Nonlinearity **11** (1998) 501–527. Printed in the UK PII: S0951-7715(98)77177-4

Global solutions and attractors of a Maxwell–Bloch Raman laser system in two transverse dimensions

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Received 13 August 1996, in final form 11 December 1997
Recommended by P Constantin

Abstract. We study a Maxwell–Bloch system describing the dynamics of single-longitudinal Raman lasers in the two transverse space dimensions. Raman lasing is generated by a coherent external pump laser, as a three-wave interaction involving two optical and one material wave. On the other hand, a two-level laser involves incoherent external pumping through, for example, an electrical discharge or flashlamp. Raman lasers have the advantage of being tunable and display a novel explicit nonlinear detuning between the pump and laser emission frequencies. Consequently these lasers exhibit much richer nonlinear dynamics. We establish the global existence of classical H^2 solutions, and show that for periodic domains the dynamics is governed by a global C^∞ smooth attractor of finite dimensions. We explain the structures of nonlinear interactions and couplings that lead to the time asymptotic smoothing. We also construct mild solutions with the dispersive Strichartz inequality for rough but spatially decaying data (in $H^1 \times (L^2 \cap L^p)^2$, $p \in (4, \infty)$) on the whole plane, which physically corresponds to the absorbing boundary condition.

AMS classification scheme numbers: 78Q60, 35Q53, 35Q60

1. Introduction

The Maxwell–Bloch Raman (MBR) laser system in two transverse dimensions reads:

$$E_t - \mathrm{i}a\Delta E = -\sigma E + \sigma P + \mathrm{i}\delta_1 EN, \tag{1.1}$$

$$P_t + (1 + \mathrm{i}\Omega)P = (r - N)E + \mathrm{i}\delta_2|E|^2 P, \tag{1.2}$$

$$N_t + bN = \tfrac{1}{2}(E^\star P + EP^\star), \tag{1.3}$$

where Δ is the two-dimensional Laplacian in $x = (x_1, x_2) \in R^2$ or T^2 (T^n will denote the unit n-dimensional torus), x being the transverse dimensions; the real parameter a measures the transverse diffraction. The complex variables E and P are the electric and polarization fields, and N is proportional to the difference between the atomic and initial inversion; the positive parameters σ and b are respectively the dissipation (decay) rates of the electric and population inversion both scaled to the decay rate of the polarization. The real detuning $\Omega = \Omega_0 + \delta_3 r(x)$, where Ω_0 is the difference between the atomic and the cavity frequencies divided by the polarization decay rate, and $r = r(x)$ is the external pumping function. One feature of the Raman laser is that the detuning Ω depends on the external pumping function. The real parameter δ_i's depend inversely on the magnitude of the detuning δ of the external pump laser. $\star$ denotes complex conjugate. For recent physical literature on the

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system (1.1)–(1.3), we refer to Jakobsen *et al* [14], Geddes *et al* [8], among others. See also earlier works on instabilities and chaos in the reduced spatially homogeneous ordinary differential equation (ODE) system with $a = 0$, [12, 13].

The MBR system reduces to the Maxwell–Bloch (MB) two-level system when the nonlinear interaction terms with coefficients δ_i, $i = 1, 2$, are absent. The familiar complex Lorenz system appears if we further ignore all the spatial dependence. Raman lasers are widely tunable sources, in contrast to two-level lasers, as the lasing emission frequency varies with the external coherent laser pump frequency. In addition, the nonlinear detuning exhibited by these lasers, called the AC Stark effect, can be quite pronounced and influences the bifurcation behaviour of the model equations above. The fact that the detuning depends explicitly on the external pump parameter $r(x)$ causes the fastest growing unstable mode to lie on a curve rather than a vertical straight line as in the two-level case.

The pumping function $r = r(x)$ normally has fast spatial decay in x, and its contribution is restricted to a bounded region, outside of which all the field variables can be regarded as zero. Hence, we assume that r is a smooth compactly supported function. This motivates our absorbing boundary condition which we will take to be $H^1(R^2) \times (L^p(R^2))^2$, $p \in (4, \infty)$ for convenience. The detuning Ω is then a smooth spatial function, approaching constant Ω_0 at infinities. The other is the commonly used periodic boundary condition $L^2(T^3)$. It is convenient to consider both r and Ω as constants for periodic boundary conditions, which we will assume hereafter. We will consider only these two boundary conditions in this paper.

The first work on global solutions and attractor of MB systems was performed by Constantin *et al* [7], for the one-dimensional longitudinal (z) two-level MB system. That is the system in which we replace $\mathrm{i}a\Delta E$ with E_z, and set $\delta_1 = \delta_2 = 0$ in (1.1), (1.2). They showed that the MB system admits unique global weak L^2 solution for any L^2 periodic initial data; moreover, there exists a finite-dimensional C^∞ universal attractor. The MB system in this case becomes hyperbolic with two characteristic speeds. Birnir and Xin [2] showed that the two-dimensional transverse MB system ($\delta_1 = \delta_2 = 0$ in (1.1), (1.2)) admits unique global classical solutions in $H^2(T^2)$ for periodic initial data; moreover, the solutions converge to a finite-dimensional $C^\infty(T^2)$ attractor as $t \to \infty$. Xin and Moloney [28] studied the three-dimensional two-level MB (adding E_z term in (1.1) and setting $\delta_1 = \delta_2 = 0$ in (1.1), (1.2)). The system has a combined effects of propagation in z and diffraction in x, corresponding to an interplay of hyperbolic and dispersive aspects of the problem. As in the case of the well known three-dimensional Navier–Stokes equations [6, 17, 25], one is unable to show global existence of classical solutions to the three-dimensional MB with Sobolev imbedding. Instead, mild solutions and their continuous dependence on the initial data in the L^2 sense are established using the dispersive Strichartz inequalities (see [3, 16, 28], and a precise statement of these inequalities will be presented in section 3). In contrast, the global attractor, understood in the sense of global time invariant ω-limit set, is only partially smoother than the initial data, and not C^∞. This is reminiscent of the partial regularity results of three-dimensional Navier–Stokes equations, [4].

From the studies of two-level MB, we see that the two-dimensional transverse case is the best in that there exist classical and mild solutions, and attracting sets in H^2. Neither the longitudinal nor the three-dimensional two-level MB seems to have attracting sets in H^1. The difference caused by the additional longitudinal term E_z in (1.1) is that the ODEs (1.2) and (1.3) can no longer help in lowering the order of spatial derivatives in the course of bounding $\|\nabla E\|_2$. We will show that in the transverse MB, taking advantage of the ODE coupling and reducing the order of spatial derivatives through temporal integration is the key to constructing global classical smooth solutions.

Naturally, the Raman system in two transverse dimensions should be the first to examine. We see that the Raman systems are much more nonlinear than its two-level analogues. Hence, it is more difficult to construct mild solutions for Raman systems. Our idea for establishing mild solutions is to estimate the difference of any two solutions in the L^2 norm by combining the *a priori* estimates of the spatial gradient of the electric field E, and the dispersive Strichartz inequalities, two approaches used independently before in [2, 28]. This allows us to construct unique weak solutions for initial data in $(E_0, P_0, N_0) \in H^1 \times (L^2 \cap L^p)^2$, $p \in (4, \infty)$. This space is stronger than what is needed to construct mild solutions for the two-level MB, for which the $L^2 \times (L^2 \cap L^p)^2$ space is enough. For proving classical solutions, we carry out energy estimates with the help of Gagliardo–Nirenberg inequalities, and the repeated substitution using the equations of the system. It turns out that this is more involved and delicate than a similar effort for the two-level MB system in two transverse dimensions, [2]. After mild and classical solutions are constructed, we go on to investigate usual global attractors for classical solutions. We also discuss a weak notion of attractors, more suitable for mild solutions and solutions on the plane. Here much further work is needed.

Let us state our main results. To be consistent with notations in the early works, we make the change of variables $E = X$, $P = Y$, $N = Z + r$ to rewrite (1.1)–(1.3) as:

$$X_t - \mathrm{i}a\Delta X = -\sigma X + \sigma Y + \mathrm{i}\delta_1 X(Z + r), \tag{1.4}$$

$$Y_t = -(1 + \Omega)Y - ZX + \mathrm{i}\delta_2 |X|^2 Y, \tag{1.5}$$

$$Z_t = -bZ + \mathrm{Re}\,(XY^\star) - br. \tag{1.6}$$

The initial condition for (1.4)–(1.6) is: $(X, Y, Z)|_{t=0} = (X_0, Y_0, Z_0)$. We have:

Theorem 1.1. *Let* $(X_0, Y_0, Z_0)(x_1, x_2) \in (H^2(T^2))^3$. *Then there exists a unique global classical solution*

$$(X(t, \cdot), Y(t, \cdot), Z(t, \cdot)), \qquad \cdot = (x_1, x_2)$$

of the MBR system (1.4)–(1.6) such that:

(1) $S = S(t) : (X_0, Y_0, Z_0)(x) \to (X, Y, Z)(t, x) \in (H^2(T^2))^3$ *is continuous and bounded;*

(2) equations (1.4)–(1.6) are satisfied in $(L^2(T^2))^3$*;*

(3) there is a global C^∞ *attractor* $\mathcal{A}$ *of finite Hausdorff dimension such that for any bounded set* $B \in (H^3(T^2))^3$,

$$\lim_{t\to+\infty} \mathrm{dist}_{(H^2(T^2))^3}(S(t)B, \mathcal{A}) = 0.$$

For global mild solutions, we have:

Theorem 1.2. *Let* $(X_0, Y_0, Z_0)(x_1, x_2) \in (H^1 \times (L^2 \cap L^p)^2)(R^2)$*, for some* $p \in (4, +\infty)$. *Then there exists a unique global solution:*

$$S(t, \cdot) = (X(t, \cdot), Y(t, \cdot), Z(t, \cdot)), \qquad \cdot = (x_1, x_2)$$

of the MBR system (1.4)–(1.6) such that:

(1) $S : (X_0, Y_0, Z_0)(x) \to (X, Y, Z)(t, x) \in (H^1 \times (L^2 \cap L^p)^2)(R^2)$ *is continuous and bounded;*

(2) S satisfies the integral equations:

$$X(t, \cdot) = U(t)X_0(\cdot) + \sigma \int_0^t U(t - s)Y(s, \cdot)\,\mathrm{d}s + \mathrm{i}\delta_1 \int_0^t U(t - s)X(Z + r)(s, \cdot)\,\mathrm{d}s \tag{1.7}$$

$$Y(t, \cdot) = Y_0(\cdot) - (1 + \mathrm{i}\Omega)\int_0^t Y(s, \cdot)\,\mathrm{d}s + \int_0^t -Z(s, \cdot)X(s, \cdot)\,\mathrm{d}s + \mathrm{i}\delta_2 |X|^2 Y(s, \cdot)\,\mathrm{d}s, \tag{1.8}$$

$$Z(t,\cdot) = Z_0(\cdot) - b\int_0^t Z(s,\cdot)\,\mathrm{d}s + \mathrm{Re}\int_0^t XY^\star(s,\cdot)\,\mathrm{d}s - r(\cdot)bt, \tag{1.9}$$

where $U(t) = \exp\{\mathrm{i}at\Delta - \sigma t\}$, *(1.7)–(1.9) hold in* $(L^2(R^2))^3$*;*

(3) the MBR dynamics admits a global C^∞ *attractor* $\mathcal{A}$ *in* $(H^1 \times (L^2 \cap L^p)^2)(R^2)$ *in the sense that* $\mathcal{A}$ *is the largest bounded time invariant set, and is the* ω*-limit set of any open bounded neighbourhood of* $\mathcal{A}$*;*

(4) if $S_j(0,\cdot) \overset{\mathrm{j}\to\infty}{\longrightarrow} S(0,\cdot)$ *in* $(L^2(R^2))^3$*, and*

$$\|(Y_j(0,\cdot), Z_j(0,\cdot))\|_{L^p} + \|X_j\|_{H^1} \leqslant C,$$

as $j \to \infty$*, for some finite constant* $C < \infty$*, then for any later time:*

$$S_j(t,\cdot) \overset{\mathrm{j}\to\infty}{\longrightarrow} S(t,\cdot),$$

in $(L^2)^3(R^2)$ *and*

$$\|(Y(t,\cdot), Z(t,\cdot))\|_{L^p} + \|X\|_{H^1} \leqslant C.$$

We will present similar results for mild solutions under the periodic boundary conditions, and classical H^2 solutions on R^2 in the coming sections. For the periodic boundary condition, the modified Strichartz inequality is only conjectured, see Bourgain, [3]. Under this conjecture, an extension can be made to construct mild global solutions. The H^2 classical solutions on R^2 are analogously proved as those on T^2, however, the global attractor also has to be understood in the above weak sense for now.

This paper is organized as follows. In section 2, we derive *a priori* estimates and prove the global existence of classical H^2 solutions for both planar and periodic domains. In section 3, we show the global mild solutions on the plane, and extend this result to mild solutions on T^2 under the conjectural Strichartz inequality. In section 4, we discuss C^∞ smoothness of global attractor $\mathcal{A}$, and comment on the structures of nonlinearities in the system to this end. In section 5, we give an upper bound on the dimensions of $\mathcal{A}$ computed in $H^2(T^2)$. Skew symmetries of the linearized flow are essential.

2. *A priori* estimates and global classical solutions

In this section, we first derive *a priori* estimates for the solution (X, Y, Z) to the system:

$$X_t - \mathrm{i}a\Delta X = -\sigma X + \sigma Y + \mathrm{i}\delta_1 X(Z + r(x)), \tag{2.1}$$

$$Y_t + (1 + \mathrm{i}\Omega(x))Y = -ZX + \mathrm{i}\delta_2|X|^2 Y, \tag{2.2}$$

$$Z_t + bZ = \mathrm{Re}\,(XY^\star) - br(x); \tag{2.3}$$

then use them to construct global classical solutions in Sobolev space H^2. Our estimates work for both periodic and the planar domains, and so we do not specify the domains for the remaining discussions of this section.

We start with estimates on (Y, Z) because there is a nice cancellation property due to ZX term in (2.2) and $\mathrm{Re}\,(XY^\star)$ term in (2.3). Indeed, multiplying (2.2) by $Y^\star$, (2.3) by Z, adding and taking real part, we have:

$$\tfrac{1}{2}(|Y(t,\cdot)|^2 + |Z(t,\cdot)|^2)_t = -|Y(t,\cdot)|^2 - b|Z(t,\cdot)|^2 - brZ, \tag{2.4}$$

which yields upon using Cauchy–Schwartz inequality and integrating in t that:

$$|Y(t,\cdot)|^2 + |Z(t,\cdot)|^2 \leqslant (|Y_0(\cdot)|^2 + |Z_0(\cdot)|^2 - br^2/2\beta)\mathrm{e}^{-2\beta t} + br^2/2\beta, \tag{2.5}$$

where $\beta = \min(1, \frac{b}{2})$, for any $t \geqslant 0$, and any x. Integrating (2.5) over space gives for $t \geqslant 0$:

$$\|Y\|_2^2 + \|Z\|_2^2 \leqslant (\|Y_0\|_2^2 + \|Z_0\|_2^2 - b\|r\|_2^2/2\beta)\mathrm{e}^{-2\beta t} + b\|r\|_2^2/2\beta. \tag{2.6}$$

Raising (2.6) to the power $p \in (2, \infty)$, integrating over space, and using Jensen's inequality, we have for all $t \geqslant 0$ with a p dependent constant c_p that:

$$\|(Y, Z)\|_p^p \leqslant c_p(\|(Y_0, Z_0)\|_p^p \mathrm{e}^{-p\beta t} + (b\|r\|_2^2/2\beta)^{p/2}). \tag{2.7}$$

Let us then turn to X and calculate using equation (2.1):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|X\|_2^2 = 2\mathrm{Re}\int XX_t^\star &= 2\mathrm{Re}\int X\{\mathrm{i}a\Delta X - \sigma X + \sigma Y + \mathrm{i}\delta_1 X(Z+r)\}^\star \\ &= 2\mathrm{Re}\left\{ -\mathrm{i}a\|\nabla X\|_2^2 - \sigma\|X\|_2^2 + \sigma\int XY^\star - \mathrm{i}\delta_1\int (Z+r)|X|^2\right\} \\ &= -2\sigma\|X\|_2^2 + 2\sigma\mathrm{Re}\int XY^\star \\ &\leqslant -\sigma\|X\|_2^2 + \sigma\|Y\|_2^2,\end{aligned} \tag{2.8}$$

which results in:

$$\|X\|_2^2(t) \leqslant \mathrm{e}^{-\sigma t}\|X_0\|_2^2 + \sigma\int_0^t \mathrm{e}^{-\sigma(t-s)}\|Y\|_2^2\,\mathrm{d}s. \tag{2.9}$$

Combining (2.6)–(2.9), we obtain:

$$\begin{aligned}\|X\|_2^2(t) &\leqslant \mathrm{e}^{-\sigma t}\|X_0\|_2^2 + \sigma\int_0^t \{(\|Y_0\|_2^2 + \|Z_0\|_2^2 - b\|r\|_2^2/2\beta)\mathrm{e}^{-2\beta t} + b\|r\|_2^2/2\beta\}\mathrm{e}^{-\sigma(t-s)}\,\mathrm{d}s \\ &\leqslant \mathrm{e}^{-\sigma t}\|X_0\|_2^2 + \frac{b\|r\|_2^2}{2\beta} + \sigma\|(Y_0, Z_0)\|_2^2\mathrm{e}(t) \equiv \tilde{C}(t),\end{aligned} \tag{2.10}$$

where $\mathrm{e}(t) = (\mathrm{e}^{-\beta t} - \mathrm{e}^{-\sigma t})/(\beta - \sigma)$, if $\beta \neq \sigma$, and $\mathrm{e}(t) = t\mathrm{e}^{-\sigma t}$, if $\beta = \sigma$.

Hereafter $\tilde{C}$ will be a *generic smooth function* of t such that it approaches a limit independent of initial data (only depending on the parameters in the MBR system) exponentially fast as $t \to +\infty$. Also c will denote a *generic constant* depending only on the MBR coefficients and not the norms of initial data.

Now we estimate the gradient of X component in terms of the p norms of the (Y, Z) components. We calculate:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla X|^2 = \mathrm{Re}\int\nabla X_t\nabla X^* &= \mathrm{Re}\left\{\int\nabla(\mathrm{i}a\Delta X - \sigma X + \sigma Y + \mathrm{i}\delta_1 X(Z+r))\nabla X^\star\right\} \\ &= -\sigma\|\nabla X\|_2^2 + \sigma\mathrm{Re}\int\nabla Y\cdot\nabla X^\star + \delta_1\mathrm{Re}\int \mathrm{i}X\nabla(Z+r)\cdot\nabla X^\star \\ &= -\sigma\|\nabla X\|_2^2 + \sigma\mathrm{Re}\int\nabla Y\cdot\nabla X^\star + \delta_1\mathrm{Re}\int(-\mathrm{i})X(Z+r)\Delta X^\star.\end{aligned} \tag{2.11}$$

Due to the appearance of ΔX, $\nabla(X, Y)$, we cannot directly estimate spatial derivatives and hope to find uniform bounds in time. The idea now is to use integration by parts to trade a time derivative X_t for the ΔX, and later use ODE coupling to reduce the time derivative.

Hence the last integral is continued as:

$$\begin{aligned}\mathrm{Re}\int(-\mathrm{i})X(Z+r)\Delta X^\star &= \mathrm{Re}\int \mathrm{i}X^\star(Z+r)\Delta X \\ &= \mathrm{Re}\int \mathrm{i}X^\star(Z+r)(\mathrm{i}a)^{-1}(X_t + \sigma X - \sigma Y - \mathrm{i}\delta_1 X(Z+r))\end{aligned}$$

$$= a^{-1}\mathrm{Re}\int X^\star(Z+r)(X_t+\sigma X-\sigma Y-\mathrm{i}\delta_1 X(Z+r))$$
$$= (2a)^{-1}\mathrm{Re}\int (Z+r)(|X|^2)_t+\frac{\sigma}{a}\int |X|^2(Z+r)-\frac{\sigma}{a}\mathrm{Re}\int X^\star(Z+r)Y. \tag{2.12}$$

Similarly, the second integral of (2.11) is handled as:

$$\mathrm{Re}\int \nabla Y\cdot\nabla X^\star = -\mathrm{Re}\int Y^\star\Delta X = -\mathrm{Re}\int (\mathrm{i}a)^{-1}Y^*(X_t+\sigma X-\sigma Y-\mathrm{i}\delta_1 X(Z+r))$$
$$= a^{-1}\mathrm{Re}\int \mathrm{i}Y^*X_t+\frac{\sigma}{a}\mathrm{Re}\int \mathrm{i}Y^*X+\frac{\delta_1}{a}\mathrm{Re}\int Y^*X(Z+r). \tag{2.13}$$

It follows from (2.11)–(2.13) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla X\|_2^2+2\sigma\|\nabla X\|_2^2 = \frac{2\sigma}{a}\mathrm{Re}\int \mathrm{i}(X_tY^*+\sigma Y^*X)+\frac{\delta_1}{a}\mathrm{Re}\int (Z+r)(|X|^2)_t$$
$$+\frac{2\delta_1\sigma}{a}\int |X|^2(Z+r). \tag{2.14}$$

Integrating (2.14) in t with integration factor $\mathrm{e}^{2\sigma t}$ gives:

$$\mathrm{e}^{2\sigma t}\|\nabla X\|^2(t) = \|\nabla X_0\|_2^2+\frac{2\sigma}{a}\mathrm{Re}\int_0^t \mathrm{i}\mathrm{e}^{2\sigma s}\int X_sY^*+\sigma Y^\star X$$
$$+\frac{\delta_1}{a}\int_0^t \mathrm{e}^{2\sigma s}\int (Z+r)(|X|^2)_s+\frac{2\delta_1\sigma}{a}\int_0^t \mathrm{e}^{2\sigma s}\int |X|^2(Z+r)$$
$$\leqslant \|\nabla X_0\|_2^2+\frac{2\sigma}{a}\mathrm{Re}\int_0^t \mathrm{i}\mathrm{e}^{2\sigma s}\int X_sY^*+\frac{\sigma^2}{a}\int_0^t \mathrm{e}^{2\sigma s}(\|Y\|_2^2+\|X\|_2^2)$$
$$+\frac{2\sigma\delta_1}{a}\int_0^t\int \mathrm{e}^{2\sigma s}\left(\frac{\epsilon^2|X|^4}{2}+\frac{|Z+r|^2}{2\epsilon^2}\right)+\frac{\delta_1}{a}\int_0^t \mathrm{e}^{2\sigma s}\int (Z+r)(|X|^2)_s$$
$$\leqslant \|\nabla X_0\|_2^2+c\int_0^t \mathrm{e}^{2\sigma s}(\epsilon^2\|X\|_4^4+\epsilon^{-2}\|(X,Y,Z+r)\|_2^2)$$
$$+\frac{2\sigma}{a}\mathrm{Re}\int_0^t \mathrm{i}\mathrm{e}^{2\sigma s}\int X_sY^*+\frac{\delta_1}{a}\int_0^t \mathrm{e}^{2\sigma s}\int (Z+r)(|X|^2)_s, \tag{2.15}$$

where c and ϵ are positive constants depending only on $(\sigma,\delta_1,a,\|r\|_\infty)$, and ϵ is to be chosen. Finally, as we just mentioned, integration by parts in time t and using (2.2) give:

$$\int_0^t \mathrm{e}^{2\sigma s}\int X_sY^* = \mathrm{e}^{2\sigma s}\int XY^*|_{s=0}^{s=t}-2\sigma\int_0^t \mathrm{e}^{2\sigma s}\int XY^*$$
$$-\int_0^t \mathrm{e}^{2\sigma s}\int X\cdot((-1+\mathrm{i}\Omega)Y^*-ZX^*-\mathrm{i}\delta_2|X|^2Y^*).$$

So:

$$\left|\int_0^t \mathrm{e}^{2\sigma s}\int X_sY^*\right| \leqslant \mathrm{e}^{2\sigma t}\|X\|_2\|Y\|_2+\|X_0\|_2\|Y_0\|_2+2\sigma\int_0^t \mathrm{e}^{2\sigma s}\|X\|_2\|Y\|_2$$
$$+(1+\|\Omega\|_\infty^2)^{1/2}\int_0^t \mathrm{e}^{2\sigma s}\|X\|_2\|Y\|_2+\frac{1}{2}\int_0^t \mathrm{e}^{2\sigma s}\int (\epsilon^2|X|^4+\epsilon^{-2}Z^2)$$
$$+\delta_2\int_0^t \mathrm{e}^{2\sigma s}\int\left(\frac{3}{4}\epsilon^{4/3}|X|^4+\frac{1}{4}\epsilon^{-4}|Y|^4\right). \tag{2.16}$$

Similarly:

$$\int_0^t e^{2\sigma s}\int (Z+r)(|X|^2)_s = (e^{2\sigma s}\int (Z+r)|X|^2)|_{s=0}^{s=t} - 2\sigma\int_0^t e^{2\sigma s}\int (Z+r)|X|^2$$
$$-\int_0^t e^{2\sigma s}\int |X|^2(-bZ+\mathrm{Re}\,(XY^*)-br)$$
$$\leqslant e^{2\sigma t}\|Z+r\|_2\|X\|_4^2 + \|Z_0+r\|_2\|X_0\|_4^2 + 2\sigma\int_0^t e^{2\sigma s}\|Z+r\|_2\|X\|_4^2$$
$$+b\int_0^t e^{2\sigma s}\|Z\|_2\|X\|_4^2 + b\|r\|_\infty\int_0^t e^{2\sigma s}\|X\|_2^2$$
$$+\int_0^t e^{2\sigma s}\int\left(\frac{3}{4}\epsilon^{4/3}|X|^4+\frac{1}{4}\epsilon^{-4}|Y|^4\right). \tag{2.17}$$

Combining (2.15)–(2.17), we obtain:

$$e^{2\sigma t}\|\nabla X\|_2^2(t) \leqslant \|\nabla X_0\|_2^2 + c\int_0^t e^{2\sigma s}(\epsilon^{-2}\|(X,Y,Z)\|_2^2+\epsilon^{-2}\|r\|_2^2+\epsilon^{4/3}\|X\|_4^4$$
$$+\epsilon^{-4}\|Y\|_4^4)\,ds + \frac{1}{2}e^{2\sigma t}\left(\|(X,Y)\|_2^2+\frac{1}{\epsilon_1(t)}\|Z+r\|_2^2+\epsilon_1(t)\|X\|_4^4\right)$$
$$+\|X_0\|_2\|Y_0\|_2+\|Z_0+r\|_2\|X_0\|_4^2, \tag{2.18}$$

where $\epsilon_1(t)$ is a smooth function which we now choose to be $\epsilon(\tilde{C}(t)+\|r\|_2)^{-1}$. By Gagliardo–Nirenberg inequality (c_0 a dimensional constant):

$$\epsilon_1(t)\|X\|_4^4 \leqslant c_0\epsilon_1(t)\|X\|_2^2\|\nabla X\|_2^2 \leqslant c_0\epsilon\|\nabla X\|_2^2. \tag{2.19}$$

We will select ϵ so small that $c_0\epsilon < \frac{1}{2}$. Plug (2.19) into (2.18), and use the earlier p norm bounds (2.7) to obtain:

$$\tfrac{1}{2}e^{2\sigma t}\|\nabla X\|_2^2 \leqslant C(\|(Y_0,Z_0)\|_2, \|X_0\|_{H^1}) + C\epsilon^{-4}(\|(X_0,Y_0,Z_0)\|_2)e^{2\sigma t}(1+\tilde{C}(t))$$
$$+c\epsilon^{\frac{4}{3}}\int_0^t e^{2\sigma s}\|X\|_4^4\,ds \leqslant \epsilon^{-4}e^{2\sigma t}\tilde{C}(t)+\epsilon^{\frac{4}{3}}c\int_0^t e^{2\sigma s}\|X\|_2^2\|\nabla X\|_2^2,$$
$$\leqslant \epsilon^{-4}e^{2\sigma t}\tilde{C}(t)+\epsilon^{\frac{4}{3}}c\int_0^t e^{2\sigma s}\tilde{C}(s)\|\nabla X\|_2^2. \tag{2.20}$$

Now by a generalized Gronwall inequality, see for example, lemma 4.2 of Mielke and Schneider [20], for $\epsilon=\epsilon_0$ small enough, independent of initial data,

$$\|\nabla X\|_2 \leqslant \tilde{C}\epsilon_0^{-2}. \tag{2.21}$$

Now we derive further *a priori* estimates of solutions to construct global classical solutions in $C([0,\infty);(H^2)^3)$. Our estimates will also imply an absorbing ball in $(H^2)^3$. To summarize, by (2.5), (2.6), and (2.21), we have the following bounds:

$$\begin{aligned}&\|\nabla X\|_2^2(t) \leqslant c + C(\|\nabla X_0\|_2^2, \|(Y_0,Z_0)\|_{L^2\cap L^4})e^{-ct},\\ &\|(Y,Z)\|_\infty(t) \leqslant c + C(\|(Y_0,Z_0)\|_\infty)e^{-ct},\end{aligned} \tag{2.22}$$

where c is a generic positive constant depending only on the MBR coefficients, and C is a generic positive constant depending on both the MBR coefficients and the initial data.

We look for gradient bounds of (Y,Z) by taking gradient of both Y and Z equations. We find:

$$(\nabla Y)_t + (1+i\Omega)(\nabla Y) + (iY\nabla\Omega) = -X\nabla Z - Z\nabla X + i\delta_2(\nabla|X|^2)Y + i\delta_2|X|^2\nabla Y, \tag{2.23}$$
$$(\nabla Z)_t + b(\nabla Z) = \mathrm{Re}\,\{\nabla X^*\cdot Y + X^*\cdot\nabla Y\} - b\nabla r. \tag{2.24}$$

Energy estimate (that is multiplying the complex conjugate of an unknown to its equation, integrating over the domain, and taking real part), Gagliardo–Nirenberg inequality, and Young's inequality give:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(Y,Z)\|_2^2+\|\nabla Y\|_2^2+b\|\nabla Z\|_2^2 &= -\int b\nabla r\cdot\nabla Z-\mathrm{Re}\int \mathrm{i}\nabla Y^*\nabla\Omega Y\\ &\quad+\mathrm{Re}\left\{-\int Z\nabla X\cdot\nabla Y^*+\mathrm{i}\delta_2\int(\nabla|X|^2)Y\nabla Y^*+\int Y\nabla X^*\cdot\nabla Z\right\}\\ &\leqslant \|Z\|_\infty\|\nabla X\|_2\|\nabla Y^*\|_2+\|Y\|_\infty\|\nabla X\|_2\|\nabla Z\|_2+\delta_2\|Y\|_\infty\|X\|_4\|\nabla X\|_4\|\nabla Y\|_2\\ &\quad+b\|\nabla r\|_2\|\nabla Z\|_2+\|\nabla\Omega\|_\infty\|\nabla Y\|_2\|Y\|_2\\ &\leqslant \frac{\epsilon}{2}\|\nabla Y\|_2^2+\frac{1}{2\epsilon}(\|Z\|_\infty\|\nabla X\|_2)^2+\frac{\epsilon}{2}\|\nabla Z\|_2^2\\ &\quad+\frac{1}{2\epsilon}(\|Y\|_\infty\|\nabla X\|_2)^2+c\delta_2\|Y\|_\infty\|X\|_4\|\nabla X\|_2^{\frac12}\|\Delta X\|_2^{\frac12}\|\nabla Y\|_2\\ &\quad+\frac{\epsilon b}{2}\|\nabla Z\|_2^2+\frac{b}{2\epsilon}\|\nabla r\|_2^2+\frac{\epsilon}{2}\|\nabla Y\|_2^2+\frac{1}{2\epsilon}(\|\nabla\Omega\|_\infty\|Y\|_2)^2. \qquad (2.25)\end{aligned}$$

We see that the gradient estimate of (Y, Z) also relies on ΔX, while in the two-level laser case ($\delta_2 = 0$), this complication is not present. The idea is to couple (2.25) with the equation on ΔX which is:

$$(\Delta X)_t-\mathrm{i}a\Delta(\Delta X)=-\sigma\Delta X+\sigma\Delta Y+\mathrm{i}\delta_1((Z+r)\Delta X+2\nabla X\cdot\nabla(Z+r)+X\Delta(Z+r)). \qquad (2.26)$$

The energy estimate shows:

$$\begin{aligned}\tfrac12(\|\Delta X\|_2^2)_t &= -\sigma\|\Delta X\|_2^2+\sigma\mathrm{Re}\int\Delta Y\cdot\Delta X^*+2\delta_1\mathrm{Re}\int\mathrm{i}\nabla X\cdot\nabla(Z+r)\Delta X^*\\ &\quad+\delta_1\mathrm{Re}\int\mathrm{i}X\Delta(Z+r)\Delta X^*.\end{aligned}$$

As before, we shall trade X_t for ΔX, then integrate in time to reduce the derivatives with the help of ODE coupling. Although a long but straightforward calculation, we end up with:

$$\begin{aligned}\frac12 e^{-2\sigma t}(e^{2\sigma t}\|\Delta X\|_2^2)_t &= \frac{\sigma}{a}\mathrm{Re}\int\mathrm{i}\Delta Y X_t^*+\frac{\delta_1}{a}\int(Z+r)(|\nabla X|^2)_t-\frac{\delta_1}{2a}\int\Delta(Z+r)(|X|^2)_t\\ &\quad+\frac{2\delta_1}{a}\mathrm{Re}\int\Delta X(Z+r)(-\sigma X^*+\sigma Y^*)+\frac{2\delta_1^2}{a}\mathrm{Re}\int(-\mathrm{i})\Delta X\cdot X^*(Z+r)^2\\ &\quad-\frac{\sigma^2}{a}\mathrm{Re}\int\mathrm{i}\nabla Y\cdot\nabla X^*+\frac{\sigma\delta_1}{a}\mathrm{Re}\int(\nabla Y\cdot\nabla X^*)(Z+r)\\ &\quad-\frac{\sigma\delta_1}{a}\mathrm{Re}\int(Z+r)\Delta|X|^2-\frac{\sigma\delta_1}{a}\mathrm{Re}\int\nabla(Z+r)\cdot(\nabla X)Y^*\\ &\quad-\frac{2\delta_1}{a}\mathrm{Re}\int\nabla X\cdot\nabla(Z+r)(\sigma X^*-\sigma Y^*+\mathrm{i}\delta_1X^*(Z+r)). \qquad (2.27)\end{aligned}$$

We observe that the first three terms on the right-hand side of (2.27), labelled as I, II, and III hereafter, contain time derivatives. The remaining terms involve only spatial derivatives, and can be estimated using Sobolev inequalities as:

$$\tfrac12 e^{-2\sigma t}(e^{2\sigma t}\|\Delta X\|_2^2)_t\leqslant I+II+III+\epsilon(\|\Delta X\|_2^2+\|\nabla Y\|_2^2+\|\nabla Z\|_2^2)+\epsilon^{-3}\tilde{C}(t), \qquad (2.28)$$

where ϵ is a convenient small number just like before. Multiplying $e^{2\sigma t}$ and integrating (2.28) in t gives:

$$\frac{1}{2}(e^{2\sigma t}\|\Delta X\|_2^2 - \|\Delta X_0\|_2^2) \leqslant \int_0^t e^{2\sigma s} I \, ds + \int_0^t e^{2\sigma s} II \, ds + \int_0^t e^{2\sigma s} III \, ds$$
$$+\epsilon \int_0^t e^{2\sigma s}\|(\Delta X, \nabla Y, \nabla Z)\|_2^2 \, ds + \epsilon^{-3} \int_0^t e^{2\sigma s}\tilde{C}(s) \, ds. \tag{2.29}$$

We estimate the first three terms by plugging the Y and Z equations to replace the time derivatives of (Y, Z), and so lower the order of derivatives.

$$\int_0^t \int e^{2\sigma s}\Delta Y X_s^* = -\int\int_0^t \Delta Y_s e^{2\sigma s} X^* - \int\int_0^t (2\sigma)e^{2\sigma s}\Delta Y X^* + \int \Delta Y e^{2\sigma t} X^*$$
$$-\int \Delta Y_0 X_0^* = -\int_0^t \int Y_s e^{2\sigma s}\Delta X^* + 2\sigma \int_0^t \int e^{2\sigma s}\nabla Y \cdot \nabla X^*$$
$$-e^{2\sigma t}\int \nabla Y \cdot \nabla X^* - \int X_0^*\Delta Y_0$$
$$= -\int_0^t \int ((-1 - i\Omega)Y - ZX + i\delta_2|X|^2 Y)e^{2\sigma s}\Delta X^*$$
$$+2\sigma \int_0^t \int e^{2\sigma s}\nabla Y \cdot \nabla X^* - e^{2\sigma t}\int \nabla Y \cdot \nabla X^* - \int X_0^*\Delta Y_0,$$

hence:

$$|I| \leqslant c\int_0^t e^{2\sigma s}(\|\nabla Y\|_2 + \|Y\|_2)\|\nabla X^*\|_2 + c\int_0^t \|Z\|_\infty \|X\|_2 \|\Delta X\|_2 e^{2\sigma s}$$
$$+c\int_0^t e^{2\sigma s}\|Y\|_\infty \|\Delta X\|_2 \|X\|_4^2 + c\int_0^t e^{2\sigma s}\|\nabla Y\|_2\|\nabla X\|_2$$
$$+\|X_0\|_2\|\Delta Y_0\|_2 + e^{2\sigma t}\|\nabla Y\|_2\|\nabla X\|_2. \tag{2.30}$$

With similar inequalities on II and III, we arrive at:

$$\|\Delta X\|_2^2 \leqslant e^{-2\sigma t}\{\|\Delta X_0\|_2^2 + \|X_0\|_2\|\Delta Y_0\|_2 + \tilde{C}(t)\|\nabla X_0\|_2^2 + \|X_0\|_4^2\|\Delta(Z_0 + r)\|_2\}$$
$$+c\epsilon \int_0^t \|(\Delta X, \nabla Y, \nabla Z)\|_2^2 \tilde{C}(s)e^{2\sigma(s-t)} \, ds + c\epsilon^{-1}\int_0^t \tilde{C}(s)e^{2\sigma(s-t)}$$
$$+\epsilon\|(\Delta X, \nabla Y)\|_2^2 + \epsilon^{-1}\tilde{C}(t) + \epsilon^{-3}c\int_0^t \tilde{C}(s)e^{2\sigma(s-t)} \, ds, \tag{2.31}$$

which gives:

$$e^{-2\min(1,b)t}(e^{2\min(1,b)t}\|\nabla(Y, Z)\|_2^2)_t \leqslant c\epsilon\|\nabla(Y, Z)\|_2^2 + \tilde{C}(t)\epsilon^{-1} + \epsilon^{-1}\tilde{C}(t)\|\Delta X\|_2$$
$$\leqslant c\epsilon\|\nabla(Y, Z)\|_2^2 + \tilde{C}(t)\epsilon^{-1} + c\epsilon\|\Delta X\|_2^2 + \epsilon^{-3}\tilde{C}(t),$$

and integrating in t implies:

$$\|\nabla(Y, Z)\|_2^2 - e^{-2\min(1,b)t}\|\nabla(Y_0, Z_0)\|_2^2 \leqslant c\epsilon\int_0^t e^{2\min(1,b)(s-t)}\|(\nabla Y, \nabla Z, \Delta X)\|_2^2 \, ds$$
$$+\epsilon^{-3}\int_0^t e^{2\min(1,b)(s-t)}\tilde{C}(s) \, ds. \tag{2.32}$$

Let $\gamma = \min(2, 2b, 2\sigma)$ and add (2.31) and (2.32) to obtain:

$$(1-\epsilon)e^{\gamma t}\|(\Delta X, \nabla Y, \nabla Z)\|_2^2 \leqslant C(\|(X_0, Y_0, Z_0)\|_{H^2}) + \epsilon^{-3}\tilde{C}(t)e^{\gamma t}$$
$$+c\epsilon\int_0^t \tilde{C}(s)e^{\gamma s}\|(\Delta X, \nabla Y, \nabla Z)\|_2^2 \, ds. \tag{2.33}$$

For ϵ small enough, we obtain the bound with Gronwall inequality as before:

$$\|(\Delta X, \nabla Y, \nabla Z)\|_2^2 \leqslant \tilde{C}(t), \tag{2.34}$$

or:

$$\limsup_{t\to\infty} \|(\Delta X, \nabla Y, \nabla Z)\|_2^2 \leqslant c. \tag{2.35}$$

So the dynamics has an absorbing ball in H^1.

We continue to bound $\|(\Delta Y, \Delta Z)\|_2$. Taking Δ of the Y and Z equations, we obtain:

$$(\Delta Y)_t + (1 + \mathrm{i}\Omega)(\Delta Y) + 2\mathrm{i}\nabla\Omega \cdot \nabla Y + \mathrm{i}Y\Delta\Omega = -(X\Delta Z + Z\Delta X + 2\nabla Z \cdot \nabla X)$$
$$+\mathrm{i}\delta_2(Y\Delta|X|^2 + 2\nabla|X|^2 \cdot \nabla Y + |X|^2\Delta Y), \tag{2.36}$$

$$(\Delta Z)_t + b(\Delta Z) = \mathrm{Re}\,\{Y\Delta X^* + 2\nabla X^* \cdot \nabla Y + X^*\Delta Y\} - \Delta(br). \tag{2.37}$$

Energy equalities are:

$$\tfrac{1}{2}(\|\Delta Y\|_2^2)_t + \|\Delta Y\|_2^2 + \tfrac{1}{2}(\|\Delta Z\|_2^2)_t + b\|\Delta Z\|_2^2 = -\mathrm{Re}\int \Delta Y^*(Z\Delta X + 2\nabla Z \cdot \nabla X)$$
$$-2\mathrm{Re}\int \mathrm{i}\nabla\Omega \cdot \nabla Y\Delta Y^* - \mathrm{Re}\int \mathrm{i}\Delta\Omega Y\Delta Y^* - \int \Delta(br)\Delta Z$$
$$+\delta_2\mathrm{Re}\int \mathrm{i}\Delta Y^*(Y\Delta|X|^2 + 2\nabla|X|^2 \cdot \nabla Y)$$
$$+\mathrm{Re}\int (Y\Delta X^* + 2\nabla X^* \cdot \nabla Y)\Delta Z. \tag{2.38}$$

By (2.35), Sobolev inequalities, we deduce the bound:

$$\limsup_{t\to\infty} \|(\Delta Y, \Delta Z)\|_2^2 \leqslant c < \infty. \tag{2.39}$$

Summarizing (2.35) and (2.39), we conclude that the MBR dynamics has an absorbing ball in $(H^2)^3$.

Proof of theorem 1.1, (1) and (2). It is straightforward to show using the contraction mapping principle that the original MB system has a local in time solution $(X, Y, Z) \in C([0, t^*), (H^k)^3)$ for initial data $(X_0, Y_0, Z_0) \in (H^k)^3$, for any $k \geqslant 2$. Nonlinearities are easily controlled using Sobolev inequalities in such spaces. For a large enough k, we have enough regularity to justify our derivations of the *a priori* estimates. Approximating H^2 solutions by a sequence of such H^k solutions (k large enough), we then have global bounds on H^2 norms of local H^2 solutions. Since solutions are imbedded into L^∞, it is easy to show the Lipschitz continuity of solutions on the initial data in L^2 by Gronwall inequality. It is then clear that the local H^2 solutions can be extended to global ones uniquely, and they flow into an absorbing ball in $(H^2)^3$. The proof of parts (1) and (2) is complete. □

We remark that if the initial data belong to higher Sobolev spaces, we can use the same procedure to show that the corresponding solutions have absorbing balls in these more regular spaces. This turns out to be more direct since we already have $(X, Y, Z) \in (L^\infty)^3$. The derivatives of X component will be first bounded, then those of (Y, Z). It is no longer necessary to couple the next-order X derivative with the estimates of (Y, Z) derivative as we have just done. So at this stage of estimates, the Raman system is like the two-level system, and the additional nonlinear terms do not matter much. We omit the details but state that if initial data are in H^k, $k \geqslant 3$, then for all later time,

$$\|(X, Y, Z)\|_{H^k} \leqslant \tilde{C}(t). \tag{2.40}$$

3. Global mild solutions

3.1. Dimension equal to one

Since lasers come out of noise, it is of both mathematical and physical interest to investigate the well posedness of MBR when less regular data are prescribed, especially when the data do not lie in L^∞. Our strategy is to approximate less regular data with smooth data, such as H^2 data, and show that in the limit we are able to define weak solutions uniquely. To this end, we estimate the difference of any two solutions for different initial data, as in [7]. We will also establish the continuous dependence of solutions on the initial data. There is an interesting distinction between one and two dimensions as we will see. In one dimension, the continuous dependence is Lipschitz, while in two dimensions we can only show continuity. This is due to H^1 being marginal or not being continuously imbedded into L^∞. For any two initial data $(X_i^{(0)}, Y_i^{(0)}, Z_i^{(0)})$, $\|X_i^{(0)}\|_{H^1} \leqslant C$, $(Y_i^{(0)}, Z_i^{(0)}) \in (L^p)^2$, for all $p \in [2, p_0]$, where the number $p_0 \in (4, \infty)$ will be clear soon, estimates (2.7), (2.10), and (2.21) hold for both solutions. The generic constants of the rest of this section will depend on norms of initial data. We will first consider weak solutions on the plane, then extend them to periodic cases.

Now we introduce the new variables:

$$\begin{aligned} &\xi = X_1 - X_2, \qquad \eta = Y_1 - Y_2, \qquad \zeta = Z_1 - Z_2, \\ &\tilde{X} = \tfrac{1}{2}(X_1 + X_2), \qquad \tilde{Y} = \tfrac{1}{2}(Y_1 + Y_2), \qquad \tilde{Z} = \tfrac{1}{2}(Z_1 + Z_2). \end{aligned} \tag{3.1}$$

Then (ξ, η, ζ) satisfies the equations:

$$\xi_t - \mathrm{i}a\Delta\xi = -\sigma\xi + \sigma\eta - \mathrm{i}\delta_1 r\xi + \mathrm{i}\delta_1(\tilde{X}\zeta + \tilde{Z}\xi), \tag{3.2}$$

$$\eta_t = -(1 + \mathrm{i}\Omega)\eta - \tilde{X}\zeta - \tilde{Z}\xi + \mathrm{i}\delta_2\left[\eta\left(|\tilde{X}|^2 + \frac{|\xi|^2}{4}\right) + \tilde{Y}(\tilde{X}\xi^* + \tilde{X}^*\xi)\right], \tag{3.3}$$

$$\zeta_t = -b\zeta + \mathrm{Re}\,(\tilde{X}\eta^* + \xi\tilde{Y}^*). \tag{3.4}$$

Multiplying η^* to (3.3) and taking real part, we obtain:

$$\tfrac{1}{2}(|\eta|^2)_t = -|\eta|^2 - \mathrm{Re}\,(\eta^*\tilde{X}\zeta + \eta^*\tilde{Z}\xi) + \mathrm{Re}\,\{\mathrm{i}\delta_2(\tilde{X}\xi^* + \tilde{X}^*\xi)\tilde{Y}\eta^*\}. \tag{3.5}$$

Similarly, we find by multiplying ζ by (3.4) that:

$$\tfrac{1}{2}(\zeta^2)_t = -b\zeta^2 + \mathrm{Re}\,(\zeta\tilde{X}\eta^*) + \mathrm{Re}\,(\zeta\xi\tilde{Y}^*). \tag{3.6}$$

Adding (3.5) and (3.6) shows:

$$\begin{aligned} \tfrac{1}{2}(|\eta|^2 + \zeta^2)_t &= -b\zeta^2 - |\eta|^2 + \mathrm{Re}\,(\zeta\xi\tilde{Y}^*) - \mathrm{Re}\,(\eta^*\tilde{Z}\xi) + \mathrm{Re}\,\{\mathrm{i}\delta_2(\tilde{X}\xi^* + \tilde{X}^*\xi)\tilde{Y}\eta^*\} \\ &\leqslant -\frac{b}{2}\zeta^2 - \frac{1}{2}|\eta|^2 + c(|\xi\tilde{Y}|^2 + |\tilde{Z}\xi|^2 + |\tilde{X}\xi\tilde{Y}|^2) = \frac{-b}{2}\zeta^2 - \frac{1}{2}|\eta|^2 + cH|\xi|^2, \end{aligned} \tag{3.7}$$

where:

$$H \equiv |\tilde{Y}|^2 + |\tilde{Z}|^2 + |\tilde{X}\tilde{Y}|^2. \tag{3.8}$$

In the one-dimensional case, integrating (3.7) over R^1 gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\eta, \zeta)\|_2^2 \leqslant -b\|\zeta\|_2^2 - \|\eta\|_2^2 + c\int H|\xi|^2, \tag{3.9}$$

and integrating in t to obtain:

$$\begin{aligned}\|(\eta,\zeta)\|_2^2 &\leqslant e^{-\beta t}\|(\eta_0,\zeta_0)\|_2^2 + c\int_0^t ds\, e^{-\beta(t-s)}\int H|\xi|^2(x,s)\\ &\leqslant e^{-\beta t}\|(\eta_0,\zeta_0)\|_2^2 + c\left(\int_0^t ds\int H^2\right)^{\frac{1}{2}}\left(\int_0^t\int|\xi|^4\right)^{\frac{1}{2}}\\ &\leqslant e^{-\beta t}\|(\eta_0,\zeta_0)\|_2^2 + c\left(\int_0^t ds\|(\tilde Y,\tilde Z)\|_4^4(1+\|\tilde X\|_\infty^2)\right)^{\frac{1}{2}}\left(\int_0^t ds\int|\xi|^4\right)^{\frac{1}{2}}\\ &\leqslant e^{-\beta t}\|(\eta_0,\zeta_0)\|_2^2 + C(t)\left(\int_0^t ds\int|\xi|^8\right)^{\frac{1}{4}}.\end{aligned}\tag{3.10}$$

Recall (3.2) to write:

$$\begin{aligned}\xi(x,t) &= \sigma\int_0^t e^{(i\Delta-\sigma)(t-s)}\eta(x,s)\,ds - i\delta_1\int_0^t e^{(i\Delta-\sigma)(t-s)}(r(x)\xi)\,ds\\ &\quad + i\delta_1\int_0^t e^{(i\Delta-\sigma)(t-s)}(\xi\tilde Z+\tilde X\zeta)\,ds + e^{(i\Delta-\sigma)t}\xi_0.\end{aligned}\tag{3.11}$$

Let us recall the Strichartz inequalities for the free Schrödinger operator on the entire space R^m (see Kato [16]):

$$\|e^{iat\Delta}\varphi(x)\|_{L^{p+1,r}} \leqslant C\|\varphi\|_{L^2}, \qquad \forall t\geqslant 0,\tag{3.12}$$

and

$$\left\|\int_0^t e^{ia(t-s)\Delta}v(s,x)\,ds\right\|_{L^{p+1,r}} \leqslant C\int_0^t\|v(s)\|_{L^2}\,ds, \qquad \forall t\geqslant 0,\tag{3.13}$$

where C is a universal constant; and $r=\frac{4(p+1)}{m(p-1)}$, $r\in(2,+\infty)$. The space $L^{p,q}$ means taking L^p norm in space and L^q norm in time, $L^{p,q}=L^q([0,T];L^p)$.

By Strichartz inequality for R^1 ($L^2\to L^{p+1,r'}$, $r'=\frac{4(p+1)}{p-1}$, $r'\in(2,\infty)$) with $p=3$, $r'=8$, we find after taking $\|\cdot\|_4^8$ of both sides of (3.11) and integrating t that:

$$\begin{aligned}\int_0^t\|\xi\|_4^8\,ds &\leqslant c\int_0^t\left\|\int_0^\tau e^{(i\Delta-\sigma)(\tau-s)}\eta\,ds\right\|_4^8 d\tau + c\int_0^t\left\|\int_0^\tau e^{(i\Delta-\sigma)(\tau-s)}(r(x)\xi)\right\|_4^8 d\tau\\ &\quad + c\int_0^t\left\|\int_0^\tau e^{(i\Delta-\sigma)(\tau-s)}(\xi\tilde Z+\tilde X\zeta)\right\|_4^8 d\tau + c\int_0^t\|e^{(i\Delta-\sigma)t}\xi_0\|_4^8\\ &\leqslant c\left(\int_0^t\|\eta\|_2\,ds\right)^8 + c\left(\int_0^t\|\xi\|_2\,ds\right)^8 + c\left(\int_0^t\|\xi\tilde Z+\tilde X\zeta\|_2\,ds\right)^8 + c\|\xi_0\|_2^8.\end{aligned}\tag{3.14}$$

Taking $\|\cdot\|_2$ of (3.11), we have:

$$\|\xi\|_2 \leqslant \sigma\int_0^t\|\eta\|_2\,ds + \delta_1 c\int_0^t\|\xi\|_2\,ds + c\int_0^t\|\xi\tilde Z+\tilde X\zeta\|_2\,ds + c\|\xi_0\|_2,\tag{3.15}$$

where c depends only on MBR coefficients, and upon integrating in t:

$$\int_0^t\|\xi\|_2\,ds \leqslant \sigma t\int_0^t\|\eta\|_2\,ds + ct\int_0^t\|\xi\|_2\,ds + ct\int_0^t\|\xi\tilde Z+\tilde X\zeta\|_2\,ds + ct\|\xi_0\|_2.\tag{3.16}$$

So for $t\in[0,t^*]$, with $t^*=t^*(c)$ small enough, we obtain:

$$\int_0^t\|\xi\|_2\,ds \leqslant ct\int_0^t\|\eta\|_2\,ds + ct\int_0^t\|\xi\tilde Z+\tilde X\zeta\|_2\,ds + ct\|\xi_0\|_2.\tag{3.17}$$

Plug (3.17) into (3.14) for $t \in [0, t^*]$ to yield:

$$\int_0^t \|\xi\|_4^8 \,\mathrm{d}s \leqslant c\left(\int_0^t \|\eta\|_2 \,\mathrm{d}s\right)^8 + c\left(\int_0^t \|\xi\tilde{Z} + \tilde{X}\zeta\|_2 \,\mathrm{d}s\right)^8 + c\|\xi_0\|_2^8$$
$$\leqslant c\left(\int_0^t \|\eta\|_2 \,\mathrm{d}s\right)^8 + c\left(\int_0^t \|\xi\|_4\|\tilde{Z}\|_4 + \|\tilde{X}\|_\infty \int_0^t \|\zeta\|_2 \,\mathrm{d}s\right)^8 + c\|\xi_0\|_2^8$$
$$\leqslant C(\|\tilde{X}_0\|_{H^1})\left(\int_0^t \|(\eta,\zeta)\|_2 \,\mathrm{d}s\right)^8 + C(\|(\tilde{Y}_0, \tilde{Z}_0)\|_4)\left(\int_0^t \|\xi\|_4^8\right) t^7 + c\|\xi_0\|_2^8.$$

If t^* is small enough, but now also depending on $\|(\tilde{Y}_0, \tilde{Z}_0)\|_4$, we obtain:

$$\int_0^t \|\xi\|_4^8 \,\mathrm{d}s \leqslant C(\|\tilde{X}_0\|_{H^1})\left(\int_0^t \|(\eta,\xi)\|_2 \,\mathrm{d}s\right)^8 + c\|\xi_0\|_2^8$$
$$\left(\int_0^t \|\xi\|_4^8 \,\mathrm{d}s\right)^{\frac{1}{4}} \leqslant C(\|\tilde{X}_0\|_{H^1})\left(\int_0^t \|(\eta,\xi)\|_2 \,\mathrm{d}s\right)^2 + c\|\xi_0\|_2^2. \tag{3.18}$$

Combining (3.10)–(3.18), we obtain (for $t \in [0, t^*]$):

$$\|(\eta,\zeta)\|_2^2(t) \leqslant C\|(\xi_0, \eta_0, \zeta_0)\|_2^2 + C\int_0^t \|(\eta,\zeta)\|_2^2 \,\mathrm{d}s. \tag{3.19}$$

Rewriting (3.2) as:

$$\xi(x,t) = U(t)\xi_0 + \sigma\int_0^t U(t,s)\eta \,\mathrm{d}s + \mathrm{i}\delta_1\int_0^t U(t,s)(\tilde{X}\zeta)\,\mathrm{d}s, \tag{3.20}$$

where $U(t,s)$ is the evolution operator corresponding to the linear equation:

$$\hat{\xi}_t = (\mathrm{i}a\Delta + \mathrm{i}\delta_1\tilde{Z} - \mathrm{i}\delta_1 r(x) - \sigma)\hat{\xi}. \tag{3.21}$$

Direct calculation shows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\xi}\|_2^2 = -2\sigma\int |\hat{\xi}|^2,$$

therefore $\|U\|_2 \leqslant 1$. Taking $\|\cdot\|_2$ of (3.20) shows that:

$$\|\xi\|_2 \leqslant \|\xi_0\|_2 + \sigma\int_0^t \|\eta\|_2 \,\mathrm{d}s + \delta_1\int_0^t \|\tilde{X}\zeta\|_2 \,\mathrm{d}s \leqslant \|\xi_0\|_2$$
$$+\sigma\int_0^t \|\eta\|_2 \,\mathrm{d}s + \delta_1\|\tilde{X}\|_\infty\int_0^t \|\zeta\|_2 \,\mathrm{d}s. \tag{3.22}$$

Combining (3.19)–(3.22), we obtain for $t \in [0, t^*]$:

$$\|(\xi,\eta,\zeta)\|_2^2 \leqslant \|(\xi_0, \eta_0, \zeta_0\|_2^2 + C\int_0^t \|(\eta,\zeta)\|_2^2 \,\mathrm{d}s,$$
$$\|(\xi,\eta,\zeta)\|_2 \leqslant \|(\xi_0, \eta_0, \zeta_0)\|_2 \mathrm{e}^{Ct}. \tag{3.23}$$

Iterating (3.23) over intervals of length t^* shows that (3.23) is valid for any finite time T with a constant $C = C(T) > 0$.

3.2. Dimension equal to two

Now consider the two-dimensional case, where we cannot pull out $\|\tilde{X}\|_\infty$ as above. We integrate (3.9) over $R^2 \times [0,t]$ again, and use Cauchy–Schwartz inequality as in (3.10) to obtain:

$$\|(\eta,\zeta)\|_2^2 \leqslant \mathrm{e}^{-\beta t}\|(\eta_0,\zeta_0)\|_2^2 + c\left(\int_0^t\int H^2\right)^{\frac{1}{2}}\left(\int_0^t\int|\xi|^4\right)^{\frac{1}{2}},$$

then by Hölder inequality in terms of a small positive number $\delta > 0$:

$$\leqslant \mathrm{e}^{-\beta t}\|(\eta_0,\zeta_0)\|_2^2 + c\left(\int_0^t \|(\tilde{Y},\tilde{Z})\|_4^4 + \|\tilde{X}\|^4_{\frac{16+4\delta}{\delta}}\|\tilde{Y}\|^4_{4+\delta}\right)^{\frac{1}{2}}\left(\int_0^t\int|\xi|^4\right)^{\frac{1}{2}},$$

or:

$$\|(\eta,\zeta)\|_2^2 \leqslant \|(\eta_0,\zeta_0)\|_2^2 + ct^{\frac{1}{2}}\left(\int_0^t\int|\xi|^4\right)^{\frac{1}{2}}. \tag{3.24}$$

Applying Strichartz inequality in (3.11) to obtain ($L^2 \to L^{p+1,r'}$, $r' = \frac{2(p+1)}{p-1}$; with a choice of $p = 3$, $r' = 4$):

$$\begin{aligned}\int_0^t \|\xi\|_4^4\,\mathrm{d}s \leqslant c\int_0^t\left\|\int_0^\tau \mathrm{e}^{(\mathrm{i}\Delta-\sigma)(\tau-s)}\eta\,\mathrm{d}s\right\|_4^4 + c\int_0^t\left\|\int_0^\tau \mathrm{e}^{(\mathrm{i}\Delta-\sigma)(\tau-s)}(r\xi)\right\|_4^4\mathrm{d}\tau\\
+c\int_0^t\left\|\int_0^\tau \mathrm{e}^{(\mathrm{i}\Delta-\sigma)(\tau-s)}(\xi\tilde{Z}+\tilde{X}\zeta)\right\|_4^4\mathrm{d}\tau + c\int_0^t\|\mathrm{e}^{(\mathrm{i}\Delta-\sigma)t}\xi_0\|_4^4\\
\leqslant c\left(\int_0^t\|\eta\|_2\,\mathrm{d}s\right)^4 + c\left(\int_0^t\|\xi\|_2\,\mathrm{d}s\right)^4 + c\left(\int_0^t\|(\xi\tilde{Z}+\tilde{X}\zeta)\|_2\right)^4 + C\|\xi_0\|_2^4.\end{aligned}$$

Using (3.17) for $t \in [0,t^*]$ and that $\|(\tilde{Y},\tilde{Z})\|_4 \leqslant \tilde{C}$ to continue as:

$$\begin{aligned}\left(\int_0^t\|\xi\|_4^4\,\mathrm{d}s\right)^{\frac{1}{2}} &\leqslant c\left(\int_0^t\|\eta\|_2\,\mathrm{d}s\right)^2 + c\left(\int_0^t\|(\xi\tilde{Z}+\tilde{X}\zeta)\|_2\,\mathrm{d}s\right)^2 + c\|\xi_0\|_2^2\\
&\leqslant c\int_0^t\|\eta\|_2^2\,\mathrm{d}s + c\left(\int_0^t\|\xi\|_4\|\tilde{Z}\|_4 + \|\tilde{X}\|_2 p\|\zeta\|_{2q}\right)^2 + c\|\xi_0\|_2^2\\
&\leqslant c\int_0^t\|\eta\|_2^2\,\mathrm{d}s + ct^{\frac{3}{2}}\left(\int_0^t\|\xi\|_4^4\right)^{\frac{1}{2}} + c\left(\int_0^t\|\tilde{X}\|_{2p}\|\zeta\|_{2q}\right)^2 + c\|\xi_0\|_2^2;\end{aligned}$$

and if t^* small enough, we obtain:

$$\left(\int_0^t\|\xi\|_4^4\,\mathrm{d}s\right)^{\frac{1}{2}} \leqslant c\int_0^t\|\eta\|_2^2\,\mathrm{d}s + c\left(\int_0^t\|\tilde{X}\|_{2p}\|\zeta\|_{2q}\right)^2 + c\|\xi_0\|_2^2, \tag{3.25}$$

for any $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Now for a $p \gg 1$, we bound:

$$\begin{aligned}\left(\int_0^t\|\tilde{X}\|_{2p}\|\zeta\|_{2q}\right)^2 &\leqslant \left(\int_0^t\|\tilde{X}\|_{2p}^2\right)\left(\int_0^t\|\zeta\|_{2q}^2\right)\\
&\leqslant \left(\int_0^t\left[a^{\frac{1}{p}-\frac{1}{2}}\left(\frac{p}{2}\right)^{\frac{1}{2}}\|\tilde{X}\|_{H^1}^{1-\frac{2}{p}}\|\tilde{X}\|_2^{\frac{2}{p}}\right]^2\right)\left(\int_0^t\|\zeta\|_{2q}^2\right) \leqslant tpC\left(\int_0^t\|\zeta\|_{2q}^2\right),\end{aligned} \tag{3.26}$$

where in the second inequality we have used the Gagliardo–Nirenberg inequality with the best constant, see [22, p 533] among others. Here a is an absolute constant.

Let us write $2q = 2\lambda + m(1-\lambda)$, where $\lambda \in (0,1)$, $m > 2q > 2$. Then:

$$\|\zeta\|_{2q}^2 = \left(\int |\zeta|^{2q}\right)^{\frac{1}{q}} = \left(\int |\zeta|^{2\lambda}|\zeta|^{m(1-\lambda)}\right)^{\frac{1}{q}} \leqslant \left(\int |\zeta|^2\right)^{\frac{\lambda}{q}} \left(\int |\zeta|^m\right)^{\frac{1-\lambda}{q}}. \tag{3.27}$$

We choose $q = 1 + \frac{\epsilon}{2}$, $\lambda = q(1-\epsilon) = (1+\frac{\epsilon}{2})(1-\epsilon) = 1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{2}$, with $\epsilon \ll 1$. Then:

$$\begin{aligned} m &= \frac{2q-2\lambda}{1-\lambda} = \frac{2q-2q(1-\epsilon)}{1-\lambda} = \frac{2q\epsilon}{1-q(1-\epsilon)} = \frac{2q\epsilon}{1+q\epsilon-q} = \frac{2(1+\frac{\epsilon}{2})\epsilon}{1+\epsilon+\frac{\epsilon^2}{2}-1-\frac{\epsilon}{2}} \\ &= \frac{2(1+\frac{\epsilon}{2})}{\frac{1}{2}+\frac{\epsilon}{2}} = \frac{4(1+\frac{\epsilon}{2})}{1+\epsilon} < 4. \end{aligned} \tag{3.28}$$

It follows from (3.27) that:

$$\|\zeta\|_{2q}^2 \leqslant \left(\int |\zeta|^2\right)^{1-\epsilon} (\|\zeta\|_m^m)^{\frac{1-\lambda}{q}} = \|\zeta\|_2^{2(1-\epsilon)} (\|\zeta\|_m^m)^{\frac{\epsilon+\epsilon^2}{2+\epsilon}} \leqslant \tilde{C}\|\zeta\|_2^{2(1-\epsilon)}. \tag{3.29}$$

Now (3.26)–(3.29) yield:

$$\left(\int_0^t \|\tilde{X}\|_{2p}\|\zeta\|_{2q}\right)^2 \leqslant tpC\int_0^t \|\zeta\|_2^{2(1-\epsilon)} = Ct\left(1+\frac{2}{\epsilon}\right)\int_0^t \|\zeta\|_2^{2(1-\epsilon)},$$

and by (3.25):

$$\left(\int_0^t \|\xi\|_4^4\,\mathrm{d}s\right)^{\frac{1}{2}} \leqslant c\|\xi\|_0^2 + c\int_0^t \|\eta\|_2^2\,\mathrm{d}s + Ct\epsilon^{-1}\int_0^t \|\zeta\|_2^{2(1-\epsilon)}. \tag{3.30}$$

Finally, let us plug in (3.24) to obtain:

$$\|(\eta,\zeta)\|_2^2 \leqslant C\|(\xi_0,\eta_0,\zeta_0)\|_2^2 + C\int_0^t \|(\eta,\zeta)\|_2^2 + Ct\epsilon^{-1}\int_0^t \|(\eta,\zeta)\|_2^{2(1-\epsilon)}, \tag{3.31}$$

where C is a constant independent of ϵ. Let us define:

$$G = \|(\xi_0,\eta_0,\zeta_0)\|_2^2 + \int_0^t \|(\eta,\zeta)\|_2^2,$$

then by Hölder inequality:

$$\int_0^t \|(\eta,\zeta)\|_2^{2(1-\epsilon)} \leqslant t^\epsilon\left(\int_0^t \|(\eta,\zeta)\|_2^2\right)^{1-\epsilon} \leqslant t^\epsilon G^{1-\epsilon}. \tag{3.32}$$

It follows from (3.31) that:

$$G_t \leqslant CG + C\epsilon^{-1}t^{1+\epsilon}G^{1-\epsilon},$$

which can be directly integrated for a small time interval $t \in [0,t^*]$, $t^* \in (0,1)$:

$$\begin{aligned} &G \leqslant (G_0^\epsilon \mathrm{e}^{\epsilon Ct} + Ct^2\mathrm{e}^{\epsilon Ct})^{\frac{1}{\epsilon}} = (\|(\xi_0,\eta_0,\zeta_0)\|_2^{2\epsilon} + Ct^2)^{\frac{1}{\epsilon}}\mathrm{e}^{Ct}, \\ &\|(\xi_0,\eta_0,\zeta_0)\|_2^2 + \int_0^t \|(\eta,\xi)\|_2^2\,\mathrm{d}s \leqslant (\|(\xi_0,\eta_0,\zeta_0)\|_2^{2\epsilon} + Ct^2)^{\frac{1}{\epsilon}}\mathrm{e}^{Ct}, \end{aligned} \tag{3.33}$$

for $t \in [0,t^*] \subset [0,1)$, where t^* is independent of ϵ, and depends only on data and coefficients of the original MBR system.

For the continuous dependence of (X,Y,Z) on the initial data (X_0,Y_0,Z_0) in the L^2 sense, we show that if $\|(\xi_0^{(\alpha)},\eta_0^{(\alpha)},\zeta_0^{(\alpha)})\|_2 \to 0$, as $\alpha \to \infty$, then $\|(\xi^{(\alpha)},\eta^{(\alpha)},\zeta^{(\alpha)})\|_2 \to 0$, as $\alpha \to \infty$, for any finite $t > 0$.

Suffices to prove this for $t \in [0, t^*]$. It follows from (3.33) that:

$$\|(\xi_0^{(\alpha)}, \eta_0^{(\alpha)}, \zeta_0^{(\alpha)})\|_2^2 + \int_0^t \|(\eta^{(\alpha)}, \xi^{(\alpha)})\|_2^2 \, \mathrm{d}s \leqslant (\|(\xi_0^{(\alpha)}, \eta_0^{(\alpha)}, \zeta_0^{(\alpha)})\|_2^{2\epsilon} + Ct^2)^{\frac{1}{\epsilon}} \mathrm{e}^{Ct}, \quad (3.34)$$

and sending $\alpha \to \infty$ shows:

$$\limsup_{\alpha\to\infty} \int_0^t \|(\eta^{(\alpha)}, \zeta^{(\alpha)})\|_2^2 \, \mathrm{d}s \leqslant (Ct^2)^{\frac{1}{\epsilon}} \mathrm{e}^{Ct}. \quad (3.35)$$

Next, in view of ϵ independence of the left-hand side of (3.35) we pass to the $\epsilon \to 0$ limit of the right-hand side of (3.35) for $t \in [0, t^*]$, $t^* < 1$ to yield:

$$\limsup_{\alpha\to\infty} \int_0^t \|(\eta^{(\alpha)}, \zeta^{(\alpha)})\|_2^2 \, \mathrm{d}s = 0, \quad (3.36)$$

for $t \in [0, t^*]$. By (3.31) and Hölder inequality, it follows that:

$$\limsup_{\alpha\to\infty} \|(\eta^{(\alpha)}, \zeta^{(\alpha)})\|_2^2 = 0, \quad (3.37)$$

uniformly for $t \in [0, t^*]$. By (3.20) and (3.21), and Hölder inequality, we have:

$$\limsup_{\alpha\to\infty} \|\xi^{(\alpha)}\|_2 = 0, \quad (3.38)$$

for $t \in [0, t^*]$. We proved that (X, Y, Z) depends on (X_0, Y_0, Z_0) continuously in L^2 over any finite time interval. However, compared with the one-dimensional result (3.23), we see that we lose (at least a proof of) Lipschitz continuity in two dimensions.

Proof of theorem 1.2, (1), (2) and (4). Let us take a sequence of initial data $(X_j^{(0)}, Y_j^{(0)}, Z_j^{(0)}) \in (H^2)^3$, so that

$$\|\nabla X_j^{(0)}\|_{H^1} + \|(Y_j^{(0)}, Z_j^{(0)})\|_{(L^2\cap L^p)^2} \leqslant C < \infty,$$

and

$$(X_j^{(0)}, Y_j^{(0)}, Z_j^{(0)}) \to (X_0, Y_0, Z_0),$$

in $(L^2)^3$ as $j \to \infty$. Here p is some finite number larger than 4. By the *a priori* bounds, the H^1 norm of X_j and $(L^2 \cap L^p)^2$ norm of (Y_j, Z_j) are uniformly bounded in time. Also, by continuity of solutions on the initial data in L^2 for any finite time, we deduce that the initial convergent sequence implies the sequence (X_j, Y_j, Z_j) is Cauchy in $(L^2)^3$. The limit (X, Y, Z) then satisfies the integral equations in L^2 sense, and obey the bounds of part (4). Moreover, the evolution operator $S(t)$ is continuous and bounded from $[0, \infty)$ to $H^1 \times (L^2 \cap L^p)^2$, which is a consequence of weak continuity and continuity of the norms. The proof of parts (1), (2), (4) of theorem 1.2 is complete. □

Finally, let us comment on how to construct the weak solution on periodic domains. The major difference is that one has to use the modified Strichartz inequalities, which are weaker than those on the whole plane. We refer to Bourgain [3] for details, and only list those inequalities for our analysis. If $f(x) \in L^2(T^1)$, then:

$$\|\mathrm{e}^{\mathrm{i}t\partial_{xx}} f\|_{L^4(T^2)} \leqslant 2\|f\|_2, \quad (3.39)$$

where $L^4(T^2)$ is the spacetime L^4 norm. If $f \in L^2(T^2)$, then it is conjectured that:

$$\|\mathrm{e}^{\mathrm{i}t\Delta} f\|_{L^{p,p}(T^3)} \leqslant c_p\|f\|_2, \quad (3.40)$$

if $p \in (2,4)$. It is known that (3.40) is false if $p = 4$. It is not hard to use Minkowsky inequality to deduce from (3.39) or (3.40) that:

$$\left\| \int_0^t \mathrm{e}^{\mathrm{i}(t-s)\Delta} v(s,x)\,\mathrm{d}s \right\|_{L^p(T^d\times[0,t])} \leqslant \frac{(4\pi+t)^2}{8\pi^2} c_p \int_0^t \|v\|_2\,\mathrm{d}s, \tag{3.41}$$

where $d = 1$ is true, and $d = 2$ is still a conjecture. See [28] for a derivation.

Let us start from (3.9) again. In case of T^1, we bound:

$$\int_0^t \int H|\xi|^2 \leqslant \left(\int_0^t \int H^2 \right)^{\frac{1}{2}} \left(\int_0^t \int |\xi|^4 \right)^{\frac{1}{2}},$$

then apply (3.41) with $d = 1$ on the last integral of ξ. The rest is the same. The initial data $(Y_0, Z_0) \in (L^4)^2$ remains the same as well. In case of T^2, we use Hölder inequality to obtain:

$$\int_0^t \int H|\xi|^2 \leqslant \left(\int_0^t \int H^q \right)^{\frac{1}{q}} \left(\int_0^t \int |\xi|^{2p} \right)^{\frac{1}{p}}$$
$$\leqslant \left(\int_0^t \|(\tilde{Y},\tilde{Z})\|_{2q}^{2q} + \|\tilde{X}\|_{2qp_1}^{2q} \|\tilde{Y}\|_{2qq_1}^{2q} \right)^{\frac{1}{q}} \left(\int_0^t \int |\xi|^{2p} \right)^{\frac{1}{p}}, \tag{3.42}$$

where $p_1^{-1} + q_1^{-1} = 1$, $p_1 \gg 1$, q_1 is close to 1; $p^{-1} + q^{-1} = 1$, $2p = p^*$, $p^* \in (2,4)$ is the exponent for which the conjectural Strichartz inequality is valid. So if $p = \frac{p^*}{2} \in (1,2)$, $q \in (2,\infty)$, then $(Y_0, Z_0) \in (L^{2q+\epsilon})^2$ is sufficient, for any small number $\epsilon > 0$. Applying (3.41) to the last ξ integral of (3.42) and going through the same steps as before, we end up with the same conclusion that the solutions depend on the initial data in the L^2 sense at any finite later time. We omit the details and the analogous formulation of theorem 1.2 in the periodic case.

4. Smooth global attractors

Results of previous sections establish the well posedness of MBR system, either the classical H^2 solutions or weak solutions. It also follows that the evolution map denoted by $S(t)$: (1) depends continuously on the initial data in the L^2 sense at any finite time; (2) forms a one-parameter group: $S(0) = \mathrm{Id}$, $S(t+s) = S(t)S(s)$, $t, s \in R^1$. Moreover, there exists an absorbing ball:

$$B_{\rho_0} = \{(X,Y,Z) : \|(X,Y,Z)\|_{H^1\times(L^2\cap L^p)^2} \leqslant \rho_0\}, \tag{4.1}$$

for $S(t)$, where p is a positive number larger than 4, and ρ_0 depends only on the coefficients of MBR system. Based on these properties, we define, as in [7], a set $\mathcal{A}$ to be:

$$\cap_{t>0} S(t) B_{\rho_0}. \tag{4.2}$$

By property (1) of $S(t)$, we see that $S(t)B_{\rho_0}$ is closed under $(L^2)^3$. The set $\mathcal{A}$ is nonempty, because $(X,Y,Z) = (0,0,-r)$ is a steady-state solution belonging to $\mathcal{A}$. We show that:

Lemma 4.1. *$\mathcal{A}$ is invariant under $S(t)$: $S(t)\mathcal{A} = \mathcal{A}$, $\forall\, t \in R^1$. Moreover, $\mathcal{A}$ is the largest closed bounded time invariant set that is bounded in $H^1 \times (L^2 \cap L^p)^2$; $\mathcal{A}$ contains the ω limit set of any closed (in the L^2 sense) bounded set (in $H^1 \times (L^2 \cap L^p)^2$).*

Proof. Step 1. We show that $u \in \mathcal{A}$ if and only if there are sequences $\{u_n\} \in B_{\rho_0}$, $\{t_n\} \to +\infty$, as $n \to \infty$ such that $S(t_n)u_n \to u$. Obviously, if $u \in \mathcal{A}$, then we can find such sequences so that $u = S(t_n)u_n$. Conversely, we can write $S(t_n)u_n = S(t)S(t_n - t)u_n$, for any $t > 0$. If n is large enough so that $t_n > t$, then $S(t_n - t)u_n \in B_{\rho_0}$, and so $S(t_n)u_n \in S(t)B_{\rho_0}$. Since $S(t_n)u_n \to u$, and $S(t)B_{\rho_0}$ is closed in L^2, we have: $u \in S(t)B_{\rho_0}$, $\forall\, t > 0$. Thus $u \in \mathcal{A}$.

Step 2. We show that $S(t)\mathcal{A} \subset \mathcal{A}$ for any $t \in R$. Any element of $S(t)\mathcal{A}$ is of the form $S(t)u$, $u \in \mathcal{A}$, where u is equal to the L^2 limit of $S(t_n)u_n$ for some sequences $u_n \in B_{\rho_0}$, $t_n \to +\infty$. By properties (1) and (2),

$$S(t)u = \lim_{n\to\infty} S(t)S(t_n)u_n = \lim_{n\to\infty} S\left(\frac{t_n}{2}\right) S\left(\frac{t_n}{2} + t\right) u_n.$$

If n is large enough, $\frac{t_n}{2} + t > 0$, therefore $S(\frac{t_n}{2} + t)u_n \equiv v_n \in B_{\rho_0}$, implying

$$S(t)u = \lim_{n\to\infty} S\left(\frac{t_n}{2}\right) v_n.$$

It follows that $S(t)u \in \mathcal{A}$, and so $S(t)\mathcal{A} \subset \mathcal{A}$, for any $t \in R$.

Step 3. Now $\mathcal{A} = S(t)S(-t)\mathcal{A} \subset S(t)\mathcal{A}$ for any $t \in R$, by step 2. It follows that $S(t)\mathcal{A} = \mathcal{A}$, for any $t \in R$. The additional statements in the lemma are straightforward in view of step 1. The proof is complete. □

We show next:

Proposition 4.1. *The set $\mathcal{A}$ consists of C^∞ functions.*

Proof. Step 1. We show that if $V = (X, Y, Z) \in \mathcal{A}$, then $(Y, Z) \in (L^\infty)^2$. This step uses the estimate (2.5) and so the cancellation property of Y and Z equations.

By (4.2), there is $V_0 \in B_{\rho_0}$ such that $V = S(t)V_0$ for any $t > 0$. From our early estimate:

$$|(Y, Z)|^2 \leqslant M^2 + \mathrm{e}^{-2\beta t}(|Y_0|^2 + |Z_0|^2),$$

for M, β depending only on MB coefficients. Let $\Sigma = \{(x_1, x_2) \in R^2 : |(Y, Z)|^2 \geqslant 2M\}$. Then on Σ:

$$3M^2 \leqslant \mathrm{e}^{-2\beta t}(|Y_0|^2 + |Z_0|^2). \tag{4.3}$$

Integrating (4.3) over $\Sigma \cap B(0, R)$, where $B(0, R)$ is the ball of radius R, we have (in the case of periodic domains, no intersection is necessary):

$$3M^2|\Sigma \cap B(0, R)| \leqslant \mathrm{e}^{-2\beta t}\|(Y_0, Z_0)\|_2^2,$$

implying as $t \to \infty$: $|\Sigma \cap B(0, R))| = 0$, or $|\Sigma| = 0$, by arbitrariness of R. It follows that $(Y, Z) \in (L^\infty)^2$. In particular, $(Y, Z) \in (L^p)^2$, for any $p \in [2, +\infty)$.

Step 2. Show that if $V \in \mathcal{A}$, then $X \in H^2$. Let us still write $V = S(t)V_0$, for some $V_0 \in B_{\rho_0}$. Then:

$$V(s) = S(s)V_0 = S(s - t)V,$$

lies in $\mathcal{A}$.

First we are going to use the (Y, Z) ODEs and also the results of step 1 to show that $(Y_t, Z_t) \in L^2$ on the attractor. The Y equation now reads:

$$Y(s, \cdot) = V_0 - (1 + \mathrm{i}\Omega)\int_0^s Y(\tau, \cdot)\,\mathrm{d}\tau - \int_0^s Z(\tau, \cdot)X(\tau, \cdot)\,\mathrm{d}\tau + \mathrm{i}\delta_2\int_0^s |X|^2(\tau, \cdot)Y(\tau, \cdot)\,\mathrm{d}\tau. \tag{4.4}$$

Taking the s derivative of Y in L^2, using $(Y, Z) \in L^\infty$ on $\mathcal{A}$, we obtain:

$$\|Y_s\|_2 \leqslant (1 + \|\Omega\|_\infty^2)^{\frac{1}{2}} \|Y\|_2 + \|Z\|_\infty \|X\|_2 + \delta_2 \|X\|_2^2 \|Y\|_\infty \leqslant M_1 < \infty, \tag{4.5}$$

where M_1 is a positive constant like M. Similarly, the Z equation gives $\|Z_s\|_2 \leqslant M_2 < \infty$, and that:

$$\|Z_s\|_4 \leqslant b\|Z\|_4 + \|Y\|_\infty \|X\|_4 + \|br\|_4 \leqslant M_3 < \infty.$$

Next we go to the X equation to show that X_t is bounded in L^2 as well based on bounds of (Y_t, Z_t). The X equation can be written as:

$$X_t = (\mathrm{i}a\Delta - \sigma + \mathrm{i}\delta_1(Z + r))X + \sigma Y. \tag{4.6}$$

It follows that:

$$X(x_1, x_2) = U(t, 0)X_0 + \int_0^t \sigma U(t, s)Y(s)\,\mathrm{d}s, \tag{4.7}$$

where $U(t, s)$ is the evolution operator for the linear problem:

$$\begin{aligned} \tilde{X}_t &= (\mathrm{i}a\Delta - \sigma + \mathrm{i}\delta_1(Z + r))\tilde{X}, \\ \tilde{X}|_{t=0} &= \tilde{X}_0. \end{aligned} \tag{4.8}$$

It is easy to check that $\|U(t, s)\|_2 \leqslant \mathrm{e}^{-\sigma(t-s)}$. In (4.7), smoothing in time comes from the second term while the first term decays and has no effect on regularity of the attractor. We note that $\int_0^t U(t, s)Y(s)\,\mathrm{d}s$ is the solution to the inhomogeneous linear problem:

$$\begin{aligned} \bar{X}_t &= (\mathrm{i}a\Delta - \sigma + \mathrm{i}\delta_1(Z + r))\bar{X} + Y, \\ \bar{X}|_{t=0} &= 0. \end{aligned} \tag{4.9}$$

We show that $\bar{X}_t \in L^2$ if $Y_t \in L^2$, and $Z_t \in L^4$. For ease of showing smoothness, let us introduce the finite difference operator:

$$D_\tau = \frac{u(t + \tau, \cdot) - u(t, \cdot)}{\tau}, \tag{4.10}$$

for $\tau > 0$, $t \geqslant 0$. Then $D_\tau \bar{X}$ satisfies:

$$\begin{aligned} (D_\tau \bar{X})_t &= (\mathrm{i}a\Delta - \sigma + \mathrm{i}\delta_1(Z + r))(D_\tau \bar{X}) + \mathrm{i}\delta_1 D_\tau Z \bar{X} + D_\tau Y, \\ D_\tau \bar{X}|_{t=0} &= \frac{\bar{X}(\tau, \cdot)}{\tau}, \end{aligned} \tag{4.11}$$

where:

$$\frac{\bar{X}(\tau, \cdot)}{\tau} = \frac{1}{\tau} \int_0^\tau U(\tau, s)Y(s)\,\mathrm{d}s \to Y(0),$$

in the L^2 sense as $\tau \to 0$. It follows that

$$D_\tau \bar{X} = U(t, \cdot)\frac{\bar{X}(\tau, \cdot)}{\tau} + \int_0^t U(t, s)(\mathrm{i}\delta_1 D_\tau Z \cdot \bar{X} + D_\tau Y)(s)\,\mathrm{d}s. \tag{4.12}$$

Using the fact that (Y, Z) already has L^2 time derivative, and also Z has L^4 time derivative, we pass to $\tau \to 0$ limit in (4.12) to find that $\lim_{\tau \to 0} D_\tau \bar{X}$ exists in L^2. Moreover, taking $\|\cdot\|_2$ of (4.12) and sending $\tau \to 0$ gives:

$$\begin{aligned} \|\bar{X}_t\|_2 &\leqslant \|Y_0\|_2 + \int_0^t \mathrm{e}^{-\sigma(t-s)}(\delta_1 \|D_s Z\|_4 \cdot \|X\|_4 + \|D_s Y\|_2)\,\mathrm{d}s, \\ &\leqslant \|Y_0\|_2 \int_0^t \mathrm{e}^{-\sigma(t-s)}(\delta_1 M_3 M + M_2)\,\mathrm{d}s \equiv M_5 < \infty, \end{aligned} \tag{4.13}$$

where M_5 depends only on $\mathcal{A}$. So:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t U(t,s)Y(s)\,\mathrm{d}s \in L^2,$$

or

$$(\mathrm{i}a\Delta - \sigma + \mathrm{i}\delta_1(Z-r))\int_0^t U(t,s)Y(s)\,\mathrm{d}s \in L^2,$$

implying

$$\left\|\Delta\left(\int_0^t U(t,s)Y(s)\,\mathrm{d}s\right)\right\|_2 \leqslant M_6,$$

or

$$\|\Delta(X - U(t,0)X_0)\|_2 \leqslant M_6.$$

Now the smoothness of X in t transfers to a weak spatial derivative in space. We upgrade it to strong derivative using decay of $U(t,0)$. For any smooth compactly supported function φ, we have for weak derivative ΔX:

$$\begin{aligned}\int \Delta X\cdot\varphi = \int X\Delta\varphi = \int (X - U(t,0)X_0)\Delta\varphi + \int \varphi U(t,0)X_0,\\ = \int \varphi\Delta(X-U(t,0)X_0) + \int \varphi U(t,0)X_0.\end{aligned} \tag{4.14}$$

So:

$$\left|\int \Delta X\cdot\varphi\right| \leqslant (\|\Delta(X-U(t,0)X_0)\|_2 + \|U(t,0)X_0\|_2)\|\varphi\|_2. \tag{4.15}$$

Letting $t\to\infty$ shows: $\|X\|_{H^2} \leqslant M_7 < \infty$.

Step 3. Now we go back to the (Y,Z) equations to gain one more spatial derivative. Let $D_h u$ be the forward finite difference quotient approximation of ∇u, with $h=(h_1,h_2)$. Then apply D_h to both Y and Z equations to obtain:

$$\begin{aligned}&(D_hY)_t + (1+\mathrm{i}\Omega)D_hY + \mathrm{i}YD_h\Omega = -D_hZ\cdot X - Z\cdot D_hX + \mathrm{i}\delta_2 XYD_hX^*\\ &\qquad + \mathrm{i}\delta_2X^*YD_hX + \mathrm{i}\delta_2|X|^2D_hY,\\ &(D_hZ)_t + b(D_hZ) = \mathrm{Re}\,\{D_hX^*\cdot Y + X^*D_hY\} - D_h(br).\end{aligned} \tag{4.16}$$

Energy estimates give:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D_hY\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D_hZ\|_2^2 + \|D_hY\|_2^2 + b\|D_hZ\|_2^2\\ &\quad = \mathrm{Re}\left\{-\int D_hZ\cdot XD_hY^* - \int D_hX\cdot ZD_hY^*\right\} - \mathrm{Re}\int \mathrm{i}(D_h\Omega)YD_hY^*\\ &\qquad + \mathrm{Re}\left\{\mathrm{i}\delta_2\int YXD_hX^*\cdot D_hY^* + \mathrm{i}\delta_2\int X^*YD_hX\cdot D_hY^*\right\}\\ &\qquad - \int D_h(br)D_hZ + \mathrm{Re}\left\{\int D_hX^*\cdot YD_hZ + X^*D_hY\cdot D_hZ\right\}.\end{aligned}$$

The first and last terms on the right-hand side cancel, and we continue the equality:

$$\begin{aligned}&= \mathrm{Re}\left\{-\int D_hX\cdot ZD_hY^*\right\} - \mathrm{Re}\int \mathrm{i}(D_h\Omega)YD_hY^* + \mathrm{Re}\left\{\mathrm{i}\delta_2\int YXD_hX^*\cdot D_hY^*\right.\\ &\qquad \left. + \mathrm{i}\delta_2\int X^*YD_hX\cdot D_hY^*\right\} - \int D_h(br)D_hZ + \mathrm{Re}\left\{\int D_hX^*\cdot YD_hZ\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(D_hY, D_hZ)\|_2^2 + \|D_hY\|_2^2 + b\|D_hZ\|_2^2 &\leqslant \delta\|D_hY^*\|_2^2 + \delta^{-1}\|ZD_hX\|_2^2 \\
&\quad +\delta_2\delta\|D_hY\|_2^2 + \delta^{-1}\delta_2\|YXD_hX\|_2^2 + \delta_2\delta\|D_hY\|_2^2 + \delta_2\delta^{-1}\|YXD_hX\|_2^2 \\
&\quad +\delta\|D_hY^*\|_2^2 + \delta\|D_hZ\|_2^2 + \delta^{-1}\|(D_h\Omega)Y\|_2^2 + \delta^{-1}\|D_h(br)\|_2^2 \\
&\leqslant 2(1+\delta_2)\delta\|(D_hY, D_hZ)\|_2^2 + \delta^{-1}\|Z\|_\infty^2\|D_hX\|_2^2 \\
&\quad +2\delta_2\delta^{-1}\|Y\|_\infty\|X\|_\infty\|D_hX\|_2^2 + \delta^{-1}\|Y\|_\infty^2\|D_h\Omega\|_2^2 + \delta^{-1}\|D_h(br)\|_2^2 \\
&\leqslant 2(1+\delta_2)\delta\|(D_hY, D_hZ)\|_2^2 + \delta^{-1}M_8. \qquad (4.17)
\end{aligned}$$

With δ small enough, integrating in t gives:

$$\|(D_hY, D_hZ)\|_2^2 \leqslant \mathrm{e}^{-\beta t}\|(D_hY_0, D_hZ_0)\|_2^2 + \frac{2\delta^{-1}}{\beta}M_8. \qquad (4.18)$$

Sending $h \to 0$, $t \to \infty$ shows that:

$$\|\nabla(Y, Z)\|_2^2 \leqslant \frac{2\delta^{-1}}{\beta}M_8, \qquad (4.19)$$

or $(Y, Z) \in H^1$.

We can then go back to the X equation and show that $X \in H^3$, which in turn implies that $(Y, Z) \in (H^2)^2$. Iterating this procedure yields $(X, Y, Z) \in H^{n+1} \times (H^n)^2$, for any $n \geqslant 1$; or $(X, Y, Z) \in (C^\infty)^3$. We omit the details. The proof is complete. □

Remark 4.1. The cancellation property of the Y and Z equations, that Y (but not ∇Y) appearing as a forcing in the X equation, and that the nonlinear interaction terms $\mathrm{i}\delta_1(Z+r)X$, $\mathrm{i}\delta_2|X|^2Y$ having pure imaginary coefficients, are all essential structures we explored to deduce time asymptotic smoothing. These interaction terms do not cause growth or sharpening or focusing of solutions. Instead, they tend to create coherence and smoothing in the solutions. In fact, it is known [14, 18] that for r above a threshold value, the system admits planar solutions of the form $A\mathrm{e}^{\mathrm{i}(k\cdot x-ct)}$, all of which are smooth solutions in the attractors.

For $\mathcal{A}$ to be a usual global attractor, see definitions 1.2 and 1.3 of [25, pp 21, 22], we need to show that $\mathcal{A}$ attracts any bounded set of initial data that lie in a space containing smooth functions. For initial data in $(H^3(T^2))^3$, we have shown that there is an absorbing ball of dynamics. Define $\mathcal{A}$ as in (4.2), but with $B_{\rho_0} = \{(X, Y, Z) : \|(X, Y, Z)\|_{(H^2(T^2))^3} \leqslant \rho_0\}$. Since $(H^3(T^2))^3$ is compact in $(H^2(T^2))^3$ uniformly in t, by the definition of $\mathcal{A}$ and the compactness of solutions in time, we have:

$$\lim_{t\to\infty} \mathrm{dist}_{(H^2(T^2))^3}(S(t)B, \mathcal{A}) = 0, \qquad (4.20)$$

for any bounded set, denoted by B, of $(H^3(T^2))^3$. In other words, $\mathcal{A}$ is the global attractor in $(H^2(T^2))^3$. This is part (3) of theorem 1.1.

For mild solutions in $H^1(T^2) \times (L^p(T^2))^2$, as well as mild and strong solutions in the whole plane, we do not know $(L^2)^3$ compactness of dynamics in time either due to lack of derivative control of (Y, Z) components, or unbounded domain. The former may be overcome by decomposing $S(t) = S_1(t) + S_2(t)$, where S_1 is uniformly compact in time, S_2 decays in time. However, for the MB system, this decompostion is not as clear as in damped driven wave equations, [25]. Simply extracting the linear part and defining it to be S_2 does not seem to work. Even the two ODEs prevent us from applying such an argument.

Nevertheless, similar decompositions remain to be explored. The latter unboundedness of domain may be circumvented by proving the well posedness and the existence of an attacting ball of dynamics in a spatially weighted Sobolev space. See [5, 19] for such studies on parabolic systems. However, we will not pursue either of the above undertakings here. Instead, we will interpret the attractivity of $\mathcal{A}$ of the above cases in a weaker sense. Following the alternative definition of attractor, introduced by Sell, see the footnote on page 21 of Temam [25], $\mathcal{A}$ is an attractor if $\mathcal{A}$ is the ω-limit set of one of its open neighbourhoods $\mathcal{U}_0$. The basin of attraction of $\mathcal{A}$ is the union of the open sets $\mathcal{U}_0$ such that $\mathcal{A} \subset \mathcal{U}_0$, and $\omega(\mathcal{U}_0) = \mathcal{A}$. By lemma 1, $\mathcal{A}$ is clearly such an attractor, and the basin of attraction is any bounded set in $H^1 \times (L^p \cap L^2)^2$, hence the attractor is global. We remark that this weaker definition of attractor is a useful notion for us to initiate the understanding of the subtle smoothing property of MB dynamics. It is an interesting problem for the future to investigate the attractivity of $\mathcal{A}$ for weak solutions in the L^2 sense.

5. Finite dimensions of the global attractor on T^2

We carry out the dimensional bounds of $\mathcal{A}$ in the space $C([0,\infty); (H^2(T^2)^3)$. Let us consider the flow around any vector function of the attractor $\mathcal{A}$, say $U \in (C^\infty(T^2))^3$ and study the contraction of finite-dimensional volume elements formed by the linearized flows. Due to the weak nature of MBR damping, it is necessary to use the intrinsic skew symmetries of the linearized operator, and measure volume elements in suitable new coordinates. The direct approach of estimating the quadratic form of the linearized system, as often used for damped nonlinear wave equations, does not seem to work here. This difference has already been observed in Constantin *et al* [7]. We will adapt the method of [7], while incorporating additional skew symmetries. For technical convenience, we consider both r and Ω to be real constants.

First let us shift the variables, and define:

$$(u_1, u_2, u_3) = (X, Y, Z - \sigma).$$

Then the new system reads:

$$u_{1,t} = \mathrm{i}a\Delta u_1 - \sigma u_1 + \sigma u_2 + \mathrm{i}\delta_1 u_1(u_3 + \sigma - r), \tag{5.1}$$

$$u_{2,t} = -(1 + \mathrm{i}\Omega)u_2 - \sigma u_1 - u_3 u_1 + \mathrm{i}\delta_2 |u_1|^2 u_2, \tag{5.2}$$

$$u_{3,t} = -bu_3 + \mathrm{Re}\,(u_1 u_2^*) - b(r + \sigma). \tag{5.3}$$

Linearized system in terms of $V = (v_1, v_2, v_3)$, along $U = (u_1, u_2, u_3) \in \mathcal{A}$ is:

$$v_{1,t} = \mathrm{i}a\Delta v_1 - \sigma v_1 + \sigma v_2 + \mathrm{i}\delta_1 v_1(u_3 + \sigma - r) + \mathrm{i}\delta_1 u_1 v_3, \tag{5.4}$$

$$v_{2,t} = -(1 + \mathrm{i}\Omega)v_2 - \sigma v_1 - v_3 u_1 - u_3 v_1 + \mathrm{i}\delta_2 |u_1|^2 v_2 + 2\mathrm{i}\delta_2 u_2 \mathrm{Re}\,(u_1^* v_1), \tag{5.5}$$

$$v_{3,t} = -bv_3 + \mathrm{Re}\,(u_1 v_2^* + v_1 u_2^*). \tag{5.6}$$

Notice that after the shift by σ, the two terms σv_2 and $-\sigma v_1$ become skew symmetric if we write down the 5×5 matrix for the real vector $(\mathrm{Re}\,(v_1), \mathrm{Im}\,(v_1), \mathrm{Re}\,(v_2), \mathrm{Im}\,(v_2), v_3)$. To overcome the weak damping, we change variable $W = (w_1, w_2, w_3) = \Delta v$. Then taking Laplacian of (5.4)–(5.6) we see that W is a solution of the following system:

$$\begin{aligned} w_{1,t} = {} & \mathrm{i}a\Delta w_1 - \sigma w_1 + \sigma w_2 + \mathrm{i}\delta_1 (u_3 + \sigma - r) w_1 + 2\mathrm{i}\delta_1 \nabla v_1 \cdot \nabla u_3 + \mathrm{i}\delta_1 v_1 \Delta u_3 \\ & + \mathrm{i}\delta_1 v_3 \Delta u_1 + 2\mathrm{i}\delta_1 \nabla u_1 \cdot \nabla v_3 + \mathrm{i}\delta_1 u_1 w_3, \end{aligned} \tag{5.7}$$

$$\begin{aligned} w_{2,t} = {} & -(1 + \mathrm{i}\Omega) w_2 - \sigma w_1 - w_3 u_1 - 2\nabla v_3 \cdot \nabla u_1 - v_3 \Delta u_1 - w_1 u_3 - 2\nabla v_1 \cdot \nabla u_3 \\ & - v_1 \Delta u_3 + \mathrm{i}\delta_2 v_2 \Delta |u_1|^2 + 2\mathrm{i}\delta_2 \nabla |u_1|^2 \cdot \nabla v_2 + \mathrm{i}\delta_2 w_2 |u_1|^2 \end{aligned}$$

$$+2\mathrm{i}\delta_2\Delta u_2 \mathrm{Re}\,(u_1^* v_1) + 2\mathrm{i}\delta_2\nabla u_2\cdot \mathrm{Re}\,\nabla(u_1^* v_1)$$
$$+2\mathrm{i}\delta_2 u_2 \mathrm{Re}\,(v_1\Delta u_1^* + 2\nabla u_1^*\cdot\nabla v_1 + u_1^* w_1) \tag{5.8}$$
$$w_{3,t} = -bw_3 + \mathrm{Re}\,\{\delta_1 w_1(\mathrm{i}u_1^*)\} + \mathrm{Re}\,\{u_1 w_2^* + w_1(u_2^* - \mathrm{i}\delta_1 u_1^*)$$
$$+v_2^*\Delta u_1 + 2\nabla u_1\cdot\nabla v_2^* + 2\nabla v_1\cdot\nabla u_2^* + v_1\Delta u_2^*\}. \tag{5.9}$$

In (5.7)–(5.9), the pairs of terms $-w_3u_1$ and $\mathrm{Re}\,\{u_1w_2^*\}$, $\mathrm{i}\delta_1u_1w_3$ and $\mathrm{Re}\,\{\delta_1w_1(\mathrm{i}u_1^*)\}$ turn out to be skew symmetric. Note that the term $\mathrm{Re}\,\{\delta_1w_1(\mathrm{i}u_1^*)\}$ is added and subtracted in (5.9) to create the latter skew symmetry. The remaining skew symmetric terms are diagonal, and those already mentioned, σw_2 and $-\sigma w_1$. The damping terms, $-\sigma w_1$, $-(1+\mathrm{i}\Omega)w_2$, and $-bw_3$, are obviously diagonal. Besides all these, the other terms involve only (u, v), and $\nabla(u, v)$, with U and its derivatives of order less than or equal to two as coefficients, except for the three w_1 terms. They are the three terms: $-w_1u_3$, $2\mathrm{i}\delta_2u_2\mathrm{Re}\,(u_1^*w_1)$, and $\mathrm{Re}\,(w_1(u_2^* - \mathrm{i}\delta_1u_1^*))$. The skew symmetries will be explicitly demonstrated in matrix form later.

Since the diagonal damping terms are zeroth order just like these three, we have to rewrite these three terms using the v_1 equation (5.4), and transfer time derivative from V to U. Upon substitution, we see that:

$$-w_1u_3 = \frac{\mathrm{i}}{a}u_3(v_{1,t} + \sigma v_1 - \sigma v_2 - \mathrm{i}\delta_1 v_1(u_3+\sigma-r) - \mathrm{i}\delta_1 u_1 v_3) = \frac{\mathrm{i}}{a}(u_3v_1)_t$$
$$+\frac{1}{a}[-\mathrm{i}v_1u_{3,t} + \mathrm{i}\sigma u_3 v_1 - \sigma \mathrm{i} v_2 u_3 + \delta_1 v_1 u_3(u_3+\sigma-r) + \delta_1 u_1 u_3 v_3]. \tag{5.10}$$

Also

$$2\mathrm{i}\delta_2u_2\mathrm{Re}\,\{u_1^*w_1\} = 2\mathrm{i}\delta_2u_2\mathrm{Re}\left\{u_1^*\frac{1}{\mathrm{i}a}(v_{1,t} + \sigma v_1 - \sigma v_2 - \mathrm{i}\delta_1 v_1(u_3+\sigma-r) - \mathrm{i}\delta_1 u_1 v_3)\right\}$$
$$= 2\mathrm{i}\delta_2 u_2\left(\frac{u_1^* v_{1,t}}{\mathrm{i}a} - \frac{u_1 v_{1,t}^*}{\mathrm{i}a}\right)$$
$$+2\mathrm{i}\delta_2u_2\mathrm{Re}\left\{u_1^*\frac{1}{\mathrm{i}a}(\sigma v_1 - \sigma v_2 - \mathrm{i}\delta_1 v_1(u_3+\sigma-r) - \mathrm{i}\delta_1 u_1 v_3)\right\}$$
$$= \frac{2\delta_2}{a}[(u_2u_1^*v_1)_t - (u_2u_1^*)_t v_1 - (u_1u_2v_1^*)_t + (u_1u_2)_t v_1^*]$$
$$+2\mathrm{i}\delta_2u_2\mathrm{Re}\left\{u_1^*\frac{1}{\mathrm{i}a}(\sigma v_1 - \sigma v_2 - \mathrm{i}\delta_1 v_1(u_3+\sigma-r) - \mathrm{i}\delta_1 u_1 v_3)\right\} \tag{5.11}$$

and

$$\mathrm{Re}\,(w_1(u_2^* - \mathrm{i}\delta_1u_1^*))$$
$$= \mathrm{Re}\left\{\frac{1}{\mathrm{i}a}(v_{1,t} + \sigma v_1 - \sigma v_2 - \mathrm{i}\delta_1 v_1(u_3+\sigma-r) - \mathrm{i}\delta_1 u_1 v_3)(u_2^* - \mathrm{i}\delta_1 u_1^*)\right\}$$
$$= \mathrm{Re}\left\{\frac{1}{\mathrm{i}a}(v_1(u_2^* - \mathrm{i}\delta_1 u_1^*))_t - \frac{v_1}{\mathrm{i}a}(u_2^* - \mathrm{i}\delta_1 u_1^*)_t\right\}$$
$$+\mathrm{Re}\left\{\frac{1}{\mathrm{i}a}(\sigma v_1 - \sigma v_2 - \mathrm{i}\delta_1 v_1(u_3+\sigma-r) - \mathrm{i}\delta_1 u_1 v_3)(u_2^* - \mathrm{i}\delta_1 u_1^*)\right\}. \tag{5.12}$$

Using (5.10)–(5.12), we recast the W system as:

$$w_{1,t} = -\sigma w_1 + (\mathrm{i}a\Delta w_1 + \mathrm{i}\delta_1(u_3+\sigma-r)w_1 + \sigma w_2 + \mathrm{i}\delta_1 u_1 w_3) + I, \tag{5.13}$$

where

$$I = I(V, \nabla V) = 2\mathrm{i}\delta_1 \nabla u_3 \cdot \nabla v_1 + 2\mathrm{i}\delta_1 \nabla u_1 \cdot \nabla v_3 + \mathrm{i}\delta_1 v_1 \Delta u_3 + \mathrm{i}\delta_1 v_3 \Delta u_1; \tag{5.14}$$

$$\left(w_2 - \frac{\mathrm{i}}{a} u_3 v_1 - \frac{2\delta_2}{a} u_2 u_1^* v_1 + \frac{2\delta_2}{a} u_1 u_2 v_1^*\right)_t = -(1 + \mathrm{i}\Omega) w_2 + (-\sigma w_1 - w_3 u_1 + \mathrm{i}\delta_2 |u_1|^2 w_2) + II, \tag{5.15}$$

where

$$\begin{aligned} II = II(V, \nabla V) = {} & -2\nabla v_3 \cdot \nabla u_1 - 2\nabla v_1 \cdot \nabla u_3 + 2\mathrm{i}\delta_2 \nabla |u_1|^2 \cdot \nabla v_2 \\ & + 2\mathrm{i}\delta_2 \nabla u_2 \cdot \mathrm{Re}\,(u_1^* \nabla v_1) + 2\mathrm{i}\delta_2 \nabla u_2 \cdot \mathrm{Re}\,(\nabla u_1^* \cdot v_1) + 4\mathrm{i}\delta_2 u_2 \mathrm{Re}\,\{\nabla u_1^* \cdot \nabla v_1\} \\ & - v_3 \Delta u_1 - v_1 \Delta u_3 + \mathrm{i}\delta_2 v_2 \Delta |u_1|^2 + 2\mathrm{i}\delta_2 \mathrm{Re}\,(u_1^* v_1) + 2\mathrm{i}\delta_2 u_2 \mathrm{Re}\,(v_1 \Delta u_1^*) \\ & + 2\mathrm{i}\delta_2 \Delta u_2 \mathrm{Re}\,(u_1^* v_1) + \frac{1}{a}[-\mathrm{i}u_{3,t} v_1 + \mathrm{i}\sigma u_3 v_1 - \mathrm{i}\sigma v_2 u_3 + \delta_1 u_3 (u_3 + \sigma - r) v_1 \\ & + \delta_1 u_1 u_3 v_3] - \frac{2\delta_2}{a}[(u_2 u_1^*)_t v_1 - (u_1 u_2)_t v_1^*] \\ & + 2\mathrm{i}\delta_2 u_2 \mathrm{Re}\left\{\frac{u_1^*}{\mathrm{i}a}(\sigma v_1 - \sigma v_2 - \mathrm{i}\delta_1 v_1 (u_3 + \sigma - r) - \mathrm{i}\delta_1 u_1 v_3)\right\}; \end{aligned} \tag{5.16}$$

and

$$\left(w_3 + \frac{\mathrm{i}}{a} v_1 (u_1^* - \mathrm{i}\delta_1 u_1^*)\right)_t = -b w_3 + \mathrm{Re}\,(u_1 w_2^* + \mathrm{i}\delta_1 w_1 u_1^*) + III, \tag{5.17}$$

where

$$\begin{aligned} III = {} & III(V, \nabla V) \\ = {} & \mathrm{Re}\,\{v_2^* \Delta u_1 + 2\nabla u_1 \cdot \nabla v_2^* + 2\nabla v_1 \cdot \nabla u_2^* + v_1 \Delta u_2^* + \frac{\mathrm{i}}{a} v_1 (u_2^* - \mathrm{i}\delta_1 u_1^*)_t \\ & - \frac{\mathrm{i}}{a}(\sigma v_1 - \sigma v_2 - \mathrm{i}\delta_1 v_1 (u_3 + \sigma - r) - \mathrm{i}\delta_1 u_1 v_3)(u_2^* - \mathrm{i}\delta_1 u_1^*)\}. \end{aligned} \tag{5.18}$$

Now let us introduce the change of variables:

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = D(U)V \equiv \begin{pmatrix} \Delta v_1 - \gamma_0 v_1 \\ \Delta v_2 - \gamma_0 v_2 - \frac{\mathrm{i}}{a} u_3 v_1 - \frac{\delta_2}{a} u_2 u_1^* v_1 + \frac{\delta_2}{a} u_1 u_2 v_1^* \\ \Delta v_3 - \gamma_0 v_3 + \frac{\mathrm{i}}{a} v_1 (u_2^* - \mathrm{i}\delta_1 u_1^*) \end{pmatrix}, \tag{5.19}$$

where γ_0 is a large positive constant to be chosen so that $D^{-1}(u)$ exists. Multiplying (5.4)–(5.6) by $-\gamma_0$, adding the resulting system to (5.16)–(5.18), and replacing $(V, \nabla V)$ by $(D^{-1}(U)\boldsymbol{\eta}, \nabla(D^{-1}(U)\boldsymbol{\eta}))$, we derive the system for $\boldsymbol{\eta}$:

$$\eta_{1,t} = -\sigma \eta_1 + (\mathrm{i}\delta_1 (u_3 + \sigma - r)\eta_1 + \mathrm{i}a \Delta \eta_1 + \sigma \eta_2 + \mathrm{i}\delta_1 u_1 \eta_3) + IV, \tag{5.20}$$

$$\eta_{2,t} = -(1 + \mathrm{i}\Omega)\eta_2 + (-\sigma \eta_1 - \eta_3 u_1 + \mathrm{i}\delta_2 |u_1|^2 \eta_2) + V, \tag{5.21}$$

$$\eta_{3,t} = -b\eta_3 + \mathrm{Re}\,(u_1 \eta_2^* + \mathrm{i}\delta_1 \eta_1 u_1^*) + VI, \tag{5.22}$$

where IV, V, and VI are all differentiable functions of $(\boldsymbol{\eta}, \nabla\boldsymbol{\eta})$ with coefficients being U and its derivatives of the order no more than two. We can further write (5.20)–(5.22) into the following compact form:

$$\begin{aligned} \boldsymbol{\eta}_t = {} & A\boldsymbol{\eta} + J(u)\boldsymbol{\eta} + C_1(\Delta U, \nabla U, U) D^{-1}(U)\boldsymbol{\eta} + C_2(\Delta U, \nabla U, U)[D^{-1}(U)\boldsymbol{\eta}]^* \\ & + \sum_{i=1}^{2} E_i(\nabla U, U)(D^{-1}(U)\boldsymbol{\eta})_{x_i} + \sum_{i=1}^{2} F_i(\nabla U, U)[D^{-1}(U)\boldsymbol{\eta}]^*_{x_i}, \end{aligned} \tag{5.23}$$

where $C_i(\Delta U, \nabla U, U)$, $i = 1, 2$, are 3×3 complex matrices, linear in ΔU, smooth in ∇U, and U; $E_i(\nabla U, U)$, $F_i(\nabla U, U)$, $i = 1, 2$, are 3×3 complex matrices, linear in ∇U, smooth in U;

$$A\boldsymbol{\eta} = \mathrm{diag}(-\sigma, -1, -b)\boldsymbol{\eta},$$

is the damping part, and

$$J(U)\boldsymbol{\eta} = \begin{pmatrix} \mathrm{i}a\Delta\eta_1 + \mathrm{i}\delta_1(u_3 + \sigma - r)\eta_1 + \sigma\eta_2 + \mathrm{i}\delta_1 u_1\eta_3 \\ -\sigma\eta_1 - \eta_3 u_1 + \mathrm{i}\delta_2|u_1|^2\eta_2 \\ \mathrm{Re}\,(u_1\eta_2^* + \mathrm{i}\delta_1\eta_1 u_1^*) \end{pmatrix}. \tag{5.24}$$

It is easy to check that $J(U)$ is the 5×5 skew symmetric matrix:

$$\begin{pmatrix} 0 & -a\Delta - \delta_1(u_3 + \sigma - r) & \sigma & 0 & -\delta_1\mathrm{Im}\,\{u_1\} \\ a\Delta + \delta_1(u_3 + \sigma - r) & 0 & 0 & \sigma & \delta_1\mathrm{Re}\,\{u_1\} \\ -\sigma & 0 & 0 & -\delta_2|u_1|^2 & -\mathrm{Re}\,\{u_1\} \\ 0 & -\sigma & \delta_2|u_1|^2 & 0 & -\mathrm{Im}\,\{u_1\} \\ \delta_1\mathrm{Im}\,\{u_1\} & -\delta_1\mathrm{Re}\,\{u_1\} & \mathrm{Re}\,\{u_1\} & \mathrm{Im}\,\{u_1\} & 0 \end{pmatrix},$$

when acting on the real vector

$$(\mathrm{Re}\,(\eta_1), \mathrm{Im}\,(\eta_1), \mathrm{Re}\,(\eta_2), \mathrm{Im}\,(\eta_2), \eta_3),$$

which we still denote by $\boldsymbol{\eta}$. We first show that:

Proposition 5.1. *The operators $C_1 D^{-1}(U)\cdot$, and $C_2 D^{-1}(U)\cdot$ are Hilbert–Schmidt when acting on the real $\boldsymbol{\eta} \in (L^2)^5$; while the operators $E_i(D^{-1}(U)\cdot)_{x_i}$, and $F_i(D^{-1}(U)\cdot)^*_{x_i}$, are in the trace ideals $g_p, \forall p > 2$.*

Proof. For γ_0 large enough depending only on $\|U\|_{H^2}$, it is not hard to show via Lax–Milgram theorem that $D^{-1}(U)$ exists and is a bounded map from $(L^2)^5$ into $(H^2)^5$. We can write:

$$C_1 D^{-1}(U) = C_1(\Delta - 1)^{-1}(\Delta - 1)D^{-1}(U).$$

Since $(\Delta - 1)^{-1}$ is a Hilbert–Schmidt operator on $L^2(T^2)$, C_1 and $(\Delta - 1)D^{-1}(U)$ are bounded, we infer that $C_1 D^{-1}(U)$ is Hilbert–Schmidt. Similarly, $C_2(D^{-1}(U))^*$ is also Hilbert–Schmidt. Now we write:

$$E_i\partial_{x_i}(D^{-1}(U)\cdot) = E_i\partial_{x_i}(\Delta - 1)^{-1}(\Delta - 1)D^{-1}(U).$$

The singular values of $\partial_{x_i}(\Delta - 1)^{-1}$ are $\frac{|k_i|}{(1+|k|^2)}$, $k = (k_1, k_2) \in Z^2$, which form a convergent sequence in l^p, for any $p > 2$. Hence $\partial_{x_i}(\Delta - 1)^{-1} \in g_p$, $p > 2$, and so $E_i\partial_{x_i}(D^{-1}(U)\cdot) \in g_p$, $p > 2$, by theorem 1.6 of Simon [24]. Similarly, $F_i\partial_{x_i}(D^{-1}(U))^* \in g_p$. The proof is complete. □

Now let us put the $\boldsymbol{\eta}$ system into the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta} = A\boldsymbol{\eta} + J(U)\boldsymbol{\eta} + K_1(U)\boldsymbol{\eta} + K_2(U)\boldsymbol{\eta}, \tag{5.25}$$

where K_1 is Hilbert–Schmidt, K_2 is in trace class g_p, $p > 2$. Let $\Gamma_1 = \|K_1\|_{HS}$, $\Gamma_p = \|K_2\|_{g_p}$, $p > 2$. There is a constant $\Gamma_3 > 0$ depending on γ_0 and $\mathcal{A}$, such that:

$$\Gamma_3^{-1}\|v\|_{H^2} \leqslant \|D^{-1}(U)v\|_2 \leqslant \Gamma_3\|v\|_{H^2}. \tag{5.26}$$

For the upper bound of the Hausdorff and fractal dimensions of $\mathcal{A}$ (still denoting the shifted attractor), we study the volume elements

$$\|V_1(t) \wedge \cdots \wedge V_n(t)\|_{\Lambda^n H^2}$$

formed by n linearly independent solutions of the linearized problem. By (5.26), there exists an n dependent constant a_n (for all $U \in \mathcal{A}$) such that:

$$a_n^{-1}\|V_1(t)\wedge\cdots\wedge V_n(t)\|_{\Lambda^n H^2} \leqslant \|\boldsymbol{\eta}_1(t)\wedge\cdots\wedge\boldsymbol{\eta}_n(t)\|_{\Lambda^n L^2} \leqslant a_n\|V_1(t)\wedge\cdots\wedge V_n(t)\|_{\Lambda^n H^2}. \tag{5.27}$$

Thus, it is enough to show the decay of $\|\boldsymbol{\eta}_1(t)\wedge\cdots\wedge\boldsymbol{\eta}_n(t)\|_{\Lambda^n L^2}$. We have the well known identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|\boldsymbol{\eta}_1(t)\wedge\cdots\wedge\boldsymbol{\eta}_n(t)\|_{\Lambda^n L^2} = \mathrm{Tr}\,((A+J+K_1+K_2)Q_n)\,, \tag{5.28}$$

where Q_n is the orthogonal projector from $(L^2)^5$ onto the span of $\boldsymbol{\eta}_1(t),\ldots,\boldsymbol{\eta}_n(t)$. We have by skew symmetry:

$$\mathrm{Tr}(AQ_n) = -(2\sigma+2+b)n, \qquad \mathrm{Tr}(JQ_n) = 0, \tag{5.29}$$

and by Cauchy–Schwartz inequality:

$$|\,\mathrm{Tr}(K_1Q_n)| \leqslant n^{\frac{1}{2}}\|K_1\|_{HS} \leqslant \Gamma_1 n^{\frac{1}{2}}. \tag{5.30}$$

Let $\{\tilde{\boldsymbol{\eta}}_j\}$, $j=1,\ldots,n$, be an orthogonal basis of the linear span of $\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_n$. Then:

$$\begin{aligned}|\,\mathrm{Tr}(K_2Q_n)| &\leqslant \sum_{j=1}^{n}|(K_2\tilde{\boldsymbol{\eta}}_j(t),\tilde{\boldsymbol{\eta}}_j(t))| \leqslant \sum_{j=1}^{n}\|K_2\tilde{\boldsymbol{\eta}}_j\|_{L^2} \leqslant n^{\frac{1}{q}}\left(\sum_{j=1}^{n}\|K_2\tilde{\boldsymbol{\eta}}_j\|^p\right)^{\frac{1}{p}} \\ &\quad \text{Theorem 1.18 of [24]} \\ &\leqslant n^{\frac{1}{q}}\left(\sum_{j=1}^{\infty}\mu_j(K_2)^p\right)^{\frac{1}{p}} = n^{\frac{1}{q}}\|K_2\|_{g_p} \leqslant n^{\frac{1}{q}}\Gamma_p,\end{aligned} \tag{5.31}$$

where $q\in(1,2)$, and μ_j denotes the jth singular value. Combining (5.28)–(5.31), we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|\boldsymbol{\eta}_1(t)\wedge\cdots\wedge\boldsymbol{\eta}_n(t)\|_{\Lambda^n L^2} \leqslant n^{\frac{1}{2}}[-(2\sigma+2+b)n^{\frac{1}{2}}+\Gamma_1+n^{\frac{1}{q}-\frac{1}{2}}\Gamma_p]. \tag{5.32}$$

It follows that

$$\begin{aligned}\|V_1(t)\wedge\cdots\wedge V_n(t)\|_{\Lambda^n H^2} &\leqslant a_n\Gamma_3^n\|V_1(0)\wedge\cdots\wedge V_n(0)\|_{\Lambda^n H^2} \\ &\quad\times\exp(-tn^{\frac{1}{2}}[(2\sigma+2+b)n^{\frac{1}{2}}-\Gamma_1-n^{\frac{1}{2}-\frac{1}{p}}\Gamma_p]),\end{aligned} \tag{5.33}$$

for any finite $p>2$. We deduce that all volume elements will contract exponentially if:

$$n > \max\left(\frac{\Gamma_1^2}{(2\sigma+2+b)^2},\frac{\Gamma_p^p}{(2\sigma+2+b)^p}\right). \tag{5.34}$$

By standard results, we have the upper bounds on Hausdorff and fractal dimensions of $\mathcal{A}$:

$$\dim_H(\mathcal{A}) \leqslant \max\left(\frac{\Gamma_1^2}{(2\sigma+2+b)^2},\frac{\Gamma_p^p}{(2\sigma+2+b)^p}\right), \tag{5.35}$$

$$\dim_F(\mathcal{A}) \leqslant c_0\dim_H(\mathcal{A}), \tag{5.36}$$

for any $p\in(2,\infty)$, where c_0 is an absolute constant.

Proof of theorem 1.1, (3) and theorem 1.2, (3). Summarizing the results of sections 4 and 5, we complete the proofs of theorems 1.1 and 1.2. For classical solutions on the whole plane, theorem 1.1 also holds, but part (3) should be rephrased as part (3) of theorem 1.2. Theorem 1.2, holds for weak solutions in the periodic domain under the conjectural Strichartz inequality for two dimensions.

Acknowledgments

The work of JX was partially supported by NSF grants DMS-9302830 and DMS-9625680. Both authors wish to thank the Arizona Center for Mathematical Sciences (ACMS) for its stimulating environment and support. ACMS is sponsored by AFOSR under contract F49620-94-1-0144DEF.

References

[1] Arechi F T, Boccaletti S and Ramazza P L 1993 *Phys. Rev. Lett.* **70** 2277
[2] Birnir B and Xin J 1996 The global attractor of the transverse Maxwell–Bloch equations *Preprint*
[3] Bourgain J 1993 Fourier transform restriction phenomenon for certain lattice subsets and applications to nonlinear evolution equations. Part I, Schrödinger equations *J. Geom. Funct. Anal.* **3** 107–56
[4] Caffarelli L, Kohn R and Nirenberg L 1982 Partial regularity of suitable weak solutions of the Navier–Stokes equations *Commun. Pure Appl. Math* **35** 771–831
[5] Collet P 1995 Nonlinear dynamics of extended systems *Newton Inst. Lect. Notes*
[6] Constantin P and Foias C 1988 *Navier–Stokes Equations* (Chicago, IL: University of Chicago Press)
[7] Constantin P, Foias C and Gibbon J D 1989 Finite dimensional attractor for the laser equations *Nonlinearity* **2** 241–69
[8] Geddes J B *et al* 1994 Pattern selection in passive and active nonlinear optical systems *Chaos, Solitons and Fractals* vol 4 pp 1261–74
[9] Ginibre J and Velo G 1984 On the global Cauchy problem for some nonlinear Schrödinger equations *Ann Inst. H Poincare* **1** 309–23
[10] Hale J 1980 *Ordinary Differential Equations* 2nd edn (Huntington, NY: Kreiger)
[11] Hale J 1988 Asymptotic behavior and dynamics in infinite dimensions *Math. Surveys and Monographs* vol 25 (Providence, RI: American Mathematical Society)
[12] Harrison R G, Lu W, Gupta P K 1989 Origin of periodic, chaotic, and bistable emission of Raman lasers *Phys. Rev. Lett.* **63** 1372
[13] Lu W and Harrison R G 1991 Nonlinear dynamics of Raman lasers *Phys. Rev.* A **43** 6358
[14] Jakobson P K *et al* 1994 Nonlinear transverse modes of large aspect ratio homogeneously broadened lasers I: analysis and numerics *Phys. Rev.* A **49** 4189–200
[15] Jakobsen P K, Moloney J V, Newell A C and Indik R 1992 Space-time dynamics of wide-gain-section lasers *Phys. Rev.* A **45** 8129
[16] Kato T 1987 On the nonlinear Schrödinger equations *Ann. Inst. H. Poincare* **46** 123–9
[17] Ladyzhenskaya O A 1969 *The Mathematical Theory of Viscous Incompressible Flows* 2nd edn (New York: Gordon and Breach)
[18] Lega J, Jakobsen P K, Moloney J V and Newell A C 1994 Nonlinear transverse modes of large aspect ratio homogeneously broadened lasers II. Pattern analysis near and beyond threshold *Phys. Rev.* A **49** 4201–12
[19] Mielke A and Schneider G 1995 Attractors for modulation equations on unbounded domains, existence and comparison *Preprint*
[20] Mielke A and Schneider G 1996 Derivation and justification of the complex Ginzburg–Landau equation as a modulation equation *Dynamical Systems and Probabilistic Methods in PDEs (AMS Lectures in Applied Math 31)* ed P Deift *et al* pp 191–216
[21] Newell A C and Moloney J M 1992 *Nonlinear Opt.* (Reading, MA: Addison-Wesley)
[22] Ogawa T and Ozawa T 1991 Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problems *J. Math. Anal. Appl.* **155** 531–40
[23] Sell G 1996 *Private communication*
[24] Simon B 1979 Trace ideals and their applications *London Math. Soc. Lect. Note Series* vol 35 (Cambridge: Cambridge University Press)
[25] Temam R 1988 Infinite dimensional dynamical systems in mechanics and physics *Appl. Math. Sci.* **68**
[26] Tsutsumi Y 1987 L^2 solutions for nonlinear Schrödinger equations and nonlinear groups *Funkcialaj Ekvacioj* **30** 115–25
[27] Whitham G B 1979 *Linear and Nonlinear Waves* (New York: Wiley)
[28] Xin J and Moloney J 1996 Global weak solutions and attractors of the three dimensional Maxwell–Bloch two level laser systems *Commun. Math. Phys.* **179** 511–28