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Compressible Navier–Stokes Equations with Degenerate Viscosity Coefficient and Vacuum

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Abstract: In this paper, we consider the compressible Navier–Stokes equations for isentropic flow of finite total mass when the initial density is either of compact or infinite support. The viscosity coefficient is assumed to be a power function of the density so that the Cauchy problem is well-posed. New global existence results are established when the density function connects to the vacuum states continuously. For this, some new *a priori* estimates are obtained to take care of the degeneracy of the viscosity coefficient at vacuum. We will also give a non-global existence theorem of regular solutions when the initial data are of compact support in Eulerian coordinates which implies singularity forms at the interface separating the gas and vacuum.

Contents

1. Introduction

Consider the one-dimensional compressible Navier–Stokes equations for isentropic flow in Eulerian coordinates,

$$\begin{cases} \rho_\tau + (\rho u)_\xi = 0, \\ (\rho u)_\tau + (\rho u^2 + P(\rho))_\xi = (\mu u_\xi)_\xi, \end{cases} \tag{1.1}$$

with initial data

$$\rho(\xi, 0) = \rho_0(\xi), \quad u(\xi, 0) = u_0(\xi), \quad a \le \xi \le b, \tag{1.2}$$

where $\xi \in \mathbf{R^1}$ and $\tau > 0$, and $\rho = \rho(\xi, \tau)$, $u = u(\xi, \tau)$ and $P(\rho)$ denote respectively the density, velocity and the pressure; $\mu \ge 0$ is the viscosity coefficient. For simplicity of presentation, we consider only the polytropic gas, i.e., $P(\rho) = A\rho^\gamma$ with $\gamma > 1$, $A > 0$ being constants.

We will consider this hyperbolic-parabolic system when the initial data is of compact support, i.e., connecting to the vacuum state. Our main concern here is the global existence of solutions and the evolution of the vacuum boundary when the viscous gas connects to vacuum continuously and the viscosity coefficient depends on the density. The physical consideration of the dependence of the viscosity coefficient on density and the difficulty from the degeneracy of the vanishing viscosity at vacuum will be addressed in the following. Right now, let's notice that one of the important features of this problem is that the interface separating the gas and vacuum propagates with finite speed if the initial data are of compact support. For the physical significance of this kind of phenomenon, we refer the readers to the survey paper [16]. It is interesting to notice that the proof of this finite speed propagation is obtained after the lower bound of the density function is given in the form of a power function in Lagrangian coordinates. In other words, this finite speed propagation property is difficult to be justified without the estimate on the density function.

Let's first review some of the previous works in this direction. When the viscosity coefficient μ is a constant, the study in [6] shows that there is no continuous dependence on the initial data of the solutions to the Navier–Stokes equations (1.1) with vacuum. The main reason for this non-continuous dependence at the vacuum comes from the kinetic viscosity coefficient being independent of the density. It is motivated by the physical consideration that in the derivation of the Navier–Stokes equations from the Boltzmann equations through the Chapman-Enskog expansion to the second order, cf. [4], the viscosity is not constant but depends on the temperature. For isentropic flow, this dependence is translated into the dependence on the density by the laws of Boyle and Gay-Lussac for ideal gas as discussed in [13]. In particular, the viscosity of gas is proportional to the square root of the temperature for hard sphere collision. Under this hypothesis, the temperature is of the order of $\rho^{\gamma-1}$ for the perfect gas where the pressure is proportional to the product of the density and the temperature. Therefore, for the hard sphere model where $\gamma = \frac{5}{3}$ for monotomic gas, the viscosity μ is proportional to ρ^θ with $\theta = \frac{1}{3}$. Notice that the following theorem on global existence requires that $0 < \theta < \frac{2}{9}$ which does not include this case. Hence, further investigation on this problem is needed.

This non-continuous dependence result leads people to study the initial boundary value problem instead of the initial value problem. For this, the free boundary problem of one dimensional Navier–Stokes equations with one boundary fixed and the other connected to vacuum was investigated in [18], where the global existence of the weak solutions was proved. Similar results were obtained in [19] for the equations of spherically symmetric motion of viscous gases. Furthermore, the free boundary problem of the one-dimensional viscous gases which expand into the vacuum has been studied by many people, see [18, 19, 22] and reference therein. A further understanding of the regularity and behavior of solutions near the interfaces between the gas and vacuum was given in [14].

The above non-continuous dependence on the initial data for constant viscosity with vacuum is also a motivation for the works on the case when the viscosity function is a function of density, such as $\mu = c\rho^\theta$, where c and θ are positive constants. Notice that now the viscosity coefficient vanishes at vacuum and this property yields the well-posedness of the Cauchy problem when the initial density is of compact support. In this situation, the local existence of weak solutions to Navier–Stokes equations with vacuum was studied in [13], where the initial density was assumed to be connected to vacuum with discontinuities. This property, as shown in [13], can be maintained for some finite time. And the authors in [20] obtained the global existence of weak solutions when $0 < \theta < 1/3$ with the same property. This result was later generalized to the cases when $0 < \theta < \frac{1}{2}$ and $0 < \theta < 1$ in [24] and [10] respectively.

It is noticed that the above analysis is based on the uniform positive lower bound of the density with respect to the construction of the approximate solutions. This estimate is crucial because the other estimates for the convergence of a subsequence of the approximate solutions and the uniqueness of the solution thus obtained will follow from the estimation by standard techniques as long as the vacuum does not appear in the solutions in finite time. And this uniform positive lower bound on the density function can only be obtained when the density function connects to vacuum with discontinuities. In this situation, the density function is positive for any finite time and thus the viscosity coefficient never vanishes. This good property of the solution was obtained and used to prove global existence of solutions to (1.1) when the initial data is of compact support, cf. [10, 20, 24].

If the density function connects to vacuum continuously, there is no positive lower bound for the density and the viscosity coefficient vanishes at vacuum. This degeneracy in the viscosity coefficient gives rise to new analysis difficulties because of the less regularizing effect on the solutions. To our knowledge, only a local existence result has been obtained in this case so far, cf. [25]. Another difficulty comes from the singularity at the vacuum boundary when the density function connects to the vacuum continuously. This can be seen from the analysis in [22] on the non-global existence of the regular solution to Navier–Stokes equations when the density function is of compact support when the viscosity coefficient is constant. The proof there is based on the estimation of the growth rate of the support in terms of time t. If the growth rate is sub-linear, then a nonlinear functional was introduced in [22] which yields the non-global existence of regular solutions. The intuitive explanation of this phenomena comes from the consideration of the pressure in the gas. No matter how smooth the initial data is, the pressure of the gas will build up at the vacuum boundary in finite time and it will push the gas into the vacuum region. This effort can not be compensated by the dissipation from the viscosity so that the support of the gas stays unchanged. This is different from the system of Euler-Possion equations for gaseous stars where the pressure and the gravitational force can become balanced to have stationary solutions. This kind of singularity at the vacuum boundary is discussed in [23] and references therein. In the case of compressible Navier–Stokes equation, the pressure will have the effort on the evolution of the vacuum boundary in finite time so that the density function at the interface will not be smooth. This singularity at the derivatives, maybe of second order for the one-dimensional case, cf. [22], gives some analytic difficulty, but it can be overcome by introducing some appropriate weights in the energy estimates. Notice that these weight functions vanish at the vacuum boundary.

In summary, the main task of this paper is to introduce some weight functions and prove some new a priori estimates on the solutions. Two new global existence results are established: the first one is for the case when the density function is of compact support in Eulerian coordinates, and the second is when the support of gas in Eulerian coordinates is infinite but the total mass is finite. We will also give a non-global existence theorem for regular solutions when the initial data is of compact support.

Precisely, for the case when the density function of compact support is in both Eulerian and Lagrangian coordinates, the restriction on the solution coming from the boundedness of the support is

$$\int_0^1 \frac{1}{\rho(x,t)}dx < \infty, \tag{1.3}$$

in Lagrangian coordinates (x, t). However, the straightforward energy estimate does not guarantee this condition. To overcome this difficulty, we introduce a weight function in applying the energy method and succeed in obtaining a global existence result when $0 < \theta < \frac{2}{9}$. And (1.3) which is a consequence of the boundedness of velocity in the L^∞ norm is justified after the *a priori* estimates on the density function are obtained. The second case is for the density function of infinite support in Eulerian coordinates. Even though the total mass is assumed to be finite, no restriction like (1.3) will be imposed. Some new *a priori* estimates are established in this case so that the global existence of the weak solution is also obtained when $\frac{1}{3} < \theta < \frac{3}{7}$. Notice that the intervals for θ are disjoint for these two cases in our analysis.

The theorem on non-global existence of regular solutions generalizes the one for a constant viscosity coefficient in [22] to the case when the viscosity coefficient depends on density. We think that this will shed some light on the study of the vacuum problem to the full Navier–Stokes equation for non-isentropic gas when the viscosity and heat conductivity coefficients depend on the temperature. It is noticed that the corresponding vacuum problem for this full Navier–Stokes equation is still open. We should also mention that the non-global existence theorems of regular solutions for inviscid compressible flow, such as the system of Euler equations with frictional damping and the system of Euler-Possion equations for gaseous stars, are also based on the estimate on the growth rate of the support of the density function.

There has been a lot of investigation on the Navier–Stokes equations when the initial density is away from vacuum, both for smooth initial data or discontinuous initial data, and one dimensional or multidimensional problems. For these results, please refer to [5, 7, 9, 11, 12, 21] and references therein. And recently, the non-appearance of vacuum in the solutions for any finite time if the initial data does not contain vacuum was proved in [8].

The rest of this paper is organized as follows. In Sect. 2, we give the definition of the weak solution and then state the main theorems of this paper. In Sect. 3, we will give the sketch of the proof of global existence in the above two cases. The details of the proofs for those *a priori* estimates and the construction of the weak solutions will be given in the Appendices. In Sect. 4, we will give a non-global existence theorem on regular solutions when the initial data of compact supports in Eulerian coordinates.

2. The Main Theorems

To solve the free boundary problem (1.1), it is convenient to convert the free boundaries to fixed boundaries by using Lagrangian coordinates. By assuming that the weak solution under consideration has the regularity properties stated in (2.4)–(2.5) below, we know that there exist two curves $\xi = a(\tau)$ and $\xi = b(\tau)$ issuing from $\xi = a$ and $\xi = b$ respectively which separate the gas and the vacuum if the support of the density function is compact. Let

$$x = \int_{a(\tau)}^{\xi} \rho(z,\tau)dz, \quad t = \tau.$$

Then the free boundaries $\xi = a(\tau)$ and $\xi = b(\tau)$ become $x = 0$ and $x = \int_{a(\tau)}^{b(\tau)} \rho(z,\tau) dz = \int_a^b \rho_0(z)dz$ by the conservation of mass, where $\int_a^b \rho_0(z)dz$ is the total mass. We normalize $\int_a^b \rho_0(z)dz = 1$. If the support is infinite, then $x = \int_{-\infty}^{\xi} \rho(z,\tau)dz$ without any ambiguity.

Hence, in the Lagrangian coordinates, the free boundary problem (1.1) becomes

$$\begin{cases} \rho_t + \rho^2 u_x = 0, \\ u_t + P(\rho)_x = (\mu\rho u_x)_x, \quad 0 < x < 1, \ \ t > 0, \end{cases} \tag{2.1}$$

with the boundary conditions

$$\rho(0,t) = \rho(1,t) = 0, \tag{2.2}$$

and initial data

$$(\rho, u)(x, 0) = (\rho_0(x), u_0(x)), \ \ 0 \le x \le 1, \tag{2.3}$$

where $P(\rho) = A\rho^{\gamma}, \ \mu = c\rho^{\theta}$. We normalize $A = 1$ and $c = 1$.

Throughout this paper, the assumptions on the initial data, θ and γ can be stated as follows:

(A1) For any positive integer n, $0 \le \rho_0(x) \le C(x(1-x))^{\alpha}$ with $0 < \alpha < 1$, $(\rho_0(x))^{-1} \in L^1([0,1])$, for some k_1 with $1 < k_1 < \min\{1 + (1-3\theta)\alpha, \ (2\gamma - 3\theta + 1)\alpha, \ \frac{5-15\theta}{1+3\theta}, \ \frac{3-5\theta}{1+\theta}, \alpha(4-2\theta)\}$, such that $(x(1-x))^{k_1}\rho_0^{2\theta-2}(x) \in L^1([0,1])$, $(x(1-x))^{\frac{1}{2}}(\rho_0^{\theta}(x))_x \in L^2([0,1])$, and $(\rho_0^{\gamma}(x))_x \in L^{2n}([0,1])$;

(A2) $u_0(x) \in L^{\infty}([0,1])$ and $(\rho_0^{1+\theta}(x)u_{0x})_x \in L^{2n}([0,1])$;

(A3) $0 < \theta < \frac{2}{9}, \ \ \gamma > 1$.

Under the assumptions (A1)–(A3), we will prove the existence of a global weak solution to the initial boundary value problem (2.1)–(2.3). The weak solution defined below is similar to the one in [20].

Definition 2.1. *A pair of functions $(\rho(x,t), u(x,t))$ is called a global weak solution to the initial boundary value problem (2.1)–(2.3), if for any $T > 0$,*

$$\rho, \ u \in L^{\infty}([0,1] \times [0,T]) \cap C^1([0,T]; L^2([0,1])), \tag{2.4}$$

$$\rho^{1+\theta}u_x \in L^\infty([0,1]\times[0,T]) \cap C^{1/2}([0,T];L^2([0,1])). \tag{2.5}$$

Furthermore, the following equations hold:

$$\int_0^\infty \int_0^1 (\rho\phi_t - \rho^2 u_x \phi)dxdt + \int_0^1 \rho_0(x)\phi(x,0)dx = 0, \tag{2.6}$$

and

$$\int_0^\infty \int_0^1 (u\psi_t + (P(\rho) - \mu\rho u_x)\psi_x)dxdt + \int_0^1 u_0(x)\psi(x,0)dx = 0,$$

for any test function $\phi(x,t)$ *and* $\psi(x,t) \in C_0^\infty(\Omega)$ *with* $\Omega = \{(x,t) : 0 < x < 1, t \geq 0\}$.

In what follows, we always use C $(C(T))$ to denote a generic positive constant depending only on the initial data (or the given time T).

We now state the main theorems in this paper. The first one is the global existence theorem when the density function is of compact support in Eulerian coordinates.

Theorem 2.2. *Under the conditions (A1)–(A3), the free boundary problem has a weak solution* $(\rho(x,t), u(x,t))$ *with* $\rho,\ u \in C^1([0,T];H^1([0,1]))$ *and* $\rho(x,t)$ *satisfies*

$$C(T)(x(1-x))^{\frac{k_2}{1-2\theta}} \leq \rho(x,t) \leq C(T)(x(1-x))^\alpha, \tag{2.7}$$

where $k_2 = \frac{1}{2}(1+k_1)$.

Remark 2.3. It is noticed that the set of initial data $(\rho_0(x), u_0(x))$ verifying all the assumptions in Theorem 2.2 contain a quite general family of functions. For example, if we choose $\rho_0(x) = A(x(1-x))^\alpha$ with the exponent α satisfying

$$\frac{1}{\gamma} < \alpha < 1,$$

then it satisfies all assumptions on density.

Notice also that when the initial data is given in the form of $A(x(1-x))^\alpha$, the condition (A1) implies that $\frac{k_2}{1-2\theta} > \alpha$.

The second global existence theorem is for the weak solution to (1.1) and (1.2) with the support of the initial density $\rho_0(x)$ being infinite. The assumptions on the initial data, θ and γ in this case can be stated as follows:

- (H1) $0 \leq \rho_0(x) \in L^\infty([0,1])$, and for any positive integer n, $(x(1-x))^{1-\frac{1}{2n}}\rho_0^{2(\theta-1)}(x) \in L^1([0,1])$, $(\rho_0^\theta(x))_x \in L^2([0,1])$, $(\rho_0^\gamma(x))_x \in L^{2n}([0,1])$;
- (H2) $u_0(x) \in L^\infty([0,1])$ and $(\rho_0^{1+\theta}(x)u_{0x})_x \in L^{2n}([0,1])$;
- (H3) $\frac{1}{3} < \theta < \frac{3}{7}$, $\gamma > 1$.

Under the assumptions (H1)–(H3), we will prove the following existence theorem of a global weak solution to the initial boundary value problem (2.1)–(2.3).

Theorem 2.4. *Under the conditions (H1)–(H3), the free boundary problem has a weak solution* $(\rho(x,t), u(x,t))$ *and* $\rho(x,t)$ *satisfies*

$$\rho(x,t) \geq C(T)(x(1-x))^{\frac{1}{2(1-2\theta)}}. \tag{2.8}$$

Remark 2.5. It is noticed that the set of initial data $(\rho_0(x), u_0(x))$ verifying all the assumptions in Theorem 5.2 contain a quite general family of functions. For example, if we choose $\rho_0(x) = A(x(1-x))^\alpha$ with the exponent α satisfying

$$\frac{1}{2\theta} < \alpha < \frac{1}{1-\theta},$$

which implies $\theta > \frac{1}{3}$, then it satisfies all assumptions on the density function. And it is easy to see that the integral $\int_0^1 \frac{1}{\rho(x,t)}dx = \infty$ because $\theta < \frac{3}{7}$ and thus the support of the density is infinite even though it is of finite total mass.

Finally in Sect. 4, we will give a non-global existence theorem on regular solutions when the initial data is of compact support. The corresponding theorem on Navier–Stokes equations for compressible fluid with constant viscosity and heat conductivity coefficients was obtained in [22]. Here we generalize the above theorem to the case when the viscosity coefficient depends on density for the isentropic gas flow.

We give first the definition of the regular solution of (1.1) and (1.2) as follows.

Definition 2.6. *A solution of (1.1) and (1.2) is called a regular solution in* $\mathbf{R} \times [0, T]$ *if*

(i) $\rho(\xi, \tau) \in C^1(\mathbf{R} \times [0, T])$, $\rho \geq 0$, *and* $u(\xi, \tau) \in C^2(\mathbf{R} \times [0, T])$*;*

(ii) $\rho^{\frac{\theta-1}{2}}(\xi, \tau) \in C^1(\mathbf{R} \times [0, T])$.

Now the non-global existence theorem can be stated as follows.

Theorem 2.7. *Let* $1 < \theta \leq \gamma$ *and* $(\rho(\xi, \tau), u(\xi, \tau))$ *be a regular solution of (1.1) and (1.2) on* $0 \leq \tau \leq T$. *If the support of the initial data* $(\rho_0(\xi), u_0(\xi))$ *is compact and* $(\rho_0(\xi), u_0(\xi)) \not\equiv 0$, *then* T *must be finite.*

3. Sketch of the Proof

3.1. The case of compact support. In this subsection, we will consider the case when the density function is of compact support in Eulerian coordinates. That is, besides the assumption on finite total mass, the solution obtained in Lagrangian coordinates should satisfy

$$\int_0^1 \frac{1}{\rho(x,t)}dx < \infty.$$

This restriction gives rise to some difficulties in analysis because the straightforward energy estimates violate this assumption as discussed in Sect. 5. For this, we introduce weight function as power functions of $x(1-x)$ in applying the energy method. The weights seem to be optimal in our case if one wants to use the weight in the form of $x^\alpha(1-x)^\beta$.

For simplicity of presentation, we establish certain *a priori* estimates in the continuous version to the initial boundary value problem (2.1)–(2.3). The corresponding estimates in discrete version will be given in Sect. 5. First, we list some useful identities as follows.

The proof of the following Lemma 3.1 is straightforward and is omitted, cf. [20, 24].

Lemma 3.1 (Some identities and standard energy estimates). *Under the conditions of Theorem 2.2, we have for* $0 < x < 1, t > 0$ *that*

$$\frac{d}{dt}\int_0^x u(y,t)dy = -\frac{d}{dt}\int_x^1 u(y,t)dy, \tag{3.1}$$

$$\left(\rho^{1+\theta}u_x\right)(x,t) = \rho^\gamma(x,t) + \int_0^x u_t(y,t)dy = \rho^\gamma(x,t) - \int_x^1 u_t(y,t)dy, \tag{3.2}$$

$$\begin{aligned}\rho^\theta(x,t) + \theta\int_0^t \rho^\gamma(x,s)ds &= \rho_0^\theta(x) - \theta\int_0^t\int_0^x u_t(y,s)dyds \\ &= \rho_0^\theta(x) + \theta\int_0^t\int_x^1 u_t(y,s)dyds,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{\gamma-1}\rho^{\gamma-1}\right)dx + \int_0^t\int_0^1 \rho^{1+\theta}u_x^2 dxdt \\ &= \int_0^1\left(\frac{1}{2}u_0^2(x) + \frac{1}{\gamma-1}\rho_0^{\gamma-1}(x)\right)dx \le C,\end{aligned} \tag{3.4}$$

$$\int_0^1 u^{2n}dx + n(2n-1)\int_0^t\int_0^1 u^{2n-2}\rho^{1+\theta}u_x^2 dxds \le Ce^{(n-1)(2n-1)t} \le C(T). \tag{3.5}$$

The following lemma gives us the upper bound for density function $\rho(x,t)$. It is noted that an upper bound in the form of a power function of $x(1-x)$ is required in later analysis, cf. (5.33), (5.35), (5.43), (5.46) and (5.51).

Lemma 3.2. *Under the conditions of Theorem 2.2, we have*

$$\rho(x,t) \le C(T)(x(1-x))^\alpha. \tag{3.6}$$

Proof. From (3.3), Assumption (A1) and Lemma 3.1, we have

$$\begin{aligned}\rho^\theta(x,t) &\le \rho_0^\theta(x) - \theta\int_0^x u(y,t)dy + \theta\int_0^x u_0(y)dy \\ &\le \rho_0^\theta(x) + C\left(\int_0^1 u^{2n}(x,t)dx\right)^{\frac{1}{2n}}(x(1-x))^{\frac{2n-1}{2n}} + Cx(1-x) \\ &\le C(x(1-x))^{\theta\alpha} + C(T)(x(1-x))^{\frac{2n-1}{2n}},\end{aligned}$$

which implies

$$\rho(x,t) \le C(x(1-x))^\alpha + C(T)(x(1-x))^{\frac{2n-1}{2n\theta}}.$$

Noticing $0 < \alpha < 1$, Lemma 3.2 follows.

Now we will give a weighted energy estimate on the function $(\rho^\theta)_x$. And as discussed in [24] for the case when the density function connects to vacuum with discontinuity, one can lower the power of ρ in $\int_0^t\int_0^1 \rho^{1+\theta}u_x^2 dxds$ to $\int_0^t\int_0^1 \rho^{1+\theta+\alpha_1}u_x^2 dxds$ for some $\alpha_1 < 0$ to have a better estimate on the lower bound of ρ with a weight function $(x(1-x))^{k_1}$.

Lemma 3.3. *For any positive integer m, if the assumptions in Theorem 2.2 are satisfied, then for any* $k_1 > 1$*, when* $0 < \theta < 1$*, we have*

$$\int_0^1 x(1-x)\rho^{2\theta-2}\rho_x^2 dx \le C(T), \tag{3.7}$$

$$\int_0^1 (x(1-x))^{k_1}\rho^{\alpha_1}u^2 dx + \int_0^t\int_0^1 (x(1-x))^{k_1}\rho^{1+\theta+\alpha_1}u_x^2 dxds \le C(T), \tag{3.8}$$

and

$$\int_0^1 (x(1-x))^{k_1}\rho^{\beta_1} dx \le C(T), \tag{3.9}$$

where $\alpha_1 = \left(1-\frac{1}{2^m}\right)(\theta-1) < 0$ *and* $\beta_1 = \left(2-\frac{1}{2^m}\right)(\theta-1) < 0$.

The proof can be found in Appendix 5.1.

Based on the above lemma, we can obtain the lower bound of density function in the following lemma. With this crucial estimate on the lower bound for density function, we can now study the other property of the solution (ρ, u) for compactness of the sequence of the approximate solutions given in Appendix 5.6.

Lemma 3.4. *For any* $0 < \theta < \frac{1}{2}$, $k_1 > 1$ *and* $k_2 = \frac{1}{2} + \frac{k_1}{2}$*, there exists* $C(T) > 0$ *such that*

$$\rho(x,t) \ge C(T)(x(1-x))^{\frac{k_2}{1-2\theta}}. \tag{3.10}$$

Proof. Let

$$\beta_2 = \theta + \left(1 - \frac{1}{2^{m+1}}\right)(\theta-1). \tag{3.11}$$

Then for sufficiently large m, $\beta_2 < 0$. Now by using Sobolev's embedding theorem $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$ and Young's inequality, we have from Lemma 3.3 that

$$\begin{aligned}
&(x(1-x))^{k_2}\rho^{\beta_2}(x,t)\\
&\le C\int_0^1 (x(1-x))^{k_2}\rho^{\beta_2}(x,t)dx + C\int_0^1 \left|\left((x(1-x))^{k_2}\rho^{\beta_2}\right)_x\right| dx\\
&\le C\int_0^1 (x(1-x))^{k_2-1}\rho^{\beta_2}(x,t)dx + C\int_0^1 (x(1-x))^{k_2}\rho^{\beta_2-1}|\rho_x| dx\\
&\le C\int_0^1 (x(1-x))^{k_1}\rho^{\beta_1}(x,t)dx + C\int_0^1 (x(1-x))^{\left(\frac{k_1}{2}-\frac{1}{2}-\frac{k_1\beta_2}{\beta_1}\right)\frac{\beta_1}{\beta_1-\beta_2}} dx\\
&\quad + C\int_0^1 x(1-x)\rho^{2\theta-2}\rho_x^2 dx + C\int_0^1 (x(1-x))^{k_1}\rho^{2\beta_2-2\theta} dx\\
&\le C(T) + C\int_0^1 (x(1-x))^{\left(\frac{k_1}{2}-\frac{1}{2}-\frac{k_1\beta_2}{\beta_1}\right)\frac{\beta_1}{\beta_1-\beta_2}} dx.
\end{aligned} \tag{3.12}$$

When $0 < \theta < \frac{1}{2}$, we have for sufficiently large m

$$\left(\frac{k_1}{2} - \frac{1}{2} - \frac{k_1\beta_2}{\beta_1}\right)\frac{\beta_1}{\beta_1 - \beta_2} > -1.$$

Therefore (3.12) implies

$$(x(1-x))^{k_2}\rho^{\beta_2}(x,t) \le C(T). \tag{3.13}$$

Since $2\theta - 1 < \beta_2 < 0$, (3.13) implies

$$\rho(x,t) \ge C(T)(x(1-x))^{-\frac{k_2}{\beta_2}} \ge C(T)(x(1-x))^{\frac{k_2}{1-2\theta}}.$$

This completes the proof of Lemma 3.4.

Lemma 3.5. *Under the assumptions of Theorem 2.2, if* $0 < \theta < \frac{2}{9}$*, then there exists* k_1 *with* $1 < k_1 < \min\left\{\frac{5-15\theta}{1+3\theta},\ 1+(1-3\theta)\alpha,\ (2\gamma-3\theta+1)\alpha\right\}$ *such that for any positive integer n,*

$$\int_0^1 u_t^{2n}dx + n(2n-1)\int_0^t\int_0^1 \rho^{1+\theta}u_{xt}^2u_t^{2n-2}dxds \le C(T). \tag*{(3.14)$_n$}$$

The proof can be found in Appendix 5.2.

Lemma 3.6. *Under the conditions in Theorem 2.2, if* $1 < k_1 < \min\{\alpha(4-2\theta), 1+(1-3\theta)\alpha, (2\gamma-3\theta+1)\alpha\}$*, then we have that*

$$\int_0^1 |\rho_x(x,t)|dx \le C(T), \tag{3.15}$$

$$\left\|\rho^{1+\theta}(x,t)u_x(x,t)\right\|_{L^\infty([0,1]\times[0,T])} \le C(T), \tag{3.16}$$

$$\int_0^1 \left|(\rho^{1+\theta}u_x)_x(x,t)\right|dx \le C(T), \tag{3.17}$$

$$\int_0^1 |u_x(x,t)|dx \le C(T), \tag{3.18}$$

$$||u(x,t)||_{L^\infty([0,1]\times[0,T])} \le C(T), \tag{3.19}$$

and for $0 < s < t \le T$*,*

$$\int_0^1 |\rho(x,t)-\rho(x,s)|^2dx \le C(T)|t-s|, \tag{3.20}$$

$$\int_0^1 |u(x,t)-u(x,s)|^2dx \le C(T)|t-s|, \tag{3.21}$$

$$\int_0^1 \left|\left(\rho^{1+\theta}u_x\right)(x,t) - \left(\rho^{1+\theta}u_x\right)(x,s)\right|^2 dx \le C(T)|t-s|. \tag{3.22}$$

The proof can be found in Appendix 5.3.

The construction of a weak solution to the initial boundary value problem (2.1)–(2.3) with the corresponding estimates for the approximate solutions is given in Appendix 5.6. With the above *a priori* estimates, we can show that the subsequence of the sequence of approximate solutions converges to a weak solution defined by Definition 2.1.

3.2. The case of infinite support. The corresponding *a priori* estimates to infinite support are listed later. The proofs are given in Appendices 5.4 and 5.5. These estimates guarantee the convergence of approximate solutions to a weak solution defined by Definition 2.1.

Lemma 3.7. *Under the conditions of Theorem 2.4, if* $\frac{1}{3} < \theta < \frac{3}{7}$*, then for any positive integers m and n, we have*

$$\int_0^1 \rho^{2\theta-2}\rho_x^2 dx \leq C(T), \tag{3.23}$$

$$\int_0^1 \rho^{\alpha_1}u^2 dx + \int_0^t\int_0^1 \rho^{1+\theta+\alpha_1}u_x^2 dxds \leq C(T), \tag{3.24}$$

$$\int_0^1 (x(1-x))^{1-\frac{1}{2n}}\rho^{\beta_1}dx \leq C(T), \tag{3.25}$$

$$\rho(x,t) \geq C(T)(x(1-x))^{\frac{1}{2(1-2\theta)}}, \tag{3.26}$$

and

$$\int_0^1 u_t^{2n}dx + n(2n-1)\int_0^t\int_0^1 \rho^{1+\theta}u_{xt}^2u_t^{2n-2}dxds \leq C(T), \tag{3.27$_n$}$$

where $\alpha_1 = \left(1-\frac{1}{2^m}\right)(\theta-1)$, $\beta_1 = \left(2-\frac{1}{2^m}\right)(\theta-1)$.

Lemma 3.8. *Under the assumptions in Theorem 2.4, we get*

$$\int_0^1 |\rho_x(x,t)|dx \leq C(T), \tag{3.28}$$

$$\left\|\rho^{1+\theta}(x,t)u_x(x,t)\right\|_{L^\infty([0,1]\times[0,T])} \leq C(T), \tag{3.29}$$

$$\int_0^1 \left|(\rho^{1+\theta}u_x)_x(x,t)\right| dx \leq C(T), \tag{3.30}$$

and for $0 < s < t \leq T$*, we have*

$$\int_0^1 |\rho(x,t)-\rho(x,s)|^2dx \leq C(T)|t-s|, \tag{3.31}$$

$$\int_0^1 |u(x,t)-u(x,s)|^2dx \leq C(T)|t-s|, \tag{3.32}$$

$$\int_0^1 \left|\left(\rho^{1+\theta}u_x\right)(x,t)-\left(\rho^{1+\theta}u_x\right)(x,s)\right|^2 dx \leq C(T)|t-s|. \tag{3.33}$$

4. Non-global Existence of Regular Solutions

In this section, we will prove the non-global existence of regular solutions to the compressible Navier–Stokes equations (1.1) when the initial data $(\rho_0(\xi), u_0(\xi))$ have compact supports, i.e. Theorem 2.7. The proof is based on the non-growth of the support for the density function and the estimation on the nonlinear functional introduced in [22] for constant viscosity and heat conductivity coefficients.

Proof of Theorem 2.7. We first prove that the supports of any regular solution of (1.1) and (1.2) with compact initial data will not change in time. That is, we want to prove $\Omega(\tau) = \Omega(0)$, where $\Omega(\tau) = \text{supp}(\rho(\xi,\tau), u(\xi,\tau))$.

To do this, we let $w = \rho^{\frac{\theta-1}{2}}$ and rewrite (1.1) as

$$\begin{cases} w_\tau + u w_\xi + \dfrac{\theta-1}{2} w u_\xi = 0, \\ u_\tau + u u_\xi + \frac{2\gamma}{\theta-1} w^{\frac{2(\gamma-\theta)}{\theta-1}} w w_\xi = \frac{2\theta}{\theta-1} w w_\xi u_\xi + w^2 u_{\xi\xi}. \end{cases} \tag{4.1}$$

Let

$$M = \sup(|w| + |u| + |w_\xi| + |u_\xi| + |u_{\xi\xi}|). \tag{4.2}$$

Then

$$\left|\frac{\partial w}{\partial \tau}\right| + \left|\frac{\partial u}{\partial \tau}\right| \le CM(|w| + |u|),$$

which implies by Gronwall's inequality

$$|w(\xi,\tau)| + |u(\xi,\tau)| \le (|w_0(\xi)| + u_0(\xi)|)e^{CM\tau}. \tag{4.3}$$

Equation (4.3) immediately implies that $\Omega(\tau) \subset \Omega(0)$.

On the other hand, it is easy to see $\Omega(\tau) \supset \Omega(0)$. Therefore, $\Omega(\tau) = \Omega(0)$.

Now we introduce the following functional as in [22]:

$$\begin{aligned} H(\tau) &= \int_{\mathbf{R}} (\xi - (1+\tau)u(\xi,\tau))^2 \rho(\xi,\tau) d\xi + \frac{2}{\gamma-1}(1+\tau)^2 \int_{\mathbf{R}} \rho^\gamma(\xi,\tau) d\xi \\ &= \int_{\mathbf{R}} \xi^2 \rho(\xi,\tau) d\xi - 2(1+\tau) \int_{\mathbf{R}} \xi \rho(\xi,\tau) u(\xi,\tau) d\xi \\ &\quad + (1+\tau)^2 \int_{\mathbf{R}} \left(\rho(\xi,\tau) u^2(\xi,\tau) + \frac{2}{\gamma-1}\rho^\gamma(\xi,\tau)\right) d\xi. \end{aligned} \tag{4.4}$$

By using (1.1) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} H'(\tau) &= \frac{2(3-\gamma)}{\gamma-1}(1+\tau) \int_{\mathbf{R}} \rho^\gamma d\xi + 2(1+\tau) \int_{\mathbf{R}} \rho^\theta u_\xi d\xi - 2(1+\tau)^2 \int_{\mathbf{R}} \rho^\theta u_\xi^2 d\xi \\ &\le \frac{2(3-\gamma)}{\gamma-1}(1+\tau) \int_{\mathbf{R}} \rho^\gamma d\xi + \int_{\mathbf{R}} \rho^\theta d\xi - (1+\tau)^2 \int_{\mathbf{R}} \rho^\theta u_\xi^2 d\xi \\ &\le \frac{2(3-\gamma)}{\gamma-1}(1+\tau) \int_{\mathbf{R}} \rho^\gamma d\xi + \int_{\mathbf{R}} \rho^\theta d\xi. \end{aligned} \tag{4.5}$$

Case 1. When $\theta = \gamma$, we have from (4.4) and (4.5),

$$H'(\tau) \leq \frac{3-\gamma}{1+\tau} H(\tau) + \frac{\gamma - 1}{2(1+\tau)^2} H(\tau),$$

which implies

$$H(\tau) \leq H(0)(1+\tau)^{3-\gamma} e^{-\frac{\gamma-1}{2(1+\tau)}}. \tag{4.6}$$

Equations (4.4) and (4.6) show

$$\int_{\mathbf{R}} \rho^{\gamma}(\xi, \tau) d\xi \leq \frac{\gamma-1}{2} H(0)(1+\tau)^{1-\gamma} e^{-\frac{\gamma-1}{2(1+\tau)}}. \tag{4.7}$$

By conservation of mass and Hölder's inequality, we have

$$\begin{aligned} \int_{\Omega(0)} \rho_0(\xi) d\xi &= \int_{\Omega(\tau)} \rho(\xi, \tau) d\xi \\ &\leq \left(\int_{\Omega(\tau)} \rho^{\gamma}(\xi, \tau) d\xi \right)^{\frac{1}{\gamma}} (\Omega(\tau))^{\frac{\gamma-1}{\gamma}} \\ &\leq (\Omega(0))^{\frac{\gamma-1}{\gamma}} \left(\frac{\gamma-1}{2} H(0) \right)^{\frac{1}{\gamma}} (1+\tau)^{\frac{1-\gamma}{\gamma}} e^{-\frac{\gamma-1}{2\gamma(1+\tau)}}. \end{aligned} \tag{4.8}$$

Equation (4.8) implies that T must be finite.

Case 2. When $1 < \theta < \gamma$, we can rewrite (4.5) as follows by using Young's inequality:

$$H'(\tau) \leq \frac{2(3-\gamma)}{\gamma-1}(1+\tau) \int_{\mathbf{R}} \rho^{\gamma} d\xi + \frac{\theta}{\gamma} \int_{\mathbf{R}} \rho^{\gamma} d\xi + \frac{\gamma-\theta}{\gamma} \Omega(\tau). \tag{4.9}$$

Equations (4.4) and (4.9) show

$$H'(\tau) \leq \frac{3-\gamma}{1+\tau} H(\tau) + \frac{\theta(\gamma-1)}{2\gamma(1+\tau)^2} H(\tau) + \frac{\gamma-\theta}{\gamma} \Omega(0). \tag{4.10}$$

Solving the inequality (4.10), we have

$$H(\tau) \leq (1+\tau)^{3-\gamma} e^{-\frac{\theta(\gamma-1)}{2\gamma(1+\tau)}} \left\{ H(0) + \frac{\gamma-\theta}{\gamma} \Omega(0) \int_0^{\tau} (1+s)^{\gamma-3} e^{\frac{\theta(\gamma-1)}{2\gamma(1+s)}} ds \right\}. \tag{4.11}$$

When $\gamma \neq 2$, we have

$$\begin{aligned} H(\tau) \leq &\left(H(0) - \frac{\gamma-\theta}{\gamma(\gamma-2)} \Omega(0) e^{\frac{\theta(\gamma-1)}{2\gamma}} \right) (1+\tau)^{3-\gamma} e^{-\frac{\theta(\gamma-1)}{2\gamma(1+\tau)}} \\ &+ \frac{\gamma-\theta}{\gamma(\gamma-2)} \Omega(0) e^{\frac{\theta(\gamma-1)}{2\gamma}} (1+\tau) e^{-\frac{\theta(\gamma-1)}{2\gamma(1+\tau)}}. \end{aligned} \tag{4.12}$$

When $\gamma = 2$, we have

$$H(\tau) \leq H(0)(1+\tau)^{3-\gamma} e^{-\frac{\theta(\gamma-1)}{2\gamma(1+\tau)}} + \frac{\gamma-\theta}{\gamma} \Omega(0) e^{\frac{\theta(\gamma-1)}{2\gamma}} (1+\tau)^{3-\gamma} e^{-\frac{\theta(\gamma-1)}{2\gamma(1+\tau)}} \ln(1+\tau). \tag{4.13}$$

Similar to the estimates (4.6)–(4.8), (4.12) and (4.13) also imply that T must be finite.

This completes the proof of Theorem 2.7.

Remark 4.1. Even though the non-global existence Theorem 2.7 is for one dimensional Navier–Stokes equations, it is straightforward to generalize it to the case in multi-dimensions, cf. [22].

5. Appendices

5.1. Proof of Lemma 3.3. First we show that (3.7) holds. From (2.1), we have

$$\left(\rho^{\theta}\right)_{xt} = -\theta\left(u_t + (\rho^{\gamma})_x\right). \tag{5.1}$$

Multiplying (5.1) by $x(1-x)(\rho^{\theta})_x$ and integrating it over $[0,1]\times[0,t]$, we have

$$\begin{aligned}
&\frac{\theta^2}{2}\int_0^1 x(1-x)\rho^{2\theta-2}\rho_x^2dx\\
&= \frac{\theta^2}{2}\int_0^1 x(1-x)\rho_0^{2\theta-2}\left((\rho_0)_x\right)^2dx - \theta\int_0^t\int_0^1 x(1-x)u_t\left(\rho^{\theta}\right)_x dxds\\
&\quad -\theta^2\gamma\int_0^t\int_0^1 x(1-x)\rho^{\gamma+\theta-2}\rho_x^2dxds\\
&\le C - \theta\int_0^1 x(1-x)u\left(\rho^{\theta}\right)_x dx + \theta\int_0^1 x(1-x)u_0\left(\rho_0^{\theta}\right)_x dx\\
&\quad +\theta\int_0^t\int_0^1 x(1-x)u\left(\rho^{\theta}\right)_{xt}dxds - \theta^2\gamma\int_0^t\int_0^1 x(1-x)\rho^{\gamma+\theta-2}\rho_x^2dxds.
\end{aligned} \tag{5.2}$$

By using the Cauchy-Schwartz inequality, (5.1) and (5.2) implies

$$\begin{aligned}
&\frac{\theta^2}{2}\int_0^1 x(1-x)\rho^{2\theta-2}\rho_x^2dx + \theta^2\gamma\int_0^t\int_0^1 x(1-x)\rho^{\gamma+\theta-2}\rho_x^2dxds\\
&\le C + \frac{\theta^2}{4}\int_0^1 x(1-x)\rho^{2\theta-2}\rho_x^2dx + \theta^2\int_0^1 x(1-x)u^2dx\\
&\quad -\theta^2\int_0^t\int_0^1 x(1-x)uu_tdxds - \theta^2\gamma\int_0^t\int_0^1 x(1-x)\rho^{\gamma-1}u\rho_xdxds\\
&\le C + \frac{\theta^2}{4}\int_0^1 x(1-x)\rho^{2\theta-2}\rho_x^2dx + \theta^2\int_0^1 x(1-x)u^2dx\\
&\quad -\frac{\theta^2}{2}\int_0^1 x(1-x)u^2dx + \frac{\theta^2}{2}\int_0^1 x(1-x)u_0^2dx\\
&\quad +\frac{\theta^2\gamma}{2}\int_0^t\int_0^1 x(1-x)\rho^{\gamma+\theta-2}\rho_x^2dxds\\
&\quad +\frac{\theta^2\gamma}{2}\int_0^t\max_{[0,1]}\rho^{\gamma-\theta}\int_0^1 x(1-x)u^2dxds.
\end{aligned} \tag{5.3}$$

Then Lemma 3.1, Lemma 3.2, and (5.3) yield

$$\frac{\theta^2}{4}\int_0^1 x(1-x)\rho^{2\theta-2}\rho_x^2dx + \frac{1}{2}\theta^2\gamma\int_0^t\int_0^1 x(1-x)\rho^{\gamma+\theta-2}\rho_x^2dxds \le C(T), \tag{5.4}$$

which implies (3.7).

Now we turn to prove (3.8). For any positive integer m, we have from (2.1),

$$\begin{aligned}
&\left((x(1-x))^{k_1}\rho^{\alpha_m}u^{2^m}\right)_t\\
&\quad=-\alpha_m(x(1-x))^{k_1}\rho^{1+\alpha_m}u^{2^m}u_x+2^m(x(1-x))^{k_1}\rho^{\alpha_m}u^{2^m-1}\left(\rho^{1+\theta}u_x\right)_x\\
&\qquad-2^m(x(1-x))^{k_1}\rho^{\alpha_m}u^{2^m-1}P(\rho)_x,
\end{aligned}\tag{5.5}$$

where α_m is a constant to be defined later.

By integrating (5.5) over $[0, 1] \times [0, t]$, we have

$$\begin{aligned}
&\int_0^1(x(1-x))^{k_1}\rho^{\alpha_m}u^{2^m}dx+2^m(2^m-1)\int_0^t\int_0^1(x(1-x))^{k_1}\rho^{1+\theta+\alpha_m}u^{2^m-2}u_x^2dxds\\
&\quad=\int_0^1(x(1-x))^{k_1}\rho_0^{\alpha_m}u_0^{2^m}dx-\alpha_m\int_0^t\int_0^1(x(1-x))^{k_1}\rho^{1+\alpha_m}u^{2^m}u_xdxds\\
&\qquad-2^m\alpha_m\int_0^t\int_0^1(x(1-x))^{k_1}\rho^{\theta+\alpha_m}u^{2^m-1}\rho_xu_xdxds\\
&\qquad-2^mk_1\int_0^t\int_0^1(x(1-x))^{k_1-1}\rho^{1+\theta+\alpha_m}u^{2^m-1}u_xdxds\\
&\qquad-2^m\gamma\int_0^t\int_0^1(x(1-x))^{k_1}\rho^{\gamma+\alpha_m-1}u^{2^m-1}\rho_xdxds\\
&\quad=\sum_{i=1}^5 I_i^m.
\end{aligned}\tag{5.6}$$

Now we estimate $I_i^m,\ i = 1, 2, \ldots, 5$ as follows:

$$\begin{aligned}
I_3^m&=-2^m\alpha_m\int_0^t\int_0^1(x(1-x))^{k_1}\rho^{\theta+\alpha_m}u^{2^m-1}\rho_xu_xdxds\\
&\le C\int_0^t\int_0^1\rho^{1+\theta}u^{2^{m+1}-2}u_x^2dxds+C\int_0^t\int_0^1(x(1-x))^{2k_1}\rho^{\theta+2\alpha_m-1}\rho_x^2dxds.
\end{aligned}\tag{5.7}$$

Let

$$\theta+2\alpha_m-1=2\theta-2,$$

i.e.

$$\alpha_m=\frac{\theta-1}{2}.\tag{5.8}$$

Then by Lemma 3.2 and (3.7) and noticing that $k_1 > 1$, we have

$$I_3^m\le C(T).\tag{5.9}$$

Furthermore, since $u_0 \in L^\infty([0, 1])$ and $(x(1-x))^{k_1}\rho_0^{2\theta-2} \in L^1([0, 1])$, we have

$$I_1^m=\int_0^1(x(1-x))^{k_1}\rho_0^{\alpha_m}u_0^{2^m}dx\le C,\tag{5.10}$$

and

$$
\begin{aligned}
I_2^m &= -\alpha_m \int_0^t \int_0^1 (x(1-x))^{k_1} \rho^{1+\alpha_m} u^{2^m} u_x dx ds \\
&\le C \int_0^t \int_0^1 (x(1-x))^{2k_1} u^{2^{m+1}} dx ds + C \int_0^t \max_{[0,1]} \rho^{1-\theta+\alpha_m} \int_0^1 \rho^{1+\theta} u_x^2 dx ds \\
&\le C(T), \qquad (5.11)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4^m &= -2^m k_1 \int_0^t \int_0^1 (x(1-x))^{k_1-1} \rho^{1+\theta+\alpha_m} u^{2^m-1} u_x dx ds \\
&\le C \int_0^t \int_0^1 \rho^{1+\theta} u^{2^{m+1}-2} u_x^2 dx ds + C \int_0^t \max_{[0,1]} \rho^{1+\theta+2\alpha_m} \int_0^1 (x(1-x))^{2k_1-2} dx ds \\
&\le C(T). \qquad (5.12)
\end{aligned}
$$

Noticing that $2\gamma + 2\alpha_m - 2 \ge 2\theta - 2$, we have

$$
\begin{aligned}
I_5^m &= -2^m \gamma \int_0^t \int_0^1 (x(1-x))^{k_1} \rho^{\gamma+\alpha_m-1} u^{2^m-1} \rho_x dx ds \\
&\le C \int_0^t \int_0^1 u^{2^{m+1}-2} dx ds + C \int_0^t \int_0^1 (x(1-x))^{2k_1} \rho^{2\gamma+2\alpha_m-2} \rho_x^2 dx ds \\
&\le C(T). \qquad (5.13)
\end{aligned}
$$

Now from (5.6) and (5.9)–(5.13), we have

$$
\begin{aligned}
&\int_0^1 (x(1-x))^{k_1} \rho^{\alpha_m} u^{2^m} dx + 2^m(2^m-1) \int_0^t \int_0^1 (x(1-x))^{k_1} \rho^{1+\theta+\alpha_m} u^{2^m-2} u_x^2 dx ds \\
&\quad \le C(T), \qquad (5.14)
\end{aligned}
$$

where α_m is given in (5.8).

By applying (5.6) again, we have

$$
\begin{aligned}
&\int_0^1 (x(1-x))^{k_1}\rho^{\alpha_{m-1}}u^{2^{m-1}}dx \\
&\quad + 2^{m-1}(2^{m-1}-1)\int_0^t\int_0^1 (x(1-x))^{k_1}\rho^{1+\theta+\alpha_{m-1}}u^{2^{m-1}-2}u_x^2 dxds \\
&= \int_0^1 (x(1-x))^{k_1}\rho_0^{\alpha_{m-1}}u_0^{2^{m-1}}dx - \alpha_{m-1}\int_0^t\int_0^1 (x(1-x))^{k_1}\rho^{1+\alpha_{m-1}}u^{2^{m-1}}u_x dxds \\
&\quad - 2^{m-1}\alpha_{m-1}\int_0^t\int_0^1 (x(1-x))^{k_1}\rho^{\theta+\alpha_{m-1}}u^{2^{m-1}-1}\rho_x u_x dxds \\
&\quad - 2^{m-1}k_1\int_0^t\int_0^1 (x(1-x))^{k_1-1}\rho^{1+\theta+\alpha_{m-1}}u^{2^{m-1}-1}u_x dxds \\
&\quad - 2^{m-1}\gamma\int_0^t\int_0^1 (x(1-x))^{k_1}\rho^{\gamma+\alpha_{m-1}-1}u^{2^{m-1}-1}\rho_x dxds \\
&= \sum_{i=1}^5 I_i^{m-1}. \qquad (5.15)
\end{aligned}
$$

Similar to the estimates of I_i^m, we can estimate for I_i^{m-1} as follows:

$$
\begin{aligned}
I_3^{m-1} &= -2^{m-1}\alpha_{m-1}\int_0^t\int_0^1 (x(1-x))^{k_1}\rho^{\theta+\alpha_{m-1}}u^{2^{m-1}-1}\rho_x u_x dxds \\
&\le C\int_0^t\int_0^1 (x(1-x))^{k_1}\rho^{1+\theta+\alpha_m}u^{2^m-2}u_x^2 dxds \\
&\quad + C\int_0^t\int_0^1 (x(1-x))^{k_1}\rho^{\theta-1+2\alpha_{m-1}-\alpha_m}\rho_x^2 dxds. \qquad (5.16)
\end{aligned}
$$

Let

$$\theta - 1 + 2\alpha_{m-1} - \alpha_m = 2\theta - 2,$$

i.e.,

$$\alpha_{m-1} = \frac{\alpha_m}{2} + \frac{\theta-1}{2}. \qquad (5.17)$$

Then we have from (5.14), (5.16) and Lemma 3.2, (3.7) and using $k_1 > 1$ that

$$I_3^{m-1} \le C(T). \qquad (5.18)$$

Similarly, for α_{m-1} given in (5.17), we have

$$I_i^{m-1} \le C(T), \quad i = 1, 2, 5. \qquad (5.19)$$

We can estimate I_4^{m-1} as follows. Since $0 < \theta < 1$ and $k_1 > 1$, we have from Lemma 3.2 and (5.14),

$$\begin{aligned} I_4^{m-1} &= -2^{m-1}k_1 \int_0^t \int_0^1 (x(1-x))^{k_1-1} \rho^{1+\theta+\alpha_{m-1}} u^{2^{m-1}-1} u_x dxds \\ &\leq C \int_0^t \int_0^1 (x(1-x))^{k_1} \rho^{1+\theta+\alpha_m} u^{2^m-2} u_x^2 dxds \\ &\quad + C \int_0^t \max_{[0,1]} \rho^{1+\theta+2\alpha_{m-1}-\alpha_m} \int_0^1 (x(1-x))^{k_1-2} dxds \\ &\leq C(T). \end{aligned} \tag{5.20}$$

Thus, (5.15) and (5.18)–(5.20) give

$$\begin{aligned} &\int_0^1 (x(1-x))^{k_1} \rho^{\alpha_{m-1}} u^{2^{m-1}} dx \\ &\quad + 2^{m-1}(2^{m-1}-1) \int_0^t \int_0^1 (x(1-x))^{k_1} \rho^{1+\theta+\alpha_{m-1}} u^{2^{m-1}-2} u_x^2 dxds \\ &\quad \leq C(T). \end{aligned} \tag{5.21}$$

By solving the recurrence relation (5.17), we have

$$\int_0^1 (x(1-x))^{k_1} \rho^{\alpha_1} u^2 dx + \int_0^t \int_0^1 (x(1-x))^{k_1} \rho^{1+\theta+\alpha_1} u_x^2 dxds \leq C(T), \tag{5.22}$$

where

$$\alpha_1 = \left(1 - \frac{1}{2^m}\right)(\theta - 1). \tag{5.23}$$

This completes the proof of (3.8).

Finally, we prove (3.9). From the first equation of (2.1), we have

$$\left((x(1-x))^{k_1} \rho^{\beta_1}\right)_t = -\beta_1 (x(1-x))^{k_1} \rho^{1+\beta_1} u_x. \tag{5.24}$$

By integrating (5.24) over $[0, 1] \times [0, t]$ and applying the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \int_0^1 (x(1-x))^{k_1} \rho^{\beta_1} dx \leq& \int_0^1 (x(1-x))^{k_1} \rho_0^{\beta_1} dx \\ &+C \int_0^t \int_0^1 (x(1-x))^{k_1} \rho^{1+\theta+\alpha_1} u_x^2 dxds \\ &+C \int_0^t \int_0^1 (x(1-x))^{k_1} \rho^{1+2\beta_1-\theta-\alpha_1} dxds. \end{aligned} \tag{5.25}$$

By noticing that

$$\int_0^1 (x(1-x))^{k_1} \rho_0^{\beta_1} dx \leq C \int_0^1 (x(1-x))^{k_1} \rho_0^{2(\theta-1)} dx \leq C$$

and

$$1 + 2\beta_1 - \theta - \alpha_1 = \beta_1,$$

we have from (5.25) and (3.8)

$$\int_0^1 (x(1-x))^{k_1} \rho^{\beta_1} dx \le C(T) + C \int_0^t \int_0^1 (x(1-x))^{k_1} \rho^{\beta_1} dx ds. \tag{5.26}$$

Equation (5.26) implies (3.9) by Gronwall's inequality and the proof of Lemma 3.3 is completed.

5.2. Proof of Lemma 3.5. By differentiating $(2.1)_2$ with respect to the time t and then integrating it after multiplying $2nu_t^{2n-1}$ with respect to x and t over $[0, 1] \times [0, t]$, we deduce

$$\begin{aligned} &\int_0^1 u_t^{2n} dx + 2n \int_0^t \int_0^1 (\rho^\gamma)_{xt} u_t^{2n-1} dx ds \\ &= \int_0^1 u_{0t}^{2n} dx + 2n \int_0^t \int_0^1 \left(\rho^{1+\theta} u_x\right)_{xt} u_t^{2n-1} dx ds. \end{aligned} \tag{5.27}$$

Since

$$u_{0t} = \left(\rho_0^{1+\theta} u_{0x}\right)_x - \left(\rho_0^\gamma\right)_x,$$

we have from Assumptions (A1) and (A2) that

$$\int_0^1 u_{0t}^{2n}(x) dx \le C. \tag{5.28}$$

On the other hand, using integration by parts, we have from $(2.1)_1$,

$$\begin{aligned} &2n \int_0^t \int_0^1 \left(\rho^{1+\theta} u_x\right)_{xt} u_t^{2n-1} dx ds \\ &= 2n \int_0^t \int_0^1 \left\{\left(\rho^{1+\theta} u_x\right)_t u_t^{2n-1}\right\}_x dx ds \\ &\quad -2n \int_0^t \int_0^1 \left(\rho^{1+\theta} u_x\right)_t \left((u_t)^{2n-1}\right)_x dx ds \\ &= -2n(2n-1) \int_0^t \int_0^1 \rho^{1+\theta} u_t^{2n-2} u_{xt}^2 dx ds \\ &\quad +2n(2n-1)(1+\theta) \int_0^t \int_0^1 \rho^{2+\theta} u_x^2 u_t^{2n-2} u_{xt} dx ds. \end{aligned} \tag{5.29}$$

As to the second term in the left-hand side of (5.27), we can get similarly

$$\begin{aligned} &2n \int_0^t \int_0^1 \left(\rho^\gamma\right)_{xt} u_t^{2n-1} dx ds \\ &= 2n \int_0^t \int_0^1 \left\{\left(\rho^\gamma\right)_t u_t^{2n-1}\right\}_x dx ds - 2n(2n-1) \int_0^t \int_0^1 \left(\rho^\gamma\right)_t u_t^{2n-2} u_{xt} dx ds \\ &= 2n(2n-1)\gamma \int_0^t \int_0^1 \rho^{1+\gamma} u_x u_{xt} u_t^{2n-2} dx ds. \end{aligned} \tag{5.30}$$

Here in (5.29) and (5.30), we have used the boundary condition (2.2) and Eqs. (2.1).

Substituting (5.28)–(5.30) into (5.27), we have

$$\int_0^1 u_t^{2n}dx + 2n(2n-1)\int_0^t\int_0^1 \rho^{1+\theta}u_t^{2n-2}u_{xt}^2 dxds$$
$$\le C + 2n(2n-1)(1+\theta)\int_0^t\int_0^1 \rho^{2+\theta}u_x^2u_t^{2n-2}u_{xt}dxds$$
$$-2n(2n-1)\gamma\int_0^t\int_0^1 \rho^{1+\gamma}u_xu_{xt}u_t^{2n-2}dxds.$$

Furthermore, we have

$$2n(2n-1)(1+\theta)\int_0^t\int_0^1 \rho^{2+\theta}u_x^2u_t^{2n-2}u_{xt}dxds$$
$$\le \tfrac{1}{2}n(2n-1)\int_0^t\int_0^1 \rho^{1+\theta}u_t^{2n-2}u_{xt}^2dxds$$
$$+2n(2n-1)(1+\theta)^2\int_0^t\int_0^1 \rho^{3+\theta}u_x^4u_t^{2n-2}dxds,$$

and

$$-2n(2n-1)\gamma\int_0^t\int_0^1 \rho^{1+\gamma}u_xu_{xt}u_t^{2n-2}dxds$$
$$\le \tfrac{1}{2}n(2n-1)\int_0^t\int_0^1 \rho^{1+\theta}u_t^{2n-2}u_{xt}^2dxds$$
$$+2n(2n-1)\gamma^2\int_0^t\int_0^1 \rho^{2\gamma+1-\theta}u_x^2u_t^{2n-2}dxds.$$

Therefore

$$\int_0^1 u_t^{2n}dx + n(2n-1)\int_0^t\int_0^1 \rho^{1+\theta}u_t^{2n-2}u_{xt}^2dxds$$
$$\le 2n(2n-1)(1+\theta)^2\int_0^t\int_0^1 \rho^{3+\theta}u_x^4u_t^{2n-2}dxds$$
$$+2n(2n-1)\gamma^2\int_0^t\int_0^1 \rho^{2\gamma+1-\theta}u_x^2u_t^{2n-2}dxds$$
$$= 2n(2n-1)(1+\theta)^2I_n^{(1)} + 2n(2n-1)\gamma^2I_n^{(2)}. \tag{5.31}$$

Now we will give the proof of $(5.14)_n$. First we consider the case of $n=1$. To do this, we need to estimate $I_1^{(1)}$ and $I_1^{(2)}$.

In fact, by Hölder's inequality, we have

$$I_1^{(1)} = \int_0^t\int_0^1 \rho^{3+\theta}u_x^4dxds \le \max_{[0,1]}\left((x(1-x))^{-k_1}\rho^{2-\alpha_1}u_x^2\right)\int_0^t V(s)ds, \tag{5.32}$$

where

$$V(s) = \int_0^1 (x(1-x))^{k_1}\rho^{1+\theta+\alpha_1}u_x^2(x,s)dx.$$

On the other hand, from Lemma 3.1, Lemma 3.2 and noticing that $-\alpha_1 - 2\theta > 0$ when $0 < \theta < \frac{1}{3}$ and sufficiently large m, we have

$$\begin{aligned}(x(1-x))^{-k_1}\rho^{2-\alpha_1}u_x^2 &= (x(1-x))^{-k_1}\rho^{-\alpha_1-2\theta}\left(\rho^{1+\theta}u_x\right)^2\\ &= (x(1-x))^{-k_1}\rho^{-\alpha_1-2\theta}\left(\int_0^x u_t(y,t)dy+\rho^\gamma\right)^2\\ &\leq C(x(1-x))^{-k_1}\rho^{-\alpha_1-2\theta}x(1-x)\int_0^1 u_t^2dx\\ &\quad + C(x(1-x))^{-k_1}\rho^{2\gamma-\alpha_1-2\theta}\\ &\leq C(T)(x(1-x))^{1-k_1-\alpha(\alpha_1+2\theta)}\int_0^1 u_t^2dx\\ &\quad + C(T)(x(1-x))^{\alpha(2\gamma-\alpha_1-2\theta)-k_1}. \end{aligned} \tag{5.33}$$

When $0 < \theta < \frac{1}{3}$ and $1 < k_1 < \min\left\{\frac{5-15\theta}{1+3\theta},\ \frac{3-5\theta}{1+\theta},\ 1+(1-3\theta)\alpha,\ (2\gamma-3\theta+1)\alpha\right\}$, for sufficiently large m, we have

$$\begin{cases} 1-k_1-\alpha(\alpha_1+2\theta) \geq 0,\\ \alpha(2\gamma-\alpha_1-2\theta)-k_1 \geq 0, \end{cases}$$

which implies

$$\max_{[0,1]}\left((x(1-x))^{-k_1}\rho^{2-\alpha_1}u_x^2\right) \leq C(T)\int_0^1 u_t^2dx + C(T).$$

Therefore,

$$I_1^{(1)} \leq C(T)\int_0^t V(s)\int_0^1 u_t^2dxds + C(T)\int_0^t V(s)ds. \tag{5.34}$$

Similarly, we have

$$\begin{aligned} I_1^{(2)} &= \int_0^t\int_0^1 \rho^{2\gamma+1-\theta}u_x^2dxds\\ &\leq C\int_0^t \max_{[0,1]}\left((x(1-x))^{-k_1}\rho^{2\gamma-2\theta-\alpha_1}\right)V(s)ds\\ &\leq C(T)\int_0^t \max_{[0,1]}(x(1-x))^{\alpha(2\gamma-\alpha_1-2\theta)-k_1}V(s)ds\\ &\leq C(T)\int_0^t V(s)ds. \end{aligned} \tag{5.35}$$

From (5.31), (5.34) and (5.35) and Lemma 3.3, we have

$$\int_0^1 u_t^2dx + \int_0^t\int_0^1 \rho^{1+\theta}u_{xt}^2dxds \leq C(T)\left(1+\int_0^t V(s)\int_0^1 u_t^2dxds\right). \tag{5.36}$$

Gronwall's inequality and Lemma 3.3 give

$$\int_0^1 u_t^2 dx \le C(T) \exp\left(C(T) \int_0^t V(s) ds \right) \le C(T). \tag{5.37}$$

Combining (5.36) with (5.37), we can get $(3.14)_1$ immediately.

Now we consider the case of $n > 1$. Assume Lemma 3.5 holds for $n-1$, i.e.,

$$\int_0^1 u_t^{2n-2} dx + (n-1)(2n-3) \int_0^t \int_0^1 \rho^{1+\theta} u_{xt}^2 u_t^{2n-4} dx ds \le C(T). \tag{5.38}$$

Now we need to prove Lemma 3.5 holds also for n, i.e., $(3.14)_n$ is true. To do this, we first estimate $I_n^{(1)}$ and $I_n^{(2)}$ as follows: By the assumption (5.38), we have

$$I_n^{(1)} = \int_0^t \int_0^1 \rho^{3+\theta} u_x^4 u_t^{2n-2} dx ds \le C(T) \int_0^t \max_{[0,1]} \left(\rho^{3+\theta} u_x^4 \right) ds.$$

On the other hand, from Lemma 3.1 and Lemma 3.4, we have

$$\begin{aligned} \rho^{3+\theta} u_x^4 &= \rho^{-1-3\theta} \left(\rho^{1+\theta} u_x \right)^4 \\ &= \rho^{-1-3\theta} \left(\int_0^x u_t(y,t) dy + \rho^\gamma \right)^4 \\ &\le C(x(1-x))^{\frac{4n-2}{n}} \rho^{-1-3\theta} \left(\int_0^1 u_t^{2n} dx \right)^{\frac{4}{2n}} + C\rho^{4\gamma-1-3\theta} \\ &\le C(T)(x(1-x))^{4-\frac{2}{n}} \rho^{-1-3\theta} \left(\int_0^1 u_t^{2n} dx \right)^{\frac{2}{n}} + C(T). \end{aligned}$$

When $0 < \theta < \frac{2}{9}$ and $1 < k_1 < \min\left\{ \frac{5-15\theta}{1+3\theta},\ \frac{3-5\theta}{1+\theta},\ 1+(1-3\theta)\alpha,\ (2\gamma-3\theta+1)\alpha \right\}$, for sufficiently large m and any $n > 1$, we have

$$4 - \frac{2}{n} - \frac{k_2(1+3\theta)}{1-2\theta} \ge 0,$$

which implies

$$\max_{[0,1]} (x(1-x))^{4-\frac{2}{n}} \rho^{-1-3\theta} \le C(T)(x(1-x))^{4-\frac{2}{n}-\frac{k_2(1+3\theta)}{1-2\theta}} \le C(T).$$

Therefore,

$$I_n^{(1)} \le C(T) \left(1 + \int_0^t \left(\int_0^1 u_t^{2n} dx \right)^{\frac{2}{n}} ds \right).$$

By Young's inequality, we have for $n > 1$,

$$\left(\int_0^1 u_t^{2n} dx \right)^{\frac{2}{n}} \le \frac{2}{n} \int_0^1 u_t^{2n} dx + \frac{n-2}{n},$$

which implies

$$I_n^{(1)} \le C(T)\left(1+\int_0^t\int_0^1 u_t^{2n}(x,s)dxds\right). \tag{5.39}$$

Similarly, we have

$$\begin{aligned} I_n^{(2)} &= \int_0^t\int_0^1 \rho^{2\gamma+1-\theta}u_x^2u_t^{2n-2}dxds \\ &\le C\int_0^t \max_{[0,1]}\left(\rho^{2\gamma+1-\theta}u_x^2\right)\left(\int_0^1 u_t^{2n}dx\right)^{\frac{n-1}{n}} \\ &\le C(T)\left(1+\int_0^t\int_0^1 u_t^{2n}dxds\right). \end{aligned} \tag{5.40}$$

From (5.31), (5.39) and (5.40), we have

$$\begin{aligned} &\int_0^1 u_t^{2n}dx + n(2n-1)\int_0^t\int_0^1 \rho^{1+\theta}u_t^{2n-2}u_{xt}^2dxds \\ &\quad\le C(T)\left(1+\int_0^t\int_0^1 u_t^{2n}dxds\right). \end{aligned} \tag{5.41}$$

Gronwall's inequality gives

$$\int_0^1 u_t^{2n}dx \le C(T).$$

This and (5.41) show $(3.14)_n$. This completes the proof of Lemma 3.5.

5.3. Proof of Lemma 3.6. Since

$$\begin{cases} \left(\rho^{1+\theta}u_x\right)(x,t) = \int_0^x u_t(y,t)dy + \rho^\gamma(x,t), \\ \left(\rho^{1+\theta}u_x\right)_x(x,t) = u_t(x,t) + (\rho^\gamma)_x(x,t), \end{cases} \tag{5.42}$$

(3.16) and (3.17) follow from Lemma 3.2, Lemma 3.3 and Lemma 3.5.

On the other hand, from Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} \int_0^1 |\rho_x(x,t)|dx &\le \int_0^1 x(1-x)\rho^{2\theta-2}\rho_x^2dx + \int_0^1 (x(1-x))^{-1}\rho^{2-2\theta}dx \\ &\le C(T) + C(T)\int_0^1 (x(1-x))^{-1+\alpha(2-2\theta)}dx \\ &\le C(T). \end{aligned} \tag{5.43}$$

This gives (3.15).

In addition, from (3.2), we have

$$u_x(x,t) = \rho^{\gamma-1-\theta}(x,t) + \rho^{-1-\theta}\int_0^x u_t(y,t)dy. \tag{5.44}$$

From Lemma 3.5 and by using Hölder's inequality, we have

$$\begin{aligned}\int_0^1 |u_x(x,t)|dx &\le \int_0^1 \rho^{\gamma-1-\theta}(x,t)dx + \int_0^1 \rho^{-1-\theta}(x,t)\int_0^x |u_t(y,t)|dydx\\ &\le \int_0^1 \rho^{\gamma-1-\theta}(x,t)dx\\ &\quad + \int_0^1 \rho^{-1-\theta}(x(1-x))^{\frac{2n-1}{2n}}dx\left(\int_0^1 u_t^{2n}(x,t)dx\right)^{\frac{1}{2n}}\\ &\le \int_0^1 \rho^{\gamma-1-\theta}(x,t)dx + C(T)\int_0^1 (x(1-x))^{\frac{2n-1}{2n}}\rho^{-1-\theta}dx.\end{aligned}$$

Case 1. If $\gamma-1-\theta<0$, then $\frac{\beta_1}{\gamma-1-\theta}>1$. By Young's inequality, we have

$$\begin{aligned}\int_0^1 \rho^{\gamma-1-\theta}(x,t)dx &= \int_0^1 (x(1-x))^{\frac{k_1(\gamma-1-\theta)}{\beta_1}}\rho^{\gamma-1-\theta}(x,t)(x(1-x))^{\frac{k_1(1+\theta-\gamma)}{\beta_1}}dx\\ &\le C\int_0^1 (x(1-x))^{k_1}\rho^{\beta_1}dx + C\int_0^1 (x(1-x))^{\frac{k_1(1+\theta-\gamma)}{\beta_1+1+\theta-\gamma}}dx.\end{aligned}$$

Similarly, noticing that $-\frac{\beta_1}{1+\theta}>1$, we have

$$\begin{aligned}\int_0^1 (x(1-x))^{\frac{2n-1}{2n}}\rho^{-1-\theta}dx &= \int_0^1 (x(1-x))^{-\frac{k_1(1+\theta)}{\beta_1}}\rho^{-1-\theta}(x(1-x))^{\frac{2n-1}{2n}+\frac{k_1(1+\theta)}{\beta_1}}dx\\ &\le C\int_0^1 (x(1-x))^{k_1}\rho^{\beta_1}dx\\ &\quad + \int_0^1 (x(1-x))^{\left[\frac{2n-1}{2n}+\frac{k_1(1+\theta)}{\beta_1}\right]\frac{\beta_1}{\beta_1+1+\theta}}dx.\end{aligned}$$

When $0<\theta<\frac{2}{9}$, $1<k_1<\min\left\{\frac{1+\gamma-3\theta}{1+\theta-\gamma},\ \frac{3-5\theta}{1+\theta}\right\}=\frac{3-5\theta}{1+\theta}$, it is easy to see that for sufficiently large n,

$$\frac{k_1(1+\theta-\gamma)}{\beta_1+1+\theta-\gamma}>-1,$$

and

$$\left[\frac{2n-1}{2n}+\frac{k_1(1+\theta)}{\beta_1}\right]\frac{\beta_1}{\beta_1+1+\theta}>-1.$$

Therefore we have in this case

$$\int_0^1 |u_x(x,t)|dx \le C(T). \tag{5.45}$$

Case 2. If $\gamma-1-\theta\ge 0$, we can also obtain (5.45). This proves (3.18).

On the other hand, by Sobolev's embedding theorem $W^{1,1}([0,1])\hookrightarrow L^\infty([0,1])$ and Young's inequality, we have from (5.45) and Lemma 3.1,

$$||u(x,t)||_{L^\infty([0,1]\times[0,T])}\le C(T),$$

which implies (3.19).

Now we prove (3.20). To do so, from $(2.1)_1$ and Hölder's inequality, we deduce by using Lemma 3.2,

$$\begin{aligned}\int_0^1 |\rho(x,t)-\rho(x,s)|^2 dx &= \int_0^1 \left|\int_s^t \rho_t(x,\eta)d\eta\right|^2 dx \\ &= \int_0^1 \left|\int_s^t (\rho^2 u_x)(x,\eta)d\eta\right|^2 dx \\ &\le |t-s| \int_s^t \int_0^1 \left(\rho^4 u_x^2\right)(x,\eta)dxd\eta \\ &\le |t-s| \int_s^t \max_{[0,1]} \left((x(1-x))^{-k_1}\rho^{3-\theta-\alpha_1}\right) V(\eta)d\eta \\ &\le C(T)|t-s| \int_0^t \max_{[0,1]} (x(1-x))^{\alpha(3-\theta-\alpha_1)-k_1} V(\eta)d\eta. \end{aligned} \tag{5.46}$$

Therefore, for sufficiently large n, when $k_1 < \alpha(4-2\theta)$, we have

$$\alpha(3-\theta-\alpha_1)-k_1 \ge 0,$$

which implies (3.20).

Since

$$\begin{aligned}\int_0^1 |u(x,t)-u(x,s)|^2 dx &= \int_0^1 \left|\int_s^t u_t(x,\eta)d\eta\right|^2 dx \\ &\le |t-s| \int_s^t \int_0^1 u_t^2(x,\eta)dxd\eta \\ &\le C(T)|t-s|, \end{aligned} \tag{5.47}$$

(3.21) follows.

Finally, we prove (3.22). For this, we first obtain from Hölder's inequality that

$$\begin{aligned}&\int_0^1 \left|\left(\rho^{1+\theta}u_x\right)(x,t)-\left(\rho^{1+\theta}u_x\right)(x,s)\right|^2 dx \\ &= \int_0^1 \left|\int_s^t \left(\rho^{1+\theta}u_x\right)_t (x,\eta)d\eta\right|^2 dx \\ &\le |t-s| \int_s^t \int_0^1 \left[\left(\rho^{1+\theta}u_x\right)_t (x,\eta)\right]^2 dxd\eta. \end{aligned} \tag{5.48}$$

On the other hand, from (2.1) and (3.2), we get

$$\begin{aligned}\left(\rho^{1+\theta}u_x\right)_t (x,t) &= \left(\rho^{1+\theta}u_{xt}\right)(x,t) + (1+\theta)\left(\rho^{\theta}\rho_t u_x\right)(x,t) \\ &= \left(\rho^{1+\theta}u_{xt}\right)(x,t) - (1+\theta)\left(\rho^{2+\theta}u_x^2\right)(x,t). \end{aligned} \tag{5.49}$$

From $(3.14)_n$ with $n = 1$, we have

$$\begin{aligned}
&\int_s^t \int_0^1 \left[\left(\rho^{1+\theta} u_x\right)_t (x, \eta)\right]^2 dx d\eta \\
&\le \int_0^t \int_0^1 \rho^{2+2\theta} u_{xt}^2 dx ds + C \int_0^t \int_0^1 \rho^{4+2\theta} u_x^4 dx ds \\
&\le C(T) + C \int_0^t \left\{ V(s) \max_{[0,1]} \left((x(1-x))^{-k_1} \rho^{3+\theta-\alpha_1} u_x^2 \right) \right\} ds. \qquad (5.50)
\end{aligned}$$

Similar to (5.33), we have for $1 < k_1 < \min\{1 + (1 - 3\theta)\alpha, (2\gamma - 3\theta + 1)\alpha\}$,

$$\begin{aligned}
(x(1-x))^{-k_1} \rho^{3+\theta-\alpha_1} u_x^2 &= (x(1-x))^{-k_1} \rho^{1-\theta-\alpha_1} \left(\rho^{1+\theta} u_x\right)^2 \\
&= (x(1-x))^{-k_1} \rho^{1-\theta-\alpha_1} \left\{ \int_0^x u_t(y,t) dy + \rho^\gamma \right\}^2 \\
&\le \max_{[0,1]} \rho^{1+\theta} (x(1-x))^{-k_1} \rho^{-\alpha_1-2\theta} \left\{ \int_0^x u_t(y,t) dy + \rho^\gamma \right\}^2 \\
&\le C(T).
\end{aligned} \qquad (5.51)$$

Therefore

$$\int_s^t \int_0^1 \left[\left(\rho^{1+\theta} u_x\right)_t (x, \eta)\right]^2 dx d\eta \le C(T) \left(1 + \int_0^t V(s) ds\right) \le C(T). \qquad (5.52)$$

This and (5.48) imply (3.22) and then we complete the proof of Lemma 3.6.

5.4. Proof of Lemma 3.7. The proofs of (3.23) and (3.24) can be found in [24].

Now we prove (3.25). From the first equation of (2.1), we have

$$\left((x(1-x))^{1-\frac{1}{2n}} \rho^{\beta_1} \right)_t = -\beta_1 (x(1-x))^{1-\frac{1}{2n}} \rho^{1+\beta_1} u_x. \qquad (5.53)$$

Integrating (5.53) over $[0, 1] \times [0, t]$ and applying the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\int_0^1 (x(1-x))^{1-\frac{1}{2n}} \rho^{\beta_1} dx &\le \int_0^1 (x(1-x))^{1-\frac{1}{2n}} \rho_0^{\beta_1} dx + C \int_0^t \int_0^1 \rho^{1+\theta+\alpha_1} u_x^2 dx ds \\
&\quad + C \int_0^t \int_0^1 (x(1-x))^{2(1-\frac{1}{2n})} \rho^{1+2\beta_1-\theta-\alpha_1} dx ds.
\end{aligned} \qquad (5.54)$$

By noticing that

$$\int_0^1 (x(1-x))^{1-\frac{1}{2n}} \rho_0^{\beta_1} dx \le C \int_0^1 (x(1-x))^{1-\frac{1}{2n}} \rho_0^{2(\theta-1)} dx \le C$$

and

$$1 + 2\beta_1 - \theta - \alpha_1 = \beta_1,$$

we have from (3.24) and (5.54) that

$$\int_0^1 (x(1-x))^{1-\frac{1}{2n}}\rho^{\beta_1}dx \le C(T) + C\int_0^t\int_0^1 (x(1-x))^{1-\frac{1}{2n}}\rho^{\beta_1}dxds. \tag{5.55}$$

Equation (5.55) implies (3.25) by Gronwall's inequality.

Now we turn to prove (3.26). Let

$$\beta_2 = \theta + \left(1 - \frac{1}{2^{m+1}}\right)(\theta - 1). \tag{5.56}$$

Then for sufficiently large m, $\beta_2 < 0$. Now by using Sobolev's embedding theorem $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$ and Young's inequality, we have

$$\begin{aligned}
&(x(1-x))^{\frac{1}{2}(1-\frac{1}{2n})}\rho^{\beta_2}(x,t)\\
&\le C\int_0^1 (x(1-x))^{\frac{1}{2}(1-\frac{1}{2n})}\rho^{\beta_2}(x,t)dx + C\int_0^1 \left|\left((x(1-x))^{\frac{1}{2}(1-\frac{1}{2n})}\rho^{\beta_2}\right)_x\right|dx\\
&\le C\int_0^1 (x(1-x))^{1-\frac{1}{2n}}\rho^{2\beta_2}(x,t)dx + C\int_0^1 (x(1-x))^{-\frac{1}{2}(1+\frac{1}{2n})}\rho^{\beta_2}(x,t)dx\\
&\quad + C\int_0^1 (x(1-x))^{\frac{1}{2}(1-\frac{1}{2n})}\rho^{\beta_2-1}|\rho_x|dx\\
&\le C\max_{[0,1]}\rho^{2\beta_2-\beta_1}\int_0^1 (x(1-x))^{1-\frac{1}{2n}}\rho^{\beta_1}(x,t)dx\\
&\quad + C\int_0^1 (x(1-x))^{1-\frac{1}{2n}}\rho^{\beta_1}(x,t)dx\\
&\quad + C\int_0^1 (x(1-x))^{-\left[\frac{1}{2}(1+\frac{1}{2n})+(1-\frac{1}{2n})\frac{\beta_2}{\beta_1}\right]\frac{\beta_1}{\beta_1-\beta_2}}dx\\
&\quad + C\int_0^1 \rho^{2\theta-2}\rho_x^2dx + C\int_0^1 (x(1-x))^{1-\frac{1}{2n}}\rho^{2\beta_2-2\theta}dx\\
&\le C(T) + C\int_0^1 (x(1-x))^{-\left[\frac{1}{2}(1+\frac{1}{2n})+(1-\frac{1}{2n})\frac{\beta_2}{\beta_1}\right]\frac{\beta_1}{\beta_1-\beta_2}}dx.
\end{aligned} \tag{5.57}$$

When $\theta > \frac{1}{3}$, we have for sufficiently large n,

$$-\left[\frac{1}{2}\left(1+\frac{1}{2n}\right)+\left(1-\frac{1}{2n}\right)\frac{\beta_2}{\beta_1}\right]\frac{\beta_1}{\beta_1-\beta_2} > -1.$$

Therefore

$$(x(1-x))^{\frac{1}{2}(1-\frac{1}{2n})}\rho^{\beta_2}(x,t) \le C(T). \tag{5.58}$$

Since $2\theta - 1 < \beta_2 < 0$, (5.58) implies

$$\rho(x,t) \ge C(T)(x(1-x))^{-\frac{1}{2\beta_2}(1-\frac{1}{2n})} \ge C(T)(x(1-x))^{\frac{1}{2(1-2\theta)}}.$$

This proves (3.26).

Finally, we prove $(3.27)_n$. Similar to the proof of Lemma 3.5, we can get

$$\int_0^1 u_t^{2n}dx + n(2n-1)\int_0^t\int_0^1 \rho^{1+\theta}u_t^{2n-2}u_{xt}^2 dxds \\ \le 2n(2n-1)(\theta+1)^2 I_n^{(1)} + 2n(2n-1)\gamma^2 I_n^{(2)}. \tag{5.59}$$

Now we prove $(3.27)_n$. When $n=1$, we estimate $I_1^{(1)}$ and $I_1^{(2)}$ as follows: In fact

$$I_1^{(1)} = \int_0^t\int_0^1 \rho^{3+\theta}u_x^4 dxds \le \int_0^t \max_{[0,1]}\left(\rho^{2-\alpha_1}u_x^2\right)V_1(s)ds,$$

where

$$V_1(s) = \int_0^1 \rho^{1+\theta+\alpha_1}u_x^2(x,s)dx.$$

On the other hand, we have

$$\begin{aligned}\rho^{2-\alpha_1}u_x^2 &= \rho^{-\alpha_1-2\theta}\left(\rho^{1+\theta}u_x\right)^2 \\ &= \rho^{-\alpha_1-2\theta}\left(\int_0^x u_t(y,t)dy + \rho^\gamma\right)^2 \\ &\le C\rho^{-\alpha_1-2\theta}x(1-x)\int_0^1 u_t^2dx + C\rho^{2\gamma-\alpha_1-2\theta}.\end{aligned}$$

From (3.26), we have for $\frac{1}{3} < \theta < \frac{3}{7}$ and sufficiently large m,

$$x(1-x)\rho^{-\alpha_1-2\theta} \le C(T)(x(1-x))^{1-\frac{\alpha_1+2\theta}{2(1-2\theta)}} \le C(T),$$

which implies

$$\max_{[0,1]}\rho^{2-\alpha_1}u_x^2 \le C(T)\int_0^1 u_t^2dx + C.$$

Therefore,

$$I_1^{(1)} \le C(T)\int_0^t V_1(s)\int_0^1 u_t^2dxds + C\int_0^t V_1(s)ds. \tag{5.60}$$

Similarly, we have

$$I_1^{(2)} = \int_0^t\int_0^1 \rho^{2\gamma+1-\theta}u_x^2dxds \le C\int_0^t \max_{[0,1]}\rho^{2\gamma-\alpha_1-2\theta}V_1(s)ds \le C(T)\int_0^t V_1(s)ds. \tag{5.61}$$

From (5.59), (5.60) and (5.61), we have

$$\int_0^1 u_t^2dx + \int_0^t\int_0^1 \rho^{1+\theta}u_{xt}^2dxds \le C(T)\left(1+\int_0^t V_1(s)\int_0^1 u_t^2dxds\right). \tag{5.62}$$

Gronwall's inequality gives

$$\int_0^1 u_t^2dx \le C(T)\exp\left(C\int_0^t V_1(s)ds\right) \le C(T). \tag{5.63}$$

Combining (5.62) with (5.63), we can get $(3.27)_1$ immediately.

When $n > 1$, similar to (5.39) and (5.40), we have

$$I_n^{(1)} = \int_0^t \int_0^1 \rho^{3+\theta} u_x^4 u_t^{2n-2} dxds \le C(T) \left(1 + \int_0^t \int_0^1 u_t^{2n}(x,s) dxds\right), \quad (5.64)$$

and

$$I_n^{(2)} = \int_0^t \int_0^1 \rho^{2\gamma+1-\theta} u_x^2 u_t^{2n-2} dxds \le C(T) \left(1 + \int_0^t \int_0^1 u_t^{2n}(x,s) dxds\right). \quad (5.65)$$

From (5.59), (5.64) and (5.65), we have

$$\begin{aligned} &\int_0^1 u_t^{2n} dx + n(2n-1) \int_0^t \int_0^1 \rho^{1+\theta} u_t^{2n-2} u_{xt}^2 dxds \\ &\le C(T) \left(1 + \int_0^t \int_0^1 u_t^{2n}(x,s) dxds\right). \end{aligned}$$

Gronwall's inequality gives

$$\int_0^1 u_t^{2n} dx \le C(T).$$

This completes the proof of Lemma 3.7.

5.5. Proof of Lemma 3.8. Since

$$\begin{cases} \left(\rho^{1+\theta} u_x\right)(x,t) = \int_0^x u_t(y,t) dy + \rho^\gamma(x,t), \\ \left(\rho^{1+\theta} u_x\right)_x (x,t) = u_t(x,t) + (\rho^\gamma)_x (x,t), \end{cases} \quad (5.66)$$

(3.29) and (3.30) follow from Lemma 3.7.

On the other hand, we have

$$\int_0^1 |\rho_x(x,t)| dx \le \int_0^1 \rho^{2\theta-2} \rho_x^2 dx + \int_0^1 \rho^{2-2\theta} dx \le C(T). \quad (5.67)$$

This proves (3.28).

Now we prove (3.31). To do so, from $(2.1)_1$ and Hölder's inequality, we deduce

$$\begin{aligned} \int_0^1 |\rho(x,t) - \rho(x,s)|^2 dx &= \int_0^1 \left| \int_s^t \rho_t(x,\eta) d\eta \right|^2 dx \\ &= \int_0^1 \left| \int_s^t (\rho^2 u_x)(x,\eta) d\eta \right|^2 dx \\ &\le |t-s| \int_s^t \int_0^1 \left(\rho^4 u_x^2\right)(x,\eta) dxd\eta \\ &\le C(T)|t-s|. \end{aligned} \quad (5.68)$$

This is (3.31).

The proof of (3.32) is similar to (3.21), and thus it is omitted.

At last, we prove (3.33). For this, we first obtain from Hölder's inequality that

$$\begin{aligned}
&\int_0^1 \left|\left(\rho^{1+\theta}u_x\right)(x,t) - \left(\rho^{1+\theta}u_x\right)(x,s)\right|^2 dx \\
&\quad = \int_0^1 \left|\int_s^t \left(\rho^{1+\theta}u_x\right)_t (x,\eta)d\eta\right|^2 dx \\
&\quad \le |t-s| \int_s^t \int_0^1 \left[\left(\rho^{1+\theta}u_x\right)_t (x,\eta)\right]^2 dxd\eta. \qquad (5.69)
\end{aligned}$$

On the other hand, from (2.1), (3.2), we can get

$$\begin{aligned}
\left(\rho^{1+\theta}u_x\right)_t (x,t) &= \left(\rho^{1+\theta}u_{xt}\right)(x,t) + (1+\theta)\left(\rho^{\theta}\rho_t u_x\right)(x,t) \\
&= \left(\rho^{1+\theta}u_{xt}\right)(x,t) - (1+\theta)\left(\rho^{2+\theta}u_x^2\right)(x,t). \qquad (5.70)
\end{aligned}$$

From $(3.27)_n$, we have

$$\begin{aligned}
&\int_s^t \int_0^1 \left[\left(\rho^{1+\theta}u_x\right)_t (x,\eta)\right]^2 dxd\eta \\
&\quad \le \int_0^t \int_0^1 \rho^{2+2\theta}u_{xt}^2 dxds + C\int_0^t \int_0^1 \rho^{4+2\theta}u_x^4 dxds \\
&\quad \le C(T) + C\int_0^t \left\{V_1(s)\max_{[0,1]}\left(\rho^{3+\theta-\alpha_1}u_x^2\right)\right\} ds. \qquad (5.71)
\end{aligned}$$

On the other hand, we have from Lemma 3.7 and Hölder's inequality

$$\begin{aligned}
\rho^{3+\theta-\alpha_1}u_x^2 &= \rho^{1-\theta-\alpha_1}\left(\rho^{1+\theta}u_x\right)^2 \\
&= \rho^{1-\theta-\alpha_1}\left\{\int_0^x u_t(y,t)dy + \rho^{\gamma}\right\}^2 \\
&\le C(T). \qquad (5.72)
\end{aligned}$$

Therefore

$$\int_s^t \int_0^1 \left[\left(\rho^{1+\theta}u_x\right)_t (x,\eta)\right]^2 dxd\eta \le C(T)\left(1 + \int_0^t V_1(s)ds\right) \le C(T). \qquad (5.73)$$

This and (5.69) give (3.33) and then complete the proof of Lemma 3.8.

5.6. Construction of weak solutions. To construct a weak solution to the initial boundary value problem (2.1)–(2.3), we apply the line method as in [17], which can be described as follows. For any given positive integer N, let $h = \frac{1}{N}$. Discretizing the derivatives with respect to x in (2.1), we obtain the system of $2N$ ordinary differential equations

$$\begin{cases} \dfrac{d}{dt}\rho_{2n}^h(t) + \left(\rho_{2n}^h(t)\right)^2 \dfrac{u_{2n+1}^h(t) - u_{2n-1}^h(t)}{h} = 0, \\ \dfrac{d}{dt}u_{2n-1}^h(t) + \dfrac{P(\rho_{2n}^h(t)) - P(\rho_{2n-2}^h(t))}{h} = \dfrac{1}{h^2}\Big\{G(\rho_{2n}^h(t))(u_{2n+1}^h(t) - u_{2n-1}^h(t)) \\ - G(\rho_{2n-2}^h(t))(u_{2n-1}^h(t) - u_{2n-3}^h(t))\Big\}, \end{cases} \tag{5.74}$$

with the boundary conditions

$$\rho_0^h(t) = \rho_{2N}^h(t) = 0, \tag{5.75}$$

and initial data

$$\begin{cases} \rho_{2n}^h(0) = \rho_0\left(2n \cdot \dfrac{h}{2}\right), \\ u_{2n-1}^h(0) = u_0\left((2n-1) \cdot \dfrac{h}{2}\right), \end{cases} \tag{5.76}$$

where $n = 1, 2, \dots, N$, $G(\rho) = \mu(\rho)\rho$. And for $n = 1$ and N, we set $u_{-1}^h(t) = u_{2N+1}^h(t) = 0$.

In the following, we will use (ρ_{2n}, u_{2n-1}) to replace $(\rho_{2n}^h, u_{2n-1}^h)$ without any ambiguity.

By using the arguments in [20], we can prove the following lemmas for obtaining the uniform estimate of the approximate solutions to (5.74)–(5.76) with respect to h. Since they are the same as or similar to those in [20], we omit the proofs for brevity. Interested readers please refer to [20]. In the following, we consider the solutions to (5.74)–(5.76) for the case of compact support for $0 \le t \le T$ where $T > 0$ is any constant.

Lemma 5.1. *Let $(\rho_{2n}(t), u_{2n-1}(t))$, $n = 1, 2, \dots, N$, be the solution to (5.74)–(5.76). Then we have*

$$\sum_{n=1}^{N}\left(\frac{1}{2}u_{2n-1}^2(t) + \frac{1}{\gamma-1}\rho_{2n}^{\gamma-1}(t)\right)h + \int_0^t \sum_{n=1}^{N} G(\rho_{2n}(s))\left(\tfrac{u_{2n+1}(s)-u_{2n-1}(s)}{h}\right)^2 h ds = \sum_{n=1}^{N}\left(\frac{1}{2}u_{2n-1}^2(0) + \frac{1}{\gamma-1}\rho_{2n}^{\gamma-1}(0)\right)h. \tag{5.77}$$

As a consequence of (5.77), the problem (5.74)–(5.76) has a unique global solution for any given h.

Lemma 5.2. *There exist C and $C(T)$ independent of h such that*

$$\rho_{2n}(t) \leq C(T)(nh(1-nh))^{\alpha}, \tag{5.78}$$

and

$$\sum_{n=1}^{N} nh(1-nh)\left(\frac{\rho_{2n}^{\theta}(t)-\rho_{2n-2}^{\theta}(t)}{h}\right)^2 h \leq C(T). \tag{5.79}$$

Lemma 5.3. *For any positive integers k and m, we have*

$$\sum_{n=1}^{N}(nh(1-nh))^{k_1}\rho_{2n}^{\beta_1}(t)h \leq C(T), \tag{5.80}$$

$$\sum_{n=1}^{N} u_{2n-1}^{2k}(t)h + k(2k-1)\int_0^t \sum_{n=1}^{N} u_{2n-1}^{2k-2}(s)\rho_{2n}^{1+\theta}(s)\left(\frac{u_{2n-1}(s)-u_{2n-3}(s)}{h}\right)^2 hds \\ \leq C(T), \tag{5.81}$$

$$\sum_{n=1}^{N}(nh(1-nh))^{k_1}\rho_{2n}^{\alpha_1}(t)u_{2n-1}^2(t)h \\ + \int_0^t \sum_{n=1}^{N}(nh(1-nh))^{k_1}\rho_{2n}^{1+\theta+\alpha_1}(s)\left(\frac{u_{2n-1}(s)-u_{2n-3}(s)}{h}\right)^2 hds \\ \leq C(T), \tag{5.82}$$

and

$$\sum_{n=1}^{N}\left[\frac{d}{dt}u_{2n-1}(t)\right]^{2k} h + \int_0^t \sum_{n=1}^{N}\rho_{2n}^{1+\theta}(s)\left[\frac{d}{dt}u_{2n-1}(s)\right]^{2k-2} \\ \left(\frac{\frac{d}{dt}u_{2n-1}(s)-\frac{d}{dt}u_{2n-3}(s)}{h}\right)^2 hds \leq C(T), \tag{5.83}$$

where $\alpha_1 = \left(1-\frac{1}{2^m}\right)(\theta-1)$, $\beta_1 = \left(2-\frac{1}{2^m}\right)(1-\theta)$. Furthermore, we have

$$\rho_{2n}(t) \geq C(T)(nh(1-nh))^{\frac{k_2}{1-2\theta}}. \tag{5.84}$$

Based on Lemma 5.1, Lemma 5.2 and Lemma 5.3, similar to the arguments in [16] and those in the proof of Lemma 3.8, we can get the following estimates:

Lemma 5.4. *There exists $C(T)$ such that the following estimates hold:*

$$\sum_{n=1}^{N}|\rho_{2n}(t)-\rho_{2n-2}(t)| \leq C(T), \tag{5.85}$$

$$\sum_{n=1}^{N} |u_{2n+1}(t) - u_{2n-1}(t)| \le C(T), \tag{5.86}$$

$$|u_{2n+1}(t)| \le C(T), \tag{5.87}$$

$$\left| \rho_{2n}^{1+\theta}(t) \frac{u_{2n+1}(t) - u_{2n-1}(t)}{h} \right| \le C(T), \tag{5.88}$$

$$\sum_{n=1}^{N} \left| G(\rho_{2n+2}(t)) \frac{u_{2n+1}(t) - u_{2n-1}(t)}{h} - G(\rho_{2n}(t)) \frac{u_{2n-1}(t) - u_{2n-3}(t)}{h} \right| \le C(T), \tag{5.89}$$

$$\sum_{n=1}^{N} |\rho_{2n}(t) - \rho_{2n}(s)|^2 \, h \le C(T)|t - s|, \tag{5.90}$$

$$\sum_{n=1}^{N} |u_{2n-1}(t) - u_{2n-1}(s)|^2 \, h \le C(T)|t - s|, \tag{5.91}$$

and

$$\sum_{n=1}^{N} \left| G(\rho_{2n}(t)) \frac{u_{2n-1}(t) - u_{2n-3}(t)}{h} - G(\rho_{2n}(s)) \frac{u_{2n-1}(s) - u_{2n-3}(s)}{h} \right|^2 h \le C(T)|t - s|. \tag{5.92}$$

Now we can define the sequence of approximate solutions $(\rho_h(x, t), u_h(x, t))$ for $(x, t) \in [0, 1] \times [0, T]$ as follows:

$$\begin{cases} \rho_h(x, t) = \rho_{2n}(t), \\ u_h(x, t) = \dfrac{1}{h} \left\{ \left(x - \left(n - \dfrac{1}{2} \right) h \right) u_{2n+1}(t) + \left(\left(n + \dfrac{1}{2} \right) h - x \right) u_{2n-1}(t) \right\}, \end{cases} \tag{5.93}$$

for $\left(n - \frac{1}{2} \right) h < x < \left(n + \frac{1}{2} \right) h$. Then we have for $\left(n - \frac{1}{2} \right) h < x < \left(n + \frac{1}{2} \right) h$

$$\partial_x u_h(x, t) = \frac{1}{h} \left(u_{2n+1}(t) - u_{2n-1}(t) \right), \tag{5.94}$$

and

$$\begin{cases} C(T)(x(1-x))^{\frac{k_2}{1-2\theta}} \leq \rho_h(x,t) \leq C(T)(x(1-x))^{\alpha}, \\ |u_h(x,t)| \leq C(T), \quad \int_0^1 |\partial_x u_h(x,t)|\, dx \leq C(T), \\ |G(\rho_h(x,t))\partial_x u_h(x,t)| \leq C(T), \\ \int_0^1 |\partial_x \left(G(\rho_h(x,t))\partial_x u_h(x,t)\right)|\, dx \leq C(T). \end{cases} \tag{5.95}$$

By using Helly's theorem and arguments in one of the references [13, 14, 18–20, 24], we complete the proof of Theorem 2.2.

Similarly, we can construct a weak solution to the initial boundary value problem (2.1)–(2.3) for the case of infinite support and complete the proof of Theorem 2.4.

Remark 5.5. The lower bound of the density function $\rho(x,t)$ obtained above is not optimal. In order to obtain the detailed description of the evolution of the interface separating the vacuum and gas, optimal decay rate of the density function is desired. However, such a decay rate estimate has not been obtained even for the case when the density function connects to vacuum with discontinuities.

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