c)}\int_0^\infty \frac{d^p}{dy^p} H(y,0) L_p^{c-1}(y)\ dy \\ &= \frac{1}{\Gamma(c)}\int_0^\infty \Gamma(y,0) L_p^{c-1}(y) y^{c-1} dy = c_p. \end{aligned} \tag{4.76}$$

Since the integral is absolutely convergent, we may choose M sufficiently large that

$$\frac{1}{\Gamma(c)}\int_M^\infty |H(y,0)|\ dy < \epsilon. \tag{4.77}$$

Let us define a decomposition $H(y,0) = H^{(M)}(y,0) + \overline{H}^{(M)}(y,0)$ via $H^{(M)}(y,0) = \varphi^{(M)}(y) H(y,0)$ and $\overline{H}^{(M)}(y,0) = \overline{\varphi}^{(M)}(y) H(y,0)$ for a smooth decomposition of unity $\varphi^{(M)}(y) + \overline{\varphi}^{(M)}(y) = 1$, with $\varphi^{(M)}(y), \overline{\varphi}^{(M)}(y) \geq 0$ and $\varphi^{(M)}(y) = 0$ for $y > M+1$, $\overline{\varphi}^{(M)}(y) = 0$ for $0 < y < M$. We may then define a corresponding decomposition $\Gamma(y,0) = \Gamma^{(M)}(y,0) + \overline{\Gamma}^{(M)}(y,0)$ via (4.74), and likewise for $A(y,0) = A^{(M)}(y,0) + \overline{A}^{(M)}(y,0)$ and $c_p = c_p^{(M)} + \overline{c}_p^{(M)}$. Next we can employ the transition probability of the $\alpha = d$ process in a triangle inequality:

$$\begin{aligned}
&\left| e^{p\tau} \int_0^\infty dy\ A(y,0) P(y,\tau|x,0) - \frac{p!}{\Gamma(c+p)} L_p^{c-1}(x) \int_0^\infty dy\ H(y,0) \right| \\
&\quad \le \quad \left| e^{p\tau} \int_0^\infty dy\ A^{(M)}(y,0) P(y,\tau|x,0) - c_p^{(M)} \frac{p!}{(c)_p} L_p^{c-1}(x) \right| \\
&\qquad\quad + e^{p\tau} \left| \int_M^\infty dy\ \overline{A}^{(M)}(y,0) P(y,\tau|x,0) \right| \\
&\qquad\quad + \frac{p!}{\Gamma(c+p)} |L_p^{c-1}(x)| \int_M^\infty dy\ \left| \overline{H}^{(M)}(y,0) \right|. \qquad (4.78)
\end{aligned}$$

As before, $\|A^{(M)}(0)\|_{L^\infty} \le \|\Gamma(0)\|_{L^\infty} e^{(M+1)}$. Thus, we can identify the first term as $\left| A_{\alpha_p}^{(M)}(x,\tau) - c_p^{(M)} \right| \cdot \frac{p!}{(c)_p} |L_p^{c-1}(x)|$ and appeal to our preliminary result for L^∞ initial data to conclude that this goes to zero uniformly on compact sets of x. Of course, the third term is less than $\frac{p!}{(c)_p} |L_p^{c-1}(x)| \cdot \epsilon$ by the assumption (4.77). The main problem is to control the middle term.

If we substitute the expression (4.74) for $\overline{\Gamma}^{(M)}(y,0)$ in terms $\overline{H}^{(M)}(y,0)$ and integrate by parts p times, we obtain

$$\int_M^\infty dy\ \overline{A}^{(M)}(y,0) P(y,\tau|x,0) = \int_M^\infty dy\ \overline{H}^{(M)}(y,0) \left(-\frac{d}{dy} \right)^p \left[\frac{P(y,\tau|x,0)}{y^{c-1} e^{-y}} \right]. \qquad (4.79)$$

Employing the formula (4.33) for the transition probability gives

$$\int_M^\infty dy\ \overline{A}^{(M)}(y,0) P(y,\tau|x,0) = \frac{2^{c-1}}{(1-e^{-\tau})^c} \int_M^\infty dy\ \overline{H}^{(M)}(y,0) \left(-\frac{d}{dy} \right)^p \left[\exp\left\{ -\frac{x+y}{e^\tau - 1} \right\} z^{-(c-1)} I_{c-1}(z) \right], \qquad (4.80)$$

with $z = \frac{2(xye^{-\tau})^{1/2}}{1-e^{-\tau}}$. The derivative can be evaluated by the generalized product rule $D^p(uv) = \sum_{r=0}^p \binom{p}{r} D^r u \cdot D^{p-r} v$ and the relation

$$\frac{d}{dy} = \frac{2xe^{-\tau}}{(1-e^{-\tau})^2} \frac{d}{zdz}. \qquad (4.81)$$

We note that

$$\frac{d^r}{dy^r} \exp\left\{ -\frac{x+y}{e^\tau - 1} \right\} = (-1)^r (e^\tau - 1)^{-r} \cdot \exp\left\{ -\frac{x+y}{e^\tau - 1} \right\}. \qquad (4.82)$$

Likewise, defining $\xi = x/(1-e^{-\tau})$,

$$\frac{d^r}{dy^r} \left[z^{-(c-1)} I_{c-1}(z) \right] = \frac{2^r \xi^r}{(e^\tau - 1)^r} \left[z^{-(c+r-1)} I_{c+r-1}(z) \right] \qquad (4.83)$$

using (4.81) and $\left(\frac{d}{zdz}\right)^r [z^{-\nu}I_\nu(z)] = z^{-(\nu+r)}I_{\nu+r}(z)$, [25], 7.11(20). Summing over all the contributions and using the estimate (4.46) for the Bessel function gives

$$\left|\frac{d^p}{dy^p}\left[\exp\left\{-\frac{x+y}{e^\tau-1}\right\}z^{-(c-1)}I_{c-1}(z)\right]\right| \le \frac{p!L_p^{c-1}(-\xi)}{2^{c-1}\Gamma(c+p)(e^\tau-1)^p}\exp\left\{\frac{-(x+y)+2\,(xye^\tau)^{1/2}}{e^\tau-1}\right\}. \tag{4.84}$$

We employed the relation

$$L_p^{c-1}(x) = \sum_{r=0}^{p}\binom{c+p-1}{p-r}\frac{(-x)^r}{r!} \tag{4.85}$$

in [25], 10.12(7). Thus, $|L_p(x)| \le L_p^{c-1}(-x)$. The exponential term is the same as was shown before to be bounded over all y by e^x. Thus, we obtain finally from (4.80) that

$$\left|e^{p\tau}\int_M^\infty dy\ \overline{A}^{(M)}(y,0)P(y,\tau|x,0)\right| \le \frac{p!L_p^{c-1}(-\xi)e^x}{(1-e^{-\tau})^{c+p}}\cdot\frac{1}{\Gamma(c+p)}\int_M^\infty dy\ |H(y,0)| \tag{4.86}$$

Thus,

$$\limsup_{\tau\to\infty}\left|e^{-p\tau}\int_M^\infty dy\ \overline{A}^{(M)}(y,0)P(y,\tau|x,0)\right| \le \frac{p!}{(c)_p}L_p^{c-1}(-x)e^x\cdot\epsilon, \tag{4.87}$$

by the definition (4.77) of M. Since $A^{(M)}(0)$ is bounded for a fixed M, our preliminary convergence result applies and we obtain

$$\limsup_{\tau\to\infty}\left|e^{p\tau}\int_0^\infty dy\ A(y,0)P(y,\tau|x,0) - c_p\frac{p!}{(c)_p}L_p^{c-1}(x)\right| \le (e^x+1)\frac{p!}{(c)_p}L_p^{c-1}(-x)\cdot\epsilon. \tag{4.88}$$

Since ϵ is arbitrary, we get

$$\lim_{\tau\to\infty} e^{p\tau}A(x,\tau) = c_p\frac{p!}{(c)_p}L_p^{c-1}(x) \tag{4.89}$$

uniformly on compact subsets of $(0,\infty)$ and likewise

$$\lim_{\tau\to\infty}\Gamma_{\alpha_p}(x,\tau) = c_p\Phi_p(x). \tag{4.90}$$

The most surprising consequence, in view of traditional beliefs, is the implied breakdown of spatial PLE on length-scales comparable to the integral scale $L(t)$. Naively one would expect that PLE holds in space for all long-range powers laws, $\sim Ar^{-\alpha}$ with any $\alpha > 0$. Instead, the memory of the initial conditions is through the invariant J_p and not through the initial amplitude A of the power-law tail, when $\alpha > d + p\gamma$ and $J_p \neq 0$. Of course, one expects that PLE still holds on the logarithmically larger length-scale $L_*(t)$. We shall consider that below. We caution that the above are pure PDE results, and do not necessarily describe statistically realizable situations. As already discussed in Section 4.2, it is not possible that $J_0 = J_1 = 0$ but $J_p \neq 0, \infty$ for some $p > 1$ is consistent with realizability. Thus, only the $p = 0, 1$ cases of the above theorems are relevant to solutions arising from the Kraichnan model. If realizable initial data have a long-range power $\alpha > d + \gamma$ and if $J_0 = 0$, then it must be that $J_1 \neq 0$, so the second theorem with $p = 1$ applies and the appropriately scaled solution converges to Φ_1. This agrees with the physical picture presented in Section 3.

It would be interesting to prove spectral analogues of these theorems. We shall not attempt to do so here. However, we note that the conditions of the present theorems can be implied by spectral conditions on initial data. In fact, an initial condition with power-law spectrum $\sim Ak^{\alpha-1}$ for an exponent $\alpha = d+\nu,\ \nu > \gamma$ has spatial decay $r^{-\alpha}$ at large r (except when $\nu = 2j$ an even integer, when the decay may be even faster). Thus, $J_p \neq \infty$ for $p = 0, 1$. However, $J_0 = 0$, because the Corrsin invariant is proportional to the coefficient of the k^{d-1} term of the low-wavenumber spectrum, which is assumed to vanish. Thus, the conditions of the second theorem apply, for $J_1 \neq 0$. In that case, the long-time limit for the scaled scalar spectrum $F(\kappa, \tau(t)) := E(\kappa/L(t), t)/(\vartheta^2(t)L(t))$ is presumably $c_1 F_1(\kappa)$ with $c_1 = J_1/\widetilde{J}_1$. Although the initial spectrum $E(k, t_0)$ had the low-wavenumber form $\sim Ak^{\alpha-1}$ for k somewhat smaller than $L^{-1}(t_0)$, at sufficiently long times the spectrum is $\sim cA(\nu, d)L^{\zeta}(t)k^{d+1}$ for wavenumbers k somewhat smaller than $L^{-1}(t)$, with the same constant $A(\nu, d)$ as in (3.68).

(4.4) Initial Data with Slow Decay & No Finite Invariant

The last case to consider is that all of the invariants $J_\ell,\ \ell = 0, 1, 2, ...$ of the initial data $\Gamma(x, 0)$ are either zero or infinite. It is important to appreciate that this is the case for the nonexceptional scaling solutions Φ themselves. In fact, for $p\gamma < \nu < (p+1)\gamma$, formula (3.40) implies that Φ has the invariants $\tilde{J}_\ell = 0,\ \ell = 0, ..., p-1$ and $\tilde{J}_\ell = \pm\infty$ for all $\ell \geq p$. Because these are conserved by the dynamics, only initial data with the same pattern of invariants can exhibit dynamical scaling at large times, governed by a nonexceptional Φ. From this point of view the failure of PLE that was discussed in the last section is not so surprising. To observe a long-range power-law in the scaling limit, the initial data must have the same invariants as the final scaling solution.

We prove now some theorems which establish convergence to the nonexceptional scaling solutions. We consider first the easiest case, when $-d < \nu < 0$. We show the following: *Suppose that* $\Theta(0) \in L^{\infty}$ *and that with* $0 < \alpha < d$

$$\Theta(r,0) \sim Ar^{-\alpha},\ r \gg L_0. \tag{4.91}$$

Then, if $\Gamma_{(\alpha)}(x,\tau)$ *is the solution rescaled corresponding to the parameter* α,

$$\lim_{\tau\to\infty} \Gamma_{(\alpha)}(x,\tau) = c_\alpha \Phi_{(\alpha)}(x) \tag{4.92}$$

uniformly on compact sets. The constant $c_\alpha = A/\tilde{A}$ *where* $\tilde{A} = \gamma^{\alpha/\gamma}\frac{\Gamma\left(\frac{d}{\gamma}\right)}{\Gamma\left(-\frac{\nu}{\gamma}\right)}$ *is the constant prefactor in the asymptotic power-law (3.23) for the scaling solution* $\Phi_{(\alpha)}(\rho) = \Phi\left(\frac{\alpha}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right)$. *Thus, spatial PLE holds.* This seems to be the optimal result that could be expected. The proof depends upon the observation that $\Gamma(x,0)$ may be written in the form

$$\Gamma(x,0) = A(x,0)\Phi_{(\alpha)}(x) \tag{4.93}$$

where $A(0) \in L^{\infty}$ and $\lim_{x\to\infty} A(x,0) = c_\alpha < \infty$. In fact, it follows from the integral representation (3.45) that $\Phi_{(\alpha)}(x)$ has no zeros and decreases monotonically from the value 1 at $x = 0$. This, along with asymptotic power-law behavior which matches that in the initial data, gives the existence of a multiplicative perturbation $A(x,0)$ with the required properties.

The proof uses the stochastic representation by the diffusion with drift $a(x)$ given by (4.25) and diffusion $b(x)$ by (4.26). By (3.45)

$$\frac{\Phi'_{(\alpha)}(x)}{\Phi_{(\alpha)}(x)} = -\frac{\int_0^1 ue^{-ux}u^{a-1}(1-u)^{c-a-1}du}{\int_0^1 e^{-ux}u^{a-1}(1-u)^{c-a-1}du} \in (-1,0). \tag{4.94}$$

Furthermore, $\frac{\Phi'_{(\alpha)}(x)}{\Phi_{(\alpha)}(x)} \sim -\frac{a}{x}$ for $x \gg 1$ by (3.22). Thus, we see that $|a(x) - c| < x$ and that for $x \gg 1$

$$a(x) \sim c - 2a + x + O(x^{-1}) \tag{4.95}$$

Therefore, the drift is nonsingular except at the right boundary $x = \infty$ of the interval, which is infinitely attractive. The left boundary $x = 0$ is repulsive. For any x, b in $(0, \infty)$ we have $\psi(x; b) = C' \frac{x^{-c}e^{-x}}{\Phi^2_{(\alpha)}(x)}$ by (4.39). Because $\psi(x; b) \sim C'x^{-c}$ for $x \to 0+$ and $c > 1$, $L_1^- = \int_0^b \psi(x; b)\, dx = +\infty$ and the process $X_{x,\tau}$ never reaches 0 a.s. However, $\psi(x; b) \sim C''x^{2a-c}e^{-x}$ as $x \to +\infty$, and thus $L_1^+ = \int_b^\infty \psi(x; b)\, dx < +\infty$. In that case, [51], Section 16, Theorem 1 implies that

$$\lim_{\tau\to\infty} X_{x,\tau} = +\infty \;\; a.s. \tag{4.96}$$

Since $A(0) \in L^\infty$ and $\lim_{x\to\infty} A(x, 0) = c_\alpha$, we may apply Lebesgue's theorem to conclude that

$$\lim_{\tau\to\infty} A(x, \tau) = \lim_{\tau\to\infty} E\left[A(X_{x,\tau}, 0)\right] = c_\alpha. \tag{4.97}$$

The proof is complete.

We now consider the case $\nu > 0$, $\nu \neq \ell\gamma$, $\ell = 0, 1, 2, ...$ This case is more difficult because the Kummer functions $\Phi(a, c; -x)$ with $a > c$ have positive, real zeros. This is, of course, required by the vanishing of a certain number of the J-invariants. However, it presents a difficulty for the analysis. We prove the following: *Suppose that the initial data* $\Theta(r, 0)$ *is a bounded perturbation of* $\Phi_{(\alpha)}$ *for* $\alpha > d$, $\alpha \neq d + \ell\gamma$, $\ell = 0, 1, 2, ..$ *in the sense that*

$$\Gamma(\rho, 0) = A(\rho, 0)\Phi_{(\alpha)}(\rho), \tag{4.98}$$

with $A(0) \in L^\infty$ *and that* $\lim_{\rho\to\infty} A(\rho, 0) = c_p < \infty$. *Suppose also that* $J_0 = J_1 = \cdots = J_{p-1} = 0$. *Then,*

$$\lim_{\tau\to\infty} \Gamma_{(\alpha)}(x, \tau) = c_p \Phi_{(\alpha)}(x) \tag{4.99}$$

uniformly on compact sets. The conditions on $A(0)$ imply that $\Gamma(\rho, 0) \sim A\rho^{-\alpha}$ for $\rho \gg L_0$, with $A = c_p\tilde{A}$ as before. Thus, spatial PLE holds for this class of initial data. However, the result is not optimal, because the conditions also imply that $\Gamma(\rho, 0)$ has zeros at precisely the locations of the zeros of $\Phi_{(\alpha)}(\rho)$. The condition on vanishing invariants—which is necessary for the convergence result to be valid—implies the existence of such zeros, but not at precisely the location of the zeros of $\Phi_{(\alpha)}$. There will be more general initial data in the domain of attraction of the scaling solution, but there is then a difficult problem in accounting for the motion of the zeros in time. We leave that to future work.

The proof of the above result uses again the stochastic representation. We note that the Kummer function $\Phi(a, c; -z)$ is an entire function in the complex z-plane and, for real a, c has a finite number Z of zeros, all simple ([25], Section 16). It then follows from Weierstrass' product formula for entire functions that $\Phi(a, c; -z) = e^{g(z)} \prod_{k=1}^{Z}(z - z_k)$ where $g(z)$ is an entire function and the product is over the Z complex zeros. When $a, c > 0$ the real zeros of $\Phi(a, c; -x)$ are all positive and there are precisely p such positive, real zeros $x_1, ..., x_p$ when $c + p < a \leq c + p + 1$ ([25], Section 16). Thus,

$$\frac{\Phi'_{(\alpha)}(x)}{\Phi_{(\alpha)}(x)} = \sum_{k=1}^{p} \frac{1}{x - x_k} + h(x) \tag{4.100}$$

where $h(x)$ is a C^∞ function on the real line. For $x \gg 1$, $\frac{\Phi'_{(\alpha)}(x)}{\Phi_{(\alpha)}(x)} \sim -\frac{a}{x}$ because of the asymptotic power-law form of $\Phi_{(\alpha)}$. Thus, we see that the drift field is

$$a(x) = c + x + \sum_{k=1}^{p} \frac{2x}{x - x_k} + 2xh(x) \tag{4.101}$$

with p repulsive singular points at the zeros of $\Phi_{(\alpha)}(x)$ and, as before,

$$a(x) \sim c - 2a + x + O(x^{-1}) \tag{4.102}$$

for $x \gg 1$. Thus, the process decomposes into $p + 1$ simple pieces and the regular set of points consists of the disjoint intervals $I_k = (x_k, x_{k+1})$, with $x_0 = 0, x_p = \infty$. For each of the first p intervals, $L_1^- = L_1^+ = +\infty$ as before. The process is ergodic on these intervals with invariant measure $P_k(x) = w_k P_{(\alpha)}(x)$ for $P_{(\alpha)}(x) = x^{c-1} e^x \Phi_{(\alpha)}^2(x)$ and $w_k = \left[\int_{I_k} P_{(\alpha)}(x)\, dx\right]^{-1}$. However, on the final interval $L_1^- = +\infty$ but $L_1^+ < +\infty$. Thus, $\lim_{\tau\to\infty} X_{x,\tau} = +\infty$ a.s. when

$x \in I_p$. Therefore, we have that

$$\lim_{\tau\to\infty} E\left[A(X_{x,\tau},0)\right] = w_k \int_{I_k} A(y,0)P_{(\alpha)}(y)\, dy := c_k \tag{4.103}$$

when $x \in I_k$, $k = 0, ..., p-1$ but that

$$\lim_{\tau\to\infty} E\left[A(X_{x,\tau},0)\right] = A(+\infty,0) = c_p \tag{4.104}$$

when $x \in I_p$. We must show that $c_k = c_p$ for $k = 0, 1, ..., p-1$. Using the above convergence result and the vanishing of the first p invariants for the initial data, one can show as before that the c-coefficients satisfy

$$\sum_{k=0}^{p} c_k \int_{I_k} L_\ell^{c-1}(x)\Phi_{(\alpha)}(x)x^{c-1}\, dx = 0 \tag{4.105}$$

for $\ell = 0, ..., p-1$. This can be written as an equation for the vector $\mathbf{c} = (c_0, ..., c_{p-1})^\top$ of the form $\mathbf{Mc} = \mathbf{d}$ with

$$M_{\ell,k} = \int_{I_k} L_\ell^{c-1}(x)\Phi_{(\alpha)}(x)x^{c-1}\, dx \tag{4.106}$$

for $\ell, k = 0, ..., p-1$ and

$$d_\ell = -c_p \int_{I_p} L_\ell^{c-1}(x)\Phi_{(\alpha)}(x)x^{c-1}\, dx \tag{4.107}$$

for $\ell = 0, ..., p-1$. Because the first p invariants vanish also for $\Phi_{(\alpha)}(x)$, one solution is $c_0 = \cdots = c_{p-1} = c_p$. In fact, the matrix $\mathbf{M}$ is non-singular, because the same argument as before shows that a polynomial $p(x)$ of degree $p-1$ which satisfies

$$\int_{I_k} p(x)\Phi_{(\alpha)}(x)x^{c-1}\, dx = 0 \tag{4.108}$$

for $k = 0, ..., p-1$ must vanish identically, $p(x) \equiv 0$. Thus, the solution is unique, as required.

The previous results again support the physical picture proposed in Section 3. As before, the theorems of this section do not describe a statistically realizable situation when $\alpha > d + \gamma$, although, as pure PDE results, they are valid. When the initial data has a long-range power decay with exponent $\alpha < d + \gamma$, then the theorems for $p = 0, 1$ do apply. In that case, two possibilities occur depending upon the value of J_0. The additional conserved quantity J_1 plays no role because, for $\alpha < d + \gamma$, $J_1 = \infty$ always. If $0 < J_0 < \infty$, then the long-time behavior is that associated to Φ_0, as shown in the previous section. However, if $J_0 = 0$ or ∞ (the latter will always hold for $\alpha < d$), then the theorems of this section imply that the long-time behavior is that associated to the realizable scaling solution $\Phi_{(\alpha)}$ for the same α. This is the circumstance in which spatial PLE holds, in the conventional sense.

(4.5) View on a Larger Length-Scale

Finally, we will discuss briefly the scaling by $L_*(t)$ logarithmically bigger than $L(t)$. Our analysis will be only heuristic and no proofs given. The solution to the equation (3.6) is $L(t) = \gamma D_1(t - t_0^*)^{1/\gamma}$ for a virtual time origin t_0^* related to the initial data L_0. Let us define

$$L_*(t) = \gamma D_1 \left[(t - t_0^*) \log(t - t_0^*)\right]^{1/\gamma} . \tag{4.109}$$

We now study rescaled solutions

$$\Theta(r,t) = \vartheta^2(t)\Gamma_*(r/L_*(t), \tau(t)), \tag{4.110}$$

where $\vartheta(t), \tau(t)$ are as before. In particular, $\tau(t) := \log(t - t_0^*) = L_*^\gamma(t)/L^\gamma(t)$. The function Γ_* solves:

$$\gamma\frac{\partial\Gamma_*}{\partial\tau}(\rho,\tau) = \frac{1}{\tau}\frac{1}{\rho^{d-1}}\frac{\partial}{\partial\rho}\left[\rho^{d+\zeta-1}\frac{\partial\Gamma_*}{\partial\rho}(\rho,\tau)\right] + \left(1+\frac{1}{\tau}\right)\rho\frac{\partial\Gamma_*}{\partial\rho}(\rho,\tau) + \alpha\Gamma_*(\rho,\tau), \tag{4.111}$$

where $\rho = r/L_*(t)$. If we make the change of variables $x = \gamma^{-1}\rho^\gamma$, we obtain

$$\frac{\partial\Gamma_*}{\partial\tau}(x,\tau) = x\frac{\partial\Gamma_*}{\partial x}(x,\tau) + a\Gamma_*(x,\tau) + \frac{1}{\tau}\left[x\frac{\partial^2\Gamma_*}{\partial x^2}(x,\tau) + (c+x)\frac{\partial\Gamma_*}{\partial x}(x,\tau)\right]x, \tag{4.112}$$

with $a = \alpha/\gamma, c = d/\gamma$.

If the initial datum has the power-law form $\Gamma(x,0) \sim Ax^{-a}$ at large x, then we expect that $\Gamma_*(x,\tau)$ converges to that same power pointwise in x. This motivates us to define $A_*(x,\tau)$ via

$$\Gamma_*(x,\tau) = x^{-a}A_*(x,\tau). \tag{4.113}$$

Thus, we want to show that $\lim_{\tau\to\infty} A_*(x,\tau) = A(+\infty, 0)$. Substitution of (4.113) into (4.112) yields the equation

$$\frac{\partial A_*}{\partial\tau}(x,\tau) = x\frac{\partial\Gamma_*}{\partial x}(x,\tau)+\frac{1}{\tau}\left[x\frac{\partial^2\Gamma_*}{\partial x^2}(x,\tau) + (c-2a+x)\frac{\partial\Gamma_*}{\partial x}(x,\tau) + \frac{a(a-c+1-x)}{x}A_*(x,\tau)\right]. \tag{4.114}$$

Heuristically, we may expect that, at long times, the terms in the bracket can be neglected because of the factor $1/\tau$. In that case, the solution is governed by just the hyperbolic equation $\frac{\partial A_*}{\partial\tau}(x,\tau) = x\frac{\partial\Gamma_*}{\partial x}(x,\tau)$. This is solved by the method of characteristics, as $A_*(x,\tau) = A(e^\tau x, 0)$. If this crude approximation is valid, then $\lim_{\tau\to\infty} A_*(x,\tau) = A(+\infty, 0)$, as required.

A more refined estimate can be made by using a stochastic representation corresponding to the diffusion process with time-dependent drift

$$a(x,\tau) = x + \frac{c-2a+x}{\tau} \tag{4.115}$$

and diffusion

$$b(x,\tau) = \sigma^2(x,\tau) = \frac{2x}{\tau}. \tag{4.116}$$

If $X_{x,\tau}$ is the process started at x at time τ_0, the solution of (4.114) has the representation

$$A_*(x,\tau) = E\left[\exp\left(\int_{\tau_0}^{\tau} c(X_{x,\sigma},\sigma)d\sigma\right) A(X_{x,\tau},0)\right], \tag{4.117}$$

where

$$c(x,\tau) = \frac{a(a-c+1-x)}{x\cdot\tau}. \tag{4.118}$$

See [51], Section 11, Theorem 3. Because the drift $a(x,\tau)\sim x$ at large x,τ and, furthermore, $\sigma^2(x)/x^2\to 0$ as $x\to\infty$ or $\tau\to\infty$, one expects that $X_{x,\tau}\sim e^\tau$ as $\tau\to\infty$ a.s. See [51], Section 17, Corollary 2. The contribution to the integral of $c(x,\sigma)$ in the exponent for $\sigma\lesssim\tau$ may be estimated by $c(X_{x,\sigma},\sigma)\approx -a/\sigma$ when $X_{x,\sigma}$ is large, as it is with high probability when $\sigma\lesssim\tau$. This gives a term $\approx\exp(-a\log\tau) = \tau^{-a}$, which vanishes for large τ. On the other hand, the contribution from $\sigma\gtrsim\tau_0$ is of the order of $\exp(O(1)/\tau)\sim 1$ for large τ. Thus, we expect again that $\lim_{\tau\to\infty} A_*(x,\tau) = A(+\infty,0)$.

5 Conclusions

In this paper we have studied in detail the 2-point correlation function in the Kraichnan model, which, for homogeneous and isotropic conditions, is governed by the equation (1.9). Our main technical results are as follows:

(i) We have found all the possible self-similar decay solutions, parameterized by space dimension d, velocity field roughness exponent $\zeta = 2 - \gamma$, and a scaling exponent $\alpha > 0$, in terms of Kummer confluent hypergeometric functions.

(ii) We have shown that these solutions are statistically realizable for all α in the interval $(0, d+\gamma]$ and for no α in the complement $(d+\gamma, \infty)$.

(iii) We have exhibited an infinite sequence of invariants J_ℓ, $\ell = 0, 1, 2, \ldots$ of the equation (1.9), defined by the formula (4.15). J_0 coincides with the well-known Corrsin invariant. At least one of J_0, J_1 must be nonzero (possibly infinite) for realizable initial data.

(iv) In terms of these invariants, we have identified initial data in domains of attraction of the scaling solutions found in (i). Our theorems relevant to the Kraichnan model covered the following cases:

$$
\begin{array}{ll}
J_0 = J_1 = \infty & (\Gamma_{(\alpha)}(\tau) \to \Phi_{(\alpha)},\ 0 < \alpha < d) \\
J_0 = 0, J_1 = \infty & (\Gamma_{(\alpha)}(\tau) \to \Phi_{(\alpha)},\ d < \alpha < d+\gamma) \\
0 < J_0 < \infty, J_1 \text{ arbitrary} & (\Gamma_{(\alpha)}(\tau) \to \Phi_0,\ \alpha = d) \\
J_0 = 0, 0 < J_1 < \infty & (\Gamma_{(\alpha)}(\tau) \to \Phi_1,\ \alpha = d+\gamma)
\end{array}
$$

The first two results apply for initial data with a power-law decay $\sim r^{-\alpha}$ for $\alpha < d+\gamma$, whereas the last two include (among other possibilities) initial power-laws with $\alpha > d+\gamma$.

There are several questions left open in this work that deserve to be addressed. It would be worthwhile to establish convergence for more general initial data with slow decay and no finite invariants, allowing for a non-coincidence of its zeros with those of the final scaling solution.

It would be interesting as well to extend the convergence results to the spectral domain and to establish convergence to a power-law solution on the logarithmically larger scale. Perhaps the most interesting issues not treated in the present work are to consider scalar decay in the Kraichnan model with initial anisotropy and/or inhomogeneity and to consider the self-similarity (or not) of the higher-order N-point scalar correlations with $N > 2$. The latter is relevant for realistic decay problems, where no closure of the moment hierarchy is found. In fact, von Kármán and Howarth in their classic analysis [1] made a self-similarity hypothesis for the 3rd-order as well as the 2nd-order velocity correlations, in order to derive a scaling equation.

Perhaps the most novel result in this work is the discovery of the infinite sequence of invariants $J_\ell,\ \ell = 0, 1, 2, ...$ in the Kraichnan model. It would be most interesting to know whether there are analogues of such invariants for true passive scalars, besides the Corrsin invariant $K = J_0$, or, for that matter, for Navier-Stokes turbulence. In the latter case, the integral invariant $C = \int_{\mathbb{R}^d} d^d\mathbf{r}\ B_{LL}(r)$ found by Saffman [52] is analogous to the Corrsin invariant, while the integral $\Lambda = \int_{\mathbb{R}^d} d^d\mathbf{r}\ r^2 B_{LL}(r)$ found by Loitsyanskii [53] is rather analogous to our next invariant J_1. However, it is well-known that $\Lambda(t)$ is actually time-dependent, due to the effects of spatial long-range correlations [29]. The connection of the "generalized invariants" introduced here with the "zero modes" studied in [50] is very intriguing, especially as the latter exist also for anisotropic statistics and, plausibly, for higher-order N-point correlators.

Our results confirm in the Kraichnan model the "two-scale" decay picture earlier found for Burgers equation in [5]. The fact that there are no realizable scaling solutions for $\alpha > d + \gamma$, implies that initial data with such a $r^{-\alpha}$ decay at large distance cannot exhibit a self-similar decay with the same power-law behavior on the scale of the integral length $L(t)$. Instead, we have shown that a wide class of such initial data converge on that length-scale to the self-similar decay solution $\Theta_1(r, t)$ corresponding to $\alpha = d + \gamma$. A heuristic argument supports the conclusion that the initial power-law decay will be preserved on a length-scale $L_*(t)$, which is logarithmically bigger than $L(t)$. The fact that the same phenomenon occurs for two such very different models as Burgers equation and Kraichnan's passive scalar model argues for its generality. In fact, the physics behind it appears to be just the phenomenon of "backtransfer", which closure results indicate is a very general feature of turbulent decay. The specific, simplifying features of the models only enter in calculating the rates of this backtransfer.

We therefore expect that the results of this work will have relevance for real passive scalars and other turbulent decay problems. It is thus of some interest to specialize our results to that case of the Kraichnan model which most closely mimicks the true passive scalar, namely, $d = 3$ and $\zeta = \frac{4}{3}$. In terms of the scalar low-wavenumber spectral exponent $n = \alpha - 1$, there is a self-similar decay with PLE if initially $-1 < n < \frac{8}{3}$. However, if $\frac{8}{3} < n < 4$ then these traditional expectations are violated. In the whole range $n > \frac{8}{3}$ the scalar energy decay law is $E(t) \sim (t - t_0)^{-11/2}$, as traditionally expected only for the lowest value $n = \frac{8}{3}$. We have learned of unpublished work of D. Thomson who finds exactly the same behavior in a simple model of a mandoline source [47, 48]. This leads us to believe that the results presented here have some validity outside the white-noise model. In fact, on this basis, we have presented a phenomenological extension of this picture to decay of 3D incompressible fluid turbulence [54].

Although the physics will be different, similar phenomena may occur in other nonequilibrium decay processes. During the preparation of this paper we became aware of a "two-scale" picture for phase-ordering by Cahn-Hilliard dynamics with power-law correlated (or fractally clustered) initial data [55]. From a renormalization group (RG) point of view, our results correspond to a case where some of the "fixed points", i.e. the scaling solutions, are unphysical. A careful study of this example using general methodologies such as RG may help to illuminate subtleties in their application.

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APPENDIX:

SELF-SIMILAR SCALAR SPECTRA FOR A BROWNIAN VELOCITY FIELD

In general, it is not possible to evaluate explicitly the Fourier transforms of the scaling functions, $\widehat{\Phi}(\kappa)$, not even in terms of standard special functions. This is not possible even for the equipartition case $\alpha = d$, in which case the Fourier transform is the spherically symmetric Lévy stable distribution with parameter γ in d dimensions. However, it is known that the stable distributions are calculable explicitly for the parameter value $\gamma = 1$, in which case they are the *d-dimensional Cauchy distributions.* Since then also $\zeta = 1$, this case corresponds to an advecting velocity field with Hölder exponent $H = \frac{1}{2}$, or a Brownian-type random field.

In fact, the Cauchy distribution of parameter β in d-dimensions is defined by the Fourier transform

$$G(\kappa;\beta) = \frac{1}{(2\pi)^{d/2}\kappa^{(d-2)/2}} \int_0^\infty e^{-\beta\rho} J_{(d-2)/2}(\kappa\rho)\, \rho^{d/2}\, d\rho \tag{5.1}$$

with the result

$$G(\kappa;\beta) = \frac{2}{(4\pi)^{d/2}} \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \cdot \frac{\beta}{(\beta^2+\kappa^2)^{\frac{1}{2}(d+1)}}. \tag{5.2}$$

This follows from [25],7.7.3(16) with the parameter choices $\gamma = \beta, \alpha = \kappa, \mu = \frac{1}{2}(d-2), \rho = \frac{1}{2}(d+2)$ there, because the hypergeometric series then simplifies to a binomial series for the function $\left(1+\frac{\kappa^2}{\beta^2}\right)^{-\frac{1}{2}(d+1)}$. We shall exploit this fact to evaluate the Fourier transforms $\widehat{\Phi}(\kappa)$ for $\gamma = 1$, at least for all of the exceptional cases $\alpha = d+\ell$, $\ell = 0, 1, 2, \ldots.$ We may use $G(\kappa;\beta)$ as a generating function in those cases, because

$$\widehat{\Phi}_\ell(\kappa) = \frac{1}{(d)_\ell \beta^{d-1}} \frac{\partial^\ell}{\partial\beta^\ell} \left[\beta^{d+\ell-1} G(\kappa;\beta)\right]\Big|_{\beta=1}, \tag{5.3}$$

as a consequence of (3.26).

The results obtained by use of this formula are, explicitly, for $\ell = 0$,

$$\widehat{\Phi}_0(\kappa) = \frac{2}{(4\pi)^{d/2}} \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \cdot \frac{1}{(1+\kappa^2)^{\frac{1}{2}(d+1)}}, \tag{5.4}$$

for $\ell = 1$,

$$\widehat{\Phi}_1(\kappa) = \frac{2}{(4\pi)^{d/2}} \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \cdot \frac{(d+1)\kappa^2}{d(1+\kappa^2)^{\frac{1}{2}(d+3)}}, \tag{5.5}$$

for $\ell = 2$,

$$\widehat{\Phi}_2(\kappa) = \frac{2}{(4\pi)^{d/2}} \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \cdot \frac{(d+2)\kappa^4 - \kappa^2}{d(1+\kappa^2)^{\frac{1}{2}(d+5)}}, \tag{5.6}$$

for $\ell = 3$,

$$\widehat{\Phi}_3(\kappa) = \frac{2}{(4\pi)^{d/2}} \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \cdot \frac{(d+3)[(d+2)\kappa^6 - 3\kappa^4]}{d(d+2)(1+\kappa^2)^{\frac{1}{2}(d+7)}}, \tag{5.7}$$

and for $\ell = 4$,

$$\widehat{\Phi}_4(\kappa) = \frac{2}{(4\pi)^{d/2}} \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \cdot \frac{(d+4)(d+2)\kappa^8 - 6(d+4)\kappa^6 + 3\kappa^4}{d(d+2)(1+\kappa^2)^{\frac{1}{2}(d+9)}}. \tag{5.8}$$

Of course, these transforms and those for all ℓ are analytic at $\kappa = 0$ with radius of convergence of the power series equal to 1. It may be easily checked that the coefficients are equal to the B_j given by (3.37). In particular, there is no singular contribution at $\kappa = 0$.

These explicit results illustrate several of the general conclusions reached in the text. For example, we see that $\widehat{\Phi}_0(\kappa), \widehat{\Phi}_1(\kappa)$ are everywhere nonnegative and that $\widehat{\Phi}_1(\kappa) = -\frac{1}{d}\kappa\frac{\partial \widehat{\Phi}_0}{\partial \kappa}(\kappa)$. Furthermore, in agreement with the result that realizability fails whenever $\nu > \gamma$, we see that the Fourier transforms for $\ell = 2, 3, 4$ are *not* everywhere positive. In the cases $\ell = 2, 3$ the coefficient of the dominant low-wavenumber power is negative, so realizability fails for the lowest wavenumber range. However, for $\ell = 4$, realizability fails despite the coefficient of the dominant low-wavenumber power being positive. In fact, the polynomial

$$P(\kappa^2) = (d+4)(d+2)\kappa^8 - 6(d+4)\kappa^6 + 3\kappa^4 \tag{5.9}$$

has two positive roots $\kappa_\pm^2 = \frac{3}{d+2}\left[1 \pm \sqrt{\frac{2(d+5)}{3(d+4)}}\,\right]$ and is negative in the interval (κ_-^2, κ_+^2).

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