# On the Dynamical Law of the Ginzburg-Landau Vortices on the Plane

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**Abstract**

We study the Ginzburg-Landau equation on the plane with initial data being the product of $n$ well-separated $+1$ vortices and spatially decaying perturbations. If the separation distances are $O(\varepsilon^{-1})$, $\varepsilon \ll 1$, we prove that the $n$ vortices do not move on the time scale $O(\varepsilon^{-2}\lambda_\varepsilon)$, $\lambda_\varepsilon = o(\log \frac{1}{\varepsilon})$; instead, they move on the time scale $O(\varepsilon^{-2}\log\frac{1}{\varepsilon})$ according to the law $\dot{x}_j = -\nabla_{x_j} W$, $W = -\sum_{l\neq j}\log|x_l - x_j|$, $x_j = (\xi_j, \eta_j) \in \mathbb{R}^2$, the location of the $j^{\text{th}}$ vortex. The main ingredients of our proof consist of estimating the large space behavior of solutions, a monotonicity inequality for the energy density of solutions, and energy comparisons. Combining these, we overcome the infinite energy difficulty of the planar vortices to establish the dynamical law. © 1999 John Wiley & Sons, Inc.

## 1 Introduction

We consider the Ginzburg-Landau (G-L) equation,

$$u_t = \Delta_x u + (1-|u|^2)u\,, \quad x \in \mathbb{R}^2\,, \tag{1.1}$$

where $u = u(t,x)$ is a complex-valued function defined for each $t > 0$ and $x = (\xi,\eta) \in \mathbb{R}^2$; $\Delta = \partial_{\xi\xi} + \partial_{\eta\eta}$ denotes the two-dimensional Laplacian. The G-L equation (1.1) admits vortex solutions of the form,

$$\Psi_n(x) = U_n(r)e^{in\theta}\,,\ \ n = \pm1, \pm2, \dots\,, \quad U_n(0) = 0\,, \quad U_n(+\infty) = 1\,, \tag{1.2}$$

where $(r,\theta)$ denote the polar coordinates on $\mathbb{R}^2$. The functions $\Psi_n(x)$ define complex planar vector fields, whose zeros are called vortices or defects. Among them, only the degree 1 vortices are dynamically stable; see Weinstein and Xin [10] for the whole-plane case, and Mironescu [7] and Lieb and Loss [3] for the related bounded domain case. Hence it makes sense to inquire about the motion law of the degree 1 vortices. Hereafter, we shall be concerned with degree $+1$ vortices, and use $U$ to denote the profile of such a vortex.

The G-L equation (1.1) defines a continuous-in-time deformation of the complex vector field $u(\cdot,x)$. So if the initial total winding number or degree at infinity is different from zero, one expects the dynamics to be organized around the motion of the zeros of $u(t,x)$. A description of the dynamics of an ensemble of spatially separated vortices is a fundamental problem. The systematic formal asymptotic

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study was initiated by Neu [8] and was further developed in the works of Pismen and Rubinstein [9] and E [2]. In these works, the regime of small $\varepsilon$, the ratio of vortex core size to the separation distance between vortices, is considered. For $\varepsilon$ small, a solution is sought in the form of a product of degree 1 vortices plus small error terms of higher order. In the small $\varepsilon$ limit, matched asymptotic analysis was used to derive a coupled system of ordinary differential equations for the centers of the widely separated vortices.

The early formal asymptotic results suggest the initial data

$$u(0,x) = \prod_{j=1}^{n} \Psi\left(x, \frac{x_j^{in}}{\varepsilon}\right) + \hat{u}(0,x)\,, \tag{1.3}$$

where

(H1) $\Psi(x, x_j^{in}/\varepsilon)$ denotes a $+1$ vortex located at $x_j^{in}/\varepsilon$ and has the form

$$\Psi\left(x, \frac{x_j^{in}}{\varepsilon}\right) = U\left(\left|x - \frac{x_j^{in}}{\varepsilon}\right|\right) e^{i\theta_j}\,, \quad \theta_j = \arg\left(x - \frac{x_j^{in}}{\varepsilon}\right), \quad x_j^{in} \sim O(1)\,. \tag{1.4}$$

(H2) $\hat{u}(0,x)$ is a bounded, twice continuously differentiable function and decays along with all its derivatives as fast as $O(|x|^{-\gamma})$ for some $\gamma > 2$ as $|x| \to \infty$.

(H3) $|u(0,x)| \le 1$.

The Cauchy problem (1.1)–(1.3) is globally well-posed in $C([0,+\infty); C_b(\mathbb{R}^2))$, $C_b$ the space of bounded continuous functions. Moreover, by the maximum principle, $|u(t,x)| \le 1$, $\forall t \ge 0$. Our main result is the following:

THEOREM 1.1 *Let $u = u(t,x,\varepsilon)$ be the solution to the initial value problem of the Ginzburg-Landau equation* (1.1) *on the whole plane with initial data* (1.3)*. If time $t$ is on the order $O(\varepsilon^{-2}\lambda_\varepsilon)$, $\lambda_\varepsilon = o(\log\frac{1}{\varepsilon})$, then the initial vortices at $x_j^{in}$ do not move as $\varepsilon \to 0$. If time $t$ is on the order $O(\varepsilon^{-2}\log\frac{1}{\varepsilon})$ and $t \le T\varepsilon^{-2}\log\frac{1}{\varepsilon}$ for a finite $T > 0$, the rescaled solution*

$$\tilde{u}_\varepsilon = \tilde{u}_\varepsilon(\tau, X) \equiv u\left(t = \tau\varepsilon^{-2}\log\frac{1}{\varepsilon},\, x = X\varepsilon^{-1}\right)$$

*converges as $\varepsilon \to 0$ for $\tau \in [0,T]$*

$$\tilde{u}_\varepsilon(\tau, X) \to e^{i\theta_0} \prod_{j=1}^{n} \frac{X - x_j(\tau)}{|X - x_j(\tau)|}\,, \tag{1.5}$$

*weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^2 \backslash \{x_1(\tau), x_2(\tau), \dots, x_n(\tau)\})$, where $\theta_0$ is a real constant.*

*Furthermore, the vortex locations $x_j(\tau)$ are continuous in $\tau$ and satisfy the dynamical law*

$$\frac{d}{d\tau} x_j = -\nabla_{x_j} W\,, \quad \forall \tau \in (0,T]\,, \tag{1.6}$$

*with* $x_j(0) = x_j^{in}$, *and*

$$W = W(x_1, x_2, \dots, x_n) = -\sum_{l \neq j} \log |x_l - x_j| , \quad x_j = (\xi_j, \eta_j) \in \mathbb{R}^2 .$$

As natural as it seems to obtain a similar result by making rigorous the formal matched asymptotics [2, 8, 9], such a task has defied many efforts and has never been carried out yet. Our approach here is motivated by recent work of F.-H. Lin [4, 5], on the dynamical law of vortices of the rescaled G-L equation on a bounded domain with prescribed Dirichlet data

$$\begin{aligned} &\frac{1}{\log \frac{1}{\varepsilon}} u_t = \Delta u + \varepsilon^{-2}(1 - |u|^2)u , \quad (t,x) \in R_+ \times \Omega , \\ &u(0,x) = g(x) , \quad x \in R_+ \times \partial\Omega , \end{aligned} \tag{1.7}$$

where the degree of $g : \partial\Omega \to \mathbb{S}^1$ is $n$. For similar initial data with $n$ degree $+1$ vortices, the dynamical law is (1.6), with $W$ the renormalized energy given in Bethuel, Brézis, and Hélein [1]. The basic tools in [4, 5] are energy comparison and the energy inequalities

$$\int_\Omega e_\varepsilon(u) \le n\pi \log \frac{1}{\varepsilon} + C(\Omega) , \tag{1.8}$$

$$\int_0^T \int_\Omega |u_t|^2 \le C(\Omega) \log \frac{1}{\varepsilon} , \quad \forall T > 0 , \tag{1.9}$$

where $e_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2$, and $C$ is a positive constant depending on the size of the domain. See also Lin [6] for the dynamical law under the Neumann boundary condition $u_\nu|_{\partial\Omega} = 0$.

An immediate difficulty in the whole-plane case is that the degree $+1$ vortex has infinite energy, even more so the $n$ degree $+1$ vortices. Hence a renormalization of the energy is necessary. The first cure is to consider the energy on a sufficiently large ball of radius $R$ so that outside of this ball the solution $u$ is not doing much; in particular, it has no vortices. This step requires an estimate of $u$ near spatial infinity for a given time interval. The energy of initial data $u(0,x)$ on such a ball $B_R$ is

$$\int_{B_R} e_\varepsilon(u(0,x)) = n\pi \log \frac{1}{\varepsilon} + n^2 \pi \log R + O(1) , \tag{1.10}$$

where the second term $n^2\pi \log R$ is due to the fact that from a large distance the $n$ degree 1 vortices look like a single degree $n$ vortex to leading order. Since $R$ is typically much larger than $\frac{1}{\varepsilon}$, we need to bound the energy locally (on subdomains of $B_R$) from above and below in order to locate the vortices and also to derive the key inequality

$$\int_0^T \int_{B_R} |u_t^\varepsilon|^2 \le C \log \frac{1}{\varepsilon} \tag{1.11}$$

for positive constants $T$ and $C$ independent of $R \to \infty$ and $\varepsilon \to 0$. Actually, we decompose $B_R$ into a union of two annuli and an inner ball of radius independent of $\varepsilon$. We use a monotonicity formula of the energy to show that there are no vortices in the outside annuli, and by a series of energy comparisons that (1.10) holds for $u(\tau, X)$ at any later time. As a result, (1.11) holds for any $R$. Then the dynamical law (1.6) follows from the weak convergence of $e_\varepsilon(u)/\log\frac{1}{\varepsilon}$ to a sum of delta functions located at $x_j(\tau)$ and the weak limit of $u^\varepsilon(\tau, X)$ of the form (1.5).

The rest of the paper is organized as follows: In Section 2 we analyze the large space behavior of solutions. We do this on the original G-L equation (1.1) and show that $u$ decays as inverse powers of $|x|$ as we move sufficiently far out ($|x| \geq R^*$, $R^*$ larger than any powers of $\varepsilon^{-1}$). We linearize about the product of the initial $n$ vortices and control the error terms based on the comparison of solution kernels with the free space heat kernel. In Section 3, we look at the rescaled G-L and derive the monotonicity formula. The monotonicity formula helps us define the two annular subdomains and the inner ball, on which detailed energy upper and lower bounds are derived. Results of Section 2 are employed to deduce the energy upper bound on $B_{R^*}$, which is needed for establishing the inequality (1.11). In Section 4 we calculate the first moment of the measure $e_\varepsilon(u)/\log\frac{1}{\varepsilon}$, derive the dynamical law (1.6), and complete the proof of Theorem 1.1.

## 2 Large Space Estimates of Solutions

In this section, we are concerned with the large space behavior of solutions to (1.1)–(1.3). Let us write

$$u(t,x) = (u_0(x) + v(t,x))e^{in\theta}, \tag{2.1}$$

where

$$u_0(x) = u_0(x,\varepsilon) = e^{-in\theta}\prod_{j=1}^{n}\Phi\left(x, \frac{x_j}{\varepsilon}\right), \quad \theta = \arg(x).$$

We calculate

$$\begin{aligned}\Delta u &= \Delta\big(u_0 e^{in\theta}\big) + \Delta\big(v e^{in\theta}\big)\\ &= e^{in\theta}\big(\Delta u_0 + 2nir^{-2}u_{0,\theta} - n^2 r^{-2} u_0\big)\\ &\quad + e^{in\theta}\big(\Delta v + 2nir^{-2}v_\theta - n^2 r^{-2} v\big)\end{aligned}$$

and

$$\big(1-|u|^2\big)u = \big(1-|u_0+v|^2\big)(u_0+v)e^{in\theta},$$

where we have used

$$\begin{aligned}e^{-in\theta}\Delta\big(ve^{in\theta}\big) &= \Delta v + 2nir^{-2}v_\theta - n^2r^{-2}v,\\ \nabla\theta &= r^{-1}(-\sin\theta, \cos\theta),\ r = |x|.\end{aligned}$$

Substituting (2.1) into (1.1) shows that $v$ satisfies the equation

$$\begin{aligned}v_t &= \Delta v + 2nir^{-2}v_\theta - n^2r^{-2}v + \big(1-|u_0+v|^2\big)(u_0+v)\\ &\quad + \Delta u_0 + 2nir^{-2}u_{0,\theta} - n^2r^{-2}u_0\end{aligned}$$

or

$$v_t = \Delta v + 2nir^{-2}v_\theta + \big(1-2|u_0|^2 - n^2r^{-2}\big)v - u_0^2 v^\star + \big(-2\,\mathrm{Re}\{u_0 v^\star\} - |v|^2\big)v - |v|^2 u_0 + \big(1-|u_0|^2\big)u_0 + \Delta u_0 + 2nir^{-2}u_{0,\theta} - n^2r^{-2}u_0 \tag{2.2}$$

with initial data $v(0,x) = \hat{u}(0,x)e^{-in\theta}$ where $x = re^{i\theta}$.

We shall consider (2.2) in the exterior of a disc of radius $R$, whose size is to be determined. Recall that the vortex profile $U$ has the properties

$$U = U(s) \sim as\left(1 - \frac{s^2}{8}\right), \quad s \to 0, \tag{2.3}$$

for some positive constant $a > 0$, and

$$U = U(s) \sim 1 - \frac{1}{2s^2} + O\big(s^{-4}\big), \quad s \to \infty. \tag{2.4}$$

For a large enough positive constant $L_0$, it follows that if $|x| = r \geq \frac{L_0}{\varepsilon} \equiv R_0$, then

$$u_0(x) = e^{-in\theta}\prod_{j=1}^{n}\left(1 - \frac{1}{2|x-\frac{x_j}{\varepsilon}|^2} + O\left(\left|x - \frac{x_j}{\varepsilon}\right|^{-4}\right)\right)e^{i\arg(x-\frac{x_j}{\varepsilon})}. \tag{2.5}$$

Note that

$$\left|x - \frac{x_j}{\varepsilon}\right|^{-2} = |x|^{-2}\Big(1 + |x|^{-2}\Big(-2\varepsilon^{-1}x\cdot x_j + \varepsilon^{-2}|x_j|^2\Big)\Big)^{-1} = \frac{1}{|x|^2}\left(1 + O\left(\frac{1}{\varepsilon|x|}\right)\right) = \frac{1}{|x|^2} + O\left(\frac{1}{\varepsilon|x|^3}\right). \tag{2.6}$$

So

$$|u_0(x)| = \prod_{j=1}^{n}\left(1 - \frac{1}{2|x|^2} + O\left(\frac{1}{\varepsilon|x|^3}\right)\right) = 1 - \frac{n}{2|x|^2} + O\left(\frac{1}{\varepsilon|x|^3}\right) \tag{2.7}$$

if $|x| \geq R_0$. It follows that

$$1 - |u_0|^2 - \frac{n^2}{r^2} = 1 - \left(1 - \frac{n}{2|x|^2} + O\left(\frac{1}{\varepsilon|x|^3}\right)\right)^2 - \frac{n^2}{r^2} = 1 - \left(1 - \frac{n}{r^2} + O\left(\frac{1}{\varepsilon r^3}\right)\right) - \frac{n^2}{r^2} = O\left(\frac{1}{\varepsilon r^3}\right) + \frac{n-n^2}{r^2}. \tag{2.8}$$

We use (2.7) to simplify $u_0(x)$ into

(2.9)
$$u_0(x) = \left(1 - \frac{n}{2|x|^2} + O\left(\frac{1}{\varepsilon|x|^3}\right)\right) \exp\left\{ i \left( \sum_{j=1}^{n} \arg\left(x - \frac{x_j}{\varepsilon}\right) - n \arg(x)\right)\right\},$$

if $|x| \geq R_0$. A direct calculation shows ($x = (\xi, \eta)$, $x_j = (\xi_j, \eta_j)$)

$$\begin{aligned}
\arg\left(x - \frac{x_j}{\varepsilon}\right) - \arg(x) &= \arctan \frac{\eta - \frac{\eta_j}{\varepsilon}}{\xi - \frac{\xi_j}{\varepsilon}} - \arctan \frac{\eta}{\xi} \\
&= \arctan \frac{1}{\varepsilon|x|} \frac{\xi_j \frac{\eta}{|x|} - \eta_j \frac{\xi}{|x|}}{1 - \frac{x \cdot x_j}{\varepsilon|x|^2}} \qquad (2.10) \\
&= \frac{1}{\varepsilon|x|}\left(\xi_j \frac{\eta}{|x|} - \eta_j \frac{\xi}{|x|}\right) + O\left(\frac{1}{\varepsilon^2|x|^2}\right)
\end{aligned}$$

if $|x| \geq R_0$, implying

$$\begin{aligned}
\sum_{j=1}^{n} \arg\left(x - \frac{x_j}{\varepsilon}\right) - n \arg(x) &= \frac{1}{\varepsilon|x|}\left(\sum_{j=1}^{n} \xi_j \frac{\eta}{|x|} - \eta_j \frac{\xi}{|x|}\right) + O\left(\frac{1}{\varepsilon^2|x|^2}\right) \qquad (2.11) \\
&= \frac{1}{\varepsilon|x|} F(\theta) + O\left(\frac{1}{\varepsilon^2|x|^2}\right),
\end{aligned}$$

where $F$ is smooth in $\theta = \arg(x)$. By (2.9) and (2.11) we have now a concise expression

$$u_0(x) = \left(1 - \frac{n}{2|x|^2} + O\left(\frac{1}{\varepsilon|x|^3}\right)\right) \exp\left\{ i \left( \frac{F(\theta)}{\varepsilon|x|} + O\left(\frac{1}{\varepsilon^2|x|^2}\right)\right)\right\} \qquad (2.12)$$

for $|x| \geq R_0$. It is straightforward to verify that

$$u_{0,\theta} = O\left(\frac{1}{\varepsilon r}\right), \quad \Delta u_0 = O\left(\frac{1}{\varepsilon r^3}\right),$$

which together with (2.8) shows that the inhomogeneous term in (2.2)

$$\begin{aligned}
\left(1 - |u_0|^2\right)u_0 + \Delta u_0 + 2nir^{-2}u_{0,\theta} - n^2 r^{-2} u_0 &= O\left(\frac{1}{\varepsilon r^3}\right) + \frac{n - n^2}{r^2} \qquad (2.13) \\
&= O\left(r^{-2}\right).
\end{aligned}$$

Upon letting $v = \alpha + i\beta$, (2.2) becomes the real system ($r \geq R_0$)

$$\begin{aligned}
\alpha_t &= \Delta\alpha - \frac{2n}{r^2}\beta_\theta + \left(-\frac{n^2}{r^2} + 1 - 3|u_0|^2\right)\alpha \qquad (2.14) \\
&\quad - \left(2\,\mathrm{Re}\{u_0(\alpha - i\beta)\} + \alpha^2 + \beta^2\right)\alpha - \left(\alpha^2 + \beta^2\right)\mathrm{Re}\{u_0\} + O\left(\frac{1}{r^2}\right),
\end{aligned}$$

$$\begin{aligned}\beta_t &= \Delta\beta + \frac{2n}{r^2}\alpha_\theta + \left(-\frac{n^2}{r^2} + 1 - |u_0|^2\right)\beta \\ &\quad - \left(2\,\mathrm{Re}\{u_0(\alpha - i\beta)\} + \alpha^2 + \beta^2\right)\beta - \left(\alpha^2+\beta^2\right)\mathrm{Im}\{u_0\} + O\left(\frac{1}{r^2}\right).\end{aligned} \tag{2.15}$$

Let us now consider the initial boundary value problem for (2.14)–(2.15) with boundary data

$$(\alpha + i\beta)(t, R_0) = u(t, R_0)e^{-in\theta} - u_0(R_0) \equiv (\alpha_1 + i\beta_1)(t)$$

and initial data $(\alpha + i\beta)(0,x) = \hat{u}(0,x)e^{-in\theta}$. By the maximum principle, $(\alpha_1^2 + \beta_1^2) \le 4$ for any $t \ge 0$. It is convenient to subtract the nonzero boundary data off. There exists a smooth function $b(t,x) = b_1(t,x) + ib_2(t,x)$, $b_1$, $b_2$ real, such that $b_1(t,|x| = R_0) = \alpha_1(t)$, $b_2(t,|x| = R_0) = \beta_1(t)$, and that $b(t,x)$ is supported in the annulus $R_0 \le |x| \le R_0 + 1$. Now setting $\alpha = \tilde{\alpha} + b_1$, $\beta = \tilde{\beta} + b_2$, (2.14)–(2.15) give

(2.16)

$$\begin{aligned}\tilde{\alpha}_t &= \Delta\tilde{\alpha} - \frac{2n}{r^2}\tilde{\beta}_\theta + \left(-\frac{n^2}{r^2} + 1 - 3|u_0|^2\right)\tilde{\alpha} \\ &\quad - \left(2\,\mathrm{Re}\{u_0(\tilde{\alpha} - i\tilde{\beta})\} + \tilde{\alpha}^2 + \tilde{\beta}^2\right)\tilde{\alpha} - \left(\tilde{\alpha}^2+\tilde{\beta}^2\right)\mathrm{Re}\{u_0\} + O\left(\frac{1}{r^2}\right) + e_1,\end{aligned}$$

(2.17)

$$\begin{aligned}\tilde{\beta}_t &= \Delta\tilde{\beta} + \frac{2n}{r^2}\tilde{\alpha}_\theta + \left(-\frac{n^2}{r^2} + 1 - |u_0|^2\right)\tilde{\beta} \\ &\quad - \left(2\,\mathrm{Re}\{u_0(\tilde{\alpha} - i\tilde{\beta})\} + \tilde{\alpha}^2 + \tilde{\beta}^2\right)\tilde{\beta} - \left(\tilde{\alpha}^2+\tilde{\beta}^2\right)\mathrm{Im}\{u_0\} + O\left(\frac{1}{r^2}\right) + e_2,\end{aligned}$$

where $e_1$ and $e_2$ depend on $b$ and are zero if $r > R_0 + 1$. The initial data become $(\tilde{\alpha},\tilde{\beta})(0,x) = \hat{u}(0,x)e^{-in\theta} - b(0,x)$. Note that $(\tilde{\alpha},\tilde{\beta})(t,x) = (\alpha,\beta)(t,x)$ if $|x| > R_0 + 1$. Skipping the tildes of (2.16)–(2.17) and using (2.7) and (2.8), we write (2.16)–(2.17) as

(2.18)

$$\begin{aligned}\alpha_t &= \Delta\alpha - \frac{2n}{r^2}\beta_\theta + \left(-2 + \frac{3n - n^2}{r^2}\right)\alpha \\ &\quad - \left(2\,\mathrm{Re}\{u_0(\alpha - i\beta)\} + \alpha^2 + \beta^2\right)\alpha - \left(\alpha^2+\beta^2\right)\mathrm{Re}\{u_0\} + O\left(\frac{1}{r^2}\right) + e_1,\end{aligned}$$

(2.19)

$$\begin{aligned}\beta_t &= \Delta\beta + \frac{2n}{r^2}\alpha_\theta \\ &\quad - \left(2\,\mathrm{Re}\{u_0(\alpha - i\beta)\} + \alpha^2 + \beta^2\right)\beta - \left(\alpha^2+\beta^2\right)\mathrm{Im}\{u_0\} + O\left(\frac{1}{r^2}\right) + e_2.\end{aligned}$$

$\|r^{-1}(\beta_\theta,\alpha_\theta)\|_\infty \le \|\nabla(\alpha,\beta)\|_\infty \le C$ for all $t \ge 0$, by the maximum principle and parabolic regularity. Let $\Omega_{R_0} = \mathbb{R}^2 \backslash B(0,R_0)$. From (2.18), we get

$$\begin{aligned}\alpha(t,x) = &\int_{\Omega_0} K_1(t,x,y)\alpha(0,y)dy\\ &+\int_0^t ds\int_{\Omega_0}\Bigg[\Bigg(-\frac{2n}{|y|^2}\beta_\theta - \big(2\,\mathrm{Re}\{u_0(\alpha - i\beta)\} + \alpha^2+\beta^2\big)\alpha\\ &\qquad\qquad - \big(\alpha^2+\beta^2\big)\,\mathrm{Re}\{u_0\}\Bigg) + O\left(\frac{1}{|y|^2}\right) + e_1\Bigg]\\ &\times(s,y)K_1(t-s,x,y)dy\end{aligned}$$

or

$$\begin{aligned}|\alpha(t,x)| \le &\int_{\Omega_0} K_1(t,x,y)|\alpha(0,y)|dy + 9\int_0^t ds\int_{\Omega_0}\big(\alpha^2+\beta^2\big)K_1\,dy\\ &+C\int_0^t ds\int_{\Omega_0}\left(|y|^{-1} + O\left(\frac{1}{|y|^2}\right) + e_1\right)K_1\,dy\,,\end{aligned}\tag{2.20}$$

where $K_1$ is the solution kernel of the linear exterior parabolic equation

$$\alpha_t = \Delta\alpha + \left(-2 + \frac{3n-n^2}{r^2}\right)\alpha\,,\quad \alpha(t,|x|=R_0)=0\,,\quad \alpha(0,x)=\delta(y)\,.$$

By the comparison principle

$$0 \le K_1(t,x,y) \le K(t,x,y)e^{-t}\,,\quad \forall(t,x,y)\,,\tag{2.21}$$

where $K$ is the two-dimensional heat kernel on $\mathbb{R}^2$. It follows from (2.20) that

$$\begin{aligned}|\alpha(t,x)| \le &\, e^{-t}\int_{\Omega_0} K(t,x,y)|\alpha(0,y)|\,dy\\ &+9\int_0^t ds\int_{\Omega_0} e^{s-t}\big(\alpha^2+\beta^2\big)K\,dy\\ &+C\int_0^t ds\int_{\Omega_0} e^{s-t}\left[|y|^{-1} + O\left(\frac{1}{|y|^2}\right) + e_1\right]K\,dy\,.\end{aligned}\tag{2.22}$$

Let us show two lemmas before proceeding with estimating (2.22).

LEMMA 2.1 *If $v_0(x)$ is bounded and decays as $|x|\to\infty$, then*

$$\left|\int_{\Omega_0} v_0(y)K(t,x,y)dy\right| \le \sup_{\{y:|y-x|\le|x|/2\}\cap\Omega_0}|v_0(y)| + C\|v_0\|_\infty e^{-|x|^2/4t}\tag{2.23}$$

*for a positive constant $C$.*

PROOF:

$$\begin{aligned}\left|\int_{\Omega_0} v_0(y)K(t,x,y)dy\right| \le &\left|\int_{\{y:|y-x|\le|x|/2\}\cap\Omega_0} v_0(y)K(t,x,y)dy\right|\\ &+\left|\int_{\{y:|y-x|\ge|x|/2\}\cap\Omega_0} v_0(y)K(t,x,y)dy\right|\end{aligned}$$

$$\begin{aligned}&\le \sup_{\{y:|y-x|\le|x|/2\}\cap\Omega_0}|v_0(y)|\\&\quad + C\int_{\{y':|y'|\ge|x|/2\}} v_0(x+y')t^{-1}e^{-|y'|^2/4t}\,dy'\\&\le \sup_{\{y:|y-x|\le|x|/2\}\cap\Omega_0}|v_0(y)|\\&\quad + C\|v_0\|_\infty\int_{\{y':|y'|\ge|x|/2\}} t^{-1}e^{-|y'|^2/4t}\,dy'\,.\end{aligned}$$

Changing variable $\xi = y't^{-1/2}/2$, we find the last integral is

$$4\int_{|\xi|\ge|x|t^{-1/2}/2} e^{-|\xi|^2}\,d\xi = 16\pi e^{-|x|^2/4t}\,,$$

hence

$$\left|\int_{\Omega_0} v_0(y)K(t,x,y)dy\right| \le \sup_{\{y:|y-x|\le|x|/2\}\cap\Omega_0}|v_0(y)| + C\|v_0\|_\infty e^{-|x|^2/4t}\,.$$

The proof is complete. □

LEMMA 2.2 *Let $f(t,x)$ be a bounded and spatially decaying function. Then*

$$\begin{aligned}\left|\int_0^t ds\int_{\Omega_0} f(s,y)K(t-s,x,y)dy\right| &\le t\sup_{s\in[0,t],\{y:|y-x|\le|x|/2\}\cap\Omega_0}|f(s,y)|\\&\quad + Cte^{-|x|^2/4t}\sup_{s\in[0,t],y\in\Omega_0}|f(s,y)|\,.\end{aligned}\tag{2.24}$$

PROOF:

$$\begin{aligned}&\left|\int_0^t ds\int_{\Omega_0} f(s,y)K(t-s,x,y)dy\right|\\&\quad\le \left|\int_0^t\int_{\{y:|y-x|\le|x|/2\}\cap\Omega_0}\cdots\right| + \left|\int_0^t\int_{\{y:|y-x|>|x|/2\}\cap\Omega_0}\cdots\right|\\&\quad\le \sup_{s\in[0,t],\{y:|y-x|\le|x|/2\}\cap\Omega_0}|f(s,y)|\int_0^t ds\int_{\{y:|y-x|\le|x|/2\}}(4\pi)^{-1}(t-s)^{-1}\\&\qquad\times e^{-|x-y|^2/4(t-s)}\,dy\\&\qquad + C\sup_{s\in[0,t],y\in\Omega_0}|f(s,y)|\int_0^t ds\int_{\{y:|y-x|>|x|/2\}}(t-s)^{-1}e^{-|x-y|^2/4(t-s)}\,dy\\&\quad\le t\sup_{s\in[0,t],\{y:|y-x|\le|x|/2\}\cap\Omega_0}|f(s,y)| + C\sup_{s\in[0,t],y\in\Omega_0}|f(s,y)|\int_0^t ds\,e^{-|x|^2/4(t-s)}\\&\quad\le t\sup_{s\in[0,t],\{y:|y-x|\le|x|/2\}\cap\Omega_0}|f(s,y)| + Cte^{-|x|^2/4t}\sup_{s\in[0,t],x\in\Omega_0}|f(s,y)|\,.\end{aligned}$$

The proof is complete. □

It follows from (2.22) and the lemmas that for $t \in (0,1)$

$$|\alpha(t,x)| \le \sup_{\{|y-x|\le|x|/2\}\cap\Omega_0} |\alpha(0,y)| + C\|\alpha(0,y)\|_\infty e^{-|x|^2/4t} + Ct\left[|x|^{-1} + \frac{1}{|x|^2} + \sup_{s\in[0,t],\{|y-x|\le|x|/2\}\cap\Omega_0} |e_1| + e^{-|x|^2/4t}\right] + 9 \sup_{s\in[0,t],\{|y-x|\le|x|/2\}\cap\Omega_0} \left(\alpha^2+\beta^2\right). \tag{2.25}$$

Similarly, equation (2.19) gives

$$\begin{aligned} |\beta(t,x)| &\le K\star|\beta(0,x)| + \int_0^t ds \int_{\Omega_0} \left[\frac{2n}{|y|^2}\alpha_\theta - \left(2\,\mathrm{Re}\{u_0(\alpha-i\beta)\}+\alpha^2+\beta^2\right)\beta \right. \\ &\qquad \left. - \left(\alpha^2+\beta^2\right)\mathrm{Im}\{u_0\} + O\left(\frac{1}{|y|^2}\right) + e_2\right] \\ &\qquad \times K(t-s,x,y)dy \\ &\le \sup_{\{|y-x|\le|x|/2\}\cap\Omega_0} |\beta(0,y)| + Ce^{-|x|^2/4t} \\ &\qquad + C \sup_{\{|y-x|\le|x|/2\}\cap\Omega_0} |y|^{-1} + 9t \sup_{s\in[0,t],\{|y-x|\le|x|/2\}\cap\Omega_0} \left(\alpha^2+\beta^2\right) \\ &\qquad + Ct \sup_{\{|y-x|\le|x|/2\}\cap\Omega_0} \frac{1}{|y|^2} + Ct \sup_{s\in[0,t],\{|y-x|\le|x|/2\}\cap\Omega_0} |e_2| + Cte^{-|x|^2/4t}, \end{aligned} \tag{2.26}$$

by parabolic regularity, and so

$$\begin{aligned} |\beta(t,x)| \le{}& \sup_{\{|y-x|\le|x|/2\}\cap\Omega_0} |\beta(0,y)| + C \sup_{\{|y-x|\le|x|/2\}\cap\Omega_0} |y|^{-1} \\ &+ 9t \sup_{s\in[0,t],\{|y-x|\le|x|/2\}\cap\Omega_0} \left(\alpha^2+\beta^2\right) + C \sup_{\{|y-x|\le|x|/2\}\cap\Omega_0} \frac{1}{|y|^2} \\ &+ Ct \sup_{s\in[0,t],\{|y-x|\le|x|/2\}\cap\Omega_0} |e_2| + C(1+t)e^{-|x|^2/4t}. \end{aligned} \tag{2.27}$$

Choose $R_1 \ge R_0+1$ such that $e_1(t,x) = e_2(t,x) = 0$ if $|x| \ge R_1$ for all $t \ge 0$.

Let us consider (2.25) and (2.27) with $|x| \ge 2R_1+2$, and define the norm

$$\|\alpha\| = \sup_{s\in[0,t]} \sup_{|x|\ge R_1} \left(|x|R_1^{-1}\right)^p |\alpha(t,x)|, \quad p \in \left(\tfrac{2}{3},1\right). \tag{2.28}$$

It follows by multiplying $(|x|R_1^{-1})^p$ to (2.25) and (2.27), taking the supremum over $s \in [0,t]$, and $|x| \geq 2R_1 + 2$ that

(2.29)
$$\sup_{s\in[0,t],\{|x|\geq 2R_1+2\}} \left(|x|R_1^{-1}\right)^p |\alpha(s,x)| \leq 2^p\|\alpha(0,x)\| + C\|\alpha(0,x)\|_\infty \|e^{-|x|^2/4t}\| + CR_1^{-1} + CR_1^{-2} + 9t\left(\|\alpha\|^2 + \|\beta\|^2\right) + C\|e^{-x^2/4t}\|$$

and

(2.30)
$$\sup_{s\in[0,t],\{|x|\geq 2R_1+2\}} \left(|x|R_1^{-1}\right)^p |\beta(s,x)| \leq 2^p\|\beta(0,x)\| + CR_1^{-1} + CR_1^{-2} + 9t\left(\|\alpha\|^2 + \|\beta\|^2\right) + C\|(1+t)e^{-x^2/4t}\|\,.$$

We have from (2.29)–(2.30) that

$$\begin{aligned}\|(\alpha,\beta)\| \equiv \|\alpha\| + \|\beta\| \leq &\sup_{s\in[0,t],\{R_1\leq|x|\leq 2R_1+2\}} \left(R_1^{-1}|x|\right)^p (|\alpha(s,x)| + |\beta(s,x)|) \\ &+ \sup_{s\in[0,t],\{|x|\geq 2R_1+2\}} \left(R_1^{-1}|x|\right)^p (|\alpha(s,x)| + |\beta(s,x)|) \\ \leq &\, 4(2^p+1) + 2^p\|(\alpha,\beta)(0,x)\| + 18t\left(\|\alpha\|^2 + \|\beta\|^2\right) \\ &+ CR_1^{-1} + CR_1^{-2} + C\|e^{-|x|^2/4t}\| \\ \leq &\, 4(2^p+1) + 2^p\|(\alpha,\beta)(0,x)\| + 18t\left(\|\alpha\|^2 + \|\beta\|^2\right) \\ &+ O\left(R_1^{-1}\right),\end{aligned}$$

implying

$$\|(\alpha,\beta)\| \leq 4(2^p+2) + 2^p\|(\alpha,\beta)(0,x)\| + 18t\|(\alpha,\beta)\|^2\,.$$

So if $0 < t < \frac{1}{72(4(2^p+2)+2^p\|(\alpha,\beta)(0,x)\|)} \equiv t_0$,

(2.31)
$$\|(\alpha,\beta)\| \leq 2(4(2^p+2) + 2^p\|(\alpha,\beta)(0,x)\|)\,,$$

which yields

(2.32)
$$\sup_{s\in[0,t]} |(\alpha,\beta)|(s,x) \leq \left(R_1^{-1}|x|\right)^{-p} 2(4(2^p+2) + 2^p\|(\alpha,\beta)(0,x)\|)\,,$$

if $|x| \geq R_1$, $0 \leq t \leq t_0/2 \equiv t_1$.

Let $R_{2,0} = R_1^q$, $q \gg 2$ to be chosen. Then (2.32) implies that if $|x| \geq R_{2,0}$,

(2.33)

$$\sup_{s\in[0,t]} |(\alpha,\beta)(s,x)| \leq \left(|x|^{(q-2)/q}|x|^{1/q}|x|^{1/q}R_1^{-1}\right)^{-p} 2(4(2^p+2)+2^p\|(\alpha,\beta)(0,x)\|)$$
$$\leq |x|^{-p(q-2)/q}R_1^{-p}2(4(2^p+2)+2^p\|(\alpha,\beta)(0,x)\|)$$
$$\leq |x|^{-p(q-2)/q}\|(\alpha,\beta)(0,x)\|\,.$$

Letting $R_2 \geq R_{2,0}+1$ and introducing the tilde functions, we repeat the previous analysis on (2.18) and (2.19) with $R_{2,0}$ replacing $R_0$, $R_2$ replacing $R_1$, $[t_1, 2t_1]$ replacing $[0,t_1]$, $(\alpha,\beta)(t_1)$ replacing $(\alpha_0,\beta_0)$, and $p(q-2)/q$ in place of $p$. The result is

$$\sup_{s\in[t_1,2t_1]} |(\alpha,\beta)|(s,x) \leq \left(R_2^{-1}|x|\right)^{-p(q-2)/q} 2\left[4\left(2^{p(q-2)/q}+2\right)+2^{p(q-2)/q}\|(\alpha_0,\beta_0\|\right] \tag{2.34}$$

if $|x| \geq R_2$. If we choose $R_{3,0} = R_2^q$, then (2.34) implies

$$\sup_{s\in[t_1,2t_1]} |(\alpha,\beta)|(s,x) \leq |x|^{-p(\frac{q-2}{q})^2}\|(\alpha_0,\beta_0)\| \tag{2.35}$$

if $|x| \geq R_{3,0}$. Iterating the above procedure $m$ times such that $T \in [(m-1)t_1, mt_1]$, then

$$\sup_{s\in[(m-1)t_1,mt_1]} |(\alpha,\beta)|(s,x) \leq |x|^{-p(\frac{q-2}{q})^m}\|(\alpha_0,\beta_0)\| \tag{2.36}$$

if $|x| \geq R_{m+1,0} = R_m^q$. We impose the condition that

$$p\left(\frac{q-2}{q}\right)^m \geq p', \quad p' \in [\tfrac{2}{3}, p)\,,$$

which holds if

$$p\left(\frac{q-2}{q}\right)^{\frac{T}{t_1}+1} \geq p'\,,$$

or

$$q \geq \frac{2}{1-\exp\left\{\frac{\log\frac{p'}{p}}{\frac{T}{t_1}+1}\right\}} \sim \frac{-2}{\log\frac{p'}{p}}\left(\frac{T}{t_1}+1\right) + \text{higher order terms} \tag{2.37}$$

Now for any given $T \geq 1$, selecting $q > 0$ as in (2.37), with $m$ such that $T \in [(m-1)t_1, mt_1]$, we have

$$\sup_{s\in[0,T]} |(\alpha,\beta)|(x) \leq |x|^{-p'}\|(\alpha_0,\beta_0)\|\,, \quad p' \in [\tfrac{2}{3}, 1)\,, \tag{2.38}$$

if $|x| \geq R_m^q$. By local existence and parabolic regularity, if $|x| \geq R_m^q + 1$, then

(2.39)
$$\sup_{s\in[0,T]} |(\alpha,\beta,\nabla(\alpha,\beta),\Delta(\alpha,\beta))|(s,x) \leq C|x|^{-p'}\|(\alpha_0,\beta_0,\nabla(\alpha_0,\beta_0),\Delta(\alpha_0,\beta_0))\|$$

with $p' \in [\frac{2}{3},1)$. Finally, we can use (2.38) and (2.39) in (2.14)–(2.15) to improve the decay exponent $p'$ to any number $\leq 2$. Summarizing, we have proven the following:

PROPOSITION 2.3 *Consider* (2.14)–(2.15) *and fix a* $p' \in [\frac{2}{3},2]$. *Then for any* $T > 0$, $\exists R^* = R^*(T,\varepsilon)$ *such that if* $|x| \geq R^*$, $R^*$ *is larger than any powers of* $\varepsilon^{-1}$ *if* $T = \varepsilon^{-2}$ *or* $\varepsilon^{-2}\log\varepsilon^{-1}$, *the inequalities* (2.38) *and* (2.39) *hold with* $p' \in [\frac{2}{3},2]$ *with a positive constant* $C$ *uniformly in* $T$ *and* $\varepsilon$.

## 3 Energy Comparison on the Rescaled Solutions

In this section, we scale G-L solutions in space and time $x \to \frac{X}{\varepsilon}$, $t \to \varepsilon^{-2}\lambda_\varepsilon\tau$, with $\lambda_\varepsilon = 1$ or $\log\varepsilon^{-1}$. The results of last section easily translate into analogous ones in $(\tau,X)$. If $\lambda_\varepsilon = 1$, the rescaled equation is

$$u_\tau = \Delta u + \varepsilon^{-2}(1-|u|^2)u\,, \qquad u(0,X) = u_0(X,\varepsilon)\,, \tag{3.1}$$

where $u_0(X,\varepsilon)$ converges in $L^2_{\text{loc}}(\mathbb{R}^2)$ to $\prod_{j=1}^n (X - x_j^{in})/|X - x_j^{in}|$. This is seen from (1.3)–(1.4). Each factor $\Psi(X/\varepsilon, x_j^{in}/\varepsilon)$ is bounded by one in absolute value and converges in $L^2_{\text{loc}}(\mathbb{R}^2)$ to $e^{i\theta_j}$, since the amplitude goes to 1 pointwise except at $X = x_j^{in}$. The remainder $\hat{u}(0,\frac{X}{\varepsilon})$ clearly goes to 0 in $L^2_{\text{loc}}(\mathbb{R}^2)$. The result is the desired product consisting of all initial vortex phases.

Let us define

$$\Phi(R) = \int_{\mathbb{R}^2} e_\varepsilon(u)\big(-R^2,X\big)\exp\left\{-\frac{|X-x_0|^2}{4R^2}\right\}dX\,, \tag{3.2}$$

where $R > 0$, $x_0$ is any point on the plane, and

$$e_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\big(1-|u|^2\big)^2\,,$$

the energy density. We show the monotonicity inequality

$$\Phi(R) \leq \Phi(R')\,, \quad \forall R < R'\,. \tag{3.3}$$

Calculating $\frac{d}{dR}\Phi(R)$, we have, using (3.1) and integration by parts,

(3.4)
$$
\begin{aligned}
\frac{d}{dR}\Phi(R) &= \int (-2R) e_{\varepsilon,t} \exp\left\{-\frac{|X-x_0|^2}{4R^2}\right\} + \frac{|X-x_0|^2}{2R^3} e_\varepsilon(u) \exp\left\{-\frac{|X-x_0|^2}{4R^2}\right\} \\
&= \int (-2R) \exp\left\{-\frac{|X-x_0|^2}{4R^2}\right\} \left[\nabla u \cdot \nabla u_\tau + \varepsilon^{-2}\left(1-|u|^2\right)\left(-u u_\tau\right)\right] \\
&\quad + \int \frac{|X-x_0|^2}{2R^3} e_\varepsilon(u) \exp\left\{-\frac{|X-x_0|^2}{4R^2}\right\} \\
&= \int 2R|u_\tau|^2 \exp\left\{-\frac{|X-x_0|^2}{4R^2}\right\} + \int (-2R) u_\tau \exp\left\{-\frac{|X-x_0|^2}{4R^2}\right\} \\
&\quad \times \frac{X-x_0}{2R^2} \cdot \nabla u + \int \frac{|X-x_0|^2}{2R^3} e_\varepsilon(u) \exp\left\{-\frac{|X-x_0|^2}{4R^2}\right\} \\
&\geq \int \left[2R|u_\tau|^2 - R^{-1} u_\tau (X-x_0)\cdot \nabla u + \frac{|X-x_0|^2}{4R^3}|\nabla u|^2\right] \\
&\quad \times \exp\left\{-\frac{|X-x_0|^2}{4R^2}\right\} \\
&\geq \int \left[\sqrt{2}R^{-1}|u_\tau||\nabla u||X-x_0| - R^{-1} u_\tau (X-x_0)\cdot \nabla u\right] \\
&\quad \times \exp\left\{-\frac{|X-x_0|^2}{4R^2}\right\} \geq 0\,,
\end{aligned}
$$

which gives (3.3).

From now on, let us translate $\tau \to \tau - \tau_0^2$, with a fixed constant $\tau_0 > 0$, so that the initial data $u_0$ is now at the time $\tau = -\tau_0^2$. We have from (3.2) and (3.3)

$$
\int e_\varepsilon(u)\left(-\tau_1^2, X\right) \exp\left\{-\frac{|X-x_0|^2}{4\tau_1^2}\right\} \leq \int e_\varepsilon(u_0) \exp\left\{-\frac{|X-x_0|^2}{4\tau_0^2}\right\} \tag{3.5}
$$

for any $\tau_1 \in (0, \tau_0)$. Here $\tau = -\tau_1^2$ is a later time.

Now we study the energy distribution on a series of annular domains that partition the ball $B_{R_4} = B(0, R_4)$, where $R_4 = 2\varepsilon R^*$, $R^* = R^*(T, \varepsilon)$ being the radius in Proposition 2.1 so that outside of $B(0, R_4)$, the solution $u^\varepsilon$ essentially behaves as $e^{in\theta}$ plus a small perturbation. We have the factor $\varepsilon$ because we are measuring in the scaled variable $X$. Let us define $R_3 = 2\tau_0\sqrt{\log\log(1/\varepsilon)}$ and choose $R_2$ to be larger than $2L_0$ so that expression (2.12) is valid. Note that the radius of validity $R_0$ of (2.12) has to be multiplied by $\varepsilon$ to become $L_0$ due to the scaling $x \to \frac{X}{\varepsilon}$. Finally, let $R_1$ be a finite number large enough so that $B(0, R_1)$ encloses all initial vortices, $R_1 > \sum_{j=1}^n |x_j^{in}| + 1$, and $R_1 < R_2$.

Next we define $T_e^\varepsilon$ to be the first time a vortex (say $X_j^\varepsilon(\tau)$) exits the ball $B(X_j^\varepsilon, r_0)$, where $r_0$ is half the size of the minimum of the distances between all initial vortices and between initial vortices and $\partial B_{R_1}$. The $T_e^\varepsilon$ is well-defined because vortices $X_j^\varepsilon$

move continuously, and its size will be clear from the coming energy estimates. The time interval we shall be concerned with is $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$, with the time spent being

$$T_s^\varepsilon = \min\left(T_e^\varepsilon, \frac{\tau_0^2}{2}\right). \tag{3.6}$$

We first observe that for small $\varepsilon$

$$\int_{B_{R_1}} e_\varepsilon(u_0)dX = n\pi \log\frac{1}{\varepsilon} + C_0 + o(1). \tag{3.7}$$

In fact, we can find $r_j' < r_j$, all $O(1)$, so that

$$\bigcup_{j=1}^n B\big(x_j^{in}, r_j'\big) \subset B_{R_1} \subset \bigcup_{j=1}^n B\big(x_j^{in}, r_j\big), \quad x_j^{in} \notin \bigcup_{j'\neq j} B\big(x_{j'}^{in}, r_{j'}\big).$$

On each $B(x_j^{in}, r_j')$ or $B(x_j^{in}, r_j)$, $u_0$ is well approximated by a single one vortex with core size $O(\varepsilon)$ located at $x_j^{in}$. The energy of such a single vortex is $\pi \log\frac{1}{\varepsilon} + O(1)$ by a direct calculation. In fact, the $\pi\log\frac{1}{\varepsilon}$ contribution comes from the integral of $\frac{1}{2}|\nabla u_0|^2$, in particular, the derivative with respect to the vortex phase. The contribution from the derivative of the vortex amplitude and the remaining part of the energy functional give $O(1)$. Summing up all the contributions yields (3.7). Over the annulus $R_1 \le |X| \le R_2$, $u_0$ has finite $H^1$ norm uniformly in $\varepsilon$; hence

$$\int_{B_{R_2}\setminus B_{R_1}} e_\varepsilon(u_0)dX \le C_1 = C_1\big(R_1, L_0\big). \tag{3.8}$$

Next, if $|X| = R > R_2$, by (2.1) and (2.12), $u_0 = u_0(X,\varepsilon)$ can be written as

$$\begin{aligned} u_0 &= \left(1 - \frac{n\varepsilon^2}{2|X|^2} + O\left(\frac{\varepsilon^2}{|X|^3}\right)\right)\exp\left\{in\theta + iF(\theta)|X|^{-1} + O\big(|X|^{-2}\big)\right\} \\ &\equiv A_n(R,\varepsilon)\exp\{\cdots\}. \end{aligned} \tag{3.9}$$

So

(3.10)

$$\begin{aligned} \int_{B_R\setminus B_{R_2}} e_\varepsilon(u_0)dX &= O(1) + \int_{B_R\setminus B_{R_2}} \frac{1}{2}|\nabla u_0|^2 \\ &= O(1) + \frac{1}{2}\int_{\theta\in[0,2\pi]}\int_{R_2}^R R\,dR\big[\big(A_{n,R} - iR^{-2}F(\theta) + O(R^{-3})\big)^2 \\ &\qquad\qquad + R^{-2}(n + O(R^{-1}))^2\big] \\ &= \pi n^2 \log\frac{R}{R_2} + O(1) = \pi n^2\log R + O(1). \end{aligned}$$

Summing up (3.7), (3.8), and (3.10), we find there is a positive constant $C$ independent of $\varepsilon$ such that

$$\int_{B_R} e_\varepsilon(u_0)dX = n\pi\log\varepsilon^{-1} + \pi n^2\log R + C + o(1) \tag{3.11}$$

for all $R > R_2$.

Now let us proceed with energy estimates of solution $u^\varepsilon$ on annular domains in four steps.

## 3.1 Step 1

There are no essential zeros of $u^\varepsilon$ in $B_{R_4}\backslash\overline{B_{R_3}}$ for $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$. By an essential zero, we mean the zero that carries an energy of the order $O(\log(\frac{1}{\varepsilon}))$; see [4].

Suppose otherwise, and there is an essential zero at $x_0 \in B_{R_4}\backslash\overline{B_{R_3}}$ when $\tau = -\tau_1^2$. Clearly $\tau_1^2 \in (\tau_0^2/2, \tau_0^2)$. We can find a positive number $\delta \in (0, \tau_1)$ independent of $\varepsilon$ such that $B(x_0, \delta) \subset B_{R_4+\delta}\backslash\overline{B_{R_3-\delta}}$. Then by (3.5)

$$
\begin{aligned}
&e^{-\delta^2/4\tau_1^2}\int_{|X-x_0|\le\delta} e_\varepsilon(u)(-\tau_1^2, X)\\
&\quad\le \int_{|X-x_0|\le\delta} e_\varepsilon(u)(-\tau_1^2, X)e^{-|X-x_0|^2/4\tau_1^2}\\
&\quad\le \int_{\mathbb{R}^2} e_\varepsilon(u_0)e^{-|X-x_0|^2/4\tau_0^2}\\
&\quad= \left(\int_{B_{R_2}} + \int_{\mathbb{R}^2\backslash B_{R_2}}\right) e_\varepsilon(u_0)e^{-|X-x_0|^2/4\tau_0^2}\\
&\quad\le \frac{1}{\log\frac{1}{\varepsilon}}\int_{B_{R_2}} e_\varepsilon(u_0) + \int_{\mathbb{R}^2\backslash B_{R_2}} \frac{1}{2}|\nabla u_0|^2 e^{-|X-x_0|^2/4\tau_0^2} + C\\
&\quad\le C + C\int_{\theta\in[0,2\pi]}\int_{R_2}^{\infty} R\, dR\, e^{-|X-x_0|^2/4\tau_0^2}\\
&\quad\le C_3 = C_3(\tau_0)\,.
\end{aligned}
\tag{3.12}
$$

It follows that

$$
\int_{|X-x_0|\le\delta} e_\varepsilon(u)(-\tau_1^2, X) \le C_3(\tau_0)e^{\delta^2/4\tau_1^2} \le C_3(\tau_0)e^{1/4}, \tag{3.13}
$$

from which we deduce a contradiction, since both $\tau_0$ and $\delta$ are independent of $\varepsilon$.

## 3.2 Step 2: Energy Inequality and Upper Bound on $B_{R_4}$

We calculate

$$
\begin{aligned}
\frac{d}{d\tau}\int_{B_{R_4}} e_\varepsilon(u) &= \int_{B_{R_4}} \nabla u^\varepsilon \cdot \nabla u_\tau^\varepsilon + \varepsilon^{-2}(1 - |u^\varepsilon|^2)(-u^\varepsilon u_\tau^\varepsilon)\\
&= -\int_{B_{R_4}} |u_\tau^\varepsilon|^2 + \int_{\partial B_{R_4}} u_\tau^\varepsilon u_R^\varepsilon\,.
\end{aligned}
\tag{3.14}
$$

Recall that at $\partial B_{R_4}$

$$
u = u^\varepsilon = A_n(R, \varepsilon)\exp\{in\theta + iF(\theta)R^{-1} + O(R^{-2})\} + v^\varepsilon e^{in\theta}\,,
$$

where $v^\varepsilon$ is the $v$ in Proposition 2.1 scaled according to $x \to \frac{X}{\varepsilon}$, $t \to \frac{\tau}{\varepsilon^2}$. It follows that at $|X| = R_4$

$$u_R^\varepsilon = (A_{n,R} + A_n(-iR^{-2}F(\theta) + O(R^{-3})))\exp\{\cdots\} + v_R^\varepsilon e^{in\theta}$$
$$\sim O\left(\frac{\varepsilon^2}{R_4^3}\right) + O(R_4^{-2}) + \varepsilon^{-1}O((\varepsilon^{-1}R_4)^{-p})$$
$$u_\tau^\varepsilon = \varepsilon^{-2}v_t e^{in\theta} = \varepsilon^{-2}O((\varepsilon^{-2}R_4)^{-p})\,, \tag{3.15}$$

where $p \in (\frac{2}{3}, 2]$. It follows that

$$\int_{\partial B_{R_4}} u_\tau^\varepsilon u_R^\varepsilon \sim \varepsilon^{-3+3p}O(R_4^{1-2p}) \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.16}$$

For (3.16), $p > \frac{1}{2}$ suffices. It follows from (3.14) that

$$\int_{B_{R_4}} e_\varepsilon(u) + \int_{-\tau_0^2}^{\tau}\int_{B_{R_4}} u_\tau^2 \le \int_{B_{R_4}} e_\varepsilon(u_0) + C \tag{3.17}$$

for $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$. In view of (3.11)

$$\int_{B_{R_4}} e_\varepsilon(u) + \int_{-\tau_0^2}^{\tau}\int_{B_{R_4}} u_\tau^2 \le n\pi\log\varepsilon^{-1} + \pi n^2\log R_4 + C, \tag{3.18}$$

where $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$.

### 3.3 Step 3: Energy Inequalities on $B_{R_3}$

By Step 1, there are no essential zeros in $B_{R_4}\backslash B_{R_3}$ for $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$. For $\varepsilon$ small enough, modify $u$ to $\tilde{u}$ such that $|\tilde{u}| \ge \frac{1}{2}$, $\tilde{u}$ is an energy minimizer on the annulus $B_{R_4}\backslash B_{R_3}$ with Neumann boundary condition [6]. Then writing $\tilde{u} = Ae^{i(n\theta+h)}$ for a well-defined smooth function $h$, we calculate

$$\begin{aligned}
\int_{B_{R_4}\backslash B_{R_3}} e_\varepsilon(\tilde{u}) &= \frac{1}{2}\int_{B_{R_4}\backslash B_{R_3}} |\nabla A|^2 + |A|^2(n^2|\nabla\theta|^2 + 2n\nabla\theta\cdot\nabla h + |\nabla h|^2) \\
&\quad + \int_{B_{R_4}\backslash B_{R_3}} \frac{(|A|^2-1)^2}{4\varepsilon^2} \\
&= \frac{1}{2}\int_{B_{R_4}\backslash B_{R_3}} |A|^2|\nabla h|^2 + \frac{2n}{r}h_\theta(|A|^2-1) + \frac{(|A|^2-1)^2}{2\varepsilon^2} \\
&\quad + \frac{1}{2}\int_{B_{R_4}\backslash B_{R_3}} \frac{n^2}{r^2} + \frac{(|A|^2-1)n^2}{r^2} + |\nabla A|^2 \\
&\ge \int_{B_{R_4}\backslash B_{R_3}} \left[\frac{|\nabla h|^2}{16} + \frac{(|A|^2-1)^2}{4\varepsilon^2} - c(n)(|A|^2-1)^2 + \frac{|\nabla A|^2}{2}\right] \\
&\quad + n^2\pi\log\frac{R_4}{R_3} - c(n)R_3^{-1}\left(\int_{B_{R_4}\backslash B_{R_3}} (|A|^2-1)^2\right)^{1/2} \\
&\ge n^2\pi\log\frac{R_4}{R_3} - c(n)\,.
\end{aligned}$$

It follows that

$$\int_{B_{R_4}\setminus B_{R_3}} e_\varepsilon(u) \geq \int_{B_{R_4}\setminus B_{R_3}} e_\varepsilon(\tilde{u}) \geq n^2\pi\log\frac{R_4}{R_3} - C\,. \tag{3.19}$$

On the other hand, (3.18) implies

$$\begin{aligned}\int_{B_{R_3}} e_\varepsilon(u) &= \int_{B_{R_4}} e_\varepsilon(u) - \int_{B_{R_4}\setminus B_{R_3}} e_\varepsilon \\ &\leq n\pi\log\varepsilon^{-1} + \pi n^2\log R_4 + C - \pi n^2\log\frac{R_4}{R_3} + C \\ &= n\pi\log\varepsilon^{-1} + \pi n^2\log R_3 + 2C \\ &\leq n\pi\log\varepsilon^{-1} + \frac{1}{2}\pi n^2\log\log\log\frac{1}{\varepsilon} + 3C\,.\end{aligned} \tag{3.20}$$

## 3.4 Step 4: Energy Lower Bound on $B_{R_4}$

By continuity, we know that there are at least $n$ essential zeros in $B_{R_1}$. Hence

$$\int_{B_{R_1}} e_\varepsilon(u) \geq n\pi\log\varepsilon^{-1} - C \tag{3.21}$$

for $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$; see lemma 2 of [6]. It follows from (3.20) and (3.21) that there are no essential zeros in the annulus $B_{R_3}\setminus B_{R_1}$ for $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$. As in Step 3, using an energy minimizer to modify $u$, we have the lower bound

$$\int_{B_{R_3}\setminus B_{R_1}} e_\varepsilon(u) \geq \pi n^2\log\frac{R_3}{R_1} - C\,, \quad \tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]\,. \tag{3.22}$$

Adding up (3.19), (3.21), (3.22), we find that

$$\int_{B_{R_4}} e_\varepsilon(u) \geq n\pi\log\frac{1}{\varepsilon} + n^2\pi\log R_4 - C\,, \quad \tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]\,. \tag{3.23}$$

Combining (3.23) with (3.18), we deduce that

$$\int_{-\tau_0^2}^{\tau}\int_{B_{R_4}} |u_\tau^\varepsilon|^2 \leq C\,, \quad \tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]\,. \tag{3.24}$$

It is easy to see that the above four steps can be carried out for any $R \geq R_4$ as well; hence

$$\int_{-\tau_0^2}^{\tau}\int_{B_R} |u_\tau^\varepsilon|^2 \leq C\,, \quad \forall R > 0\,,\ \tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]\,, \tag{3.25}$$

for a positive constant $C$ uniformly in $\varepsilon$.

We conclude this section with the following:

PROPOSITION 3.1 *Solution $u = u^\varepsilon$ of the rescaled Ginzburg-Landau equation* (3.1) *satisfies the energy estimate for any $R > \sum_{j=1}^n |x_j^{in}| + 1$, and $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$*

$$\int_{B_R} e_\varepsilon(u) = n\pi\log\frac{1}{\varepsilon} + n^2\pi\log R + O(1)\,, \tag{3.26}$$

*uniformly in* $\varepsilon \to 0$ *and* $R \to +\infty$. *Moreover, inequality* (3.25) *holds for the time derivative over the same time interval.*

As a corollary, there are exactly $n$ essential zeros inside $B(0,R_1)$ over the time interval $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$.

## 4 Vortex Mobility and Dynamical Law

Let us first show that vortices initially located at the $x_j^{in}$'s do not move in the time scale $t \sim O(\varepsilon^{-2})$. To this end, we look at the rescaled G-L equation (3.1). As in section 3 and theorem 3.3 of [4], let $\varphi \in C_0^\infty(B_{R_2})$, and let $\varphi$ be supported away from small neighborhoods of the $x_j^{in}$'s in $B_{R_1}$. The function $\varphi$ is independent of $\varepsilon$. We calculate

$$\begin{aligned}\frac{d}{d\tau}\int_{B_{R_2}} \varphi^2 e_\varepsilon(u) &= \int_{B_{R_2}} \varphi^2(\nabla u\cdot\nabla u_\tau + \varepsilon^{-2}(-uu_\tau)(1-|u|^2))\\ &= -\int_{B_{R_2}} \varphi^2|u_\tau|^2 - 2\int_{B_{R_2}} \varphi u_\tau \nabla u\cdot\nabla\varphi\\ &\le \int_{B_{R_2}} \varphi^2\frac{|\nabla u|^2}{2} + C(\varphi)\int_{B_{R_2}} |u_\tau|^2\\ &\le \int_{B_{R_2}} \varphi^2 e_\varepsilon(u) + C(\varphi)\int_{B_{R_2}} |u_\tau|^2,\end{aligned}$$

implying via (3.25) and upon integrating in $\tau \in [-\tau_0^2, T']$, with $T' \le -\tau_0^2 + T_s^\varepsilon$, that

$$\begin{aligned}\int_{B_{R_2}} \varphi^2 e_\varepsilon(u) &\le e^{T'+\tau_0^2}\int_{B_{R_2}} \varphi^2 e_\varepsilon(u_0) + \int_{-\tau_0^2}^{T'} ds\, e^{T'-s}C(\varphi)\int_{B_{R_2}} |u_\tau|^2\\ &\le e^{T'+\tau_0^2}C(\varphi) + e^{T'+\tau_0^2}C(\varphi)\int_{-\tau_0^2}^{T'}\int_{B_{R_2}} |u_\tau|^2 \le e^{\tau_0^2}C_1(\varphi).\end{aligned} \tag{4.1}$$

Hence vortices do not move for $\tau \sim O(1)$ as $\varepsilon \to 0$. In other words, $T_e^\varepsilon = +\infty$, $T_s^\varepsilon = \tau_0^2/2$.

In general, for time scale $t \sim O(\varepsilon^{-2}\lambda_\varepsilon)$ or $\tau = \varepsilon^2\lambda_\varepsilon^{-1}t \sim O(1)$, $\lambda_\varepsilon \to +\infty$, as $\varepsilon \to 0$, the monotonicity formula and energy comparison arguments in Section 3 apply also. The rescaled equation now becomes

$$\frac{1}{\lambda_\varepsilon}u_\tau = \Delta u + \varepsilon^{-2}(1-|u|^2)u, \tag{4.2}$$

except that the factor $e^{-|X-x_0|^2/(4R^2)}$ in the definition of $\Phi(R)$ should be

$$e^{-|X-x_0|^2/(4R^2\lambda_\varepsilon)}.$$

The $R_3$ is now $2\tau_0\lambda_\varepsilon^{1/2}\sqrt{\log\log\varepsilon^{-1}}$. The bound (3.24) is replaced by

$$\int_{-\tau_0^2}^{\tau} d\tau \int_{B_{R_4}} u_\tau^2 \le C\lambda_\varepsilon, \tag{4.3}$$

with $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$, where $C$ is independent of $\varepsilon$. A calculation similar to that of (4.1) shows that

$$\int_{B_{R_2}} \varphi^2 e_\varepsilon(u) \le \lambda_\varepsilon e^T C(\varphi) \tag{4.4}$$

over the same time interval, and thus vortices still do not move if $\lambda_\varepsilon = o(\log \varepsilon^{-1})$. We also have $T_s^\varepsilon = \tau_0^2/2$.

Now let us consider the time scale $t \sim O(\varepsilon^{-2} \log \varepsilon^{-1})$ or when $\lambda_\varepsilon = \log \varepsilon^{-1}$. The energy bounds are

$$\int_{B_R} e_\varepsilon(u) \le n\pi \log \frac{1}{\varepsilon} + \pi n^2 \log R + C \tag{4.5}$$

and

$$\frac{1}{\log \frac{1}{\varepsilon}} \int_{-\tau_0^2}^{\tau} d\tau \int_{B_R} |u_\tau|^2 \le C \tag{4.6}$$

for any $R > R_1$ and $\tau \in [-\tau_0^2, -\tau_0^2 + T_s^\varepsilon]$. As in [4, 5], a calculation on

$$\frac{d}{d\tau} \int_{B_{R_2}} \varphi^2 \frac{e_\varepsilon(u)}{\log \varepsilon^{-1}}$$

with the help of (4.6) shows that

$$\frac{e_\varepsilon(u)}{\log \varepsilon^{-1}} \rightharpoonup \pi \sum_{j=1}^{n} \delta_{x_j(\tau)} \tag{4.7}$$

in the sense of distribution (as measures), and that the $x_j(\tau)$ are continuous in time with the modulus of continuity depending on the energy estimates. Then there is a constant $\tau' = \tau'(R_1) > 0$ independent of $\varepsilon$ such that if $\tau_0 \le \tau'$, $T_s^\varepsilon = \tau_0^2/2$.

PROOF OF THEOREM 1.1: Let us first consider the small time interval $\tau \in [-\tau_0^2, -\tau_0^2/2]$. For convenience, let us translate the time forward by $\tau_0^2$ so that $\tau \in [0, \tau_0^2/2]$ with $\tau_0 \le \tau'$. Let $R \in [R_0, 2R_0]$, $R_0 > 0$ so that $x_j(t) \in B_R(x_j(0))$ for $\tau \in [0, \tau_0^2/2]$. We calculate the first moment of the measure $e_\varepsilon(u)/\log \varepsilon^{-1}$:

(4.8)

$$\frac{1}{\log \frac{1}{\varepsilon}} \frac{d}{d\tau} \int_{B_R(x_j(0))} \vec{x} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 \right) =$$
$$- \frac{1}{(\log \frac{1}{\varepsilon})^2} \int_{B_R(x_j(0))} |u_\tau|^2 \vec{x} + \frac{1}{\log \frac{1}{\varepsilon}} \int_{\partial B_R(x_j(0))} \vec{x} u_\tau u_\nu - \frac{1}{\log \frac{1}{\varepsilon}} \int_{B_R(x_j(0))} u_\tau \nabla u \,.$$

Integrating (4.8) over $\tau \in [0, \tau_0^2/2]$ and $R \in [R_0, 2R_0]$, and dividing by $R_0$, we find

$$\text{L.H.S. of } (4.8) = R_0^{-1} \int_{R_0}^{2R_0} dR \int_{B_R(x_j(0))} \vec{x} \left[ \frac{e_\varepsilon(u)}{\log \varepsilon^{-1}} - \frac{e_\varepsilon(u_0)}{\log \varepsilon^{-1}} \right] \tag{4.9}$$

and

$$\begin{aligned}\text{R.H.S. of (4.8)} = &-\frac{1}{(\log\frac{1}{\varepsilon})^2}\int_0^\tau R_0^{-1}\int_{R_0}^{2R_0}\int_{B_R(x_j(0))}\vec{x}|u_\tau|^2 \\ &+\frac{1}{\log\frac{1}{\varepsilon}}\int_0^\tau R_0^{-1}\int_{R_0}^{2R_0}\int_{\partial B_R(x_j(0))}u_\tau u_\nu\vec{x} \\ &-\frac{1}{\log\frac{1}{\varepsilon}}\int_0^\tau R_0^{-1}\int_{R_0}^{2R_0}\int_{B_R(x_j(0))}u_\tau\nabla u\,.\end{aligned}\tag{4.10}$$

Due to (4.6), the first term of (4.10) converges to zero as $\varepsilon\to 0$. Similarly, the second term of (4.10) goes to zero because of (4.6) and $\int_{R_0}^{2R_0}\int_{\partial B_R(x_j(0))}|u_\nu|^2\le C$ as a consequence of the absence of vortices in the annulus $R_0\le|x|\le 2R_0$. In view of (4.2), we have, using integration by parts,

$$\begin{aligned}\frac{1}{\log\frac{1}{\varepsilon}}\int_{B_R(x_j(0))}u_\tau\nabla u = &-\int_{\partial B_R(x_j(0))}\frac{1}{2}|\nabla u|^2\nu+\int_{\partial B_R(x_j(0))}u_\nu\nabla u \\ &-\frac{1}{4\varepsilon^2}\int_{\partial B_R(x_j(0))}(1-|u|^2)^2\nu\,;\end{aligned}$$

hence as $\varepsilon\to 0$,

(4.11)

$$\text{R.H.S. of (4.8)}\sim\int_0^\tau R_0^{-1}\int_{R_0}^{2R_0}\int_{\partial B_R(x_j(0))}\frac{1}{2}|\nabla u|^2\nu+\frac{1}{4\varepsilon^2}(1-|u|^2)^2\nu-u_\nu\nabla u\,.$$

By strong convergence of $u=u^\varepsilon$ in $H^1$ away from vortices [4],

$$\int_0^\tau R_0^{-1}\int_{R_0}^{2R_0}\int_{\partial B_R(x_j(0))}\frac{1}{4\varepsilon^2}(1-|u|^2)^2\nu\to 0\,,\tag{4.12}$$

and by the general weak convergence [4, 6],

$$u^\varepsilon\to\exp\left\{i\sum_{l=1}^n\theta_l+ih(x)\right\}\tag{4.13}$$

in $L^2_{\text{loc}}$ strong but $H^1_{\text{loc}}$ weak, where $\theta_l=\arg(x-x_l)$, $l=1,2,\dots,n$, and $h(x)$ is a harmonic function. It follows that as $\varepsilon\to 0$

$$\text{R.H.S. of (4.8)}\to\int_0^\tau R_0^{-1}\int_{R_0}^{2R_0}\int_{\partial_R B(x_j(0))}\frac{1}{2}|\nabla v|^2\nu-v_\nu\nabla v\,,\tag{4.14}$$

where $v=e^{i(H+h)}$ and

$$H=\sum_{l=1}^n\arctan\left(\frac{\eta-\eta_l}{\xi-\xi_l}\right),\tag{4.15}$$

where $(\xi,\eta)=x$, $(\xi_l,\eta_l)=x_l$.

We show that $h \equiv$ const. It follows from energy inequality (4.4)

$$\int_{B_R \setminus \bigcup_{j=1}^n B_{R_0}(x_j(0))} \frac{1}{2}|\nabla v|^2 \le n^2 \pi \log R + C \tag{4.16}$$

as $R \to +\infty$. Plugging in the weak limit $e^{i(H+h)}$, we see that

$$\int_{B_R \setminus \bigcup_{j=1}^n B_{R_0}(x_j(0))} |\nabla h|^2 \le O(\log R) \tag{4.17}$$

implying $|\nabla h| \le$ const. Since $h_\xi$ and $h_\eta$ are harmonic functions, they must all be constant. So $h$ is linear, $h = a\xi + b\eta + c$. By (4.17), $h$ must be identically a constant.

Now back to (4.14), using polar coordinates, we have as $\varepsilon \to 0$

(4.18)

$$\begin{aligned}
\text{R.H.S. of (4.8)} &\to \int_0^\tau R_0^{-1} \int_{R_0}^{2R_0} \int_{\partial B_R(x_j(0))} \frac{1}{2}|\nabla H|^2 \nu - H_\nu \nabla H \\
&= \int_0^\tau R_0^{-1} \int_{R_0}^{2R_0} \int_{\partial B_R(x_j(0))} \begin{pmatrix} \frac{1}{2}(H_\xi^2 + H_\eta^2)\cos\theta - H_\xi(H_\xi \cos\theta + H_\eta \sin\theta) \\ \frac{1}{2}(H_\xi^2 + H_\eta^2)\sin\theta - H_\eta(H_\xi \cos\theta + H_\eta \sin\theta) \end{pmatrix} \\
&= \int_0^\tau R_0^{-1} \int_{R_0}^{2R_0} \int_{\partial B_R(x_j(0))} \begin{pmatrix} \frac{1}{2}(H_\eta^2 - H_\xi^2)\cos\theta - H_\xi H_\eta \sin\theta \\ \frac{1}{2}(H_\xi^2 - H_\eta^2)\sin\theta - H_\xi H_\eta \cos\theta \end{pmatrix}.
\end{aligned}$$

The inner boundary integral is on the Hopf differential $\frac{1}{2}w(z) = \frac{1}{2}(H_\xi - iH_\eta)^2$ (see [1]) and hence is independent of the size of radius $R$. It is convenient to take the limit $R \to 0$ for evaluation.

We calculate from (4.15) that

$$\begin{aligned}
H_\xi &= \sum_{l=1}^n \frac{-(\eta - \eta_l)}{(\xi - \xi_l)^2 + (\eta - \eta_l)^2} \equiv \Big(\sum_{l \ne j} \cdots\Big)_{\mathrm{I}} + \frac{-(\eta - \eta_j)}{(\xi - \xi_j)^2 + (\eta - \eta_j)^2}, \\
H_\eta &= \sum_{l=1}^n \frac{(\xi - \xi_l)}{(\xi - \xi_l)^2 + (\eta - \eta_l)^2} \equiv \Big(\sum_{l \ne j} \cdots\Big)_{\mathrm{II}} + \frac{(\xi - \xi_j)}{(\xi - \xi_j)^2 + (\eta - \eta_j)^2}.
\end{aligned} \tag{4.19}$$

It follows, via polar coordinates, that

(4.20)

$$\begin{aligned}
\frac{1}{2}(H_\eta^2 - H_\xi^2) &= \frac{1}{2}\frac{(\xi - \xi_j)^2 - (\eta - \eta_j)^2}{((\xi - \xi_j)^2 + (\eta - \eta_j)^2)^2} + \Big(\sum_{l \ne j} \cdots\Big)_{\mathrm{II}} \frac{(\xi - \xi_j)}{(\xi - \xi_j)^2 + (\eta - \eta_j)^2} \\
&\quad + \Big(\sum_{l \ne j} \cdots\Big)_{\mathrm{I}} \frac{(\eta - \eta_j)}{(\xi - \xi_j)^2 + (\eta - \eta_j)^2} + \text{regular terms} \\
&= \frac{1}{2}\frac{\cos 2\theta}{r^2} + \Big(\sum_{l \ne j} \cdots\Big)_{\mathrm{II}} \frac{\cos\theta}{r} + \Big(\sum_{l \ne j} \cdots\Big)_{\mathrm{I}} \frac{\sin\theta}{r} + \text{regular terms.}
\end{aligned}$$

Similarly,

$$
\begin{aligned}
H_\xi H_\eta &= \frac{-(\xi-\xi_j)(\eta-\eta_j)}{((\xi-\xi_j)^2+(\eta-\eta_j)^2)^2} + \left(\sum_{l\neq j}\cdots\right)_{\mathrm{II}} \frac{-(\eta-\eta_j)}{(\xi-\xi_j)^2+(\eta-\eta_j)^2} \\
&\quad + \left(\sum_{l\neq j}\cdots\right)_{\mathrm{I}} \frac{\xi-\xi_j}{(\xi-\xi_j)^2+(\eta-\eta_j)^2} + \text{regular terms} \\
&= \frac{-\sin 2\theta}{2r^2} + \left(\sum_{l\neq j}\cdots\right)_{\mathrm{II}} \left(\frac{-\sin\theta}{r}\right) \\
&\quad + \left(\sum_{l\neq j}\cdots\right)_{\mathrm{I}} \frac{\cos\theta}{r} + \text{regular terms}\,.
\end{aligned} \tag{4.21}
$$

It follows from (4.20) that

$$
\begin{aligned}
&\lim_{R\to 0}\int_{\partial B_R(x_j(0))} \frac{1}{2}(H_\eta^2 - H_\xi^2)\cos\theta - H_\eta H_\xi \sin\theta \\
&= \left(\sum_{l\neq j}\cdots\right)_{\mathrm{II}} \int_0^{2\pi}\cos^2\theta + \left(\sum_{l\neq j}\cdots\right)_{\mathrm{II}} \int_0^{2\pi}\sin^2\theta \\
&= 2\pi\left(\sum_{l\neq j}\cdots\right)_{\mathrm{II}}\Bigg|_{x=x_j} = 2\pi\sum_{l\neq j}\frac{\xi_j-\xi_l}{(\xi_j-\xi_l)^2+(\eta_j-\eta_l)^2}\,.
\end{aligned} \tag{4.22}
$$

Similarly,

$$
\lim_{R\to 0}\int_{\partial B_R(x_j(0))} \frac{1}{2}(H_\xi^2 - H_\eta^2)\sin\theta - H_\eta H_\xi\cos\theta = 2\pi\sum_{l\neq j}\frac{\eta_j-\eta_l}{(\xi_j-\xi_l)^2+(\eta_j-\eta_l)^2}\,. \tag{4.23}
$$

Define

$$
W = W(x_1, x_2, \dots, x_n) = -\sum_{l\neq j}\log|x_j - x_l|\,, \tag{4.24}
$$

where $x_j = (\xi_j, \eta_j)$. Then (4.22) and (4.23) form the two components of $-\pi\nabla_{x_j}W$. Therefore, as $\varepsilon\to 0$, we combine (4.7), (4.9), (4.18), (4.22), and (4.23) to deduce the vortex dynamical law

$$
x_j(\tau) - x_j^{in} = -\int_0^\tau \nabla_{x_j} W
$$

or

$$
\frac{d}{d\tau}x_j = -\nabla_{x_j}W\,, \tag{4.25}
$$

with $x_j(0) = x_j^{in}$, and over the time interval $\tau\in[0,\tau_0^2/2]$. The dynamics of (4.25) implies that vortices $x_j(t)$ repel each other, and the distances between vortices increase in time. So we always have $x_i(\tau)\neq x_j(\tau)$ for $i\neq j$. Now the energy estimate at $\tau = \tau_0^2/2$ is of the same form as when $\tau = 0$ (Proposition 3.1). We can thus enlarge $R_1$ and $R_2$ by an $O(1)$ amount if necessary and repeat the above analysis over the next time interval of length $\tau_0^2/2$. Since the size of $\tau_0$ depends only on the starting vortex configuration and the energy estimates, we iterate the above procedure in time to reach $\tau = T$. The proof of Theorem 1.1 is complete. □

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