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Scaling Limits of Waves in Convex Scalar Conservation Laws Under Random Initial Perturbations

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We study waves in convex scalar conservation laws under noisy initial perturbations. It is known that the building blocks of these waves are shock and rarefaction waves, both are invariant under hyperbolic scaling. Noisy perturbations can generate complicated wave patterns, such as diffusion process of shock locations. However we show that under the hyperbolic scaling, the solutions converge in the sense of distribution to the unperturbed waves. In particular, randomly perturbed shock waves move at the unperturbed velocity in the scaling limit. Analysis makes use of the Hopf formula of the related Hamilton-Jacobi equation and regularity estimates of noisy processes.

KEY WORDS: Waves; Convex scalar conservation laws; Noise; Scaling limit.
AMS subject classifications: 35L60, 35B40, 60H15

1. INTRODUCTION

Deterministic wave solutions are well studied for the convex scalar conservation law:

$$u_t + (H(u))_x = 0, \tag{1.1}$$

H a convex function, $H(u)/u \to +\infty$ as $|u| \to \infty$. Classical results[3,5] are that the general waves are made up of shocks and rarafaction waves generated from scaling invariant initial data $u(x, 0) = \chi_{R_\pm}(x)$, the characteristic functions of the left or right half lines. Both types of waves are asymptotically stable under spatially

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localized initial perturbations[3]. Effects of random perturbations have been studied for Burgers equation ($H = u^2/2$),[2,6,7]. Complex wave patterns arise in particular from shocks. Analysis of the present authors[6,7] showed that at large times the stochastic (either from random initial data or flux) Burgers shock waves diffuse around their mean (deterministic) location. More precisely, the Burgers shock front location obeys Gaussian statistics (central limit theorem)[6,7]. Rarefaction waves are more stable under initial random perturbation, and remain close in the sense of probability to the unperturbed ones at large times[2].

In this paper, we study the limit of initially randomly perturbed waves for general convex scalar laws by means of the hyperbolic scaling $x \to \frac{x}{\epsilon}$, $t \to \frac{t}{\epsilon}$, as $\epsilon \to 0$. This turns out to be simpler than a large time asymptotic analysis of random perturbative effects on waves as seen for the Burgers equation if we are not interested in detailed information such as statistics of shock locations. Quantities such as shock speeds are captured in the scaling limit. We first consider white noise $W_x(x)$ initial perturbation, $W(x)$ the standard Wiener process. Namely, $u(x, 0) = \chi_{R_\pm}(x) + W_x(x)$. The scaled solutions then satisfy the same scalar law with initial data $u^\epsilon(x, 0) = \chi_{R_\pm}(x) + W_x(x/\epsilon)$. Due to unbounded and irregular perturbation from random process, solutions to (1.1) are understood as distributional derivative of Hopf solutions (Section 2) to the Hamilton-Jacobi equation:

$$v_t^\epsilon + H\big(v_x^\epsilon\big) = 0, \qquad v^\epsilon(x, 0) = x\chi_{R_\pm}(x) + \epsilon W(x/\epsilon). \tag{1.2}$$

The problem reduces to analysis of v^ϵ as $\epsilon \to 0$. The initial data of (1.2) is almost surely Hölder continuous and grows sublinearly in x. With the help of Hopf formula (Section 2) and its properties, we show

Theorem 1. *With probability one, u^ϵ converges in the sense of distribution to the unperturbed solution of (1.1) which is either shock (minus sign) or a rarefaction wave (plus sign).*

Consequently, both shock and rarefaction waves are stable under white noise perturbations in the hyperbolic scaling limit, and the shock wave speeds are unchanged. The robustness of Theorem 1.1 is shown through extension to other types of noise including the colored Gaussian noise (Section 3).

The paper is organized as follows. Section 2 analyzes Hopf formula and establishes compactness as well as limits of scaled Hamilton-Jacobi solutions v^ϵ in the case of white noise initial perturbation. This is followed by the proof of Theorem 1.1. Section 3 extends the results to more general noisy perturbations. Conclusions are in Section 4.

2. HOPF SOLUTIONS AND LIMITS

Let u solve the one dimensional scalar convex Hamilton-Jacobi equation:

$$u_t + H(u_x) = 0, \tag{2.1}$$

with convex Hamiltonian H and initial data $u(0, x) = G(x)$. The function G is in the form: $G(x) = \Gamma + g(x)$, where $g(x)$ is locally Hölder continuous, and grows no faster than linearly in x, i.e. for some positive constant $C(g)$:

$$|g(x) - g(x')| \leq C(g)(|x - x'|^{\beta} + |x - x'|), \tag{2.2}$$

with $\beta \in (0, 1)$; $\Gamma(x) = x\chi_{R_{\pm}}(x)$. Clearly, G satisfies (2.2) with a constant $C(G)$.

Assume that:

$$\frac{H(p)}{|p|} \to +\infty \text{ as } p \to \infty, \qquad H \text{is convex.} \tag{2.3}$$

Under (2.3), it is known[1] that the Lagrangian function L exists as Legendre transform of H, and satisfies (2.3) as well. Moreover, the Hopf's formula:

$$u(x, t) = \min_{y \in R} \left\{ tL\left(\frac{x - y}{t}\right) + G(y) \right\}, \tag{2.4}$$

is well-defined in the sense that the minimum is achieved. Provided that $u(x, t)$ is continuous in (x, t) which we prove below, Hopf solution (2.4) is the unique viscosity solution [1] by convexity of H and L. For Hölder continuous initial data with at most linear growth (2.2), Hölder continuity easily persists in x, however this is not as clear in time. In contrast, for Lipschitz continuous initial data [1], $u(x, t)$ is space-time Lipschitz continuous over $x \in R$ and $t \geq 0$.

Lemma 1. *The Hopf solution $u(x, t)$ is uniformly continuous over any compact space-time region $D = [x_1, x_2] \times [t_1, t_2]$, $t_1 \geq 0$, $x_1, x_2 \in R$. Moreover, $\forall \hat{x}, x, t \geq 0$,*

$$|u(\hat{x}, t) - u(x, t)| \leq C(G)(|\hat{x} - x|^{\beta} + |\hat{x} - x|). \tag{2.5}$$

Proof. Fix $t > 0, x, \hat{x} \in R$. Pick $y \in R$ such that

$$tL\left(\frac{x - y}{t}\right) + G(y) = u(x, t), \qquad y \text{ depending on } x, t.$$

Then

$$u(\hat{x}, t) - u(x, t) = -tL\left(\frac{x - y}{t}\right) - G(y) + \inf_z \left\{ tL\left(\frac{\hat{x} - z}{t}\right) + G(z) \right\}$$

$$\underset{z = \hat{x} - x + y}{\leq} G(\hat{x} - x + y) - G(y) \leq C(G)(|\hat{x} - x|^{\beta} + |\hat{x} - x|).$$

Interchanging $\hat{x}$ and x gives:

$$|u(\hat{x}, t) - u(x, t)| \leq C(G)(|\hat{x} - x|^\beta + |\hat{x} - x|), \tag{2.6}$$

which is (2.5).

Taking $y = x$ in (2.4), we bound u from above

$$u(x, t) \leq tL(0) + G(x). \tag{2.7}$$

Now consider small time t. For any $\epsilon > 0$, there is $\delta_1 > 0$, such that if $|x - y| \leq \delta_1$, $|G(y) - G(x)| \leq \epsilon/2$. Due to superlinear growth of L for large argument, the minimizer of Hopf formula (2.4) is attained. We denote the minimizer by y_* and observe that $|y_* - x| \leq \delta_1$ if $x \in [-M, M]$ and $t \leq t_1 = t_1(\delta_1, M) < 1$. Suppose otherwise, then $tL((x - y_*)/t)$ in (2.4) grows faster than $2C(G)|x - y_*|$, if t is small enough depending only on δ_1 and M. Inequality (2.7) implies that

$$2C(G)|x - y_*| + G(y_*) \leq tL(0) + \max_{x \in [-M,M]} G(x) \leq C_1 = C_1(M),$$

and so $|y_*| \leq C_2 = C_2(M)$. By (2.4), $u(x, t)$ would diverge as $t \to 0$, contradicting (2.7).

Using the fact that $L(\cdot) \geq -C(L)$, for some positive constant $C(L)$, we have:

$$\begin{aligned} u(x, t) &= G(x) + tL((x - y_*)/t) + G(y_*) - G(x) \\ &\geq G(x) - tC(L) - \epsilon/2, \end{aligned}$$

and together with (2.7):

$$|u(x, t) - G(x)| \leq t \max(C(L), L(0)) + \epsilon/2 \leq \epsilon,$$

if $t \leq t_1(\delta_1, M) < \epsilon/(2 \max(C(L), L(0)))$. This proves the uniform continuity of u in t as $t \to 0$ over any compact interval of x.

In terms of any positive time $s \in (0, t)$, thanks to convexity of L, Hopf formula reads [1]:

$$u(x, t) = \min_{y \in R} \{(t - s)L((x - y)/(t - s)) + u(y, s)\}.$$

As $C(u(x, s)) \leq C(G)$, we repeat the above analysis to obtain the uniform continuity at $t = s$. The proof is complete. □

Lemma 2. *Let (W_t), $t \geq 0$ be a one-dimensional Wiener process starting from 0, that is, a Gaussian process with $\mathbf{E}[W_t] = 0$ and $\mathbf{E}[W_s W_t] = \min(s, t)$. Let $0 < \alpha < \frac{1}{2}$, $p > 1/2$. Then for any sequence $\epsilon_n \to 0$, there exists a subsequence $\epsilon_{n_j} \to 0$ such that with probability one the functions $t \mapsto \epsilon_{n_j}^p W(\frac{t}{\epsilon_{n_j}})$ satisfy the Hölder condition with the exponent α uniformly in n_j. That is with probability* *one, there exists a C such that for every n_j and for every $s, t \in [0, 1]$*

$$\epsilon_{n_j}^p \left| W\left(\frac{t}{\epsilon_{n_j}}\right) - W\left(\frac{s}{\epsilon_{n_j}}\right)\right| \le C|t-s|^\alpha, \tag{2.8}$$

where the constant C depends on the realization of the process W. Together with the almost sure bounds on the growth of the Wiener process for large arguments, this implies for any n_j, t, s:

$$\epsilon_{n_j}^p \left| W\left(\frac{t}{\epsilon_{n_j}}\right) - W\left(\frac{s}{\epsilon_{n_j}}\right)\right| \le C'(|t-s|^\alpha + |t-s|), \tag{2.9}$$

with probability one, and with a realization dependent constant C'.

Proof. the classical theorem of Kolmogorov (page 53, [4]) asserts that the random variable

$$L = \sup_{s,t\in[0,1]} \frac{|W_t - W_s|}{|t-s|^\alpha}$$

is finite with probability one and, consequently,

$$\mathbf{P}[L \ge k] \to 0$$

as $k \to \infty$. We can thus choose integers $k_n \to \infty$ so that

$$\sum_{n=1}^{\infty} \mathbf{P}[L \ge k_n] < \infty.$$

Choosing subsequence n_j such that $\epsilon_{n_j}^{\frac{1}{2}-p} \ge k_n$, we have, since $(\sqrt{\epsilon_{n_j}} W_{\frac{t}{\epsilon_{n_j}}})_{0\le t\le 1}$ is another Wiener process,

$$\mathbf{P}\left[\sup_{s,t\in[0,1]} \frac{\epsilon_{n_j}^p |W_{\frac{t}{\epsilon_{n_j}}} - W_{\frac{s}{\epsilon_{n_j}}}|}{|t-s|^\alpha} \ge 1\right] \le \mathbf{P}\left[L \ge k_n\right],$$

so, by the first Borel-Cantelli lemma, we have

$$\sup_{s,t\in[0,1]} \frac{\left|\epsilon_{n_j}^p W_{\frac{t}{\epsilon_{n_j}}} - \epsilon_{n_j}^p W_{\frac{s}{\epsilon_{n_j}}}\right|}{|t-s|^\alpha} \le 1$$

for all but finitely many n_j. Since the supremum is finite for the remaining values of n_j by the Kolmogorov theorem, (2.8) follows. Invoking the law of iterated logarithm [4], we have (2.9), and the lemma is proved.

Remark 21. *Existence of the sequence ϵ_{n_j} is sufficient for the proof of our main theorem. It is possible, using arguments related to the proof of the Kolmogorov theorem together with the Gaussian distribution of the Wiener process increments, to prove that with probability one all functions $t \mapsto \epsilon W(\frac{t}{\epsilon})$, where $\epsilon \in (0, 1)$ satisfy the Hölder condition uniformly in ϵ. We do not include the proof, since it is longer and the result is not used in the present paper.*

Proof of Theorem 1.1: solutions to the convex scalar conservation law:

$$\begin{aligned} &u_t + H(u)_x = 0, \\ &u|_{t=0} = \Gamma_x(x) + V_x, \end{aligned} \tag{2.10}$$

that is wave plus the white noise perturbation as initial data, are intepreted as distributional derivative in x of the solutions of the Hamilton-Jacobi problem (2.1). Under the scaling $x \to x/\epsilon$, $t \to t/\epsilon$, we obtain solutions of the scaled equation:

$$\begin{aligned} &u_t^\epsilon + (H(u^\varepsilon))_x = 0, \quad t \in [0, T] \\ &u^\epsilon|_{t=0} = \Gamma_x(x) + V_x\left(\frac{x}{\epsilon}\right). \end{aligned} \tag{2.11}$$

We write $u^\epsilon = \bar{u}_x^\epsilon$, and the latter solves:

$$\begin{aligned} &\bar{u}_t^\epsilon + H\left(\bar{u}_x^\epsilon\right) = 0, \\ &\bar{u}^\epsilon|_{t=0} = \Gamma(x) + \epsilon W\left(\frac{x}{\epsilon}\right). \end{aligned} \tag{2.12}$$

The solution $\bar{u}^\epsilon$ is given by Hopf's formula. It follows from the lemmas proven above that with probability one, the sequence $\bar{u}^\epsilon$ is equi-continuous hence compact in $C([0, T], C_{loc}(\mathbb{R}^1))$, any $T > 0$. For almost all random realizations, we can choose a sequence of the values of $\epsilon \to 0$ such that $\bar{u}^\epsilon \to v$, pointwise on compact sets of $R_+ \times R$. As the scaled noisy part of the initial data converges to zero almost surely on compact set, the limit v is the unique Hopf solution of the problem:

$$v_t + H(v_x) = 0, \qquad v|_{t=0} = \Gamma(x).$$

In other words, the scaled noisy solution u^ϵ converges to the unperturbed waves $\bar{u} = v_x$, in the sense of distribution for almost all realizations. The proof is complete.

Remark 22. *In case of shock wave data, we have as $\epsilon \to 0$:*

$$u^\epsilon = u\left(\frac{x}{\epsilon}, \frac{t}{\epsilon}\right) \longrightarrow \chi_{R_-}(x - c_s t),$$

in the sense of distribution, where c_s is the unperturbed shock speed. Thus in the scaling limit, shock speeds are invariant under noisy initial perturbations.

3. GENERALIZATION

We generalize Theorem 1.1 to include non-white stationary initial noisy perturbations, which still satisfy the Hölder regularity property (2.9). Precise conditions are stated in the lemma below, which applies in particular to a class of Gaussian colored noises. As a result, the proof of Theorem 1.1 extends beyond the white noise case.

The following lemma gives a sufficient condition for existence of a modification $\tilde{X}(t)$ of the original process $X(t)$ and of a subsequence of positive numbers $\epsilon_n \downarrow 0$ such that the rescaled processes

$$\tilde{X}_n(t) = \epsilon_n \tilde{X}\left(\frac{t}{\epsilon_n}\right)$$

satisfy with probability one a Hölder condition on $[0, 1]$ uniformly in n. The modified process $\tilde{X}$ differs from X only for events of probability zero, i.e, $\mathbf{P}[\tilde{X}_t = X_t] = 1$, for any t, [4].

In order to make sure that the process X has a Hölder-continuous modification to begin with, a classical way is to assume that the increments of X satisfy the moment condition in the statement of the lemma below. Remarkably, as the lemma shows, this necessary condition turns out to be sufficient for the rescaled processes to satisfy the Hölder condition (with some choice of the sequence ϵ_n) as well.

Lemma 1. *Suppose that a process with stationary increments X_t, $t \geq 0$, satisfies the condition*

$$\mathbf{E}[|X_t - X_s|^\alpha] \leq C|t - s|^{1+\beta}$$

for some positive constants C, α and β such that $1 + \beta < \alpha$ and for all s and t. Let γ be any number satisfying $0 < \gamma < \frac{\beta}{\alpha}$. Then for any sequence $\epsilon_N \downarrow 0$, there exists a modification $\tilde{X}(t)$ and a subsequence $\epsilon_n \downarrow 0$ such that the rescaled processes $\tilde{X}_n(t)$, defined above, satisfy:

$$\sup_n \sup_{s,t \in [0,1]} \frac{\tilde{X}_n(t) - \tilde{X}_n(s)}{|t - s|^\gamma} < +\infty$$

Proof. The classical Kolmogorov-Chentsov theorem ([4], p. 53) implies the existence of a continuous modification $\tilde{x}$ of x (for all $t > 0$) such that with probability one

$$L \stackrel{\text{def}}{=} \sup_{s,t\in[0,1]} \frac{|\tilde{X}(t) - \tilde{X}(s)|}{|t-s|^{\gamma}} < +\infty.$$

for any $\gamma \in (0, \frac{\beta}{\alpha})$. Moreover, it follows from its proof that there exist constants $\delta(K) \to 0$ which only depend on C, α and β, such that

$$\mathbf{P}[L > K] \leq \delta(K)$$

for all K. Now let $\kappa = \frac{1+\beta}{\alpha}$. By our assumption $\kappa < 1$. For any $\epsilon > 0$ consider the process

$$Y_{\epsilon}(t) \stackrel{\text{def}}{=} \epsilon^{\kappa} \tilde{X}\left(\frac{t}{\epsilon}\right).$$

This process satisfies the same moment condition as X and thus the same bounds on large deviation probabilities for the analog of L:

$$\mathbf{P}\left[\sup_{s,t\in[0,1]} \frac{Y_{\epsilon}(t) - Y_{\epsilon}(s)}{|t-s|^{\gamma}} > K\right] \leq \delta(K)$$

where γ and $\delta(K)$ are as above. Note that since Y is defined in terms of the continuous process $\tilde{X}$, no further continuous modification is necessary to guarantee the above inequality. Now let us choose a subsequence $\epsilon_n \downarrow 0$ so that $K_n \equiv \epsilon_n^{-1}$, go to $+\infty$ fast enough to satisfy:

$$\sum_{n=1}^{\infty} \delta\left(K_n^{1-\kappa}\right) < +\infty.$$

Then

$$\mathbf{P}\left[\sup_{s,t\in[0,1]} \frac{|\tilde{X}_n(t) - \tilde{X}_n(s)|}{|t-s|^{\gamma}} > 1\right] = \mathbf{P}\left[\sup_{s,t\in[0,1]} \left|Y_{\epsilon_n}(t) - Y_{\epsilon_n}(s)\right| > K_n^{1-\kappa}\right] \leq \delta(K_n^{1-\kappa})$$

and it follows from the first Borel-Cantelli lemma that for all but finitely many n we have

$$\sup_{s,t\in[0,1]} \frac{\tilde{X}_n(t) - \tilde{X}_n(s)}{|t-s|^{\gamma}} < 1.$$

Since for the remaining n the above supremum is finite, the lemma follows.

Remark 31. *Inequality (2.9) follows from Lemma 3. under mild additional assumptions ensuring that for $|t-s| \geq 1$ we have $|X(t) - X(s)| \leq C|t-s|$ with a realization-dependent finite constant C. This is, for example, true for stationary-increment Gaussian processes with an appropriate decay of correlations, as in the example below.*

Example: The above lemma applies in particular to "colored Gaussian noise":

$$X_t = \int_0^t C(u)\, du,$$

where $C(u)$ is a stationary, mean-zero Gaussian process with the covariance

$$K(u) = \mathbf{E}[C(0)C(u)]$$

satisfying

$$V = \int_{\mathbf{R}} |K(u)|\, du < +\infty.$$

To see this, let us take $\alpha = 4$ and estimate (with $s < t$, $\vec{u} = (u_1, u_2, u_3, u_4)$):

$$\begin{aligned}\mathbf{E}[(X_t - X_s)^4] &= \int_{[s,t]^4} \mathbf{E}[C(u_1)C(u_2)C(u_3)C(u_4)]\, d\vec{u} \\ &= \int_{[s,t]^4} \mathbf{E}[K(u_1 - u_2)K(u_3 - u_4) + K(u_1 - u_3)K(u_2 - u_4) \\ &\quad + K(u_1 - u_4)K(u_2 - u_3)]\, d\vec{u},\end{aligned}$$

using the standard Gaussian moment formula. Since the last expression is bounded by $3|t - s|^2 V^2$, X satisfies the assumption of the lemma with $\alpha = 4$ and $\beta = 1$. A similar calculation allows to take $\alpha = 2k$ and $\beta = k - 1$ for any positive integer k, thus proving that γ in the statement of the lemma can be chosen arbitrarily close to $\frac{1}{2}$. Of course, the paths of X are smooth functions of t and therefore satisfy the Lipschitz condition (i.e. the Hölder condition with $\gamma = 1$) but the Lipschitz constant grows with the length of the interval and the statement of the lemma fails with $\gamma = 1$ or, in fact, with any $\gamma \geq \frac{1}{2}$. This is not surprising, since in this case the rescaled processes Y_ϵ converge to the Wiener process as ϵ goes to zero.

4. CONCLUSIONS

Under the hyperbolic scaling limit, the wave solutions, shock and rarefaction waves, of the convex scalar conservation laws are shown to be stable with probability one in the sense of distribution under white and colored noisy perturbations with finite correlation at initial time. Our approach is simple and effective so long as detailed structures such as shock locations are not concerned with. The central limit theorem on shock locations for convex laws – a natural extension of results on the shocks speed – requires different methods and will be addressed in a separate work.

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