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Front Quenching in G-equation Model Induced by Straining of Cellular Flow

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Abstract We study homogenization of G-equation with a flow straining term (or the strain G-equation) in two dimensional periodic cellular flow. The strain G-equation is a highly non-coercive and non-convex level set Hamilton-Jacobi equation. The main objective is to investigate how the flow induced straining (the nonconvex term) influences front propagation as the flow intensity A increases. Three distinct regimes are identified. When A is below the critical level, homogenization holds and the turbulent flame speed s_T (effective Hamiltonian) is well-defined for any periodic flow with small divergence and is enhanced by the cellular flow as $s_T \geq O(A/\log A)$. In the second regime where A is slightly above the critical value, homogenization breaks down, and s_T is not well-defined along any direction. Solutions become a mixture of fast moving part and a stagnant part. When A is sufficiently large, the whole flame front ceases to propagate forward due to the flow induced straining. In particular, along directions $p = (\pm 1, 0)$ and $(0, \pm 1)$, s_T is well-defined again with a value of zero (trapping). A partial homogenization result is also proved. If we consider a similar but relatively simpler Hamiltonian, the trapping occurs along all directions. The analysis is based on the two-player differential game representation of solutions, selection of game strategies and trapping regions, and construction of connecting trajectories.

Keywords strain G-equation, cellular flows, non-coercive, and non-convex Hamiltonian, flow induced straining, two-person zero-sum differential game

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1 Introduction

Front propagation in prescribed fluid flows has been actively studied for decades in science and engineering as well as mathematics literature [37, 41] due to its fundamental role in understanding the flow effects on reactive transport, and the existence and qualitative properties of turbulent flame speeds s_T, [38]. Two types of scalar model equations have

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been intensively investigated: one is the first principle based reaction-diffusion-advection equations (RDA), and the other is the level-set phenomenological equations (so called G-equations with details to follow). Both have their advantages and limitations, and interestingly may agree or differ in predicting s_T, [21,43]. At qualitative level, front speed enhancement occurs in both types of models consistently. Propagation failure or front quenching in the RDA context has been studied a lot to date. In case that the reaction is cubic and changes sign, quenching refers to the situation that a steady state solution appears in lieu of a traveling wave under some flow conditions, where the front speed is effectively zero, also known as wave blocking ([44] and references therein). In case of reaction with ignition cut-off [19,39,23], or with small enough reaction rate near low temperature [46,28], quenching refers to the eventual decay of solution to zero (extinction). Conditions on quenching range from absence or smallness of plateau region in shear flow profiles [19,28], the cell sizes of cellular flows [23], heat loss [9], to widths of compactly supported initial data and critical power of reaction at low temperature [46]. An essential mechanism in these results is the presence of molecular diffusion in RDA models that spreads the solution to below ignition (or low enough) temperature. Then the nonlinear evolution behaves rather close to linear advection-diffusion, leading up to decay (extinction) in the large time limit. However in turbulent combustion, propagation failure is mostly attributed to the stretching of flames by turbulent flows [37,10]. Little appears to have been rigorously analyzed on flow stretching and front speeds in the level set models. In this paper, we are interested in understanding such flow stretching mechanism in the absence of molecular diffusion, and the connection to the persistence and breakdown of homogenization of the governing equation.

A natural place to pursue this line of inquiry is the G-equation which takes the following form:

$$G_t + s_L|DG| + V(x)\cdot DG = 0, \tag{1.1}$$

where V is the velocity of the surrounding fluid, e.g, the mixture of gasoline and air in the car engine; s_L is the laminar flame speed. G-equation (1.1) was first introduced by Williams [40] and is a very popular flame propagation model in turbulent combustion [38,37]. Its derivation is based on the simple front motion law that the normal velocity of the interface (V_n) is equal to the laminar speed (s_L) plus the projection of fluid velocity along the normal $\overrightarrow{n}$. See Fig. 1 for an illustration. Let the flame front be the zero level set of a reference function $G(x,t)$, the burnt region is $G(x,t)<0$, and the unburnt region is $G(x,t)>0$. The normal direction pointing from the burnt region to the unburnt region is $DG/|DG|$, the normal velocity is $-G_t/|DG|$. The motion law immediately leads to the G-equation (1.1).

The surface of the flame front will be either stretched or compressed by the flow, which inevitably affects the reaction over the flame front. Therefore the laminar flame speed s_L in general is not constant and might depend on flame stretch due to the curvature of flame front and flow straining effect. Using two-scale asymptotic analysis of corrugated premixed flames, Pelce-Clavin [36] and Matalon-Matkowsky [31] derived an expression of s_L involving a first order correction [37]:

$$s_L = s_L^0 - s_L^0 d\,\kappa + d\,\mathbf{n}\cdot S\cdot\mathbf{n}. \tag{1.2}$$

Here s_L^0 is a positive constant representing the burning velocity of the unstretched planar flame, κ is the mean curvature of the flame surface, $\mathbf{n}=\overrightarrow{n}$ is the normal vector to the flame surface in the direction of the unburnt region, $S=\frac{DV+(DV)^\top}{2}$ is the strain rate tensor and d is the Markstein length which is very small and proportional to laminar flame thickness. Many experiments and numerical simulations show that the flame stretch effect plays an important role [10,11,37,38]. A fundamental problem is to study qualitatively and quantitatively the effect of flame stretch on turbulent flame speed

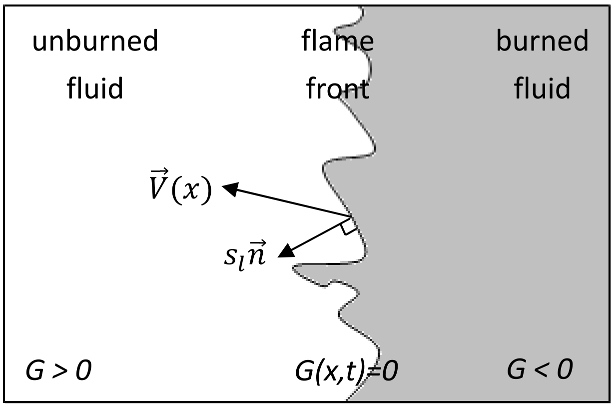

Fig. 1 Illustration of G-equation (level set) model.

(effective burning velocity) as the flow intensity increases. To determine the turbulent flame speed is one of the most important unsolved problems in turbulent combustion. In a previous paper [29], we studied the linearized curvature effect by replacing the mean curvature term with Laplacian and proved that the diffusion dramatically slows down flame propagation. Recent computation [30] suggests that the flame speed slowdown also occurs in the presence of the curvature term, though the effect is weaker than that of a regular diffusion from Laplacian. The precise speed enhancement law of the curvature G-equation in large amplitude cellular flows remains an open problem.

Hereafter, we shall focus on the effect of the strain rate (flow stretching) in the absence of curvature (i.e., $s_L = s_L^0 + d\,\mathbf{n}\cdot S\cdot\mathbf{n}$). For simplicity, we assume that $s_L^0 = 1$. Multiplying the velocity V by a positive constant amplitude A (flow intensity) and plugging the resulting expression in G-equation (1.1), we get the parameterized strain G-equation ($s_L^0 = 1$):

$$G_t + |DG| + A\,V(x)\cdot DG + A\,d\frac{DG\cdot S(x)\cdot DG}{|DG|} = 0. \tag{1.3}$$

Here $S(x) = \frac{DV+(DV)^\top}{2}$, DV is the Jacobian of V and $(DV)^\top$ its transpose. The matrix S in general has both negative and positive eigenvalues. When A is large, the above equation becomes highly non-coercive and non-convex. We intend to use this equation to investigate the effect of the strain term (the non-convex term) on flame propagation under strong flow intensity (large A). We would like to point out that the curvature and strain corrected motion law (1.2) is often derived under certain physical conditions (e.g, low flow intensity in order to validate the linear dependence on the flow strain rate and avoid a negative burning velocity). Due to the independent mathematical interest, we shall not restrict it, as has been similarly treated in [35]. We note that various modifications of s_L have been introduced in the combustion literature to avoid negative burning velocity [45,5]. One such example is $s_L = \max\{0,\ s_L^0 + d\,\mathbf{n}\cdot S\cdot\mathbf{n}\}$ in [45], which we plan to investigate in the future.

In the combustion literature [38,37], there is no universal definition of turbulent flame speed. Its existence theory remains to be established. In the strain G-equation model (1.3), for any unit vector p, we say that *the turbulent flame speed exists along the direction p and equals to a constant $s_T(p,A)$* if

$$\lim_{t\to+\infty}\frac{-G(x,t)}{t} = s_T(p,A) \quad \text{locally uniformly for } x\in\mathbb{R}^n.$$

Here $G(x,t) \in C(\mathbb{R}^n \times [0,+\infty))$ is the unique viscosity solution of equation (1.3) with planar initial data $G(x,0) = p \cdot x$ and satisfies that $G - p \cdot x$ is periodic. According to [2](see [1] for the convex case), this is equivalent to the existence of approximate corrector in the homogenization theory (cell problem), i.e., for any $\delta > 0$, there exists a continuous periodic function $w_\delta(x)$ which satisfies the following inequality in the viscosity sense

$$s_T(p,A) - \delta \le \mathcal{H}(p + Dw_\delta, x) \le s_T(p,A) + \delta$$

for $\mathcal{H}(q,x) = |q| + AV(x) \cdot q + Ad\frac{q \cdot S(x) \cdot q}{|q|}$. Note that $s_T(p,A)$ is the effective Hamiltonian. Therefore if the turbulent flame speed exists, the flame will propagate approximately with a profile $G(x,t) \approx -s_T(p,A)t + w_\delta(x) + p \cdot x$. This, by standard arguments, will also lead to homogenization of the strain G-equation ($x \to \frac{x}{\epsilon}$, $d \to d\epsilon$, ϵ is the turbulence scale and $d \ll 1$ since the flame thickness is much smaller than turbulence scale in the G-equation model). That is, as $\epsilon \to 0$, solution G^ϵ of

$$\begin{cases} G_t^\epsilon + |DG^\epsilon| + AV(\frac{x}{\epsilon}) \cdot DG^\epsilon + Ad\frac{DG^\epsilon \cdot S(\frac{x}{\epsilon}) \cdot DG^\epsilon}{|DG^\epsilon|} = 0 \\ G^\epsilon(x,0) = g(x), \end{cases}$$

converges locally uniformly to solution $\bar{G}$ of the effective equation ($\bar{H}(p) = s_T(p,A)$):

$$\begin{cases} \bar{G}_t + \bar{H}(D\bar{G}) = 0 \\ \bar{G}(x,0) = g(x). \end{cases}$$

There are not many mathematical studies on cell problems and homogenization of genuinely noncoercive and nonconvex Hamilton-Jacobi equations (e.g., [2–4, 6, 7, 13, 15], etc). The major difference between the strain G-equation (1.3) and most of equations studied in cited works is that its Hamiltonian does not have any partial coercivity and can not be written as the difference of two convex Hamiltonians in a simple and natural way. In order to derive some detailed qualitative and quantitative properties of the turbulent flame speeds (effective Hamiltonian), we should look at concrete periodic flows which are both mathematically and scientifically interesting. Throughout this paper (except Theorem 1.1), we choose V to be the following representative example of two-dimensional cellular flows which had received considerable attention in the scientific literature [12, 17, 18, 24, 27].

$$V(x) = (-H_{x_2}, H_{x_1}) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2 \text{ and } H = \sin x_1 \sin x_2. \tag{1.4}$$

The corresponding strain tensor is $S = \begin{pmatrix} -\Phi & 0 \\ 0 & \Phi \end{pmatrix}$ for $\Phi(x) = \cos x_1 \cos x_2$. The strain G-equation with linear initial data then takes form of

$$\begin{cases} G_t + |DG| + AV(x) \cdot DG - Ad\Phi(x)\frac{|G_{x_1}|^2 - |G_{x_2}|^2}{|DG|} = 0 \\ G(x,0) = p \cdot x. \end{cases} \tag{1.5}$$

Our main goal is to investigate how the flow straining influences the existence of turbulent flame speed and its dependence on A. An executive summary of our results is:

• **Propagation Range (Theorem 1.1-1.2):** When $Ad \max_{x\in\mathbb{R}^n} ||S(x)|| < 1$ (this is equivalent to saying that $A < \frac{1}{d}$ for the cellular flow (1.4)), turbulent flame speeds (effective Hamiltonian) are well defined along all directions for any periodic flow with small divergence. Homogenization holds and the flame propagates forward with an effective front. In particular, for the cellular flow (1.4), turbulent flame speeds grow like $\frac{A}{\log A}$ as A increases.

• **Local Quenching (trapping) Range (Theorem 1.3):** Assume that V is the two-dimensional cellular flow (1.4). When A is slightly above $\frac{1}{d}$ but is not too large, the flame front near hyperbolic stagnation points($\pi\mathbb{Z}^2$) will be trapped and cease to propagate, but the other parts keep moving. Stationary isolated islands of unburned area are then generated. See the left picture of Fig. 2. The turbulent flame speed is no longer well-defined along any direction which implies the breakdown of homogenization.

• **Global Quenching (trapping) Range (Theorem 1.4, 1.5):** These are our most delicate results. Assume that V is the two-dimensional cellular flow (1.4). When the A is sufficiently large, the resulting high strain rate together with the strong flow will stop the entire flame front from propagating forward. This means that flame front might either be trapped or retreat if possible. For $p = (\pm 1, 0)$ and $(0, \pm 1)$, the turbulent flame speed (effective Hamiltonian) is well-defined again and drops down to zero (trapping). A partial homogenization result is also proved. If we consider a simplified non-convex term, the associated effective Hamiltonian will be shown to be constant zero (**Theorem 1.6**).

We would like to mention that similar phenomena have been studied in [15] for equation like $u_t - \sigma \mathrm{div}(\frac{Du}{|Du|})|Du| + a(x)|Du| = 0$ with sign-changing $a(x)$ and $\sigma \geq 0$.

When $d = 0$, the G-equation is convex and the turbulent flame speed is always well-defined for any periodic flow with small divergence ([14,42]) and incompressible stationary ergodic flow [16] (see also [32] for $n = 2$). In the cellular flow, it obeys the growth law of $O\left(\frac{A}{\log A}\right)$ ([34,33]). Hence the flow straining indeed significantly slows down flame propagation. Although our result is primarily a mathematical consequence from fluid dynamics without considering heat conduction, it is consistent with combustion experimental findings in that flow straining plays an important role in flame quenching [10]. The following are precise statements. Throughout this paper, a constant is called *universal* if it does not depend on A, d and the unit vector p. We denote

$$||M|| = \max\{|\lambda| : \ \lambda \text{ is an eigenvalue of } M\}$$

as the norm of an $n \times n$ symmetric matrix M. We also denote $\mathbb{T}^n = [0, 2\pi]^n$ and $c_n > 0$ as the smallest positive number such that the following Poincaré inequality holds: for any $f \in W^{1,1}(\mathbb{T}^n)$ and $\bar{f} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f \, dx$,

$$||f - \bar{f}||_{L^{\frac{n}{n-1}}(\mathbb{T}^n)} \leq c_n ||Df||_{L^1(\mathbb{T}^n)}. \tag{1.6}$$

For convenience to deal with the cellular flow (1.4), throughout this paper, we use the cube $\mathbb{T}^n = [0, 2\pi]^n$ instead of the usual unit cube $[0, 1]^n$. The first result is a straightforward modification of that in [42] for the strain-free G-equation ($d = 0$). Our method can be easily extended to time-dependent V.

Theorem 1.1 *Let the flow velocity $V : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous and periodic function (i.e., $V(x + 2\pi\mathbf{v}) = V(x)$ for any $\mathbf{v} \in \mathbb{Z}^n$). Suppose that*

$$\tau_A = 1 - Ad \max_{x \in \mathbb{R}^n} ||S(x)|| > 0.$$

If

$$||\mathrm{div}(V)||_{L^n(\mathbb{T}^n)} < \frac{\tau_A}{Ac_n}, \tag{1.7}$$

then the turbulent flame speed $s_T(p, A)$ is well defined for any unit vector p.

In Theorems 1.2-1.6, we assume that $n = 2$ and V is the two-dimensional cellular flow (1.4). Then the general strain G-equation (1.3) has the particular form (1.5).

Theorem 1.2 *Suppose that V is the two-dimensional cellular flow (1.4). Then when $Ad < 1$, the turbulent flame speed $s_T(p, A)$ is well defined for any unit vector p. Moreover, there exists a universal positive constant C such that when $d \in (0, \frac{1}{4})$ and $A \in [4, \frac{1}{d})$*

$$s_T(p, A) \geq C\frac{A}{\log A}. \tag{1.8}$$

We want to remark that $A = \frac{1}{d}$ is not the exact transition value for the existence of turbulent flame speed when V is (1.4). By more delicate analysis based on the special structure of the cellular flow (1.4), we can actually show that $s_T(p, A)$ is still well defined if $A\,d$ is larger but is extremely close to 1. However, the following theorem says that when Ad is slightly above 1, the turbulent flame speed is no longer well-defined along any direction. So practically we can still view $A = \frac{1}{d}$ as the transition value. Now we define

$$\mathbb{Z}^2_e = \{(m, n) \in \mathbb{Z}^2 |\ m + n \text{ is even}\} \quad \text{and} \quad \mathbb{Z}^2_o = \{(m, n) \in \mathbb{Z}^2 |\ m + n \text{ is odd}\}$$

Theorem 1.3 *Suppose that V is the two-dimensional cellular flow (1.4). Let G be the unique viscosity solution of equation (1.5).*
(i) (Stationary isolated unburned area) There exists a universal constant $d_0 \in (0, \frac{1}{4})$ such that when $0 < d < d_0$ and $Ad \geq 1 + 6d^2$, for $x \in \pi\mathbf{v} + [-3d, 3d] \times [-d^3, d^3]$ and $\mathbf{v} \in \mathbb{Z}^2_o$ or $x \in \pi\mathbf{v} + [-d^3, d^3] \times [-3d, 3d]$ and $\mathbf{v} \in \mathbb{Z}^2_e$, we have that

$$G(x, t) \geq p \cdot x - 2 \quad \textit{for all } t \geq 0. \tag{1.9}$$

In particular, this implies that

$$\limsup_{t \to +\infty} \frac{-G(x, t)}{t} \leq 0.$$

(ii) (Propagation of the other part of the flame front) Assume that $d \in (0, \frac{1}{4})$, $A \geq 4$ and $Ad \leq 1 + \frac{d}{10}$. Then for $\mathcal{X}_0 = (\frac{\pi}{2}, 0)$,

$$\liminf_{t \to +\infty} \frac{-G(\mathcal{X}_0, t)}{t} \geq \frac{CA}{\log A}. \tag{1.10}$$

Here C is a universal positive constant.

Combining (i) and (ii), we have that there exists a universal constant $\tilde{d_0} \in (0, \frac{1}{60})$ such that when $0 < d < \tilde{d_0}$ and $Ad \in [1 + 6d^2, 1 + \frac{d}{10}]$, the turbulent flame speed is not well defined along any unit direction p.

The choice of $\mathcal{X}_0$ is not special, other than simplifying our calculations. (1.9) implies that the flame front will never enter regions near those hyperbolic stagnation points $\{x \in \pi\mathbb{Z}^2 |\ p \cdot x > 2\}$. When $Ad \in [1 + 6d^2, 1 + \frac{d}{10}]$, the solution actually becomes a mixture of fast moving part M and a stagnant part $\mathbb{R}^2 \backslash M$. Qualitatively, the equation in this case behaves similar to $u_t + a(x)|Du| = 0$ for $a(x)$ which is zero inside $\mathbb{R}^2 \backslash M$ and positive in M. See the left picture of Fig. 2 and the comment at the end of section 4 for more explanations. Moreover, the existence of local unburned area does not really depend on the specific form (1.4). In fact, for a stream function H, as long as the strain rate tensor S is a diagonal matrix at saddle points, results similar to (i) in the above theorem can be established in small tubular neighbourhoods around streamlines containing these points. Part (ii), however, is a global result which relies more on the specific structure of (1.4).

When A gets very large, the entire flame front ceases to propagate forward, as seen in the following two theorems.

Theorem 1.4 *Assume that V is the two-dimensional cellular flow (1.4). Let $G \in C(\mathbb{R}^2 \times [0, +\infty))$ be the unique viscosity solution of equation (1.5). Then there exists a universal constant $d_0 \in (0, 1)$ such that when $d < d_0$ and $A > \frac{8}{d^3}$*

$$G(x,t) \geq p \cdot x - 2\sqrt{2}\pi \quad \text{for all } (x,t) \in \mathbb{R}^2 \times [0, +\infty).$$

This implies that for all unit vectors p

$$\limsup_{t\to+\infty} \frac{-G(x,t)}{t} \leq 0 \quad \text{locally uniformly for } x \in \mathbb{R}^2.$$

In particular, if $p = (\pm 1, 0)$ or $(0, \pm 1)$, the flame front is actually trapped, i.e.,

$$|G(x,t) - p \cdot x| \leq 4\pi \quad \text{for all } (x,t) \in \mathbb{R}^2 \times [0, +\infty)$$

which implies that the turbulent flame speed (effective Hamiltonian) exists again and has a value of zero, i.e.,

$$s_T(p, A) = \lim_{t\to+\infty} \frac{-G(x,t)}{t} = 0 \quad \text{locally uniformly for } x \in \mathbb{R}^2. \tag{1.11}$$

Theorem 1.5 *Assume that V is the two-dimensional cellular flow (1.4) and $\Phi(x) = \cos x_1 \cos x_2$. For $\epsilon > 0$, Suppose that $G^\epsilon \in C(\mathbb{R}^2 \times [0, +\infty))$ is the unique viscosity solution of the strain G-equation*

$$\begin{cases} G^\epsilon_t + |DG^\epsilon| + AV(\frac{x}{\epsilon}) \cdot DG^\epsilon - Ad\Phi(\frac{x}{\epsilon}) \frac{|G^\epsilon_{x_1}|^2 - |G^\epsilon_{x_2}|^2}{|DG^\epsilon|} = 0 \\ G^\epsilon(x, 0) = g(x). \end{cases} \tag{1.12}$$

Here $g \in C(\mathbb{R}^2)$ is Lipschitz continuous. Then there exists a universal constant $d_0 \in (0, 1)$ such that when $d < d_0$ and $A > \frac{8}{d^3}$, for $L = 2\sqrt{2}\pi||Dg||_{L^\infty}$

$$G^\epsilon(x,t) \geq g(x) - L\epsilon \quad \text{for all } (x,t) \in \mathbb{R}^2 \times [0, +\infty), \tag{1.13}$$

$$\{x|\ G^\epsilon(x,t) \leq 0\} \subseteq \{x|\ d(x, \Omega) \leq 2\sqrt{2}\pi\epsilon\} \quad \text{for all } t \geq 0, \tag{1.14}$$

where $\Omega = \{x|\ g \leq 0\}$ is the initial burned region. Moreover,

$$\liminf_{\substack{\epsilon\to 0 \\ (y,s)\to(x,t)}} G^\epsilon(y,s) = g(x) \quad \text{for all } (x,t) \in \mathbb{R}^2 \times [0, \infty). \tag{1.15}$$

Numerical simulations suggest that (1.11) might hold for all directions p (equivalently $\lim_{\epsilon\to 0} G^\epsilon(x,t) = g(x)$), which we are not able to rigorously verify. However, this can be established for a similar but simplified Hamiltonian which is interesting in its own right and worth mentioning.

Theorem 1.6 *Assume that V is the two-dimensional cellular flow (1.4) and $\Phi(x) = \cos x_1 \cos x_2$. For $\epsilon > 0$, suppose that $\tilde{G}^\epsilon \in C(\mathbb{R}^2 \times [0, +\infty))$ is the unique viscosity solution of the following simplified equation*

$$\begin{cases} \tilde{G}^\epsilon_t + |D\tilde{G}^\epsilon| + AV(\frac{x}{\epsilon}) \cdot D\tilde{G}^\epsilon - Ad\Phi(\frac{x}{\epsilon}) \cdot (|\tilde{G}^\epsilon_{x_1}| - |\tilde{G}^\epsilon_{x_2}|) = 0 \\ \tilde{G}^\epsilon(x, 0) = g(x). \end{cases} \tag{1.16}$$

Here $g \in C(\mathbb{R}^2)$ is Lipschitz continuous. Then when $d < d_0$ and $A > \frac{8}{d^3}$, for $\tilde{L} = 4\sqrt{2}\pi||Dg||_{L^\infty}$

$$|\tilde{G}^\epsilon(x,t) - g(x)| \leq \tilde{L}\epsilon \quad \text{for all } (x,t) \in \mathbb{R}^2 \times [0, +\infty). \tag{1.17}$$

This implies that

$$\lim_{\epsilon\to 0} \tilde{G}^\epsilon(x,t) = g(x) \quad \text{uniformly in } \mathbb{R}^2 \times [0, \infty). \tag{1.18}$$

The validity and breakdown of homogenization (Theorems 1.2 and 1.3) are also true for the above simplified Hamiltonian. The d_0 in Theorems 1.4, 1.5 and 1.6 are the same as that in Lemma 5.5. When $A\,d < 1$, the strain G-equation (1.5) has a hidden coercivity structure by taking integrations. When A is large, this structure is lost and the equation becomes highly noncoercive. Proofs of Theorems 1.4, 1.5 and 1.6 are completely different from that of Theorem 1.1 (or Theorem 1.2). They are based on representation formulas in terms of suitable two-player, zero sum differential games and careful analysis of the underlying dynamics. This is the main new approach used in this paper. See [3, 4] for other interesting connections between game theory and homogenization. Due to the presence of the strong flow and competition between the two players, the overall dynamics is quite complicated and subtle. It seems to us that standard PDE techniques for viscosity solutions are sometimes too rough to derive delicate information of the solution (e.g. the global trapping). Our main idea is to show that no matter how one player moves, his opponent can always find a strategy such that the game trajectory will eventually be trapped inside a finite domain (a trapping region). The large lower bound $\frac{8}{d^3}$ might be reduced to $\frac{C}{d}$ or even $\frac{1}{d}+C$ through more sophisticated analysis of the game dynamics. We believe that at least for $p=(\pm 1,0)$ and $p=(0,\pm 1)$, there should exist a unique transition value μ_d for quenching (trapping), i.e., when $Ad \in [1+6d^2, \mu_d)$, the turbulent flame speed (effective Hamiltonian) does not exist; and when $Ad > \mu_d$, the turbulent flame speed (effective Hamiltonian) exists again and becomes zero. Owing to (1.10), $\mu_d > 1+\frac{d}{10}$ if such a threshold value does exist. Nevertheless, it is not clear to us whether $\mu_d = O(1)$ or $\mu_d = 1+O(1)d$. This will be investigated in the future.

Remark 1.1 *It is also natural to ask whether trapping and homogenization results established in this paper also hold for more general two dimensional cellular flows besides the specific one (1.4) and for some other stream functions which lead to strain G-equations like (1.5), e.g. the cat's-eye flow: $H=\sin x_1 \sin x_2 + \delta \cos x_1 \cos x_2$ for $\delta \in (0,1)$. The key is to obtain similar controls of the game trajectory as those in Lemma 5.3 and 5.4. However, the approach used in the present paper depends heavily on the particular structure of (1.4) and can not be extended to other cases via simple modifications. It remains open to find a more robust method to treat more general flows.*

The rest of the paper is organized as follows. In section 2, we revisit briefly the two player, zero sum differential game representation of solutions of non-convex Hamilton-Jacobi equations, which serves as our analytical platform. In section 3, we prove Theorem 1.1 by establishing the approximate correctors in the viscosity solution sense. Homogenization holds in spite of the lack of exact correctors. Theorem 1.2 follows immediately from Thereom 1.1 and (1.10). In section 4, we use control theory, two player game strategies and comparison principle to prove Theorem 1.3 in the regime ($A\,d \in [1+6d^2,\ 1+\frac{d}{10}]$, $d \ll 1$) of breakdown of homogenization, thus s_T is not well-defined. The solution is a disparate mixture of a fast propagating piece and a stagnant piece. In section 5, we give proofs of Theorems 1.4, 1.5 and 1.6 when A is large enough ($A > 8\,d^{-3}$, $d \ll 1$). The main ingredients are subtle modifications of Hamiltonians, judicious choices of trapping regions and connecting trajectories, and delicate bounds of solutions in the two player game representation. Concluding remarks are in section 6.

Assumptions and Notations: Throughout this paper, solutions of Hamilton-Jacobi equations are always interpreted in the viscosity sense and are uniformly continuous within any finite time. Such type of solutions are known to be unique with given initial data. We refer to the *User's Guide* [20] for precise definitions and comparison principles used in this paper. Also, we denote

- $\mathbb{T}^n = [0,2\pi]^n$ and f as periodic if $f(x+2\pi\mathbf{v}) = f(x)$ for any $\mathbf{v} \in \mathbb{Z}^n$.
- $H(x) = \sin x_1 \sin x_2$ and $\Phi(x) = \cos x_1 \cos x_2$.

- $\mathcal{H}(p,x) = |p| + AV(x)\cdot p - Ad\Phi(x)\frac{|p_1|^2-|p_2|^2}{|p|}$ for $V = (-H_{x_2}, H_{x_1})$
- If I is an interval, I^2 represents the square $\{x = (x_1,x_2)|\ x_1 \in I,\ x_2 \in I\}$.

2 Representation formula for solutions of non-convex Hamilton-Jacobi equation

Two-person, zero sum differential games were first introduced by Isaacs [26] in early 1950's. Value functions of a large class of such games are found to be equivalent to solutions of nonconvex Hamilton-Jacobi equations. For reader's convenience, we provide a quick review of the representation formula which is a key tool to prove our main results. Our presentation is mainly based on [22], in which readers may find more background and references of the game theory. Let $S_1 \in \mathbb{R}^k$ and $S_2 \in \mathbb{R}^l$ be two given compact sets, which are legal moves players I and II can make respectively. Now suppose that $u(x,t) : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is the viscosity solution of the following initial value problem

$$\begin{cases} u_t + H(Du,x) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\ u(x,0) = g(x). \end{cases}$$

Then $v(x,t) = -u(x,T-t)$ is the viscosity solution of the following terminal value problem which was used in [22]

$$\begin{cases} v_t + H(-Dv,x) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\ v(x,T) = -g(x). \end{cases}$$

Note that from initial value problem to terminal value problem, the sign needs to be reversed in the definition of viscosity solutions. For simplicity, we assume the Isaacs condition

$$H(-p,x) = \max_{\eta\in S_1}\min_{\mu\in S_2}\{f(x,\eta,\mu)\cdot p\} = \min_{\mu\in S_2}\max_{\eta\in S_1}\{f(x,\eta,\mu)\cdot p\}. \tag{2.1}$$

Here we only consider time independent f and zero running cost which is sufficient in our situation. According to Theorem 4.1 in [22], the terminal value of u is given by

$$\begin{aligned} -u(x,T) = v(x,0) &= \inf\nolimits_{\Lambda\in\Delta(T)}\sup\nolimits_{\alpha\in M(T)}\{-g(\xi(T))\} \\ &= \sup\nolimits_{\Sigma\in\Gamma(T)}\inf\nolimits_{\beta\in N(T)}\{-g(\xi(T))\}. \end{aligned} \tag{2.2}$$

Here $\xi : [0,T] \to \mathbb{R}^2$ satisfies $\xi(0) = x$ and
(i) in the inf-sup expression (player I moves first)

$$\dot{\xi}(t) = f(\xi,\alpha,\Lambda(\alpha)) \quad \text{for a.e } t \in (0,T)$$

(ii) in the sup-inf expression (player II moves first)

$$\dot{\xi}(t) = f(\xi,\Sigma(\beta),\beta) \quad \text{for a.e } t \in (0,T).$$

We also set
(1) $M(T)$ as the set of measurable functions $[0,T] \to S_1$;
(2) $N(T)$ as the set of measurable functions $[0,T] \to S_2$;
(3) $\Gamma(T)$ as the set of strategies of player I, i.e, nonanticipating mappings $\Sigma : N(T) \to M(T)$ which satisfies that for all $t < T$

$$\begin{cases} \beta(s) = \tilde{\beta}(s) \quad \text{for a.e. } 0 \le s \le t \\ \text{implies that } \Sigma(\beta)(s) = \Sigma(\tilde{\beta})(s) \quad \text{for a.e. } 0 \le s \le t. \end{cases}$$

(4) $\Delta(T)$ as as the set of strategies of player II, i.e., nonanticipating mappings $\Lambda : M(T) \to N(T)$ which satisfies that for all $t < T$

$$\begin{cases} \alpha(s) = \tilde{\alpha}(s) \quad \text{for a.e. } 0 \le s \le t \\ \text{implies that } \ \Lambda(\alpha)(s) = \Lambda(\tilde{\alpha})(s) \quad \text{for a.e. } 0 \le s \le t. \end{cases}$$

Note that our $(\Sigma, \Lambda, \alpha, \beta)$ is the similar to (α, β, y, z) in [22]. Also throughout this paper, $S_1 = [-1,1]^2$ and $S_2 = [-1,1]$. If a Hamiltonian can be written in max-min or min-max forms in (2.1) plus a possible running cost, more information of solutions can be obtained by analyzing the dynamics of the game. For the strain G-equation, the associated Hamiltonian $\mathcal{H}(p,x) = |p| + AV(x)\cdot p - Ad\Phi(x)\frac{|p_1|^2-|p_2|^2}{|p|}$ does not possess any simple and natural max-min or min-max expression. The general max-min or min-max formulation provided in [22] (or [25]) is too rough to derive delicate information like Theorem 1.4 and 1.5. Fortunately, thanks to the special structure of $\mathcal{H}$ and the equalities

$$\frac{|p_1|^2-|p_2|^2}{|p|} = |p_1|\cdot\frac{|p_1|}{|p|} - |p_2|\cdot\frac{|p_2|}{|p|} = |p_1|\cdot\frac{|p_1|+|p_2|}{|p|} - |p_2|\cdot\frac{|p_1|+|p_2|}{|p|}, \tag{2.3}$$

the nonconvex term $\Phi(x)\frac{|p_1|^2-|p_2|^2}{|p|}$ in most of our proofs behaves qualitatively similar to either $|p_1|-|p_2|$ or $|p_2|-|p_1|$ which have clear max-min and min-max forms. This is achieved by introducing nice auxiliary Hamiltonians and applying comparison principle. However, see the subtle difference between Theorem 1.5 and 1.6.

3 Proof of Theorem 1.1 and Theorem 1.2

Note that Theorem 1.2 follows immediately from Theorem 1.1 and (1.10). Hence we only need to prove Theorem 1.1. Let us assume $V(x)$ is a n-dimensional periodic, and Lipschitz continuous vector field with small divergence (1.7). The proof is a simplified version of that in [42] for the inviscid G-equation ($d=0$) by establishing the approximate corrector. It can be easily extended to time-dependent velocity field $V(x,t)$.

Step 1: For any $\lambda > 0$, let $u_\lambda \in C(\mathbb{R}^n)$ be the unique continuous periodic viscosity solution of

$$\lambda u_\lambda + |p+Du_\lambda| + AV(x)\cdot(p+Du_\lambda) + Ad\cdot\frac{(p+Du_\lambda)\cdot S(x)\cdot(p+Du_\lambda)}{|p+Du_\lambda|} = 0.$$

To establish the approximate cell problem, it suffices to show that there exists a sequence $\lambda_m \to 0$ as $m \to 0$ such that

$$\lim_{m\to+\infty} \lambda_m u_{\lambda_m} = \text{constant} \quad \text{uniformly on } \mathbb{R}^n.$$

The comparison principle implies that the limiting constant does not depend on specific convergent subsequences. This constant is $-s_T(p,A)$ (effective Hamiltonian).

Step 2: Fix $x_0 \in \mathbb{R}^n$. Choose a sequence $\lambda_m \to 0$ and $x_m \to x_0$ such that

$$\lim_{m\to+\infty} \lambda_m u_{\lambda_m}(x_m) = \liminf_{\substack{y\to x_0 \\ m\to+\infty}} u_{\lambda_m}(y).$$

Our goal is to show that

$$\lim_{m\to+\infty} \lambda_m u_{\lambda_m} = \text{constant} \quad \text{uniformly on } \mathbb{R}^n. \tag{3.1}$$

Due to the lack of coercivity, there is no uniform control of the modules of continuity of $\lambda_m u_{\lambda_m}$ as $m \to +\infty$. Since $\lambda_m |u_{\lambda_m}|$ is uniformly bounded, using a well known technique in homogenization theory, we consider

$$u^*(x) = \limsup_{\substack{y\to x\\ m\to+\infty}} \lambda_m u_{\lambda_m}(y) \quad \text{and} \quad u_*(x) = \liminf_{\substack{y\to x\\ m\to+\infty}} \lambda_m u_{\lambda_m}(y).$$

For simplification, we drop the dependence on A and write

$$\tau = \tau_A = 1 - Ad \max_{x\in\mathbb{R}^n} ||S(x)|| > 0.$$

Then u_{λ_m} is a viscosity subsolution of

$$\lambda_m u_{\lambda_m} + \tau|p + Du_{\lambda_m}| + AV(x)\cdot(p + Du_{\lambda_m}) \le 0. \tag{3.2}$$

Now we introduce a slight simplification of the argument in [42]. Since the Hamiltonian in (3.2) is convex (see [8] for instance), $v_{\lambda_m} = -u_{\lambda_m}$ is a viscosity subsolution of

$$-\lambda_m v_{\lambda_m} + \tau| - p + Dv_{\lambda_m}| - AV(x)\cdot(-p + Dv_{\lambda_m}) \le 0.$$

This is essentially due to the fact that a Lipschitz continuous function is a viscosity subsolution of a convex Hamilton-Jacobi equation if and only if it satisfies the inequality a.e. Hence it is easy to see that u^* is upper semi-continuous and a periodic viscosity subsolution of

$$\tau|Du^*| + AV(x)\cdot Du^* \le 0 \tag{3.3}$$

and $v^* = -u_*$ is upper semi-continuous and a periodic viscosity subsolution of

$$\tau|Dv^*| - AV(x)\cdot Dv^* \le 0. \tag{3.4}$$

Step 3: From (3.3) and (3.4), we will show that both u^* and $v^* = -u_*$ are constants. For $\delta > 0$, consider the sup-convolution of u^*:

$$u^*_\delta = \sup_{y\in\mathbb{R}^n}\{u^*(y) - \frac{1}{\delta}|x-y|^2\}.$$

Then it is well known in the theory of viscosity solution, u^*_δ is a Lipschitz continuous periodic viscosity subsolution of

$$(\tau - C\sqrt{\delta})|Du^*_\delta| + AV(x)\cdot Du^*_\delta \le 0.$$

Here C is an quantity depending only on V and A. Taking integration over $\mathbb{T}^n = [0, 2\pi]^n$ on both sides, we obtain that

$$(\tau - C\sqrt{\delta})\int_{\mathbb{T}^n} |Du^*_\delta|\,dx \le A\int_{\mathbb{T}^n}(\mathrm{div}V)u^*_\delta\,dx = A\int_{\mathbb{T}^n}(\mathrm{div}V)(u^*_\delta - \bar{l})\,dx$$
$$\le Ac_n||\mathrm{div}V||_{L^n(\mathbb{T}^n)}||Du^*_\delta||_{L^1(\mathbb{T}^n)}.$$

Here $\bar{l} = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} u^*_\delta\,dx$. The last inequality is due to Hölder inequality and (1.6). Owing to (1.7), we may choose δ small enough such that $\tau - C\sqrt{\delta} > Ac_n||\mathrm{div}V||_{L^n(\mathbb{T}^n)}$. Then $\int_{\mathbb{T}^n} |Du^*_\delta|\,dx = 0$. Hence u^*_δ is a constant for small δ. Therefore $u^* = \lim_{\delta\to0} u^*_\delta$ is also a constant. Similarly by considering the sup-convolution of v^*, we can show that $v^* = -u_*$ is also a constant.

Step 4: Now let us denote

$$u^*(x) \equiv c^* \quad \text{and} \quad u_*(x) \equiv c_* \quad \text{for all } x \in \mathbb{R}^n.$$

The final step is to prove that these two constants are the same, i.e., $c^* = c_*$. Since $c^* \geq c_*$, it suffices to show that $c^* \leq c_*$. We apply a simple local reachability property established in [42] (Lemma 2.1) which is true for any continuous velocity field V: for $x_0 \in \mathbb{R}^n$ from Step 2, there exists $y_0 \in \mathbb{R}^n$ and two positive numbers r_1 and r_2 such that for any $x \in B_{r_1}(x_0)$ and $y \in B_{r_2}(y_0)$, we can find a Lipschitz continuous curve $\xi : [0, t_0] \to \mathbb{R}^n$ which depends on x, y and satisfies:
(1) $t_0 \leq 1$, $\xi(0) = x$ and $\xi(t_0) = y$;
(2) $|\dot{\xi}(s) - AV(\xi(s))| \leq \tau$ for a.e. $s \in [0, t_0]$.
Then (3.2) and Lemma 2.2 in [42] immediately imply that

$$\sup_{B_{r_2}(y_0)} \lambda_m u_{\lambda_m} \leq \inf_{B_{r_1}(x_0)} \lambda_m u_{\lambda_m} + o(1).$$

Here $o(1)$ is a quantity depending only on V and A such that $\lim_{m\to+\infty} o(1) = 0$. Due to the choice of x_0, λ_m and y_m from step 2, we have that when $|x_m - x_0| < r_1$, $\inf_{B_{r_1}(x_0)} \lambda_m u_{\lambda_m} \leq \lambda_m u_{\lambda_m}(x_m)$ and $\lim_{m\to+\infty} \lambda_m u_{\lambda_m}(x_m) = u_*(x_0)$. Accordingly, we have that

$$c^* = u^*(y_0) \leq u_*(x_0) = c_*.$$

□

4 Proof of Theorem 1.3

We first establish (1.9). It suffices to prove this for $x \in I_d = [\pi - 3d, \pi + 3d] \times [-d^3, d^3]$. Proof of the other parts is similar by periodicity and symmetry. The basic idea is that when Ad exceeds 1, a kinetic balance between flow, laminar flame speed and strain rate will be achieved along upper and lower sides of I_d (4.2). Moreover, the strong flow will prevent the flame from entering I_d through left and right sides (4.1). Let us first fix d_0. Choose $d_0 \in (0, \frac{1}{4})$ small enough such that when $d \in (0, d_0)$ and $Ad \geq 1 + 6d^2$,

$$A \sin 3d \cos d^3 - 2Ad \cos 3d \cos d^3 - 1 \geq 10\, d^2 - O(d^4) > 0 \tag{4.1}$$

$$Ad \cos 3d \cos d^3 - A \cos 3d \sin d^3 - 1 \geq \frac{d^2}{2} - O(d^4) > 0. \tag{4.2}$$

In order to apply (2.2), we introduce an auxiliary Hamiltonian H_1:

$$H_1(p, x) = \begin{cases} |p_1| + |p_2| + AV(x) \cdot p - 2Ad\Phi(x)|p_1| + Ad\Phi(x)|p_2| & \text{if } \Phi(x) \leq 0 \\ (1 + Ad\Phi(x))(|p_1| + |p_2|) + AV(x) \cdot p & \text{if } \Phi(x) \geq 0. \end{cases}$$

Due to (2.3), it is clear that $H_1(p, x) \geq \mathcal{H}(p, x)$. Suppose that $U \in C(\mathbb{R}^2 \times [0, +\infty))$ is the unique viscosity solution of

$$\begin{cases} U_t + H_1(DU, x) = 0 \\ U(x, 0) = p \cdot x \end{cases} \tag{4.3}$$

such that $U - p \cdot x$ is periodic. Since G is a viscosity solution of (1.5), it is a viscosity supersolution of the above equation. Standard comparison principle implies that $G \geq U$. According to (2.2),

$$-U(x, t) = \inf_{\Lambda \in \Delta(t)} \sup_{\alpha \in M(t)} -p \cdot \xi(t) \tag{4.4}$$

for $\xi : [0, t] \to \mathbb{R}^2$ satisfying

$$\begin{cases} \dot{\xi}(s) = f(\xi, \alpha, \Lambda(\alpha)) & \text{for a.e. } 0 \leq s \leq t \\ \xi(0) = x. \end{cases}$$

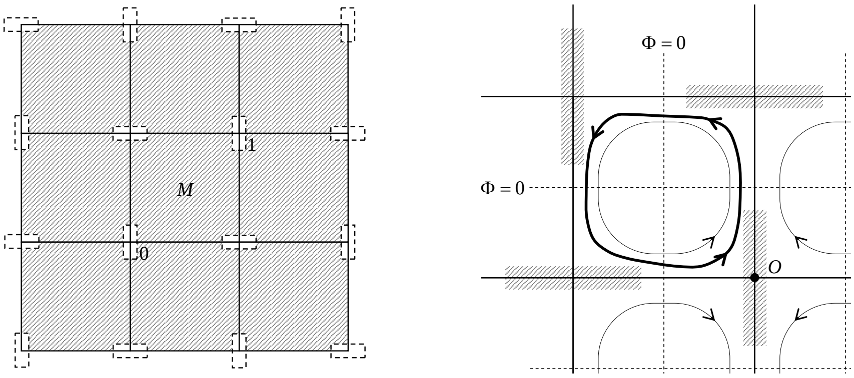

Fig. 2 Figures for Theorem 1.3 (left) and Theorems 1.4-1.5 (right)

and $f = f(x, \eta, \mu) : \mathbb{R}^2 \times [-1,1]^2 \times [-1,1] \to \mathbb{R}^2$ is given by ($\eta = (\eta_1, \eta_2)$)

$$f(x,\eta,\mu) = \begin{cases} (\eta_1 - 2Ad\Phi(x)\eta_1,\ \eta_2 - Ad\Phi(x)\mu) - AV(x) & \text{if } \Phi(x) \le 0 \\ (\eta_1 + Ad\Phi(x)\eta_1,\ \eta_2 + Ad\Phi(x)\eta_2) - AV(x) & \text{if } \Phi(x) \ge 0. \end{cases}$$

See section 2 for definitions of $M(t)$, $N(t)$, $\Sigma(t)$ and $\Delta(t)$. $S_1 = [-1,1]^2$ and $S_2 = [-1,1]$. Now fix $x \in [\pi - 3d, \pi + 3d] \times [-d^3, d^3]$. We will choose a strategy Λ_x of player II. For $\alpha = (\alpha_1(s), \alpha_2(s)) \in M(t)$, let $\xi(s) = (x_1(s), x_2(s))$ be the unique solution of

$$\begin{cases} \dot{\xi}(s) = -AV(\xi) + \Big(\alpha_1 - 2Ad\Phi(\xi)\alpha_1,\ \alpha_2 + \frac{Ax_2(s)}{d^2}\Phi(\xi)\Big) & \text{for a.e. } 0 \le s \le t \\ \xi(0) = x. \end{cases}$$

Then

$$\begin{cases} (4.1) \Rightarrow \dot{x}_1(s) > 0 & \text{when } \xi(s) \text{ is close to } \{(\pi - 3d, x_2) :\ |x_2| \le d^3\} \\ (4.1) \Rightarrow \dot{x}_1(s) < 0 & \text{when } \xi(s) \text{ is close to } \{(\pi + 3d, x_2) :\ |x_2| \le d^3\} \\ (4.2) \Rightarrow \dot{x}_2(s) < 0 & \text{when } \xi(s) \text{ is close to } \{(x_1, d^3) :\ |x_1 - \pi| \le 3d\} \\ (4.2) \Rightarrow \dot{x}_2(s) > 0 & \text{when } \xi(s) \text{ is close to } \{(x_1, -d^3) :\ |x_1 - \pi| \le 3d\} \end{cases}$$

Hence the curve must be trapped within the box $[\pi - 3d, \pi + 3d] \times [-d^3, d^3]$. Note that $\Phi < 0$ in this box. Hence $\dot{\xi}(s) = f(\xi, \alpha, \frac{-x_2(s)}{d^3})$ for a.e., $0 \le s \le t$. Therefore if player II chooses the strategy $\Lambda_x :\ M(t) \to N(t)$ as

$$\Lambda_x(\alpha)(s) = \frac{-x_2(s)}{d^3} \quad \text{for } s \in [0,t],$$

representation formula (4.4) and comparison principle imply that $G(x,t) - p \cdot x \ge U(x,t) - p \cdot x > -7d > -2$. Note that this strategy is simply saying that the player II will try his best to pull down (or pull up) the trajectory along the vertical direction when x_2 is close to d^3(or close to -d^3). □

Next we will prove (1.10). This can be reduced to a control problem where one player is inactive. Since G is a viscosity solution of (1.5), it is a viscosity subsolution of $G_t + (1 - Ad|\Phi(x)|)|DG| + AV(x) \cdot DG = 0$ and this Hamiltonian is convex in the region $\{x \in \mathbb{R}^2 :\ 1 > Ad|\Phi(x)|\}$. In order to prove (1.10), we will construct a suitable control trajectory within the valid region $\{1 > Ad|\Phi(x)|\}$. See Fig. 3.

Lemma 4.1 *Assume that* $d \in (0, \frac{1}{4})$, $A \geq 4$ *and* $Ad \leq 1 + \frac{d}{10}$. *Then there exist* $T > 0$ *and a Lipschitz continuous curve* $\xi : [0,T] \to [\frac{\pi}{2}, \pi] \times [0, \frac{\pi}{2}]$ *such that*

$$\xi(0) = \mathcal{X}_0 = (\frac{\pi}{2}, 0),\ \xi(T) = (\pi, \frac{\pi}{2}),\ T \leq \frac{C \log A}{A} \quad \textit{for a universal constant } C$$

and

$$\xi([0,T]) \subset \{x \in \mathbb{R}^2 :\ 1 > Ad|\Phi(x)|\}$$

and

$$|\dot{\xi} + AV(\xi)| \leq 1 - Ad|\Phi(\xi)| \quad \textit{for all a.e. } s \in [0,T].$$

Proof: See the left picture of Fig. 3. **Step 1:** Let $\xi_1(s) = (x_1(s), x_2(s))$ be a solution of

$$\begin{cases} \dot{\xi}_1 = -AV(\xi_1) + (1 - Ad|\Phi(\xi_1)|)^+ \frac{DH}{|DH|} & \text{for } s \geq 0 \\ \xi_1(0) = (\frac{\pi}{2}, 0). \end{cases}$$

Here r^+ is the positive part of number r. Denote $B = [\frac{\pi}{2}, \frac{2\pi}{3}] \times [-\frac{\pi}{6}, \frac{\pi}{6}]$ and

$$t_1 = \inf\{s \geq 0 |\ \xi(s) \notin B\}.$$

Since $\dot{x}_1(0) = A > 0$, we have that $t_1 > 0$. Also note that

$$\dot{x}_1(s) \geq A \sin x_1 \cos x_2 - 1 \geq \frac{3A}{4} - 1 \geq \frac{A}{2} \quad \text{when } \xi(s) \in B.$$

So $x_1(s)$ is strictly increasing in B and $t_1 < \frac{\pi}{3A}$. Moreover, since for a.e $s \in [0, t_1)$,

$$\left|\frac{dx_2(s)}{ds}\right| \leq A|\sin x_2 \cos x_1| + 1 \leq \frac{A}{4} + 1 \leq \frac{A}{2},$$

we derive that $|x_2(t_1)| \leq \frac{At_1}{2} < \frac{\pi}{6}$. This implies that $x_1(t_1) = \frac{2\pi}{3}$. Furthermore, for $x \in B$, $Ad|\Phi(x)| \leq \frac{(1+\frac{d}{10})}{2} < \frac{2}{3}$. We deduce that

$$\frac{dH(\xi_1(s))}{ds} = (1 - Ad|\Phi(\xi_1)|)|DH| > \frac{|DH|}{3} \quad \text{for a.e. } 0 \leq s \leq t_1.$$

Therefore $H(\xi_1(s)) \geq H(\xi_1(0)) = 0$ which implies that $x_2([0, t_1]) \subset [0, \frac{\pi}{6}]$ and

$$\xi_1([0, t_1]) \subset [\frac{\pi}{2}, \frac{2\pi}{3}] \times [0, \frac{\pi}{6}].$$

Since $\dot{x}_1(s) > 0$ and $\dot{x}_1(s) \leq A \sin x_1 \cos x_2 + 1$ for $s \in [0, t_1]$, by changing of variables $s \to s^{-1}(x_1)$, $x_1(s) \to x_1$ and $x_2(s) \to x_2(s^{-1}(x_1)) = x_2$(for abbreviation), we obtain

$$H(\xi_1(t_1)) \geq \frac{1}{3} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{|DH|}{A \sin x_1 \cos x_2 + 1}\, dx_1 > \frac{1}{8A} > \frac{d}{10}.$$

The first $>$ is due to $|DH| \geq \sin x_1 \cos x_2$ and $1 \leq \frac{A}{3} \sin x_1 \cos x_2$ for $x \in [\frac{\pi}{2}, \frac{2\pi}{3}] \times [0, \frac{\pi}{6}]$.

Step 2: Next we define $\xi_2 = (y_1(s), y_2(s)) : [t_1, +\infty) \to \mathbb{R}^2$ as

$$\begin{cases} \dot{\xi}_2(s) = -AV(\xi_2(s)) \\ \xi_2(t_1) = \xi_1(t_1). \end{cases}$$

Then $H(\xi_2(s)) \equiv H(\xi_1(t_1)) > \frac{d}{10}$. Since $|\Phi(x)| + |H(x)| \leq 1$, we have that

$$Ad|\Phi(\xi_2(s))| \leq (1 + \frac{d}{10})(1 - \frac{d}{10}) < 1 \quad \text{for } s \geq t_1.$$

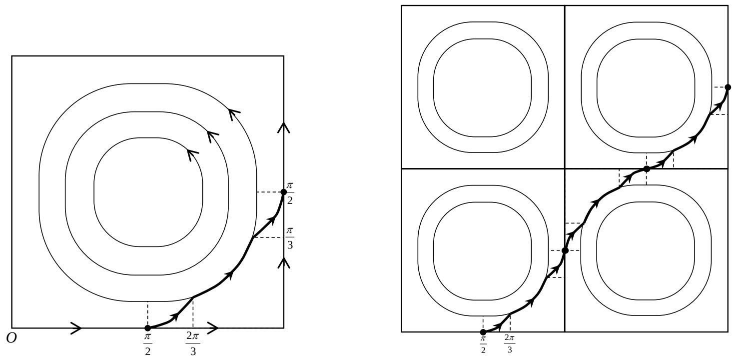

Fig. 3 Figures for Lemma 4.1 (left) and the proof of (1.10) in Theorem 1.3 (right)

According to step 1, $y_2(t_1) \in [0, \frac{\pi}{6}]$. We denote $t_2 = \min\{s \geq t_1 | \ y_2(s) = \frac{\pi}{3}\}$. Then $y_2(t_2) = \frac{\pi}{3}$ and $\xi_2([t_1, t_2]) \subset [\frac{2\pi}{3}, \pi] \times [0, \frac{\pi}{3}]$. See Fig. 3. Because $\sin\gamma \geq \frac{\gamma}{2}$ for $\gamma \in [0, \frac{\pi}{2}]$, we derive that

$$\dot{y}_2(s) = -A \sin y_2 \cos y_1 \geq \frac{Ay_2}{4} \quad \text{ for } s \in [t_1, t_2].$$

Since $y_2(t_1) \geq \sin y_1(t_1) \sin y_2(t_1) = H(\xi_2(t_1)) > \frac{d}{10}$ and $d > \frac{1}{A}$, we get that

$$t_2 - t_1 \leq \int_{\frac{d}{10}}^{\frac{\pi}{3}} \frac{4}{Ay_2} \, dy_2 \leq \frac{C \log A}{A}.$$

Also due to symmetry, $\xi_2(t_2) = \big(\pi - y_2(t_1), \ \pi - y_1(t_1)\big) = \big(\pi - x_2(t_1), \ \pi - x_1(t_1)\big)$.

Step 3 Let $\xi_1(s) = (x_1(s), x_2(s)) \subset [\frac{\pi}{2}, \frac{2\pi}{3}] \times [0, \frac{\pi}{6}]$ be the curve constructed in Step 1. For $s \in [t_2, t_2 + t_1]$, we define

$$\xi_3(s) = \big(\pi - x_2(t_2 + t_1 - s), \ \pi - x_1(t_2 + t_1 - s)\big).$$

Then $\xi_3(t_2) = \xi_2(t_2)$ and $\xi_3([t_2, t_2 + t_1]) \subseteq [\frac{5\pi}{6}, \pi] \times [\frac{\pi}{3}, \frac{\pi}{2}]$. Also it is easy to check that

$$\Phi(\xi_3(s)) = \Phi(\xi_1(t_2 + t_1 - s)) \quad \text{and} \quad \dot{\xi}_3 + AV(\xi_3)|_s = (\dot{\xi}_1 + AV(\xi_1))^{\top}|_{t_1+t_2-s}.$$

Here for $v = (v_1, v_2)$, $v^{\top} = (v_2, v_1)$. Therefore $1 - Ad|\Phi(\xi_3(s))| > 0$ and we have that $|\dot{\xi}_3(s) + AV(\xi_3(s))| \leq 1 - Ad|\Phi(\xi_3(s))|$ for $s \in [t_2, t_2 + t_1]$.

Step 4: Finally, let $T = t_1 + t_2$ and we define that

$$\xi(s) = \begin{cases} \xi_1(s) & \text{for } s \in [0, t_1] \\ \xi_2(s) & \text{for } s \in [t_1, t_2] \\ \xi_3(s) & \text{for } s \in [t_2, T]. \end{cases}$$

It is easy to see that ξ and T satisfy requirements in the statement of the lemma. □

Proof of (1.10). Due to symmetry, we may assume that the unit vector $p = (p_1, p_2)$ satisfies that $p_1 \leq 0$ and $p_2 \leq 0$. We will construct a suitable global control trajectory in the region $\{x : \ Ad|\Phi(x)| < 1\}$. See the right picture of Fig. 3. Let $\xi(s) = (x_1(s), x_2(s)) :$ $[0, T] \to [\frac{\pi}{2}, \pi] \times [0, \frac{\pi}{2}]$ be the one constructed in Lemma 4.1. Let $\tilde{\xi}(s) : [0, T] \to [\pi, \frac{3\pi}{2}] \times$ $[\frac{\pi}{2}, \pi]$ be a suitable reflection and translation of ξ, i.e

$$\tilde{\xi}(s) = (x_2(s) + \pi, x_1(s)).$$

Then $\Phi(\tilde{\xi}) = -\Phi(\xi)$ and $\dot{\tilde{\xi}} + AV(\tilde{\xi}) = (\dot{\xi} + AV(\xi))^{\top}$. Here for $v = (v_1, v_2)$, $v^{\top} = (v_2, v_1)$. Through translations, we define $\Upsilon(s) : [0, +\infty) \to \{x : \ 1 > Ad|\Phi(x)|\}$ as follows

$$\Upsilon(s) = \begin{cases} \xi(s - kT) + \frac{k}{2}(\pi, \pi) & \text{when } k \text{ is even and } s \in [kT, (k+1)T] \\ \tilde{\xi}(s - kT) + \frac{k-1}{2}(\pi, \pi) & \text{when } k \text{ is odd and } s \in [kT, (k+1)T]. \end{cases}$$

Then $|\dot{\Upsilon} + AV(\Upsilon)| \leq 1 - Ad|\Phi(\Upsilon)|$ a.e. Note that G is a viscosity subsolution of

$$G_t + (1 - Ad|\Phi(x)|)|DG| + AV(x) \cdot DG = 0.$$

Then for fixed t, $\frac{d}{ds}G(\Upsilon(t-s), s) \leq 0$ for a.e $s \in (0, t)$ since the above Hamiltonian is convex in the region $\{1 > Ad|\Phi(x)|\}$. Accordingly, $G(\mathcal{X}_0, t) \leq G(\Upsilon(t), 0) = p \cdot \Upsilon(t)$. Choose $m \in \mathbb{N}$ such that $t \in [(m-1)T, mT)$. Then $G(\mathcal{X}_0, t) \leq -\frac{(m-1)\pi}{2}$. Since $\frac{m}{t} \geq \frac{1}{T} \geq C\frac{A}{\log A}$, (1.10) holds. □

In this theorem, we did not really identify the exact range of intermediate values of A where homogenization fails. Especially, it is not clear to us whether the upper bound of those intermediate values is $\frac{C}{d}$ or $\frac{1}{d} + C$ for some universal constant C. The choice of $\mathcal{X}_0$ is not special either, other than simplifying our calculations. Actually, (1.10) is true in a connected open set M away from narrow neighborhood around hyperbolic stagnation points. See the left picture of Fig. 2. The proof is to show that any point $x \in M$ can be connected to $\mathcal{X}_0 = (\frac{\pi}{2}, 0)$ through appropriate control trajectories within the region $\{1 > Ad|\Phi(x)|\}$. Therefore stationary unburned islands are formed.

5 Proof of Theorems 1.4, 1.5 and 1.6

Throughout this section, we denote $\mathcal{H}(p, x) = |p| + AV(x) \cdot p - Ad\Phi(x)\frac{|p_1|^2 - |p_2|^2}{|p|}$ for V given by (1.4) and $\Phi(x) = \cos x_1 \cos x_2$ and

(1) $M(t)$ as the set of measurable functions $[0, t] \to [-1, 1]^2 = S_1$;
(2) $N(t)$ as the set of measurable functions $[0, t] \to [-1, 1] = S_2$;
(3) $\Sigma(t)$ as set of strategies for player I, i.e., nonanticipating mappings from $N(t)$ to $M(t)$;
(4) $\Delta(t)$ as the set of strategies for player II, i.e., nonanticipating mappings from $M(t)$ to $N(t)$.

We first prove several lemmas. The first one says that overall the flame will not move backward along vertical or horizontal directions.

Lemma 5.1 *Let $G(x, t) \in C(\mathbb{R}^2 \times [0, +\infty))$ be the viscosity solution of*

$$\begin{cases} G_t + |DG| + AV(x) \cdot DG - Ad\Phi(x)\frac{|G_{x_1}|^2 - |G_{x_2}|^2}{|DG|} = 0 \\ G(x, 0) = p \cdot x. \end{cases}$$

Assume $p = (\pm 1, 0)$ or $(0, \pm 1)$. Then when $A \geq 0$ and $d \in [0, 1)$

$$G(x, t) \leq p \cdot x + 4\pi \quad \text{for all } (x, t) \in \mathbb{R}^2 \times [0, +\infty). \tag{5.1}$$

Proof: Since $G - p \cdot x$ is periodic, it suffices to prove the above inequality for $x \in [0, 2\pi]^2$ and $p = (\pm 1, 0)$. The proof for $p = (0, \pm 1)$ is similar. Denote

$$\begin{cases} L_1 = \{(3\pi - \rho_d, x_2)| \ x_2 \in \mathbb{R}\} \\ L_2 = \{(-\pi + \rho_d, x_2)| \ x_2 \in \mathbb{R}\} \end{cases}$$

for $\rho_d \in [0, \frac{\pi}{4}]$ satisfying that $d\cos\rho_d = \sin\rho_d$. In order to apply (2.2), we introduce an auxiliary Hamiltonian H_0 as follows:

$$H_0(p,x) = \begin{cases} |p_1| + AV(x)\cdot p - Ad\Phi(x)|p_1| + 2Ad\Phi(x)|p_2|, & \text{if } \Phi(x) \le 0 \\ |p_1| + AV(x)\cdot p - Ad\Phi(x)|p_1| & \text{if } \Phi(x) \ge 0. \end{cases}$$

It is clear that H_0 is Lipschitz continuous and periodic in the x variable. Given (2.3), we also have that $H_0 \le \mathcal{H}$. Let $R \in C(\mathbb{R}^2 \times [0,+\infty))$ be the viscosity solution of

$$\begin{cases} R_t + H_0(DR, x) = 0 \\ R(x,0) = p\cdot x \end{cases}$$

such that $R - p\cdot x$ is periodic. Note that G is a viscosity subsolution of the above equation. Standard comparison principle says that $G \le R$. Owing to (2.2),

$$-R(x,t) = \sup_{\Gamma\in\Sigma(t)} \inf_{\beta\in N(t)} \{-p\cdot\xi(t)\}, \tag{5.2}$$

where

$$\begin{cases} \dot\xi(s) = f_0(\xi, \Gamma(\beta), \beta) & \text{for a.e. } 0 \le s \le t \\ \xi(0) = x. \end{cases}$$

Here $f_0 = f_0(x,\eta,\mu) : \mathbb{R}^2 \times [-1,1]^2 \times [-1,1] \to \mathbb{R}^2$ is given as follows ($\eta = (\eta_1, \eta_2)$):

$$f_0(x,\eta,\mu) = \begin{cases} \big(\eta_1 - Ad\Phi(x)\eta_1,\ -2Ad\Phi(x)\mu\big) - AV(x) & \text{if } \Phi(x) \le 0 \\ \big(\eta_1 + Ad\Phi(x)\mu,\ 0\big) - AV(x) & \text{if } \Phi(x) \ge 0 \end{cases}$$

Now fix $x_0 \in [0, 2\pi]^2$ and we will verify (5.1) for $G(x_0, t)$ by choosing a suitable strategy Σ_{x_0} of player I. For $\beta \in N(t)$, let $\xi(s) = (x_1(s), x_2(s))$ be the unique solution of

$$\begin{cases} \dot\xi(s) = f_0(\xi, \phi(\xi), \beta) & \text{for a.e. } 0 \le s \le t \\ \xi(0) = x_0. \end{cases}$$

Here for $x = (x_1, x_2)$

$$\phi(x) = \Big(\frac{2\pi - 2x_1}{4\pi - 2\rho_d}, 0\Big).$$

We claim that

$$\begin{cases} \dot x_1 < 0 & \text{when } \xi \text{ is near the line } L_1 = \{(3\pi - \rho_d, x_2) |\ x_2 \in \mathbb{R}\} \\ \dot x_1 > 0 & \text{when } \xi \text{ is near the line } L_2 = \{(-\pi + \rho_d, x_2) |\ x_2 \in \mathbb{R}\}. \end{cases}$$

It suffices to check the first one. The other one is similar. Note that $\sin x_1 + d\cos x_1 = 0$ for $x \in L_1$.

Case 1: $\xi(s) \in L_1$ and $\Phi(\xi(s)) \le 0$, then

$$\dot x_1(s) = A\sin x_1 \cos x_2 + \frac{2\pi - 2x_1}{4\pi - 2\rho_d}(1 - Ad\cos x_1 \cos x_2) = -1$$

Case 2: $\xi(s) \in L_1$ and $\Phi(\xi(s)) \ge 0$, then

$$\dot x_1(s) \le A\sin x_1 \cos x_2 + \frac{2\pi - 2x_1}{4\pi - 2\rho_d} + Ad\cos x_1 \cos x_2 = -1.$$

Hence ξ must be trapped in the strip bounded by L_1 and L_2, i.e,

$$x_1([0,t]) \subseteq (-\pi + \rho_d, 3\pi - \rho_d).$$

and $\phi(\xi) \in M(t)$. Accordingly, if player I chooses the strategy $\Sigma_{x_0} : N(t) \to M(t)$ as $\Sigma_{x_0}(\beta)(s) = \phi(\xi)(s)$, the representation formula (5.2) together with the comparison principle imply that for $p = (\pm 1, 0)$

$$-G(x_0, t) \geq -R(x_0, t) \geq -p \cdot x_0 - 4\pi.$$

□

Remark 5.1 *Here is another way to view the above lemma. Assume that the initial flame front is line L_1 (i.e., $G(x, 0) = x_1 - 3\pi + \rho_d$). Note that starting normal velocity $v_n = 1 + AV \cdot \mathbf{n} + Ad\mathbf{n} \cdot S \cdot \mathbf{n}$ is constant 1 along L_1 for $\mathbf{n} = (1, 0)$. Comparison principle of level set therefore implies that the flame front is always moving forward, i.e., for any $0 \leq t_1 \leq t_2$,*

$$\{x|\ G(x, t_1) \leq 0\} \subseteq \{x|\ G(x, t_2) \leq 0\}.$$

If we consider the simplified Hamiltonian $\tilde{\mathcal{H}} = |p| + AV(x) \cdot p - Ad\Phi(x) \cdot (|p_1| - |p_2|)$, the above lemma is true for all directions. Precisely speaking,

Lemma 5.2 *Let $\tilde{G}(x, t) \in C(\mathbb{R}^2 \times [0, +\infty))$ be the unique viscosity solution of*

$$\begin{cases} \tilde{G}_t + |D\tilde{G}| + AV(x) \cdot D\tilde{G} - Ad\Phi(x) \cdot (|\tilde{G}_{x_1}| - |\tilde{G}_{x_2}|) = 0 \\ \tilde{G}(x, 0) = p \cdot x. \end{cases}$$

Then for all unit vector p, $A \geq 0$ and $d \in [0, 1)$

$$\tilde{G}(x, t) \leq p \cdot x + 4\sqrt{2}\pi \quad \text{for all } (x, t) \in \mathbb{R}^2 \times [0, +\infty). \tag{5.3}$$

If we replace the initial data $p \cdot x$ by any Lipschitz continuous function $g(x)$, the inequality becomes

$$\tilde{G}(x, t) \leq g(x) + 4\sqrt{2}\pi ||Dg||_{L^\infty(\mathbb{R}^2)} \quad \text{for all } (x, t) \in \mathbb{R}^2 \times [0, +\infty).$$

Proof: The argument is very similar to that of Lemma 5.1. We will just give a sketch. Replace (H_0, f_0) in the proof of Lemma 5.1 by $(\tilde{H}_0, \tilde{f}_0)$ which are defined as follows:

$$\tilde{H}_0(p, x) = \frac{1}{2}(|p_1| + |p_2|) + AV(x) \cdot p - Ad\Phi(x)(|p_1| - |p_2|) \leq \tilde{\mathcal{H}}$$

and $\tilde{f}_0 = \tilde{f}_0(x, \eta, \mu) : \mathbb{R}^2 \times [-1, 1]^2 \times [-1, 1] \to \mathbb{R}^2$ is given as follows:

$$\tilde{f}_0(x, \eta, \mu) = \begin{cases} (\frac{\eta_1}{2} - \eta_1 Ad\Phi(x),\ \frac{\eta_2}{2} - \mu Ad\Phi(x)) - AV(x) & \text{if } \Phi(x) \leq 0 \\ (\frac{\eta_1}{2} + \mu Ad\Phi(x),\ \frac{\eta_2}{2} + \eta_2 Ad\Phi(x)) - AV(x) & \text{if } \Phi(x) \geq 0. \end{cases}$$

Now for fixed $x_0 \in [0, 2\pi]^2$, we will verify (5.3) by choosing a strategy Σ_{x_0} of player I. For $\beta \in N(t)$, let $\xi(s) = (x_1(s), x_2(s))$ be the unique solution of

$$\begin{cases} \dot{\xi}(s) = \tilde{f}_0(\xi, \phi(\xi), \beta) & \text{for a.e. } 0 \leq s \leq t \\ \xi(0) = x_0. \end{cases}$$

Here for $x = (x_1, x_2)$

$$\phi(x) = (\frac{2\pi - 2x_1}{4\pi - 2\rho_d}, \frac{2\pi - 2x_2}{4\pi - 2\rho_d}).$$

Almost exactly the same as the proof of Lemma 5.1, we can show that $\dot{x}_1 < 0$ near L_1, $\dot{x}_1 > 0$ near L_2, $\dot{x}_2 < 0$ near L_3 and $\dot{x}_2 > 0$ near L_4 for

$$\begin{cases} L_3 = \{(x_1, 3\pi - \rho_d)|\ x_1 \in \mathbb{R}\} \\ L_4 = \{(x_1, -\pi + \rho_d)|\ x_1 \in \mathbb{R}\}. \end{cases}$$

Hence ξ will be trapped inside of the box B bounded by L_1, L_2, L_3 and L_4. That is $\xi([0,t]) \subset B$. Accordingly, if player I chooses the strategy $\Sigma_{x_0} : N(t) \to M(t)$ as $\Sigma_{x_0}(\beta)(s) = \boldsymbol{\phi}(\xi)(s)$, then the corresponding representation formula together with the comparison principle imply (5.3). □

For the original Hamiltonian $\mathcal{H} = |p| + AV(x)\cdot p - Ad\Phi(x)\frac{|p_1|^2-|p_2|^2}{|p|}$, it is not clear to us whether part of the flame front might retreat (move backwards) through the corners of the box B when A is very large and p is neither horizontal nor vertical.

To prove that flame front will eventually stop moving forward along any direction at high flow intensity is much more subtle. Let us look at the moving flame front within the domain $[-\pi, 0] \times [-\pi, \pi]$ to demonstrate the basic idea. See the right picture of Figure 2. The strain rate along the x_1 direction is negative within $(-\frac{\pi}{2}, \frac{\pi}{2})^2$. Hence due to the same mechanism as the proof of (1.9) in Theorem 1.3, the flame front is not able to enter a narrow strip centered at the origin with width $2(d-d^2)$ and length close to π (cold block) for large A. The flame has to move to either $(-\frac{\pi}{2}, 0) \times (\frac{\pi}{2}, \pi)$ or $(-\frac{\pi}{2}, 0) \times (-\pi, -\frac{\pi}{2})$. In both regions, the strain rate becomes positive along x_1 direction and tries to push the flame front forward. However, the flow begins to bend toward negative x_1 direction. This leads to a pretty subtle competition between the flow and strain rate within a narrow distance of $d - d^2$. Delicate computations show that the flow wins and the flame front is not able to reach the line $x_1 = 0$ before it arrives at the next cold block. The flame front then fails to move further. This rough idea will be made rigorous through the game theory interpretation. By comparison principle, we only need to look at the game starting from points on the boundary. For our purpose, player II does not have to figure out the optimal strategy to minimize the final payoff. It suffices to find a strategy to steer the state of the trajectory of the game into those narrow strips of cold block (trapping region). This can be viewed as a special example of the pursuit game in [26]. If $x = (x_1, x_2)$ represents the relative position of the evader (player I) to the pursuer (player II), the goal of the pursuer is to trap the evader in a fixed region.

Lemma 5.3 *For $x \in [\frac{\pi}{2}, \frac{5\pi}{4}] \times [\frac{d^2}{2} - d, d - \frac{d^2}{2}]$ and $\alpha \in M(t)$, let ξ be the unique solution of*

$$\begin{cases} \dot{\xi} = -AV(\xi) + b(\xi, \alpha) & \text{for a.e. } 0 \le s \le t \\ \xi(0) = x \end{cases}$$

with $b : \mathbb{R}^2 \times [-1,1]^2 \to \mathbb{R}^2$ and $\eta = (\eta_1, \eta_2)$

$$b(x, \eta) = \left((-2Ad\Phi(x) + 1)\eta_1,\ \eta_2 + \frac{Adx_2}{d - \frac{d^2}{4}}\Phi(x) \right).$$

Then there exists a universal constant $d_0 \in (0, \frac{1}{17})$ such that when $d < d_0$ and $A > \frac{8}{d^3}$

$$\xi([0,t]) \subset [\frac{\pi}{2}, \frac{5\pi}{4}] \times [\frac{d^2}{4} - d, d - \frac{d^2}{4}] \tag{5.4}$$

Proof: The reason we need to extend the range of $|x_2|$ from $d - \frac{d^2}{2}$ to $d - \frac{d^2}{4}$ is because $\Phi(x)$ is zero along the line $x_1 = \frac{\pi}{2}$. For $\xi(s) = (x_1(s), x_2(s))$ and $\alpha(s) = (\alpha_1(s), \alpha_2(s))$, we rewrite the above ODE as

$$\begin{cases} \dot{x}_1(s) = A \sin x_1 \cos x_2 + (-2Ad\Phi(\xi) + 1)\alpha_1 \\ \dot{x}_2(s) = -A \sin x_2 \cos x_1 + \alpha_2 + \frac{Adx_2}{d - \frac{d^2}{4}} \cdot \Phi(\xi) \end{cases}$$

We first fix d_0. Choose $d_0 \in (0, \frac{1}{17})$ such that for $0 < d < d_0$

$$\sin(\frac{\pi}{2} + d)\cos(d - \frac{d^2}{4}) - 2d|\cos(\frac{\pi}{2} + d)| - \frac{d^3}{8} = 1 - O(d) \ge \frac{1}{2}, \tag{5.5}$$

$$\sin(\frac{\pi}{4})\cos(d-\frac{d^2}{4})-2d\cos\frac{\pi}{4}-\frac{d^3}{8}=\frac{\sqrt{2}}{2}-O(d)>0 \tag{5.6}$$

and

$$d\cos(d-\frac{d^2}{4})-\sin(d-\frac{d^2}{4})-\frac{d^3}{8\sin d}=\frac{d^2}{8}-O(d^3)>0.$$

Note that the previous inequality implies that

$$d\cos(d-\frac{d^2}{4})|\cos x_1|-\sin(d-\frac{d^2}{4})|\cos x_1|-\frac{d^3}{8}>0 \quad \text{for } x_1\in(\frac{\pi}{2}+d,\frac{5\pi}{4}). \tag{5.7}$$

Now we assume that $d<d_0$ and $A\geq\frac{8}{d^3}$. It is easy to check that

$$\begin{cases}(5.5)\Rightarrow \dot{x}_1(s)>0 & \text{if } \xi(s) \text{ is close to } \{(\frac{\pi}{2}+d,x_2)|\ \frac{d^2}{4}-d\leq x_2\leq d-\frac{d^2}{4}\}\\ (5.6)\Rightarrow \dot{x}_1(s)<0 & \text{if } \xi(s) \text{ is close to } \{(\frac{5\pi}{4},x_2)|\ \frac{d^2}{4}-d\leq x_2\leq d-\frac{d^2}{4}\}\\ (5.7)\Rightarrow \dot{x}_2(s)>0 & \text{if } \xi(s) \text{ is close to } \{(x_1,\frac{d^2}{4}-d)|\ \frac{\pi}{2}+d\leq x_1\leq\frac{5\pi}{4}\}\\ (5.7)\Rightarrow \dot{x}_2(s)<0 & \text{if } \xi(s) \text{ is close to } \{(x_1,d-\frac{d^2}{4})|\ \frac{\pi}{2}+d\leq x_1\leq\frac{5\pi}{4}\}\end{cases}$$

Hence if there exists $\bar{t}\in(0,t]$ such that $\xi(\bar{t})\in[\frac{\pi}{2}+d,\frac{5\pi}{4}]\times[\frac{d^2}{4}-d,d-\frac{d^2}{4}]$, the curve will be trapped in the region after $\bar{t}$, i.e.,

$$\xi([\bar{t},t])\subset[\frac{\pi}{2}+d,\frac{5\pi}{4}]\times[\frac{d^2}{4}-d,d-\frac{d^2}{4}]. \tag{5.8}$$

Since $A\cos(d-\frac{d^2}{4})>\frac{8\cos d}{d^3}>1$, we have that

$$\dot{x}_1(s)>0 \quad \text{if } \xi(s) \text{ is close to } \{(\frac{\pi}{2},x_2)|\ \frac{d^2}{4}-d\leq x_2\leq d-\frac{d^2}{4}\}.$$

Therefore if (5.4) is not true, there must exist $t_0\in(0,t)$ such that

$$\xi([0,t_0])\subset[\frac{\pi}{2},\frac{\pi}{2}+d]\times[\frac{d^2}{4}-d,d-\frac{d^2}{4}]$$

and $|x_2(t_0)|=d-\frac{d^2}{4}$. For a.e. $s\in[0,t_0]$, due to (5.5),

$$\dot{x}_1(s)\geq A\sin x_1\cos x_2+2Ad\Phi(\xi)-1\geq\frac{A}{2}.$$

Accordingly, $t_0\leq\frac{2d}{A}$. Due to $\sin d\leq d$, for a.e. $s\in[0,t_0]$, we also have that

$$|\dot{x}_2(s)|\leq A|\sin x_2\cos x_1|+Ad|\cos x_1||\cos x_2|+1\leq 2Ad^2+1.$$

Therefore $|x_2(t_0)|\leq|x_2(0)|+t_0(2Ad^2+1)\leq d-\frac{d^2}{2}+4d^3+\frac{d^4}{4}<d-\frac{d^2}{4}$. The last "$<$" is due to $d<d_0<\frac{1}{17}$. This is a contradiction. □

Lemma 5.4 *Let $\alpha=(\alpha_1,\alpha_2):[0,+\infty)\to[-1,1]^2$ be a measurable function. For $\theta\in[\frac{4d}{5},\frac{\pi}{2}]$, assume that $\xi=(x_1,x_2)$ is the unique solution of*

$$\begin{cases}\dot{\xi}=-AV(\xi)+(\alpha_1,0)+(0,\ Ad\Phi(\xi)\alpha_2+\alpha_2) & \text{a.e. for } s\in(0,\infty)\\ \xi(0)=(\theta,0).\end{cases}$$

Denote $t_0=\min\{s\geq 0|\ x_1(s)=\frac{\pi}{2}\}$. Then there exists a universal constant $d_0\in(0,1)$ such that when $d<d_0$ and $A>\frac{1}{d^3}$, we have that $t_0<\infty$,

$$\xi([0,t_0])\subseteq[\frac{4d}{5},\frac{\pi}{2}]\times[-\frac{\pi}{3},\frac{\pi}{3}]$$

and

$$\xi(t_0)\in\{(\frac{\pi}{2},\tau)|\ \frac{d^2}{2}-d\leq\tau\leq d-\frac{d^2}{2}\}.$$

Proof: Again for $\xi(s) = (x_1(s), x_2(s))$ and $s \in [0, +\infty)$, we rewrite the above dynamical system as

$$\begin{cases} \dot{x}_1(s) = A\sin x_1 \cos x_2 + \alpha_1(s) \\ \dot{x}_2(s) = -A\sin x_2 \cos x_1 + \alpha_2(s)\cdot(Ad\Phi(\xi)+1). \end{cases}$$

For clarity, we first pick d_0. Choose $d_0 \in (0, \frac{1}{3})$ such that when $d < d_0$ and $A > \frac{1}{d^3}$

$$d(1-\sin\frac{4d}{5}) + \int_{\frac{4d}{5}}^{\frac{\pi}{2}} \frac{8}{Ax-4}\,dx \le d - \frac{4d^2}{5} + O(d^3|\log d|) < d - \frac{3d^2}{4}. \tag{5.9}$$

$$(Ad+1)\int_{\frac{4d}{5}}^{\frac{\pi}{2}} \frac{4}{Ax-4}\,dx \le O(d|\log d|) < 1. \tag{5.10}$$

We assume that $d < d_0$ and $A > \frac{1}{d^3}$. Denote $B = [\frac{4d}{5}, \frac{\pi}{2}] \times [-\frac{\pi}{3}, \frac{\pi}{3}]$ and

$$t_0 = \inf\{s \ge 0 |\ \xi(s) \notin B\}.$$

Because $\sin\gamma \ge \frac{\gamma}{2}$ for $\gamma \in [0, \frac{\pi}{2}]$, we have that $A\sin x_1 \cos x_2 - 1 > \frac{Ax_1}{4} - 1 > 0$ for $x \in B$. Since $\dot{x}_1(s) \ge A\sin x_1\cos x_2 - 1$, $x_1(s)$ is strictly increasing when $\xi(s) \in B$. Also, $\dot{x}_1(s) > 0$ for s close to 0 which implies that $t_0 > 0$. Accordingly,

$$0 < t_0 < \int_{\frac{4d}{5}}^{\frac{\pi}{2}} \frac{4}{Ax_1 - 4}\,dx_1 < \infty$$

Since $x_2(0) = 0$, due to (5.10) and

$$\frac{d|x_2(s)|}{ds} = \dot{x}_2(s)\cdot \mathrm{sign}(x_2(s)) \le Ad+1 \quad \text{for a.e. } s \in [0, t_0],$$

we deduce that $|x_2(t_0)| \le (Ad+1)t_0 < \frac{\pi}{3}$. Therefore $x_1(t_0) = \frac{\pi}{2}$. Since $x_1(s)$ is strictly increasing for $s \in [0, t_0)$, t_0 is therefore the first moment $x_1(s)$ reaches $\frac{\pi}{2}$. The tricky part is to derive the subtle upper bound $|x_2(t_0)| \le d - \frac{d^2}{2}$. Our strategy here is to compute $|H(\xi(t_0))|$ instead of estimating x_2 directly which is hard to control. Note that for a.e. $s \in [0, t_0]$,

$$\begin{aligned} \left|\tfrac{dH(\xi)}{ds}\right| &\le |H_{x_1}| + |H_{x_2}| + Ad\cos x_1 \sin x_1 \cos^2 x_2 \\ &\le 1 + Ad\sin x_1 \cos x_2 \cos x_1 \end{aligned}$$

Since $H(\xi(0)) = 0$,

$$|H(\xi(t_0))| \le \int_0^{t_0} (1 + Ad\sin x_1\cos x_2\cos x_1)\,dt.$$

Owing to

$$\dot{x}_1(s) > A\sin x_1\cos x_2 - 1 > 0 \quad \text{for a.e } s \in [0, t_0],$$

by changing of variables $s \to s^{-1}(x_1)$, $x_1(s) \to x_1$ and $x_2(s) \to x_2(s^{-1}(x_1))$($x_2$ for abbreviation), we deduce that

$$\begin{aligned} |H(\xi(t_0))| &\le \textstyle\int_{\frac{4d}{5}}^{\frac{\pi}{2}} \frac{1+Ad\sin x_1\cos x_2\cos x_1}{A\sin x_1 \cos x_2 - 1}\,dx_1 \\ &= d\textstyle\int_{\frac{4d}{5}}^{\frac{\pi}{2}} \cos x_1\,dx_1 + \int_{\frac{4d}{5}}^{\frac{\pi}{2}} \frac{1+d\cos x_1}{A\sin x_1\cos x_2 - 1}\,dx_1 \\ &< d(1-\sin\frac{4d}{5}) + \textstyle\int_{\frac{4d}{5}}^{\frac{\pi}{2}} \frac{8}{Ax-4}\,dx_1. \end{aligned}$$

The last inequality is due to $A\sin x_1\cos x_2 \ge \frac{Ax_1}{4}$ for $x \in [0, \frac{\pi}{2}] \times [-\frac{\pi}{3}, \frac{\pi}{3}]$ and $1 + d\cos x_1 < 2$. Thanks to (5.9), $|\sin(x_2(t_0))| = |H(\xi(t_0))| < d - \frac{3d^2}{4}$. By Taylor expansion, for $\omega \in [0, \frac{\pi}{3}]$, $\sin\omega \ge \omega - \frac{\omega^3}{6}$. Hence it is easy to see that $|x_2(t_0)| < d - \frac{d^2}{2}$. □

Lemma 5.5 *Assume $g \in C(\mathbb{R}^2)$ is Lipschitz continuous. Let $G(x,t) \in C(\mathbb{R}^2 \times [0,\infty))$ be the viscosity solution of*

$$\begin{cases} G_t + |DG| + V(x) \cdot DG - Ad\Phi(x)\frac{|G_{x_1}|^2 - |G_{x_2}|^2}{|DG|} = 0 \\ G(x,0) = g(x). \end{cases}$$

Then there exists a universal constant $d_0 \in (0, \frac{1}{17})$ such that when $d < d_0$ and $A > \frac{8}{d^3}$

$$G(x,t) \geq \min_{y \in x + [-2\pi, 2\pi]^2} g(y) \quad \text{for all } (x,t) \in \mathbb{R}^2 \times [0, +\infty)$$

Proof: Let d_0 be the smaller one from Lemma 5.3 and 5.4. Due to symmetry, it suffices to show that when $x \in [0,\pi]^2$

$$G(x,t) \geq \min_{[-\pi,2\pi]^2} g \quad \text{for all } t \geq 0.$$

Note that $[-\pi, 2\pi]^2 \subset x + [-2\pi, 2\pi]^2$ for any $x \in [0,\pi]^2$. By comparison principle, we only need to establish this inequality along the boundary of $[0,\pi] \times [0,\pi]$ which consists of four line segments. Since the proof is similar, we will just prove the above bound for $x_0 \in \{(\theta, 0) |\ 0 \leq \theta \leq \pi\}$.

Case 1: $0 \leq \theta \leq \frac{4d}{5}$ or $\frac{\pi}{2} \leq \theta \leq \pi$. In order to use the representation formula (2.2), we introduce an auxiliary Hamiltonian $H_1(p,x)$

$$H_1(p,x) = \begin{cases} |p_1| + |p_2| + AV(x) \cdot p - Ad\Phi(x)|p_1| + 2Ad\Phi(x)|p_2| & \text{if } \Phi(x) \geq 0 \\ |p_1| + |p_2| + AV(x) \cdot p - 2Ad\Phi(x)|p_1| + Ad\Phi(x)|p_2| & \text{if } \Phi(x) \leq 0 \end{cases}$$

Note that H_1 is periodic and Lipschitz continuous on the x variable. Also due to (2.3), $H_1 \geq \mathcal{H}$. Suppose that $U \in C(\mathbb{R}^2 \times [0, +\infty))$ is the viscosity solution of

$$\begin{cases} U_t + H_1(DU, x) = 0 \\ U(x,0) = g(x) \end{cases}$$

given by the differential game representation formula

$$-U(x,t) = \inf_{\Lambda \in \Delta(t)} \sup_{\alpha \in M(t)} \{-g(\xi(t))\} \tag{5.11}$$

for

$$\begin{cases} \dot{\xi} = f(\xi, \alpha, \Lambda(\alpha)) & \text{for a.e. } s \in [0,t] \\ \xi(0) = x. \end{cases}$$

Here $f_1 = f_1(x, \eta, \mu)$ for $(x, \eta, \mu) \in \mathbb{R}^2 \times [-1,1]^2 \times [-1,1]$ is given as follows:

$$f_1(x,\eta,\mu) = \begin{cases} (\eta_1 + Ad\Phi(x)\mu,\ \eta_2 + 2Ad\Phi(x)\eta_2) - AV(x) & \text{if } \Phi(x) \geq 0 \\ (\eta_1 - 2Ad\Phi(x)\eta_1,\ \eta_2 - Ad\Phi(x)\mu) - AV(x) & \text{if } \Phi(x) \leq 0. \end{cases}$$

Case 1.1: For fixed $\theta_0 \in [0, \frac{4d}{5}]$, we will choose a strategy Λ_{θ_0} of player II. For $\alpha \in M(t)$, let $\xi = (x_1(s), x_2(s))$ be the unique solution of

$$\begin{cases} \dot{\xi}(s) = -AV(\xi) + \left(\alpha_1 - \frac{Adx_1(s)}{d - \frac{d^2}{4}}\Phi(\xi),\ 2Ad\Phi(\xi)\alpha_2 + \alpha_2\right) & \text{for a.e. } s \in (0,t) \\ \xi(0) = (\theta_0, 0). \end{cases}$$

Note that the flow on the strip $[\frac{d^2}{4}-d, d-\frac{d^2}{4}]\times[-\frac{\pi}{2},\frac{\pi}{2}]$ is identical to that on the strip $[\frac{\pi}{2},\frac{3\pi}{2}]\times[\frac{d^2}{4}-d, d-\frac{d^2}{4}]$ after a rotation of $\frac{\pi}{2}$ and $\Phi(x)$ changes sign. Because $\frac{4d}{5}\in(0, d-\frac{d^2}{4})$, according to (5.8) in the proof of Lemma 5.3,

$$\xi([0,t])\subseteq[\frac{d^2}{4}-d, d-\frac{d^2}{4}]\times[-\frac{\pi}{4},\frac{\pi}{2}]\subset\{\Phi\geq 0\}.$$

Then $\dot{\xi}(s)=f_1(\xi,\alpha,-\frac{x_1(s)}{d-\frac{d^2}{4}})$. So if player II chooses the strategy $\Lambda_{\theta_0}: M(t)\to N(t)$ as

$$\Lambda_{\theta_0}(\alpha)(s)=-\frac{x_1(s)}{d-\frac{d^2}{4}}\quad\text{for } s\in[0,t],$$

(5.11) together with comparison principle imply that for $x_0=(\theta_0,0)$

$$-G(x_0,t)\leq -U(x_0,t)\leq\max_{[-\frac{\pi}{2},\frac{\pi}{2}]^2}(-g)\quad\text{for all } t\geq 0.$$

Case 1.2: For a fixed $\theta_0\in[\frac{\pi}{2},\pi]$, we will choose a strategy of Λ_{θ_0} of player II. For $\alpha\in M(t)$, let $\xi(s)=(x_1(s),x_2(s))$ be unique solution of

$$\begin{cases}\dot{\xi}(s)=-AV(\xi)+\left((-2Ad\Phi(\xi)+1)\alpha_1,\ \alpha_2+\frac{Adx_2(s)}{d-\frac{d^2}{4}}\cdot\Phi(\xi)\right) & \text{for } s\in[0,t]\\ \xi(0)=(\theta_0,0).\end{cases}$$

By Lemma 5.3, $\xi([0,t])\subset[\frac{\pi}{2},\frac{5\pi}{4}]\times[\frac{d^2}{4}-d, d-\frac{d^2}{4}]\subset\{\Phi\leq 0\}$ and $\dot{\xi}=f_1(\xi,\alpha,-\frac{x_2(s)}{d-\frac{d^2}{4}})$. Hence if player II chooses the strategy $\Lambda_{\theta_0}: M(t)\to N(t)$ as

$$\Lambda_{\theta_0}(\alpha)(s)=-\frac{x_2(s)}{d-\frac{d^2}{4}}\quad\text{for } s\in[0,t],$$

(5.11) together with comparison principle imply that for $x_0=(\theta_0,0)$

$$-G(x_0,t)\leq -U(x_0,t)\leq\max_{[\frac{\pi}{2},\frac{5\pi}{4}]\times[-\frac{\pi}{2},\frac{\pi}{2}]}(-g)\quad\text{for all } t\geq 0.$$

Case 2: $\frac{4d}{5}<\theta<\frac{\pi}{2}$. We define an auxiliary Hamiltonian H_2 as follows:

$$H_2(p,x)=\begin{cases}|p_1|+|p_2|+AV(x)\cdot p+Ad\Phi(x)|p_2| & \text{if } \Phi(x)\geq 0\\ |p_1|+|p_2|+AV(x)\cdot p-2Ad\Phi(x)|p_1|+Ad\Phi(x)|p_2| & \text{if } \Phi(x)\leq 0\end{cases}$$

Note that H_2 is Lipschitz continuous and periodic in x variable and again by (2.3) $H_2\geq\mathcal{H}$. Such a relaxation of $\mathcal{H}$ in the region $\{\Phi(x)\geq 0\}$ is basically saying that: player II does not need to take any action before crossing the line $x_1=\frac{\pi}{2}$ and the flow alone is enough to beat the strain state and steer the state of the trajectory to the desired region. After crossing the line, player II just uses the same strategy as in case 1.2. Suppose that $W\in C(\mathbb{R}^2\times[0,+\infty))$ is the viscosity solution of

$$\begin{cases}W_t+H_2(DW,x)=0\\ W(x,0)=g(x)\end{cases}$$

given by the differential game representation formula

$$-W(x,t)=\inf_{\Lambda\in\Delta(t)}\sup_{\alpha\in M(t)}\{-g(\xi(t))\}\tag{5.12}$$

for

$$\begin{cases} \dot{\xi}(s) = f_2(\xi, \alpha, \Lambda(\alpha)) & \text{for a.e. } s \in [0,t] \\ \xi(0) = x. \end{cases}$$

Here $f_2 = f_2(x,\eta,\mu)$ for $(x,\eta,\mu) \in \mathbb{R}^2 \times [-1,1]^2 \times [-1,1]$ is given as follows:

$$f_2(x,\eta,\mu) = \begin{cases} (\eta_1,\ \eta_2 + Ad\Phi(x)\eta_2) - AV(x) & \text{if } \Phi(x) \geq 0 \\ (\eta_1 - 2Ad\Phi(x)\eta_1,\ \eta_2 - \mu Ad\Phi(x)) - AV(x) & \text{if } \Phi(x) \leq 0 \end{cases}$$

Fix $\theta_0 \in (\frac{4d}{5}, \frac{\pi}{2})$, we will choose a strategy Λ_{θ_0} of player II. For $\alpha \in M(t)$, let $\tilde{\xi} = (\tilde{x}_1(s), \tilde{x}_2(s))$ be the unique solution of

$$\begin{cases} \dot{\tilde{\xi}}(s) = -AV(\tilde{\xi}) + (\tilde{\alpha}_1, 0) + (Ad\Phi(\tilde{\xi}) + 1)(0, \tilde{\alpha}_2) & \text{for } s \in (0,+\infty) \\ \tilde{\xi}(0) = (\theta_0, 0). \end{cases}$$

Here $\tilde{\alpha} : [0,+\infty) \to [-1,1]^2$ is given by

$$\tilde{\alpha}(s) = (\tilde{\alpha}_1(s), \tilde{\alpha}_2(s)) = \begin{cases} \alpha(s) & \text{for } s \in [0,t] \\ (0,0) & \text{for } s > t. \end{cases}$$

Denote $t_0 = \min\{s > 0 |\ \tilde{x}_1(s) = \frac{\pi}{2}\}$. Then according to Lemma 5.4, $t_0 < \infty$ and

$$\tilde{\xi}([0,t_0]) \subseteq [\frac{4d}{5}, \frac{\pi}{2}] \times [-\frac{\pi}{3}, \frac{\pi}{3}] \subset \{\Phi \geq 0\}$$

and

$$\tilde{\xi}(t_0) \in \{(\frac{\pi}{2}, \tau)|\ \frac{d^2}{2} - d \leq \tau \leq d - \frac{d^2}{2}\}. \tag{5.13}$$

Player II chooses the strategy $\Lambda_{\theta_0} : M(t) \to N(t)$ as follows:

- If $t_0 \geq t$, player II has no influence on the trajectory. The trajectory will be contained in the rectangle $[\frac{4d}{5}, \frac{\pi}{2}] \times [-\frac{\pi}{3}, \frac{\pi}{3}]$. For convenience, we just set $\Lambda_{\theta_0} : M(t) \to N(t)$ as $\Lambda_{\theta_0}(\alpha) \equiv 0$.
- If $t_0 < t$, let $\xi(s) = (x_1(s), x_2(s)) : [t_0, t] \to \mathbb{R}^2$ be the unique solution of

$$\begin{cases} \dot{\xi} = AV(\xi) + (\alpha_1 - 2Ad\Phi(\xi)\alpha_1, \alpha_2 + \frac{Adx_2(s)}{d - \frac{d^2}{4}}\Phi(\xi)) & \text{for } t_0 \leq s \leq t \\ \xi(t_0) = \tilde{\xi}(t_0). \end{cases}$$

Owing to (5.13) and Lemma 5.3, $\xi([t_0,t]) \subseteq [\frac{\pi}{2}, \frac{5\pi}{4}] \times [\frac{d^2}{4} - d, d - \frac{d^2}{4}] \subset \{\Phi \leq 0\}$. Player II chooses the strategy $\Lambda_{\theta_0} : M(t) \to N(t)$ as

$$\Lambda_{\theta_0}(\alpha)(s) = \begin{cases} 0 & 0 \leq s < t_0 \\ -\frac{x_2(s)}{d - \frac{d^2}{4}} & \text{for } t_0 \leq s \leq t \end{cases}$$

Then

$$\bar{\xi}(s) = \begin{cases} \tilde{\xi}(s) & \text{for } s \in [0, t_0) \\ \xi(s) & \text{for } s \in [t_0, t] \end{cases}$$

is the unique solution of

$$\begin{cases} \dot{\bar{\xi}} = f_2(\bar{\xi}, \alpha, \Lambda_{\theta_0}(\alpha)) & \text{for a.e. } s \in (0,t) \\ \bar{\xi}(0) = (\theta, 0) \end{cases}$$

and we have that $\bar{\xi}([0,t]) \subset [\frac{4d}{5}, \frac{5\pi}{4}] \times [-\frac{\pi}{3}, \frac{\pi}{3}]$.

Hence (5.12) together with comparison principle imply that for $x_0 = (\theta_0, 0)$

$$-G(x_0,t) \leq -W(x_0,t) \leq \max_{[0,\frac{5\pi}{4}]\times[-\frac{\pi}{2},\frac{\pi}{2}]} \{-g\} \quad \text{for all } t \geq 0.$$

Combining all the cases together, we deduce for any $x_0 \in \{(\theta,0)|\ 0 \leq \theta \leq \pi\}$

$$G(x_0,t) \geq \min_{[-\frac{\pi}{2},\frac{3\pi}{2}]^2} g \geq \min_{[-\pi,2\pi]^2} g \quad \text{for all } (x,t) \in \mathbb{R}^2 \times [0,+\infty).$$

□

Remark 5.2 *Note that both auxiliary Hamiltonians (H_1 and H_2) used in the above proof are also $\geq$ the simplified Hamiltonian $\tilde{\mathcal{H}} = |p| + AV(x)\cdot p - Ad\Phi(x)\cdot(|p_1| - |p_2|)$. Hence Lemma 5.5 holds for this simplified Hamiltonian as well (of course, a direct proof is easier due to its game-friendly form). Precisely speaking, under the same assumptions, let $\tilde{G}(x,t) \in C(\mathbb{R}^2 \times [0,+\infty))$ be the viscosity solution of*

$$\begin{cases} \tilde{G}_t + |D\tilde{G}| + AV(x)\cdot D\tilde{G} - Ad\Phi(x)\cdot(|\tilde{G}_{x_1}| - |\tilde{G}_{x_2}|) = 0 \\ \tilde{G}(x,0) = g(x). \end{cases}$$

Then

$$\tilde{G}(x,t) \geq \min_{y\in x+[-2\pi,2\pi]^2} g(y) \quad \textit{for all } (x,t) \in \mathbb{R}^2 \times [0,+\infty)$$

when $d < d_0$ and $A > \frac{8}{d^3}$ with the same d_0 as in Lemma 5.5.

Proof of Theorem 1.4, 1.5 and 1.6: Theorem 1.4 follows immediately from Lemma 5.1 and Lemma 5.5.

As for Theorem 1.5, let $G^\epsilon = \epsilon G(\frac{x}{\epsilon}, \frac{t}{\epsilon})$. Then G is the viscosity solution of

$$\begin{cases} G_t + |DG| + AV(x)\cdot DG - Ad\Phi(x)\frac{|G_{x_1}|^2 - |G_{x_2}|^2}{|DG|} = 0 \\ G(x,0) = \frac{1}{\epsilon} g(\epsilon x). \end{cases}$$

Then (1.13) and (1.14) are immediately corollaries of Lemma 5.5. Next we will prove (1.15). Owing to (1.13), we have that for all $(x,t) \in \mathbb{R}^2 \times [0,+\infty)$

$$\liminf_{\substack{\epsilon\to 0 \\ (y,s)\to(x,t)}} G^\epsilon(y,s) \geq g(x). \tag{5.14}$$

To show the reverse "$\leq$" is simple. For convenience, we will apply an interesting general result from [15]. Note that the flow velocity $V(x)$ and the strain rate $\Phi(x)$ are both zero at $x = Q_0 = (\frac{\pi}{2}, \frac{\pi}{2})$. Hence there exists $\tau \in (0,1)$ such that $\mathcal{H}(p,x) \geq \frac{1}{2}|p|$ for $x \in B_\tau(Q_0) + \mathbb{Z}^2$. So we may construct a smooth periodic function $a(x)$ such that
(i) $a(x) > 0$ if $d(x,\ Q_0 + \mathbb{Z}^2) < \frac{\tau}{2}$
(ii) $a(x) < 0$ if $d(x,\ Q_0 + \mathbb{Z}^2) > \tau$
(iii) $\mathcal{H} \geq a(x)|p|$.

Now let $F^\epsilon \in C(\mathbb{R}^2 \times [0+\infty))$ be the viscosity solution of

$$\begin{cases} F^\epsilon_t + a(\frac{x}{\epsilon})|DF^\epsilon| = 0 \\ F^\epsilon(0) = g(x). \end{cases}$$

Comparison principle implies that $F^\epsilon \geq G^\epsilon$. Hence for all $(x,t) \in \mathbb{R}^2 \times [0,+\infty)$

$$\liminf_{\substack{\epsilon\to 0 \\ (y,s)\to(x,t)}} G^\epsilon(y,s) \leq \liminf_{\substack{\epsilon\to 0 \\ (y,s)\to(x,t)}} F^\epsilon(y,s) \leq g(x).$$

The second "≤" is due to Theorem 1.3 in [15]. Combining with (5.14), (1.15) holds.

Finally, for Theorem 1.6, let $\tilde{G}^\epsilon = \epsilon\tilde{G}(\frac{x}{\epsilon}, \frac{t}{\epsilon})$. Then $\tilde{G}$ is the viscosity solution of

$$\begin{cases} \tilde{G}_t + |D\tilde{G}| + AV(x) \cdot D\tilde{G} - Ad\Phi(x) \cdot (|\tilde{G}_{x_1}| - |\tilde{G}_{x_2}|) = 0 \\ \tilde{G}(x,0) = \frac{1}{\epsilon} g(\epsilon x). \end{cases}$$

Then (1.17) is an immediate corollary of Lemma 5.2 and Remark 5.2. □

6 Concluding Remarks

Three regimes of propagation and quenching dynamics have been established for the strain G-equation in cellular flows, corresponding to the existence, breakdown and resurgence of homogenization and effective Hamiltonian (cell problem) in suitable sense. The work performed the first homogenization analysis of a non-coercive, non-convex, inviscid level-set Hamilton-Jacobi equation arising in turbulent combustion. A future line of work is to determine whether the flame front will actually partially retreat along directions which are neither horizontal nor vertical, to study refined transition across the three regimes and the additional effects from time-dependent two-dimensional incompressible flows and three dimensional steady flows.

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