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Physica D 188 (2004) 178–192
PHYSICA D
www.elsevier.com/locate/physd

Energy method for Boltzmann equation

Tai-Ping Liu [a,b,*], Tong Yang [c], Shih-Hsien Yu [c]

[a] *Institute of Mathematics, Academia Sinica, Taipei, Taiwan, ROC*
[b] *Department of Mathematics, Stanford University, Stanford, USA*
[c] *Department of Mathematics, City University of Hong Kong, Hong Kong, China*

Received 3 February 2003; received in revised form 30 June 2003; accepted 15 July 2003
Communicated by Y. Nishiura

Abstract

A basic, simple energy method for the Boltzmann equation is presented here. It is based on a new macro–micro decomposition of the Boltzmann equation as well as the H-theorem. This allows us to make use of the ideas from hyperbolic conservation laws and viscous conservation laws to yield the direct energy method. As an illustration, we apply the method for the study of the time-asymptotic, nonlinear stability of the global Maxwellian states. Previous energy method, starting with Grad and finishing with Ukai, involves the spectral analysis and regularity of collision operator through sophisticated weighted norms.

Keywords: Boltzmann equation; H-theorem; Maxwellian states; Macro–micro decomposition

1. Introduction

Consider the Boltzmann equation:

$$f_t + \xi \cdot \nabla_x f = \frac{Q(f, f)}{\kappa}, \quad (f, x, t, \xi) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3, \tag{1.1}$$

where the positive constant κ is the Knudsen number [1]. For simplicity, we consider the hard sphere model, for which the bilinear collision operator $Q(f, g)$ is of the following form:

$$Q(f, g)(\xi) \equiv \frac{1}{2} \int_{\substack{\mathbb{R}^3 \times S^2 \\ (\xi-\xi_*)\cdot\Omega \geq 0}} (-f(\xi)g(\xi_*) - g(\xi)f(\xi_*) + f(\xi')g(\xi'_*) + g(\xi')f(\xi'_*))|(\xi - \xi_*) \cdot \Omega| \mathrm{d}\xi_* \, \mathrm{d}\Omega,$$

where

$$\xi' = \xi - [(\xi - \xi_*) \cdot \Omega]\Omega, \qquad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega]\Omega.$$

* Corresponding author. Tel.: +1-650-723-2965; fax: +1-650-725-4066.
E-mail addresses: tpliu@math.sinica.edu.tw, liu@math.stanford.edu (T.-P. Liu), matyang@cityu.edu.hk (T. Yang), mashyu@cityu.edu.hk (S.-H. Yu).

0167-2789/$ – see front matter
doi:10.1016/j.physd.2003.07.011

The main purpose of the present paper is to introduce a macro–micro decomposition of the equation. The decomposition is based on the decomposition of the solution into the macroscopic, fluid part, the local Maxwellian $\boldsymbol{M} = \boldsymbol{M}(x, t, \xi) = \boldsymbol{M}_{[\rho,u,\theta]}(\xi)$, and the microscopic, non-fluid part $\boldsymbol{G} = \boldsymbol{G}(x, t, \xi)$ of the solution:

$$f = \boldsymbol{M} + \boldsymbol{G}.$$

The local Maxwellian is constructed from the fluid variables, the five conserved quantities, the mass density $\rho(x, t)$, momentum $m(x, t) = \rho u(x, t)$ and energy $E + |u|^2/2$ of the Boltzmann equation [9]:

$$\rho(x, t) \equiv \int_{\mathbb{R}^3} f(x, t, \xi)\,\mathrm{d}\xi, \qquad m^i(x, t) \equiv \int_{\mathbb{R}^3} \psi_i f(x, t, \xi)\,\mathrm{d}\xi \quad \text{for } i = 1, 2, 3,$$
$$\rho\left(E + \frac{1}{2}|u|^2\right)(x, t) \equiv \int_{\mathbb{R}^3} \psi_4 f(x, t, \xi)\,\mathrm{d}\xi, \tag{1.2}$$

$$\boldsymbol{M} \equiv \boldsymbol{M}_{[\rho,u,\theta]}(\xi) \equiv \frac{\rho}{\sqrt{(2\pi R\theta)^3}} \exp\left(-\frac{|\xi - u|^2}{2R\theta}\right). \tag{1.3}$$

Here $\theta(x, t)$ is the temperature and is related to the internal energy E through the gas constant R, $E = (3/2)R\theta$, and $u(x, t)$ is the fluid velocity. The five fluid variables are conserved quantities because of the following property of the collision invariants ψ_α [1]:

$$\int_{\mathbb{R}^3} \psi_\alpha Q(h, g)\,\mathrm{d}\xi = 0 \quad \text{for any } \alpha = 0, 1, 2, 3, 4$$

and for any functions h, g:

$$\psi_0 \equiv 1, \qquad \psi_i \equiv \xi^i \quad \text{for } i = 1, 2, 3, \qquad \psi_4 \equiv \tfrac{1}{2}|\xi|^2. \tag{1.4}$$

With respect to the local Maxwellian, we define an inner product in $\xi \in \mathbb{R}^3$ as

$$\langle h, g\rangle \equiv \int_{\mathbb{R}^3} \frac{1}{\boldsymbol{M}} h(\xi) g(\xi)\,\mathrm{d}\xi$$

for functions h, g of ξ. The following functions are orthogonal with respect to this inner product:

$$\chi_0(\xi; \rho, u, \theta) \equiv \frac{1}{\sqrt{\rho}}\boldsymbol{M}, \qquad \chi_i(\xi; \rho, u, \theta) \equiv \frac{\xi^i - u^i}{\sqrt{R\theta\rho}}\boldsymbol{M} \quad \text{for } i = 1, 2, 3,$$
$$\chi_4(\xi; \rho, u, \theta) \equiv \frac{1}{\sqrt{6}\rho}\left(\frac{|\xi - u|^2}{R\theta} - 3\right)\boldsymbol{M}, \qquad \langle\chi_\alpha, \chi_\beta\rangle = \delta_{\alpha\beta} \quad \text{for } \alpha, \beta = 0, 1, 2, 3, 4. \tag{1.5}$$

We define the macroscopic projection $\boldsymbol{P}_0$ and microscopic projection $\boldsymbol{P}_1$ as follows:

$$\boldsymbol{P}_0 h \equiv \sum_{\alpha=0}^{4} \langle h, \chi_\alpha\rangle \chi_\alpha, \qquad \boldsymbol{P}_1 h \equiv h - \boldsymbol{P}_0 h. \tag{1.6}$$

We view the above decomposition of Boltzmann equation as the linearization around the local Maxwellian states so that the linear collision operator $L_{[\rho,u,\theta]}$ is

$$L = L_{[\rho,u,\theta]} g \equiv Q(\boldsymbol{M}_{[\rho,u,\theta]} + g, \boldsymbol{M}_{[\rho,u,\theta]} + g) - Q(g, g). \tag{1.7}$$

The operator $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ are projections, that is

$$\boldsymbol{P}_0\boldsymbol{P}_0 = \boldsymbol{P}_0, \qquad \boldsymbol{P}_1\boldsymbol{P}_1 = \boldsymbol{P}_1.$$

A function $h(\xi)$ is called non-fluid if it gives raise to zero conserved quantities, that is

$$\int_{\mathbb{R}^3} h(\xi)\psi_\alpha \,\mathrm{d}\xi = 0 \quad \text{for } \alpha = 0, 1, 2, 3, 4. \tag{1.8}$$

Note that functions in the range of the microscopic projection $\boldsymbol{P}_1$ are non-fluid. It is clear that for the solution $f(x, t, \xi)$ of the Boltzmann equation:

$$\boldsymbol{P}_0 f = \boldsymbol{M}, \qquad \boldsymbol{P}_1 f = \boldsymbol{G}.$$

From the decomposition of the solution $f = \boldsymbol{M} + \boldsymbol{G}$, the Boltzmann equation becomes

$$(\boldsymbol{M} + \boldsymbol{G})_t + \xi \cdot \nabla_x(\boldsymbol{M} + \boldsymbol{G}) = \frac{1}{\kappa}(2Q(\boldsymbol{G}, \boldsymbol{M}) + Q(\boldsymbol{G}, \boldsymbol{G})). \tag{1.9}$$

We now decompose the Boltzmann equation. The conservation laws are obtained, as usual, by integrating with respect to ξ of the Boltzmann equation times the collision invariants $\psi_\alpha(\xi)$:

$$\rho_t + \operatorname{div} m = 0, \qquad m_t^i + \left(\sum_{j=1}^{3} u^j m^i\right)_{x^j} + p_{x^i} + \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_x \boldsymbol{G})\,\mathrm{d}\xi = 0 \quad \text{for } i = 1, 2, 3,$$
$$\left[\rho\left(\frac{|u|^2}{2} + E\right)\right]_t + \sum_{j=1}^{3}\left[u^j\left[\rho\left(\frac{|u|^2}{2} + E\right) + p\right]\right]_{x^j} + \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_x \boldsymbol{G})\,\mathrm{d}\xi = 0. \tag{1.10}$$

Here p is the pressure for the monatomic gases:

$$p = \tfrac{2}{3}\rho E.$$

The microscopic equation is obtained by applying the microscopic projection $\boldsymbol{P}_1$ to the Boltzmann equation (1.9). Since the projections are based on local Maxwellian, the projections and partial differentiations in (x, t) may not commute. Nevertheless, we note that $\boldsymbol{M}_t$, as a function of ξ, is in the space spanned by χ_α, $\alpha = 1, 2, 3, 4, 5$. Thus $\boldsymbol{P}_0\boldsymbol{M}_t = \boldsymbol{M}_t$. We note that $\boldsymbol{P}_0 h = 0$ if

$$\int_{\mathbb{R}^3} h\psi_\alpha \,\mathrm{d}\xi = 0.$$

Thus the projection of collision terms under $\boldsymbol{P}_0$ is zero. We also have

$$\int_{\mathbb{R}^3} G_t\psi_\alpha \,\mathrm{d}\xi = \partial_t \int_{\mathbb{R}^3} G\psi_\alpha \,\mathrm{d}\xi = 0.$$

Thus we have $\boldsymbol{P}_0 G_t = 0$, and so $\boldsymbol{P}_1(\boldsymbol{M}_t + \boldsymbol{G}_t) = \boldsymbol{G}_t$. With these, the microscopic equation is

$$\boldsymbol{G}_t + \boldsymbol{P}_1(\xi \cdot \nabla_x \boldsymbol{G} + \xi \cdot \nabla_x \boldsymbol{M}) = \frac{1}{\kappa}L\boldsymbol{G} + \frac{1}{\kappa}Q(\boldsymbol{G}, \boldsymbol{G}). \tag{1.11}$$

This decomposition improvises that of [8], where the linearization is about the global Maxwellian. The advantage of the present one is that the nonlinear term $Q(\boldsymbol{G}, \boldsymbol{G})$ in (1.11) depends only on the microscopic part $\boldsymbol{G}$. This is convenient for the energy method.

From (1.11) we have

$$\boldsymbol{G} = \kappa L^{-1}(\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}) + L^{-1}(\kappa(\partial_t \boldsymbol{G} + \boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{G}) - Q(\boldsymbol{G}, \boldsymbol{G})) \tag{1.12}$$

and substitute this into (1.10) to result in

$$\rho_t + \operatorname{div} m = 0, \qquad m^i_t + \left(\sum_{j=1}^{3} u^j m^i\right)_{x^j} + p_{x^i} + \kappa \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_x L^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{M})\, \mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_x L^{-1})(\kappa[\boldsymbol{G}_t + \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{G}] - Q(\boldsymbol{G}, \boldsymbol{G}))\, \mathrm{d}\xi = 0 \quad \text{for } i = 1, 2, 3,$$
$$\left[\rho\left(\frac{|u|^2}{2} + E\right)\right]_t + \sum_{j=1}^{3}\left[u^j\left[\rho\left(\frac{|u|^2}{2} + E\right) + p\right]\right]_{x^j} + \kappa \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_x L^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{M})\, \mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_x L^{-1})(\kappa[\boldsymbol{G}_t + \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{G}] - Q(\boldsymbol{G}, \boldsymbol{G}))\, \mathrm{d}\xi = 0. \tag{1.13}$$

The fluid equations, the Euler and Navier–Stokes equations, are in fact part of the above equations. For instance, when the Knudsen number κ and the microscopic part $\boldsymbol{G}$ are set zero, the system (1.13) becomes the Euler equations as in the Hilbert expansion. When only the microscopic part $\boldsymbol{G}$ is set to be zero in (1.13), it becomes the Navier–Stokes equations as in the Chapman–Enskog expansion. These fluid equations as derived through the Hilbert and Chapman–Enskog expansions are approximations to the Boltzmann equation [3]. Here we derive it as part of the full Boltzmann equation. Nevertheless, our approach is consistent in spirit with these expansions in that the higher order terms beyond first order in the expansions must satisfy a solvability condition, which means that these terms are microscopic.

In the above system, the terms:

$$-\kappa \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_x L^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{M})\, \mathrm{d}\xi$$
$$= -\kappa \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_x L^{-1}_{[\rho,u,\theta]} \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{M}_{[\rho,u,\theta]})\, \mathrm{d}\xi$$
$$= -\kappa \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_x L^{-1}_{[1,u,\theta]} \boldsymbol{P}_1 \xi \cdot \nabla_x M_{[1,u,\theta]})\, \mathrm{d}\xi, \quad i = 1, 2, 3,$$
$$-\kappa \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_x L^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{M})\, \mathrm{d}\xi = -\kappa \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_x L^{-1}_{[\rho,u,\theta]} \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{M}_{[\rho,u,\theta]})\, \mathrm{d}\xi$$
$$= -\kappa \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_x L^{-1}_{[1,u,\theta]} \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{M}_{[1,u,\theta]})\, \mathrm{d}\xi$$

are the viscosity and heat conductivity terms for the Navier–Stokes equations; and they are independent of the density gradient $\nabla_x \rho$.

The Boltzmann equation as decomposed in (1.10) and (1.13) consists of the fluid equations plus the microscopic part. This allows for the use of the ideas from hyperbolic and viscous conservation laws for the energy method. For the conservation laws, there is the basic concept of entropy. For this, we discuss in Section 2 the derivation of the macroscopic entropy based on the H-theorem for the Boltzmann equation. In Section 3 we carry out the energy method for the nonlinear stability of global Maxwellian states. The energy method here is elementary and generalizes that in [8]. For other energy methods making use of the spectral properties of the linearized operator, see [7,10,11].

2. H-theorem

The H-theorem of the Boltzmann equation is based on the observation that

$$\int_{\mathbb{R}^3} Q(f,f)\log f\,\mathrm{d}\xi \le 0$$

and that the equality holds only when the solution is a Maxwellian, $f = \boldsymbol{M}$. The H-theorem is obtained by multiplying the Boltzmann equation by $\log f$ and integrating with respect to ξ:

$$\int_{\mathbb{R}^3} f\log f\,\mathrm{d}\xi + \nabla\cdot\int_{\mathbb{R}^3}\xi f\log f\,\mathrm{d}\xi = \kappa\int_{\mathbb{R}^3} Q(f,f)\log f\,\mathrm{d}\xi \le 0. \tag{2.1}$$

There are two ways to view this. The first is to ignore the transport term and study the linearized collision operator. The linearized collision operator L of (1.7) is symmetric:

$$\langle h, Lg\rangle = \langle Lh, g\rangle.$$

The null space of L contains the macroscopic variables:

$$L\chi_\alpha = 0 \quad \text{for } \alpha = 0,\ldots,4.$$

Hilbert [6] shows that the linearized operator has the form:

$$(Lh)(\xi) = -\nu(\xi)h(\xi) + \boldsymbol{M}^{1/2}(\xi)K(h\boldsymbol{M}^{-1/2})(\xi) \tag{2.2}$$

for a symmetric compact L^2-operator K. Notice that the linearized operator L around $M_{[\rho,u,\theta]}$ and the linearized operator L_1 around $M_{[1,u,\theta]}$ have the following relation:

$$L = \rho L_1.$$

The multiplicative operator satisfies, for the hard sphere model:

$$0 < \lim_{|\xi|\to\infty}\frac{\nu(\xi)}{|\xi|} < \infty.$$

With this, it was proved by Carleman [2] for hard sphere model, and by Grad [5] for cutoff hard potentials, that the linearized operator is negative definite on the space of non-fluid distributions. That is, there exists $\sigma > 0$ such that for any function satisfying (1.8):

$$\langle h, Lh\rangle \le -\sigma\langle \nu h, h\rangle. \tag{2.3}$$

In particular, this implies that the non-fluid part $\boldsymbol{G}$ of the Boltzmann solution:

$$\langle \boldsymbol{G}, L\boldsymbol{G}\rangle \le -\sigma\langle \nu\boldsymbol{G}, \boldsymbol{G}\rangle.$$

When the solution is space homogeneous and close to a Maxwellian, it follows from (2.3) that it will converge to the equilibrium exponentially in time. In other words, it gives a quantitative expression of the H-theorem. This is a microscopic version of the H-theorem and does not take into account of the space inhomogeneity of the solutions.

For later use, we notice that the projections $\boldsymbol{P}_0$, $\boldsymbol{P}_1$ are defined by the collision invariants and therefore satisfy the following basic properties:

$$\begin{gathered}\boldsymbol{P}_0\psi_\alpha\boldsymbol{M} = \psi_\alpha\boldsymbol{M},\qquad \boldsymbol{P}_1\psi_\alpha\boldsymbol{M} = 0 \quad\text{for } \alpha = 0,1,2,3,4,\qquad \boldsymbol{P}_1 L = L,\qquad \boldsymbol{P}_1 Q(h,h) = Q(h,h),\\ \boldsymbol{P}_0 L = 0,\qquad \boldsymbol{P}_0 Q(h,h) = 0,\qquad \langle\psi_\alpha\boldsymbol{M}, h\rangle = \langle\psi_\alpha\boldsymbol{M}, \boldsymbol{P}_0 h\rangle \quad\text{for } \alpha = 0,1,2,3,4,\\ \langle h, Lg\rangle = \langle \boldsymbol{P}_1 h, L\boldsymbol{P}_1 g\rangle,\qquad \langle h, L^{-1}\boldsymbol{P}_1 g\rangle = \langle L^{-1}\boldsymbol{P}_1 h, \boldsymbol{P}_1 g\rangle.\end{gathered} \tag{2.4}$$

For the decomposition (1.11) and (1.13) of the Boltzmann equation we have only used these basic properties of the projections.

We next derive the macroscopic version of the H-theorem. This version is to study the dissipative property of the Boltzmann equation with regard to the space inhomogeneity and corresponds to the notion of entropy in gas dynamics. Set the macroscopic entropy S by

$$-\frac{3}{2}\rho S \equiv \int_{\mathbb{R}^3} \boldsymbol{M}\log \boldsymbol{M}\,\mathrm{d}\xi. \tag{2.5}$$

It is easy to see that $\log \boldsymbol{M}$ and $\partial_t \boldsymbol{M}/\boldsymbol{M}$ are collision invariants and so

$$\int_{\mathbb{R}^3} \boldsymbol{G}\log \boldsymbol{M}\,\mathrm{d}\xi = \int_{\mathbb{R}^3} \boldsymbol{G}\frac{\partial_t \boldsymbol{M}}{\boldsymbol{M}}\,\mathrm{d}\xi = 0, \qquad \int_{\mathbb{R}^3} Q(f,f)\log \boldsymbol{M}\,\mathrm{d}\xi = 0. \tag{2.6}$$

Multiply (1.9) by $\log \boldsymbol{M}$ and integrate in ξ:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} (\boldsymbol{M}+\boldsymbol{G})\log \boldsymbol{M}\,\mathrm{d}\xi + \nabla_x\cdot\int_{\mathbb{R}^3} \xi(\boldsymbol{M}+\boldsymbol{G})\log \boldsymbol{M}\,\mathrm{d}\xi - \int_{\mathbb{R}^3} \frac{(\boldsymbol{M}+\boldsymbol{G})\boldsymbol{M}_t}{\boldsymbol{M}}\,\mathrm{d}\xi - \int_{\mathbb{R}^3} \frac{(\boldsymbol{M}+\boldsymbol{G})\xi\cdot\nabla_x \boldsymbol{M}}{\boldsymbol{M}}\,\mathrm{d}\xi$$
$$= \int_{\mathbb{R}^3} \frac{Q(f,f)}{\kappa}\log \boldsymbol{M}\,\mathrm{d}\xi. \tag{2.7}$$

Use (2.6) to simplify this into

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} \boldsymbol{M}\log \boldsymbol{M}\,\mathrm{d}\xi + \nabla_x\cdot\int_{\mathbb{R}^3} \xi\boldsymbol{M}\log \boldsymbol{M}\,\mathrm{d}\xi + \nabla_x\cdot\int_{\mathbb{R}^3} \xi\boldsymbol{G}\log \boldsymbol{M}\,\mathrm{d}\xi - \int_{\mathbb{R}^3} \boldsymbol{M}_t\,\mathrm{d}\xi - \nabla_x\cdot\int_{\mathbb{R}^3} \xi\boldsymbol{M}\,\mathrm{d}\xi$$
$$= \int_{\mathbb{R}^3} \frac{\boldsymbol{G}\xi\cdot\nabla_x \boldsymbol{M}}{\boldsymbol{M}}\,\mathrm{d}\xi. \tag{2.8}$$

Note that, from the continuity equation:

$$-\int_{\mathbb{R}^3} \boldsymbol{M}_t\,\mathrm{d}\xi - \nabla_x\cdot\int_{\mathbb{R}^3} \xi\boldsymbol{M}\,\mathrm{d}\xi = -\rho_t - \nabla_x\cdot(\rho u) = 0. \tag{2.9}$$

Also from (2.5):

$$\int_{\mathbb{R}^3} \xi\boldsymbol{M}\log \boldsymbol{M}\,\mathrm{d}\xi = u\int_{\mathbb{R}^3} \boldsymbol{M}\log \boldsymbol{M}\,\mathrm{d}\xi + \int_{\mathbb{R}^3} (\xi-u)\boldsymbol{M}\log \boldsymbol{M}\,\mathrm{d}\xi = -\frac{3}{2}u\rho S + 0 \tag{2.10}$$

and so we have

$$-\frac{3}{2}(\rho S)_t - \frac{3}{2}\nabla_x\cdot(\rho S u) + \nabla_x\cdot\left(\int_{\mathbb{R}^3} \xi\boldsymbol{G}\log \boldsymbol{M}\,\mathrm{d}\xi\right) = \int_{\mathbb{R}^3} \frac{\boldsymbol{G}\xi\cdot\nabla_x \boldsymbol{M}}{\boldsymbol{M}}\,\mathrm{d}\xi. \tag{2.11}$$

Plug (1.11) into the right-hand side of (2.11):

$$-\frac{3}{2}(\rho S)_t - \frac{3}{2}\nabla_x\cdot(\rho S u) + \nabla_x\cdot\left(\int_{\mathbb{R}^3} \xi\boldsymbol{G}\log \boldsymbol{M}\,\mathrm{d}\xi\right)$$
$$= \int_{\mathbb{R}^3} \frac{\xi\cdot\nabla_x \boldsymbol{M}}{\boldsymbol{M}} L^{-1}_{[\rho,u,\theta]}(\kappa \boldsymbol{P}_1\xi\cdot\nabla_x \boldsymbol{M})\,\mathrm{d}\xi + \int_{\mathbb{R}^3} \frac{\xi\cdot\nabla_x \boldsymbol{M}}{\boldsymbol{M}} L^{-1}_{[\rho,u,\theta]}(\kappa \boldsymbol{G}_t + \kappa \boldsymbol{P}_1\xi\cdot\nabla_x \boldsymbol{G} - Q(\boldsymbol{G},\boldsymbol{G}))\,\mathrm{d}\xi. \tag{2.12}$$

From (2.4) the right-hand side of the above equation is purely microscopic and so the equation becomes

$$-\frac{3}{2}(\rho S)_t - \frac{3}{2}\nabla_x\cdot(\rho S u) + \nabla_x\cdot\left(\int_{\mathbb{R}^3} \xi\boldsymbol{G}\log \boldsymbol{M}\,\mathrm{d}\xi\right)$$
$$= \int_{\mathbb{R}^3} \frac{\boldsymbol{P}_1\xi\cdot\nabla_x \boldsymbol{M}}{\boldsymbol{M}} L^{-1}_{[\rho,u,\theta]}(\kappa \boldsymbol{P}_1\xi\cdot\nabla_x \boldsymbol{M})\,\mathrm{d}\xi + \int_{\mathbb{R}^3} \frac{\boldsymbol{P}_1\xi\cdot\nabla_x \boldsymbol{M}}{\boldsymbol{M}} L^{-1}_{[\rho,u,\theta]}(\kappa \boldsymbol{G}_t + \kappa \boldsymbol{P}_1\xi\cdot\nabla_x \boldsymbol{G} - Q(\boldsymbol{G},\boldsymbol{G}))\,\mathrm{d}\xi. \tag{2.13}$$

From (1.3) and (2.5):

$$-\frac{3}{2}\rho S=\rho\log\rho-\frac{3}{2}\rho\log(2\pi R\theta)-\frac{1}{2}\rho. \tag{2.14}$$

We now construct an entropy pair (η, q) around a given state $\bar{\boldsymbol{M}}$. In the following, the fluid functions for $\bar{M}$ are denoted by $\bar{\rho}$, $\bar{u}$, etc. We may transform the base velocity $\bar{u}$ to zero by introducing the variable $v = u - \bar{u}$ and consider the new conserved quantities:

$$\begin{aligned}\alpha &\equiv (\rho, \rho v_1, \rho v_2, \rho v_3, \rho(E+\tfrac{1}{2}|v|^2))\\ &=(\rho, \rho u_1-\bar{u}_1\rho, \rho u_2-\bar{u}_2\rho, \rho u_3-\bar{u}_3\rho, \rho(E+\tfrac{1}{2}u^2)+\tfrac{1}{2}\bar{u}^2\rho-\bar{u}\cdot(\rho u_1, \rho u_2, \rho u_3)).\end{aligned}$$

Thus we may write the conservation laws (1.10) as

$$\alpha_t+\nabla_x\cdot\beta=0. \tag{2.15}$$

The entropy function (2.14) is expressed as function of the conserved quantities α:

$$-\frac{3}{2}\rho S=\frac{5}{3}\alpha_0\log\alpha_0-\frac{3}{2}\alpha_0\log\left(\alpha_4-\frac{(\alpha_1)^2+(\alpha_2)^2+(\alpha_3)^2}{2\alpha_0}\right)-\left(\frac{3}{2}\log\frac{4\pi}{3}+\frac{1}{2}\right)\alpha_0. \tag{2.16}$$

We set the entropy pair as

$$\eta=-\tfrac{3}{2}\rho S+\tfrac{3}{2}\bar{\rho}\bar{S}+\tfrac{3}{2}\nabla_\alpha(\rho S)|_{\alpha=\bar{\alpha}}\cdot(\alpha-\bar{\alpha}),\qquad q=-\tfrac{3}{2}\rho S u+\tfrac{3}{2}\bar{\rho}\bar{S}\bar{u}+\tfrac{3}{2}\nabla_\alpha(\rho S)|_{\alpha=\bar{\alpha}}\cdot(\beta-\bar{\beta}). \tag{2.17}$$

This clearly has the property that $\bar{\eta}=\eta(\bar{\boldsymbol{M}})=0$, $\nabla_\alpha\bar{\eta}=\nabla_\alpha\eta(\bar{\boldsymbol{M}})=0$. Direct calculations from (2.16) show that at the base state $\bar{\alpha}$, the Hessian $\partial^2\eta/\partial\alpha_i\partial\alpha_j$, is

$$\begin{pmatrix} \frac{5}{2\bar{\rho}} & 0 & 0 & 0 & -\frac{3}{2\bar{\rho}\bar{E}}\\ 0 & \frac{3}{2\bar{\rho}\bar{E}} & 0 & 0 & 0\\ 0 & 0 & \frac{3}{2\bar{\rho}\bar{E}} & 0 & 0\\ 0 & 0 & 0 & \frac{3}{2\bar{\rho}\bar{E}} & 0\\ -\frac{3}{2\bar{\rho}\bar{E}} & 0 & 0 & 0 & \frac{3}{2\bar{\rho}\bar{E}^2}\end{pmatrix}. \tag{2.18}$$

It is easy to check that the matrix is positive definite and so η is a convex function of the macroscopic variables α at the base state $\bar{\boldsymbol{M}}$. Thus around the base state $\bar{\alpha}$, there exist positive constants C_1 and C_2 such that

$$C_1|\alpha-\bar{\alpha}|^2\le\eta\le C_2|\alpha-\bar{\alpha}|^2. \tag{2.19}$$

Actually, it can be shown that η is positive except at the base state. Thus the above holds for α in any given bounded region.

From (2.13) and (2.15):

$$\begin{aligned}&\eta_t+\nabla_x\cdot q+\nabla_x\cdot\left(\int_{\mathbb{R}^3}\xi\boldsymbol{G}\log\boldsymbol{M}\,\mathrm{d}\xi\right)\\ &=\int_{\mathbb{R}^3}\kappa\frac{\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}L^{-1}_{[\rho,u,\theta]}(\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M})\,\mathrm{d}\xi+\int_{\mathbb{R}^3}\frac{\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}L^{-1}_{[\rho,u,\theta]}(\kappa\boldsymbol{G}_t+\kappa\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{G}-Q(\boldsymbol{G},\boldsymbol{G}))\,\mathrm{d}\xi.\end{aligned} \tag{2.20}$$

In the above identity, the term:

$$\kappa\int_{\mathbb{R}^3}\frac{\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}L^{-1}_{[\rho,u,\theta]}(\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M})\,\mathrm{d}\xi=\kappa\langle\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M},L^{-1}_{[\rho,u,\theta]}(\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M})\rangle$$

represents the entropy dissipation. Since the non-fluid functions $\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}$ belong to a finite dimensional space in the ξ variables, we have from the microscopic version of the H-theorem (2.3) that the term satisfies, for some positive constants σ_1 and σ_2:

$$\sigma_1\int_{\mathbb{R}^3}\frac{|\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}|^2}{\boldsymbol{M}}\le-\int_{\mathbb{R}^3}\frac{\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}L^{-1}_{[\rho,u,\theta]}(\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M})\,\mathrm{d}\xi\le\sigma_2\int_{\mathbb{R}^3}\frac{|\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}|^2}{\boldsymbol{M}}\,\mathrm{d}\xi. \tag{2.21}$$

Simple calculation shows that

$$\int_{\mathbb{R}^3}\frac{|\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}|^2}{\boldsymbol{M}}\,\mathrm{d}\xi=\mathrm{O}(1)\sum_{j=1}^{3}[|\partial_{x^j}u|^2+|\nabla_x\theta|^2] \tag{2.22}$$

for some positive function O(1). Notice that in the macroscopic version (2.20) of H-theorem, the dominant term on the right-hand side is the first integral, which, as we have just seen, represents the dissipation, and the second integral consists of only higher order derivatives and the quadratic term of microscopic component G. Therefore, it captures the dissipative effect of the fluid components in the solution of Boltzmann equation, and this is useful for the energy estimates in the following sections.

3. Nonlinear stability of a Maxwellian state

In this section, we will show that the macro–micro decomposition yields elementary energy estimates for stability of a global Maxwellian state. Thus, we assume that the initial value $f|_{t=0}$ is a small perturbation of a global Maxwellian state $\bar{M}$. We will show that the macroscopic component M tends to $\bar{M}$ and the microscopic component G tends to zero as t tends to infinity. There are two steps in the energy estimates. In the first step, the lower order estimate follows from the two versions of the H-theorem. For the higher order estimates, our analysis uses the techniques for the Navier–Stokes equations in treating the coupling of the fluid variables.

For simplicity, we assume from now on that the Knudsen number $\kappa=1$.

3.1. Lower order energy estimates of the fluid variables

Integrate the macroscopic H-theorem (2.20) over $0<t<\tau$ and $x\in\mathbb{R}^3$ to yield

$$\begin{aligned}\int_{\mathbb{R}^3}\eta\,\mathrm{d}x\Big|_{t=0}^{t=\tau}&=\int_0^\tau\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}L^{-1}_{[\rho,u,\theta]}(\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M})\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^\tau\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}L^{-1}_{[\rho,u,\theta]}(\boldsymbol{G}_t+\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{G}-Q(\boldsymbol{G},\boldsymbol{G}))\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.1}$$

We have from (2.21), (2.22) and the Schwartz inequality that

$$\begin{aligned}&\int_{\mathbb{R}^3}\eta\,\mathrm{d}x\Big|_{t=0}^{t=\tau}+\sigma_0\int_0^t\int_{\mathbb{R}^3}\sum_{j=1}^{3}[|\partial_{x^j}u|^2+|\nabla_x\theta|^2]\,\mathrm{d}x\,\mathrm{d}t\\&\quad\le C\int_0^\tau\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{\boldsymbol{M}}|L^{-1}_{[\rho,u,\theta]}(\boldsymbol{G}_t+\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{G}-Q(\boldsymbol{G},\boldsymbol{G}))|^2\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.2}$$

for some positive constants σ_0 and C. Here and in the following, C is used to denote a generic constant. This yields the main first order estimate for the fluid variables. The analysis is fully nonlinear and along the same line as the entropy estimate for the conservation laws.

To apply the energy estimate on the microscopic component and the higher order energy estimates, we will fix a background to be an absolute Maxwellian $\boldsymbol{M}_-$:

$$\boldsymbol{M}_- \equiv \boldsymbol{M}_{[1,u_0,\theta_0]}.$$

Here the temperature is chosen with the property that $\theta_0 < \theta$, for all θ under consideration. For definiteness, we set

$$|u_0 - \bar{u}| \ll 1, \qquad \theta_0 = \bar{\theta} - \delta \quad \text{for} \ \ \delta > 0. \tag{3.3}$$

Remark 3.1. The introduction of the weight function $\boldsymbol{M}_-$ is technical. Here, we need to require δ is small but larger than the size of the perturbation of $|(\rho, u, \theta) - (\bar{\rho}, \bar{u}, \bar{\theta})|$:

$$|\rho - \bar{\rho}| + |u - \bar{u}| + |\theta - \bar{\theta}| \le \epsilon_0 < \delta. \tag{3.4}$$

With these, we have, for any given positive constants $\bar{\alpha}$ and $\bar{\beta}$:

$$\boldsymbol{M}_- \le C\boldsymbol{M}(1 + |\xi|)^{-\bar{\alpha}}, \qquad \boldsymbol{M}^{1+\bar{\beta}} \le C\boldsymbol{M}_-(1 + |\xi|)^{-\bar{\alpha}} \tag{3.5}$$

for a constant C independent of (x, t, ξ).

By (3.5), (3.2) leads to

$$\int_{\mathbb{R}^3} \eta \,\mathrm{d}x \bigg|_{t=0}^{t=\tau} + \frac{\sigma_0}{2} \int_0^\tau \int_{\mathbb{R}^3} \sum_{j=1} [|\partial_{x^j} u|^2 + |\nabla_x \theta|^2] \,\mathrm{d}x \,\mathrm{d}t$$
$$\le C\left(\int_0^\tau \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{\boldsymbol{M}_-} (|\boldsymbol{G}_t|^2 + |\nabla_x \boldsymbol{G}|^2) \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{\boldsymbol{M}_-} |L^{-1}_{[\rho,u,\theta]} Q(\boldsymbol{G}, \boldsymbol{G})|^2 \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t \right). \tag{3.6}$$

By the using the property of the linearized operator L in (2.3), one can show that

$$\langle (1 + |\xi|) L^{-1} h, L^{-1} h \rangle \le \sigma^{-2} \langle (1 + |\xi|)^{-1} h, h \rangle$$

for any non-fluid function h. Hence, we have

$$\int_{\mathbb{R}^3} \eta \,\mathrm{d}x \bigg|_{t=0}^{t=\tau} + \frac{\sigma_0}{2} \int_0^\tau \int_{\mathbb{R}^3} \sum_{j=1}^{3} [|\partial_{x^j} u|^2 + |\nabla_x \theta|^2] \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t$$
$$\le C\left(\int_0^\tau \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{\boldsymbol{M}_-} (|\boldsymbol{G}_t|^2 + |\nabla_x \boldsymbol{G}|^2) \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{\boldsymbol{M}_-} (1 + |\xi|)^{-1} Q^2(\boldsymbol{G}, \boldsymbol{G}) \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t \right). \tag{3.7}$$

3.2. Lower order energy estimates of the non-fluid variables

Multiply (1.11) with $\boldsymbol{G}/\boldsymbol{M}_-$; and integrate it over $[0, \tau] \times \mathbb{R}^3 \times \mathbb{R}^3$ to result in that

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\boldsymbol{G}^2}{\boldsymbol{M}_-} \,\mathrm{d}\xi \,\mathrm{d}x \bigg|_{t=0}^{t=\tau} + \int_0^\tau \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{G P_1 \xi \cdot \nabla_x (\boldsymbol{M} + \boldsymbol{G})}{\boldsymbol{M}_-} \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t$$
$$= \int_0^\tau \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\boldsymbol{G} L_{[\rho,u,\theta]} \boldsymbol{G} + \boldsymbol{G} Q(\boldsymbol{G}, \boldsymbol{G})}{\boldsymbol{M}_-} \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t. \tag{3.8}$$

Since the linearized collision operator around the Maxwellian depends smoothly on the parameters ρ, u, θ, the following lemmas follow easily from the basic properties (2.2) and (2.3) of the linearized collision operator and the choice of the Mawellian state $\boldsymbol{M}_-$, (3.3), (3.4) and (35). Their proofs are therefore omitted. These lemmas are used when we replace the local Maxwellian $\boldsymbol{M}$ in the weighted energy estimates by the global Maxwellian $\boldsymbol{M}_-$. A polynomial of ξ appears in the integral because of the differentiation on the local Maxwellian $\boldsymbol{M}$ in u and θ give the first and second order polynormial in ξ. However, all of these polynomials can be controlled by the exponential decay in Maxwellian when $\boldsymbol{M}_-$ is chosen appropriately.

Lemma 3.2. *There exists $\epsilon_0 > 0$ so that, for any $K > 0$ and integer $i > 0$:*

$$\left|\int_{\mathbb{R}^3} \frac{g|\xi|^i \boldsymbol{M}}{\boldsymbol{M}_-}\,\mathrm{d}\xi\right| \le C_i\left(K\int_{\mathbb{R}^3}\frac{|g|^2}{\boldsymbol{M}_-}\,\mathrm{d}\xi + K^{-1}\right). \tag{3.9}$$

Lemma 3.3. *There exists $C > 0$ such that for any non-fluid functions $g_1(\xi)$ and $g_2(\xi)$ satisfying*

$$\int_{\mathbb{R}^3}\frac{g_i(\xi)^2}{\boldsymbol{M}_-}\,\mathrm{d}\xi < \infty \quad \textit{for}\ \ i = 1, 2,\quad x\in\mathbb{R}^3, \tag{3.10}$$

one has the following estimates for any given $K > 0$:

$$\left|\int_{\mathbb{R}^3}\frac{g_1\boldsymbol{P}_1|\xi|^j g_2}{\boldsymbol{M}_-}\,\mathrm{d}\xi - \int_{\mathbb{R}^3}\frac{g_1|\xi|^j g_2}{\boldsymbol{M}_-}\,\mathrm{d}\xi\right| \le C\int_{\mathbb{R}^3}\frac{K|g_1|^2 + K^{-1}|g_2|^2}{\boldsymbol{M}_-}\,\mathrm{d}\xi \quad \textit{for}\ \ j > 0. \tag{3.11}$$

Lemma 3.4. *For the δ satisfying (3.3), there exists $\epsilon_0 > 0$ so that for any function $g(\xi)$ with $\int_{\mathbb{R}^3}(|g|^2(1+|\xi|)/\boldsymbol{M}_-)\,\mathrm{d}\xi$ bounded, we have*

$$-(1+\mathrm{O}(1)\epsilon_0)\int_{\mathbb{R}^3}\frac{gL_{[\rho_0,u_0,\theta_0]}g}{\boldsymbol{M}_-}\,\mathrm{d}\xi \le -\int_{\mathbb{R}^3}\frac{gL_{[\rho,u,\theta]}g}{\boldsymbol{M}_-}\,\mathrm{d}\xi \le -(1-\mathrm{O}(1)\epsilon_0)\int_{\mathbb{R}^3}\frac{gL_{[\rho_0,u_0,\theta_0]}g}{\boldsymbol{M}_-}\,\mathrm{d}\xi \tag{3.12}$$

and then

$$\left|\int_{\mathbb{R}^3}\frac{g(L_{[\rho,u,\theta]} - L_{[\rho_0,u_0,\theta_0]})g}{\boldsymbol{M}_-}\,\mathrm{d}\xi\right| \le \mathrm{O}(1)\epsilon_0\int_{\mathbb{R}^3}\frac{|g|^2(1+|\xi|)}{\boldsymbol{M}_-}\,\mathrm{d}\xi. \tag{3.13}$$

By the above lemmas, we deduce from (3.8) that there exists $K > 0$ such that when $\delta \ll 1$:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\boldsymbol{G}^2}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\bigg|_{t=0}^{t=\tau} - \frac{1}{4}\int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\boldsymbol{G}L_{[\rho_0,u_0,\theta_0]}\boldsymbol{G}}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le K\int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\left(\sum_{j=1}^3|u_{x^j}|^2 + |\nabla_x\theta|^2\right)\mathrm{d}x\,\mathrm{d}t + \mathrm{O}(1)(\epsilon_0 + K^{-1})\int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\boldsymbol{G}^2(|\xi|+1)}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\sum_{j=1}^3\frac{K|\boldsymbol{G}_{x^j}|^2}{\boldsymbol{M}_-} + \frac{\boldsymbol{G}Q(\boldsymbol{G},\boldsymbol{G})}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.14}$$

Here, and also in the next estimate, we have used (2.3). From (3.7) $+\,\sigma_0/2K$ (3.14) we have by choosing K sufficiently large:

$$\int_{\mathbb{R}^3} \eta \,\mathrm{d}x + \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\sigma_0 \boldsymbol{G}^2}{2K\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\Bigg|_{t=0}^{t=\tau} + \frac{\sigma_0}{4}\int_0^\tau \int_{\mathbb{R}^3} \sum_{i=1}^3 |u_{x^i}|^2 + |\nabla_x\theta|^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{8K}\int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\boldsymbol{G}L_{[\rho_0,u_0,\theta_0]}\boldsymbol{G}}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C\left[\int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\boldsymbol{G}Q(\boldsymbol{G},\boldsymbol{G})}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\boldsymbol{G}_t|^2 + \sum_{j=1}^3 |\boldsymbol{G}_{x^j}|^2}{\boldsymbol{M}_-}\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(1+|\xi|)^{-1}Q^2(\boldsymbol{G},\boldsymbol{G})}{\boldsymbol{M}_-}\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t\right]. \tag{3.15}$$

We denote by ∂^β the differential operator:

$$\partial^\beta \equiv \partial_t^{\beta_0}\partial_{x^1}^{\beta_1}\partial_{x^2}^{\beta_2}\partial_{x^3}^{\beta_3}, \qquad |\beta| \equiv \sum_{i=0}^3 \beta_i, \quad \text{where } \beta_i \ge 0 \text{ are nonnegative integers.}$$

Consider the first-order energy estimate from the Boltzmann equation (1.1):

$$0 = \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\partial^\beta f)(\partial^\beta[f_t + \xi\cdot\nabla_x f - (L_{[\rho,u,\theta]}\boldsymbol{G} + Q(\boldsymbol{G},\boldsymbol{G}))])}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t \quad \text{with } |\beta| = 1. \tag{3.16}$$

This yields

$$\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\partial^\beta f)^2}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\Bigg|_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\partial^\beta \boldsymbol{G})L_{[\rho,u,\theta]}(\partial^\beta \boldsymbol{G})}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\partial^\beta \boldsymbol{M})(\partial^\beta L_{[\rho,u,\theta]}\boldsymbol{G}) + (\partial^\beta \boldsymbol{G})([L_{[\rho,u,\theta]},\partial^\beta]\boldsymbol{G}) + (\partial^\beta f)(\partial^\beta Q(\boldsymbol{G},\boldsymbol{G}))}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t, \tag{3.17}$$

where [,] is the Lie bracket.

From (2.3) there exists $\sigma_1 > 0$ such that

$$\sigma_1 \int_{\mathbb{R}^3} \frac{|\boldsymbol{G}_t|^2 + \sum_{j=1}^3 |\boldsymbol{G}_{x^j}|^2}{\boldsymbol{M}_-}(1+|\xi|)\,\mathrm{d}\xi \le -\sum_{|\beta|=1}\int_{\boldsymbol{R}^3} \frac{(\partial^\beta \boldsymbol{G})L_{[\rho,u,\theta]}(\partial^\beta \boldsymbol{G})}{\boldsymbol{M}_-}\,\mathrm{d}\xi.$$

Consider (3.15) + C_1 (3.17) to obtain when C_1 is sufficiently large:

$$\left[\int_{\mathbb{R}^3} \eta\,\mathrm{d}x + \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\boldsymbol{G}^2 + \sum_{|\beta|=1}(\partial^\beta f)^2}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\right]\Bigg|_{t=0}^{t=\tau} + \int_0^\tau \int_{\mathbb{R}^3} \sum_{i=1}^3 |u_{x^i}|^2 + |\nabla_x\theta|^2\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\boldsymbol{G}L_{[\rho_0,u_0,\theta_0]}\boldsymbol{G}}{\boldsymbol{M}_-} + \sum_{|\beta|=1} \frac{(\partial^\beta \boldsymbol{G})L_{[\rho,u,\theta]}(\partial^\beta \boldsymbol{G})}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C\Bigg[\int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\boldsymbol{G}Q(\boldsymbol{G},\boldsymbol{G})}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(1+\xi|)^{-1}Q^2(\boldsymbol{G})}{\boldsymbol{M}_-}\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3}$$
$$\times \sum_{|\beta|=1} \frac{(\partial^\beta \boldsymbol{M})(\partial^\beta L_{[\rho,u,\theta]}\boldsymbol{G}) + (\partial^\beta \boldsymbol{G})([L_{[\rho,u,\theta]},\partial^\beta]\boldsymbol{G}) + (\partial^\beta f)(\partial^\beta Q(\boldsymbol{G},\boldsymbol{G},\boldsymbol{G}))}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\Bigg]. \tag{3.18}$$

Remark 3.5. Notice that the order of differentiations on both sides of (3.18) is up to the first order. The left-hand side contains positive integrals only. Thus to close this energy estimate, it remains to show that the smallness in $\|\cdot\|_\infty$ for lower order derivative terms, so that the terms on the right-hand side can be absorbed by the terms on the left-hand side. For this, we need to consider the higher order estimates, which will be done in the next subsection. One also needs to keep in mind that in the energy estimates (3.18), the integral in the fluid variable contains the terms $\sum_{j=1}^{3}|u_{x^j}|^2+|\nabla_x\theta|^2$ but not the term $|\nabla_x\rho|^2$. This is a structure similar to the compressible Navier–Stokes equation.

3.3. High order energy estimates

In this subsection, we will consider the energy estimates of $\partial^\beta \boldsymbol{M}$ and $\partial^\beta G$ for $1\le|\beta|\le 5$.

Parallel to (3.1), (3.8) and (3.16), we consider the following double integrals:

$$\int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(\partial^\alpha \boldsymbol{M})(\partial^\alpha[\boldsymbol{M}_t+\boldsymbol{P}_0\xi\cdot\nabla_x(\boldsymbol{M}+\boldsymbol{G})])}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t=0, \tag{3.19}$$

$$\int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(\partial^\alpha G)(\partial^\alpha[\boldsymbol{G}_t+\boldsymbol{P}_1\xi\cdot\nabla_x(\boldsymbol{M}+\boldsymbol{G})-(L_{[\rho,u,\theta]}\boldsymbol{G}+Q(\boldsymbol{G},\boldsymbol{G}))])}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t=0, \tag{3.20}$$

$$\int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(\partial^\beta f)(\partial^\beta[f_t+\xi\cdot\nabla_x f-(L_{[\rho,u,\theta]}\boldsymbol{G}+Q(\boldsymbol{G},\boldsymbol{G}))])}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t=0, \tag{3.21}$$

where $1\le|\alpha|\le 4$ and $2\le|\beta|\le 5$.

Since the terms which are higher than quadratic order either contain lower order derivatives of the fluid variables or the product of $Q(\partial^{\gamma_1}\boldsymbol{G},\partial^{\gamma_2}\boldsymbol{G})$ for some γ_1 and γ_2, the following estimate on $Q(g,f)$, cf. [4], will be used:

$$\|(|\xi|+1)^{-1/2}Q(G,G)\|_{L^2_\xi}\le C\|(1+|\xi|)^{1/2}G\|^2_{L^2_\xi}.$$

3.3.1. Smallness assumption

$$\sup_{\substack{t\ge 0\\ x\in\mathbb{R}^3\\ |\alpha|\le 2}}|\partial^\alpha\rho|+|\partial^\alpha u|+|\partial^\alpha\theta|+\sup_{\substack{t\ge 0\\ x\in\mathbb{R}^3\\ |\gamma|\le 2}}\left\|\frac{|\partial^\gamma \boldsymbol{G}(x,t,\xi)|}{\sqrt{\boldsymbol{M}_-}}(|\xi|+1)^{1/2}\right\|_{L^2_\xi}<\epsilon_1. \tag{3.22}$$

Here, the small parameter ϵ_1 is given in terms of the initial data.

The integral (3.19) and (1.12) result in

$$\begin{aligned}0&=\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(\partial^\alpha \boldsymbol{M})^2}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\Big|_{t=0}^{t=\tau}+\int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(\partial^\alpha \boldsymbol{M})(\partial^\alpha \boldsymbol{P}_0\xi\cdot\nabla_x L^{-1}_{[\rho,u,\theta]}\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M})}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(\partial^\alpha \boldsymbol{M})(\partial^\alpha[\boldsymbol{P}_0,\xi\cdot\nabla_x]\boldsymbol{M})}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(\partial^\alpha \boldsymbol{M})(\partial^\alpha \boldsymbol{P}_0\xi\cdot\nabla_x L^{-1}_{[\rho,u,\theta]}[(\boldsymbol{G}_t+\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{G})-Q(\boldsymbol{G},\boldsymbol{G})])}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{3.23}$$

where $[\boldsymbol{P}_0,\xi\cdot\nabla_x]$ is a zeroth order differential operator. From the smallness assumption (3.22), it follows that

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(\partial^\alpha \boldsymbol{M})^2}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\Big|_{t=0}^{t=\tau}+\int_0^\tau\int_{\mathbb{R}^3}\sum_{j=1}^{3}|\partial^\alpha u_{x^j}|^2+|\nabla_x\partial^\alpha\theta|^2\,\mathrm{d}x\,\mathrm{d}t\\&\quad\le C\int_0^\tau\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\partial_t\partial^\alpha \boldsymbol{G}|^2+\sum_{j=1}^{3}|\partial_{x^j}\partial^\alpha \boldsymbol{G}|^2}{\boldsymbol{M}_-}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$+ C\epsilon_1 \int_0^\tau \int_{\mathbb{R}^3} \sum_{(|\alpha|+1)/2<|\alpha'|<|\alpha|} |\partial^{\alpha'}\rho|^2 + |\partial^{\alpha'}u|^2 + |\partial^{\alpha'}\theta|^2 \, dx\, dt$$

$$+ C\epsilon_1 \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \sum_{(|\alpha|+1)/2<|\alpha'|<|\alpha|} (1+|\xi|)\frac{|\partial^{\alpha'}\boldsymbol{G}|^2}{\boldsymbol{M}_-} \, d\xi\, dx\, dt. \tag{3.24}$$

Similarly, (3.20) and (3.21) combined with (3.22) to yield the following two estimates:

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\partial^\alpha \boldsymbol{G}|^2}{\boldsymbol{M}_-} \, d\xi\, dx \Big|_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\partial^\alpha \boldsymbol{G}) L_{[\rho_0,u_0,\theta_0]}(\partial^\alpha \boldsymbol{G})}{\boldsymbol{M}_-} \, d\xi\, dx\, dt$$
$$\le C\Bigg[\int_0^\tau \int_{\mathbb{R}^3\times\boldsymbol{R}^3} \left(\sum_{j=1}^3 |\partial^\alpha u_{x^j}|^2 + |\nabla_x \partial^\alpha \theta|^2\right) dx\, dt$$
$$+ \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \left(\sum_{j=1}^3 \frac{|\partial^\alpha G_{x^j}|^2}{\boldsymbol{M}_-} + \frac{(\partial^\alpha \boldsymbol{G})(\partial^\alpha Q(\boldsymbol{G},\boldsymbol{G}))}{\boldsymbol{M}_-}\right) d\xi\, dx\, dt$$
$$+ \epsilon_1 \int_0^\tau \int_{\mathbb{R}^3} \sum_{(|\alpha|+1)/2<|\alpha'|<|\alpha|} (|\partial^{\alpha'}\rho|^2 + |\partial^{\alpha'}u|^2 + |\partial^{\alpha'}\theta|^2)\, dx\, dt$$
$$+ \epsilon_1 \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \sum_{(|\alpha|+1)/2<|\alpha'|<|\alpha|} \frac{(1+|\xi|)|\partial^{\alpha'}\boldsymbol{G}|^2}{\boldsymbol{M}_-} \, d\xi\, dx\, dt\Bigg] \tag{3.25}$$

and

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\partial^\beta f)^2}{\boldsymbol{M}_-} \, d\xi\, dx \Big|_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\partial^\beta \boldsymbol{G}) L_{[\rho,u,\theta]}(\partial^\beta \boldsymbol{G})}{\boldsymbol{M}_-} \, d\xi\, dx\, dt$$
$$= \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{C(\delta+\epsilon_1)\sum_{|\beta|/2<|\beta'|\le|\beta|}[|\partial^{\beta'}\rho|^2 + |\partial^{\beta'}u|^2 + |\partial^{\beta'}\theta|^2 + (1+|\xi|)|\partial^{\beta'}\boldsymbol{G}|^2]}{\boldsymbol{M}_-} \, d\xi\, dx\, dt. \tag{3.26}$$

Similar to the combination leading to (3.18), we consider $(3.24)+(\mu_1)^{-1}(3.25)+\mu_2(3.26)$, where both $\mu_i, i = 1, 2$, are positive constants which are sufficiently large, to result in

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \left(\sum_{1\le|\alpha|\le4} \frac{(\partial^\alpha \boldsymbol{M})^2 + (\partial^\alpha \boldsymbol{G})^2}{\boldsymbol{M}_-} + \sum_{1\le|\beta|\le5} \frac{(\partial^\beta f)^2}{\boldsymbol{M}_-}\right) d\xi\, dx \Bigg|_{t=0}^{t=\tau}$$
$$+ \sum_{1\le|\alpha|\le4} \int_0^\tau \int_{\mathbb{R}^3} \sum_{i=1}^3 |\partial^\alpha u_{x^i}|^2 + |\partial^\alpha \nabla_x \theta|^2 \, dx\, dt + \sum_{1\le|\alpha|\le5} \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\partial^\alpha G)^2(1+|\xi|)}{\boldsymbol{M}_-} \, d\xi\, dx\, dt$$
$$\le C(\epsilon_1+\delta) \sum_{1\le|\beta'|\le5} \int_0^\tau \int_{\mathbb{R}^3} |\partial^{\beta'}\rho|^2 + |\partial^{\beta'}u|^2 + |\partial^{\beta'}\theta|^2 \, dx\, dt + C(\epsilon_1+\delta)$$
$$\times \sum_{1\le|\beta'|\le5} \int_0^\tau \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(1+|\xi|)|\partial^{\beta'}\boldsymbol{G}|^2}{\boldsymbol{M}_-} \, d\xi\, dx\, dt. \tag{3.27}$$

In the energy estimate (3.27), the highest order terms of the fluid variables on the left-hand side do not contain the term $|\partial^\beta \rho|$ with $|\beta| = 5$. However, on the right-hand side the error terms do contain $|\partial^\beta \rho|^2$ with $|\beta| = 5$ but with small coefficient. In order to bound these terms, we need to use the conservation laws (1.10). Hence, we rewrite the first two equations of (1.10) in the (ρ, u, θ, G) variables:

$$\rho_t + \sum_{j=1}^{3} (\rho u^j)_{x^j} = 0, \qquad u_t^i + \sum_{j=1}^{3} u^j u_{x^j}^i + R\left(\theta_{x^i} + \theta \frac{\rho_{x^i}}{\rho}\right) + \int_{\mathbb{R}^3} \frac{\psi_i - u^i \psi_0}{\rho} \xi \cdot \nabla_x G \, \mathrm{d}\xi = 0. \tag{3.28}$$

Now we consider

$$\sum_{i=1}^{3} \int_0^\tau \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\partial^\alpha \rho_{x^i}) \partial^\alpha \left(u_t^i + \sum_{j=1}^{3} u^j u_{x^j}^i + R\left(\theta_{x^i} + \theta \frac{\rho_{x^i}}{\rho}\right) + \int_{\mathbb{R}^3} \frac{\psi_i - u^i \psi_0}{\rho} \xi \cdot \nabla_x \boldsymbol{G} \, \mathrm{d}\xi \right) \mathrm{d}x \, \mathrm{d}t = 0, \tag{3.29}$$

where $|\alpha| = 4$. By using integration by parts and the continuity equation in (3.28), we have

$$\begin{aligned} &\sum_{i=1}^{3} \int_0^\tau \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\partial^\alpha \rho_{x^i}) \partial^\alpha \left(R\theta \frac{\rho_{x^i}}{\rho} \right) \mathrm{d}x \, \mathrm{d}t \\ &\quad = \sum_{i=1}^{3} \int_{\mathbb{R}^3} (\partial^\alpha \rho_{x^i})(\partial^\alpha u^i) \, \mathrm{d}x \Big|_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\partial^\alpha \operatorname{div}(\rho u)) \partial^\alpha u_{x^i}^i \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad - \sum_{i=1}^{3} \int_0^\tau \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\partial^\alpha \rho_{x^i}) \partial^\alpha \left(\sum_{j=1}^{3} u^j u_{x^j}^i + R\theta_{x^i} + \int_{\mathbb{R}^3} \frac{\psi_i - u^i \psi_0}{\rho} \xi \cdot \nabla_x \boldsymbol{G} \, \mathrm{d}\xi \right) \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{3.30}$$

Thus, we have

$$\begin{aligned} &\sum_{i=1}^{3} \int_0^\tau \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\partial^\alpha \rho_{x^i}|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \le C \Bigg[\int_{\mathbb{R}^3} \sum_{|\alpha'|=5} |\partial^{\alpha'} \rho|^2 + \sum_{|\alpha'|=4} |\partial^{\alpha'} u|^2 \, \mathrm{d}x \Big|_{t=0} + \int_{\mathbb{R}^3} \sum_{|\alpha'|=5} |\partial^{\alpha'} \rho|^2 + \sum_{|\alpha'|=4} |\partial^{\alpha'} u|^2 \, \mathrm{d}x \Big|_{t=\tau} \\ &\qquad + \sum_{|\alpha'|=5} \int_0^\tau \int_{\mathbb{R}^3} \left(|\partial^{\alpha'} u|^2 + |\partial^{\alpha'} \theta|^2 + \int_{\mathbb{R}^3} \frac{|\partial^{\alpha'} \boldsymbol{G}|^2}{\boldsymbol{M}_-} \mathrm{d}\xi \right) \mathrm{d}x \, \mathrm{d}t \\ &\qquad + \epsilon_1 \sum_{0 \le |\alpha'| < 5} \int_0^\tau \int_{\mathbb{R}^3} \left(|\partial^{\alpha'} u|^2 + |\partial^{\alpha'} \theta|^2 + \int_{\mathbb{R}^3} \frac{|\partial^{\alpha'} \boldsymbol{G}|^2}{\boldsymbol{M}_-} \mathrm{d}\xi \right) \mathrm{d}x \, \mathrm{d}t \Bigg]. \end{aligned} \tag{3.31}$$

The combination of (3.27) and (3.31) yields

$$\begin{aligned} &\int_{\mathbb{R}^3} \eta \, \mathrm{d}x \Big|_{t=\tau} + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left(\frac{\boldsymbol{G}^2}{\boldsymbol{M}_-} + \sum_{1 \le |\alpha| \le 4} \frac{(\partial^\alpha \boldsymbol{M})^2 + (\partial^\alpha \boldsymbol{G})^2}{\boldsymbol{M}_-} + \sum_{1 \le |\beta| \le 5} \frac{(\partial^\beta f)^2}{\boldsymbol{M}_-} \right) \mathrm{d}\xi \, \mathrm{d}x \Big|_{t=\tau} \\ &\quad + \sum_{0 \le |\alpha| \le 4} \int_0^\tau \int_{\mathbb{R}^3} \left(\sum_{i=1}^{3} |\partial^\alpha u_{x^i}|^2 + |\partial^\alpha \nabla_x \theta|^2 \right) \mathrm{d}x \, \mathrm{d}t + \sum_{1 \le |\beta| \le 5} \int_0^\tau \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(\partial^\beta \boldsymbol{G})^2 (1 + |\xi|)}{\boldsymbol{M}_-} \mathrm{d}\xi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \le C \left[\int_{\mathbb{R}^3} \eta \, \mathrm{d}x + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\boldsymbol{G}^2}{\boldsymbol{M}_-} + \sum_{1 \le |\alpha| \le 4} \frac{(\partial^\alpha \boldsymbol{M})^2 + (\partial^\alpha \boldsymbol{G})^2}{\boldsymbol{M}_-} + \sum_{1 \le |\beta| \le 5} \frac{(\partial^\beta f)^2}{\boldsymbol{M}_-} \mathrm{d}\xi \, \mathrm{d}x \right] \Bigg|_{t=0}, \end{aligned} \tag{3.32}$$

where ϵ_1 and δ are chosen to be sufficiently small.

Furthermore, (3.32) justifies the smallness assumption (3.22). The energy estimates for higher order derivatives $\partial^\alpha \boldsymbol{M}$ and $(\partial^\alpha \boldsymbol{G})$, $|\alpha| \geq 6$, are similar. We therefore obtain the stability of global Maxwellian as summarized in the following theorem.

Theorem 3.6. *For any global Maxwellian $\boldsymbol{M}$, there exists a small positive constant ϵ_0 such that, if*

$$\| f(x, 0, \xi) - \boldsymbol{M} \|_{H^s_x(L^2_\xi)} \leq \epsilon_0$$

for $s \geq 5$, then there exists $C > 0$ such that

$$\| f(x, t, \xi) - \boldsymbol{M} \|_{H^s_x(L^2_\xi)} \leq C\epsilon_0$$

and

$$\lim_{t\to\infty} \| f(x, t, \xi) - \boldsymbol{M} \|_{L^\infty_x(L^2_\xi)} = 0.$$

Acknowledgements

The research of the first author was supported by Institute of Mathematics, Academia Sinica and NSF Grant DMS-9803323. The research of the second author was supported by the Competitive Earmarked Research Grant of Hong Kong CityU 1142/01P #9040648. The research of the third author was supported by the Competitive Earmarked Research Grant of Hong Kong #9040645.

References

[1] L. Boltzmann, Lectures on Gas Theory, Dover, New York, 1964 (translated by Stephen G. Brush).
[2] T. Carleman, Sur la théorie de l'équation intégrodifférentielle de Boltzmann, Acta Math. 60 (1933) 91–142.
[3] S. Chapman, T.G. Cowling, The Mathematical Theory of Non-uniform Gases, 3rd ed., Cambridge University Press, Cambridge, 1990.
[4] F. Golse, B. Perthame, C. Sulem, On a boundary layer problem for the nonlinear Boltzmann equation, Arch. Ration. Mech. Anal. 103 (1988) 81–96.
[5] H. Grad, Asymptotic theory of the Boltzmann equation, in: J.A. Laurmann (Ed.), Rarefied Gas Dynamics, vol. 1, no. 26, Academic Press, New York, 1963, pp. 26–59.
[6] D. Hilbert, Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Chelsea, New York, 1953, Chapter 22.
[7] S. Kawashima, A. Matsumura, T. Nishida, On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier–Stokes equation, Commun. Math. Phys. 70 (2) (1979) 97–124.
[8] T.-P. Liu, S.-H. Yu, Boltzmann equation: micro–macro decompositions and positivity of shock profiles, Commun. Math. Phys., to appear.
[9] J.C. Maxwell, Dynamical theory of gases, The Scientific Papers of James Clerk Maxwell, Cambridge University Press, Cambridge, vol. II, 1980, p. 26;
J.C. Maxwell, Stresses in rarefied gases arising from inequalities of temperature, The Scientific Papers of James Clerk Maxwell, Cambridge University Press, Cambridge, p. 681.
[10] T. Nishida, Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation, Commun. Math. Phys. 61 (2) (1978) 119–148.
[11] S. Ukai, On the existence of global solutions of mixed problem for non-linear Boltzmann equation, Proc. Jpn. Acad. 50 (1974) 179–184.